On the Optimal Control of Parallel Systems and Queues

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Abstract

A continously operating system consisting of N K_i-out-of-N_i subsystems connected in parallel is considered. The components of all subsystems are assumed identical with life times independent exponentially distributed random variables and the system is maintained by a single repaiman. Repair times are also assumed identical independent exponentials. We are interested in characterizing the allocation policy of the repairman which maximizes the system reliability at any time instant t (if any). In the present paper, we give a partial characterization of the optimal policy for systems consisting of highly reliable components using dynamic programming techniques. We also compute the leading term of a power series expansion of the reliability of the system at an arbitrary time instant t under the optimal policy. Finally, these results are extended to the problem of controlling the corresponding network of parallel queues in a scheduling problem with long mean arrival times and in its dual routing problem with long mean processing times.

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1 Introduction

Consider a system consisting of N subsystems which functions when at least 1 out of its N component-subsystems are operational. Subsystem i consists of N_i identical components and functions when at least K_i of them operate. This structure will be denoted by $[1|N;(K_i|N_i)_{i=1,\ldots,N}]$. The status of all subsystems is given by a state vector $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_N(t))$ with $x_i(t)$ denoting the number of functioning components of subsystem i at the time instant t, $0 \leq x_i(t) \leq N_i$ for $i = 1, \ldots, N$. Components of the subsystems may fail. Their lifetimes are exponentially distributed, independent random variables. The rate of failure is common for all components of all subsystems and is denoted by μ. Failures may occur even when the system is not functioning. The system is maintained by a single repairman who may be assigned to any failed component. The repairman may switch from one failed component to another instantaneously, and the time it takes him to complete the repair of any failed component is considered to be an exponentially distributed random variable with parameter λ. Repair times are assumed independent among themselves and with component lifetimes. Repaired components are as good as new. We assume that preemptions are allowed.

Obviously, a change in the state vector $\mathbf{x}$ means either that a previously functioning component became non-functioning or that the non-functioning component currently under repair became functioning. Hence, if we assume that the pepairman is currently assigned to subsystem $a(\mathbf{x})$, $x_{a(\mathbf{x})} < N_{a(\mathbf{x})}$, the possible transitions from state $\mathbf{x}$ are either to a state $0_j\mathbf{x}$ for j such that $x_j > 0$ or to $1_{a(\mathbf{x})}\mathbf{x}$ The components of these states are given by

$$(0_j\mathbf{x})_i := \begin{cases} x_i - 1 & \text{if} \quad i = j \\ x_i & \text{if} \quad i \neq j \end{cases}$$

$$(1_{a(\mathbf{x})}\mathbf{x})_i := \begin{cases} x_i + 1 & \text{if} \quad i = a(\mathbf{x}) \\ x_i & \text{if} \quad i \neq a(\mathbf{x}) \end{cases}$$

The repairman is assigned to a non-functioning component whenever a state transition occurs. The problem is to construct a dynamic repair allocation policy that optimizes some performance criterion for the system (if such a policy exists). There is a close relationship of such reliability problems to problems of assigning arriving jobs to processors (each processor corresponding to a queue, possibly with finite capacity) so that some total performance criterion is optimized. For an introduction to this area see Walrand (1988).

In this paper, we examine the problem of the maximization of the reliability of the sustem under consideration at any time instant t when the system components are highly reliable (i.e. μ is sufficiently small). We give a partial characterization of any optimal policy that depends on the state and structure of the system only (i.e. $\mathbf{x}$, K_i, and N_i, $i = 1, \ldots . N$ and not on the failure and repair rates), which also enables us to compute the leading term of the value (the optimal reliability) in a series expansion in terms of the failure rate μ. It is very unlikely that a complete characterization of the optimal policy may be given except in particular cases (for two such interesting cases we do it). However, if the discounted system operation time is considered as the optimality criterion, then a complete and easy to compute, although quite complicated, index type optimal policy exists for all discount rates sufficiently small or sufficiently large (Dinopoulou and Melolidakis, 1998). That policy is a refinement of the present policy and is unique. One then may show that if a complete characterization of the optimal policy exists under the reliability criterion, then it will coincide with the optimal policy under the discounted operation time criterion.

Katehakis and Melolidakis (1994) examined this problem under the additional assumption that all subsystems are identical (i.e. N_i and K_i are constant, $i = 1, \ldots, N$) and showed that for arbitrary μ the "inequalizing" policy (the policy that tends to maximize the number of functioning components of the subsystem that is "more" operational) is optimal. The policy we construct here is a generalization of the inequalizing policy in the sense that it coincides with the inequalizing policy when all subsystems are identical. The approach used to deal with the problem takes the following steps. First the continous-time problem is discretized via uniformization (see Lippman S. A. (1975), Ross S. M. (1983)) and consideration of the embedded Markov Decision Process. This approach has been used in a similar context in Katehakis and Melolidakis (1988) and we will not repeat it in this paper. Then, we use Dynamic Programming techniques to compare different stationary policies.

D. R. Smith (1978) used a power series expansion technique to obtain the policy that maximizes the long run probability that a K-out-of-N system consisting of single non-identical components functions, when the failure rates or the repair rates of the components belong in a neighborhood of 0. M. N. Katehakis and C. Derman (1989) formulated this problem along the lines of Markov Decision Theory and obtained the discounted operation time optimal policy.

There is extensive literature concerning dynamic repair allocation policies in reliability systems. It seems however that most authors prefer coupling and sample path arguments in dealing with them as well as with the corresponding queueing network problems. We are of the opinion that dynamic programming is a much more powerfull approach and framework in dealing with asymptotic problems where bounds and estimates are involved. Also, when the optimal policy is really complicated, as for example is the case in the present problem under the discounted system operation time criterion, it is very doubtful whether sample path arguments could work. For a recent review binding together related work in this area see R. Righter (1996).

The repair allocation model we examine has the following interpretation in queueing theory and scheduling. Take N parallel queues, the buffer (capacity) of queue i being finite and equal to N_i, $i = 1, \dots, N$. To each queue a single processor is assigned which serves all customers (jobs) in the queue simultaneously. Jobs arrive according to a Poisson process with rate λ and the processing time of any job is exponentially distributed with rate μ. The problem is to which queue to assign a newly arrived job so as to optimize various performance criteria. Assumptions similar to those of the maintenance problem are made (independence of processing and interarrival times, customers that find all queues full are lost, etc). Dual to the routing problem described above is the following scheduling problem. N queues with capacities N_i, $i = 1, \dots, N$, are all served by a single processor which may process one job at a time. The time it takes for each empty position in the buffer of any queue to be occupied is exponentially distributed with rate μ and the processing time of any job is exponentially distributed with rate λ. We control the assignment of the processor to a queue, preemptions are allowed, and the usual independence assumptions for interarrival and processing times are made. Problems similar to the above have been studied extensively under various performance criteria and generalized in various directions. For related work (the queueing and routing systems there are not identical to our model) see A. Hordijk and E. Koole (1990, 1992), D. Towsley, P. D. Sparaggis and Chr. Cassandras (1992, 1993).

In the present paper we introduce control levels K_i for each queue i with $1 \leq K_i \leq N_i$, $i = 1, \dots, N$. Then, we provide a partial characterization of any policy which at any time instant t (a) For the routing problem and for jobs with long mean processing time (i.e. sufficiently small μ) maximizes (minimizes) the probability that all queues have size less than their control level (b) For the scheduling problem and for long mean arrival times maximizes (minimizes) the probability that at least one queue has a number of empty positions greater or equal to its control level.

2 The Dynamic Programming formulation of the problem

Let $I(\mathbf{x})$ denote the minimum number of components that must become non-functioning, when the system is at state $\mathbf{x}$, in order for the system to become non-functioning (e.g. if at $\mathbf{x}$ the system is non-functioning, then

$I(\mathbf{x}) = 0$). Hence

$$I(\mathbf{x}) = \sum_{x_\ell \geq K_\ell} (x_\ell - K_\ell + 1) \tag{1}$$

Let $c(\mathbf{x}(t))$ be a random process taking values 1 or 0 according to whether the system is functioning or not at $\mathbf{x}(t)$. The distribution of this process depends on the repair allocation policy π followed. To show that a policy π^* maximizes the reliability of the system at any time instant t, we have to show that $P_{\pi^*}(c(\mathbf{x}(t)) = 1) \geq P_\pi(c(\mathbf{x}(t)) = 1)$ for all policies π.

Now, consider the family P_M, $M = 0, 1, 2, \ldots$ of discrete finite horizon D. P. problems with optimality equations given by

$$\begin{aligned} v(\mathbf{x}; m) = & \min_{\pi(\mathbf{x})} \Bigg[(1 - \lambda - \mu \sum_\ell x_\ell) v(\mathbf{x}; m-1) + \lambda v(1_{\pi(\mathbf{x})}\mathbf{x}; m-1) \\ & + \mu \sum_\ell x_\ell v(0_\ell \mathbf{x}; m-1) \Bigg], \quad m = 1, \ldots, M \\ v(\mathbf{x}; 0) = & \begin{cases} 0 & \text{if} \quad I(\mathbf{x}) \geq 1 \\ 1 & \text{if} \quad I(\mathbf{x}) = 0 \end{cases} \end{aligned} \tag{2}$$

Let $w_\pi(\mathbf{x}; m)$ denote the value function of a policy π; it is the solution to the following recursive scheme

$$\begin{aligned} w_\pi(\mathbf{x}; m) = & \left(1 - \lambda - \mu \sum_\ell x_\ell\right) w_\pi(\mathbf{x}; m-1) + \lambda w_\pi(1_{\pi(\mathbf{x})}\mathbf{x}; m-1) \\ & + \mu \sum_\ell x_\ell w_\pi(0_\ell \mathbf{x}; m-1), \quad m = 1, \ldots, M \\ w_\pi(\mathbf{x}; 0) = & \begin{cases} 0 & \text{if} \quad I(\mathbf{x}) \geq 1 \\ 1 & \text{if} \quad I(\mathbf{x}) = 0 \end{cases} \end{aligned} \tag{3}$$

Using uniformization and a time rescaling so that the uniform rate is 1, one gets the following equation

$$P_\pi(c(\mathbf{x}(t)) = 0) = \sum_{M=0}^{\infty} w_\pi(\mathbf{x}; M) e^{-t} \frac{t^M}{M!} \tag{4}$$

Hence, to show that a policy π^* is optimal in the original problem, it is sufficient (but not necessary) for its discrete version to be optimal in the family $P_M, M = 0, 1, 2, \ldots$. We call this notion of optimality for π^* "strong optimality".

A necessary and sufficient condition for a stationary policy $\pi(\mathbf{x})$ to be strongly optimal is the following

$$w_\pi(1_{\pi(\mathbf{x})}\mathbf{x}; m) \leq w_\pi(1_i \mathbf{x}; m), \quad \forall i \in A(\mathbf{x}) - \{\pi(\mathbf{x})\}, \quad \forall m \geq 0 \tag{5}$$

The recursive scheme of Equation 3 shows that $w_\pi(\mathbf{x};m)$ is a polynomial in μ. We will be interested in the behavior of this polynomial in a neighborhood of 0 and hence in its leading coefficients. So, we will write

$$w_\pi(\mathbf{x};m) = \sum_{\nu=0}^{\infty} w_\pi^{(\nu)}(\mathbf{x};m)\mu^\nu \tag{6}$$

where of course the coefficients $w_\pi^{(\nu)}(\mathbf{x};m)$ vanish for $\nu > \nu_0(\mathbf{x};m)$ for some $\nu_0(\mathbf{x};m)$.

Then, for any state $\mathbf{x}$ at which a decision has to be made, substituting (6) into (3) will result to the following recurcive equations for the polynomial coefficients.

$$\begin{aligned}
w_\pi^{(\nu+1)}(\mathbf{x};m+1) &= (1-\lambda)w_\pi^{(\nu+1)}(\mathbf{x};m) + \lambda w_\pi^{(\nu+1)}(1_{\pi(\mathbf{x})}\mathbf{x};m) \\
&\quad + \sum_{\ell=1}^{N} x_\ell \left(w_\pi^{(\nu)}(0_\ell\mathbf{x};m) - w_\pi^{(\nu)}(\mathbf{x};m) \right) \\
w_\pi^{(0)}(\mathbf{x};m+1) &= (1-\lambda)w_\pi^{(0)}(\mathbf{x};m) + \lambda w_\pi^{(0)}(1_{\pi(\mathbf{x})}\mathbf{x};m) \\
w_\pi^{(\nu)}(\mathbf{x};0) &= \begin{cases} 0 & \text{if } I(\mathbf{x}) \geq 1 \\ 1 & \text{if } I(\mathbf{x}) = 0 \text{ and } \nu = 0 \\ 0 & \text{if } I(\mathbf{x}) = 0 \text{ and } \nu > 0 \end{cases}
\end{aligned} \tag{7}$$

For μ sufficiently close to 0, the optimal repair allocation policy will be determined by the leading (i.e. the first in increasing order of ν non-zero) coefficient of this power series expansion, which, as we shall see, is acquired at $\nu = I(\mathbf{x})$. Hence, (5) implies that to show that assigning the repairman to $\pi(\mathbf{x})$ is better than assigning him to i, it is sufficient to show that for all m

$$w_\pi^{(\nu^*)}(1_{\pi(\mathbf{x})}\mathbf{x};m) \leq w_\pi^{(\nu^*)}(1_i\mathbf{x};m)$$

where $\nu^* = \min(I(1_{\pi(\mathbf{x})}\mathbf{x}), I(1_i\mathbf{x}))$. In case the above relationship is an equality, we will have to compare the next to the leading coefficients, etc.

To be able to describe the optimal policy, we will need some definitions and notations. Let

$P_1(\mathbf{x}) := \{i : x_i = K_i - 1\}$,
$P_2(\mathbf{x}) := \{i : K_i - 1 < x_i < N_i\}$,
$P(\mathbf{x}) := P_1(\mathbf{x}) \cup P_2(\mathbf{x})$,
$Q(\mathbf{x}) := \{i : x_i < K_i - 1\}$,
$R(\mathbf{x}) := \{i : x_i = N_i\}$,
$A(\mathbf{x}) := P(\mathbf{x}) \cup Q(\mathbf{x})$.

For $i \in Q(\mathbf{x}) \cup P_1(\mathbf{x})$, let $n_i(\mathbf{x}) := K_i - x_i$ and $n(\mathbf{x}) := \min_{i \in Q(\mathbf{x}) \cup P_1(\mathbf{x})} \{n_i(\mathbf{x})\}$. Let also

$$s(\mathbf{x}) := \begin{cases} 0 & \text{if } P(\mathbf{x}) \neq \emptyset \\ n(\mathbf{x}) - 1 & \text{if } P(\mathbf{x}) = \emptyset \end{cases}$$

and for $m \geq I(\mathbf{x})$ let $\alpha(\mathbf{x}; m) := min\,[s(\mathbf{x}), m - I(\mathbf{x})]$ Whenever ambiguities may not arise, we will write just n instead of $n(\mathbf{x})$. Notice that if all subsystems operate at full capacity (i.e., $i \in R(\mathbf{x})$, $i = 1, \ldots, N$), then $s(\mathbf{x}) = +\infty$ and $\alpha(\mathbf{x}; m) = m - I(\mathbf{x})$.

For $i \in A(\mathbf{x})$, let $\delta_i(\mathbf{x}) := \begin{cases} 0 & \text{if} \quad i \in P(\mathbf{x}) \\ 1 & \text{if} \quad i \in Q(\mathbf{x}) \end{cases}$, $\ell_i(\mathbf{x}) = \ell_i(\mathbf{x}_i) := \begin{cases} \frac{x_i+1}{x_i-K_i+2} & \text{if} \quad i \in P(\mathbf{x}) \\ n_i(\mathbf{x}) & \text{if} \quad i \in Q(\mathbf{x}) \end{cases}$.

Finally, define

$$A_1(\mathbf{x}) := \left\{ i : \delta_i(\mathbf{x}) = \min_{j \in A(\mathbf{x})} \delta_j(\mathbf{x}) \right\} = \begin{cases} P(\mathbf{x}) & \text{if} \quad P(\mathbf{x}) \neq \emptyset \\ Q(\mathbf{x}) & \text{if} \quad P(\mathbf{x}) = \emptyset \end{cases}$$

$$A_2(\mathbf{x}) := \left\{ i : i \in A_1(\mathbf{x}) \text{ and } \ell_i(\mathbf{x}) = \min_{j \in A_1(\mathbf{x})} \ell_j(\mathbf{x}) \right\}$$

So, $A(\mathbf{x}) \supset A_1(\mathbf{x}) \supset A_2(\mathbf{x})$. Intuitevely, $P(\mathbf{x})$ is the set of all subsystems that are either functioning although they have non-functioning components or they are non-functioning but would be functioning after the repair of one more component. $Q(\mathbf{x})$ is the set of non-functioning subsystems which need at least two components repaired in order to start functioning. $R(\mathbf{x})$ is the set of subsystems with all of their components functioning. $A(\mathbf{x})$ is the set of available actions at state $\mathbf{x}$, i.e. at state $\mathbf{x}$ we must assign the repairman to a member of $A(\mathbf{x})$. A decision must be made if $A(\mathbf{x}) \neq \emptyset$ and in this case $A_1(\mathbf{x}) \neq \emptyset$ always. Notice also that $A_1(\mathbf{x})$ and $A_2(\mathbf{x})$ depend on the structure of the system only.

In what follows, we show that there exists a neighborhood of 0, $\mathcal{N}(0)$, such that for $\mu \in \mathcal{N}(0)$ the first two steps of any optimal policy are described as follows: When at state $\mathbf{x}$, the optimal choice $i_0 \in A(\mathbf{x})$ will be decided by

- **Step 1:** Restrict your attention to the subset $A_1(\mathbf{x})$ of $A(\mathbf{x})$. If this is a unique member set, then assign the repairman to that subsystem; otherwise

- **Step 2:** Restrict your attention to the subset $A_2(\mathbf{x})$ of $A_1(\mathbf{x})$. If this is a unique member set, then assign the repairman to that subsystem.

If there are two or more subsystems in $A_2(\mathbf{x})$, then we don't know which action is optimal. However, a complete characterization of the optimal policy is known in two interesting special cases for μ sufficiently small. So, let $T := \{i : K_i = 1\}$, $T(\mathbf{x}) := T \cap P(\mathbf{x})$. Then, we have: Case I: If $T(\mathbf{x}) \neq \emptyset$, then $A_2(\mathbf{x}) = T(\mathbf{x})$ and assigning the repairman to any one among the subsystems which belong in $A_2(\mathbf{x})$ is optimal. Case II: If the subsystems which belong in $A_2(\mathbf{x})$ have the same $K > 1$, then it is optimal to assign the repairman to $i_0 \in A_2(\mathbf{x})$ such that $N_{i_0} = \max_{i\in A_2(\mathbf{x})} N_i$ The proof of the optimality of the pertinent policy beyond Step 1 and Step 2 in these two cases is rather tedious (esp. for Case II) and will be omitted (for the proof see Dinopoulou and Melolidakis (1998b)).

Throughout this article we follow the usual conventions (i) $\binom{n}{k} = 0$ if $n < k$, (ii) $\sum_{i\in\emptyset} a_i = 0$ (in particular $\sum_{i=i_1}^{i_2} a_i = 0$ if $i_1 > i_2$), (iii) $\prod_{i\in\emptyset} a_i = 1$, (iv) $\min\emptyset = +\infty$

3 Establishing Steps 1 and 2

Proposition 1 *For all policies π and for $I(\mathbf{x}) > 0,\ 0 \le \nu < I(\mathbf{x})$,*

$$w_\pi^{(\nu)}(\mathbf{x}; m) = 0, \quad m = 0, 1, \ldots \tag{8}$$

Proof: The proof is by double induction on ν and m using relations (7). Let $P(\nu, m)$ denote relation (8). Then, one may easily check that, (i) $P(0,0)$ is true (from (7)), (ii) If $P(0,m)$ is true, then $P(0,m+1)$ is true, (iii) If $P(\nu,0)$ is true, then $P(\nu+1,0)$ is true (from (7)), (iv) If $P(\nu,m)$ and $P(\nu+1,m)$ are true, then $P(\nu+1,m+1)$ is true. ∎

Therefore, the first possible non zero coefficient in the polynomial expansion of $w_{\pi^*}(\mathbf{x}; m)$ is the $I(\mathbf{x})$-th coefficient. To examine if a policy π^* is optimal one then compares the $I(1_i\mathbf{x})$ terms between $w_{\pi^*}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m)$ and $w_{\pi^*}(1_i\mathbf{x}; m)$, $i \in A(\mathbf{x})$, $i \ne \pi^*(\mathbf{x})$. If these terms are equal, then one compares the $I(1_i\mathbf{x}) + 1$-th term, e.t.c.

Proposition 2 *For all policies π and for $I(\mathbf{x}) > 0,\ 0 \le m < I(\mathbf{x}) \le \nu$,*

$$w_\pi^{(\nu)}(\mathbf{x}; m) = 0, \quad \nu = I(\mathbf{x}), I(\mathbf{x}) + 1, \ldots \tag{9}$$

Proof: The proof is again by double induction on ν and m using relations (7) and Proposition 1. Let $P(\nu, m)$ denote relation (9) Then, one may easily check that, (i) $P(I(\mathbf{x}),0)$ is true (from (7)), (ii) If $P(I(\mathbf{x}),m)$ is true, then $P(I(\mathbf{x}),m+1)$ is true, (iii) If $P(\nu,0)$ is true, then $P(\nu+1,0)$ is true (from (7)), (iv) If $P(\nu,m)$ and $P(\nu+1,m)$ are true, then $P(\nu+1,m+1)$ is true. ∎

In the rest of this paper the product $\prod_{x_i\ge K_i}\binom{x_i}{K_i-1}$ appears quite often. For notational simplicity we let $\varpi(\mathbf{x}) := \prod_{x_i\ge K_i}\binom{x_i}{K_i-1}$. We observe that

$$\varpi(1_i\mathbf{x}) = \varpi(\mathbf{x})\ell_i(\mathbf{x}) \quad \text{if } i \in P(\mathbf{x}) \tag{10}$$

The next Lemma provides the leading coefficient of $w_{\pi^*}(\mathbf{x}; m)$ (it is actually the leading coefficient of any policy satisfying steps 1 and 2 of the previous section).

Lemma 1 *For $m \geq I(\mathbf{x})$,*

$$w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x}; m) = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}} \varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\alpha(\mathbf{x};m)} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \tag{11}$$

Proof: We consider the following cases:

Case 1.1: $P(\mathbf{x}) \neq \emptyset$. Then (11) is true and the proof is by double induction on $I(\mathbf{x})$ and m. Let $P(I(\mathbf{x}), m)$ denote relation (11).
(a) $P(0,0)$ is true, since $w_{\pi^*}^{(0)}(\mathbf{x}; 0) = 1$ (boundary condition (7). Notice that $n(\mathbf{x}) = 1$).
(b) if $P(0,m)$ is true, then $P(0, m+1)$ is true, since from (7), Proposition 1 and $P(0,m)$

$$\begin{aligned} w_{\pi^*}^{(0)}(\mathbf{x}; m+1) &= (1-\lambda) w_{\pi^*}^{(0)}(\mathbf{x}; m) + \lambda w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) \\ &= (1-\lambda) \sum_{z=0}^{0} \binom{m}{z} (1-\lambda)^{m-z} \lambda^z = (1-\lambda)^{m+1} \end{aligned}$$

(c) If $P(I(\mathbf{x}) = \nu, m)$ is true for $m = \nu$, then $P(I(\mathbf{x}) = \nu + 1, m+1)$ is true, since by relations (7) and (1), Propositions 1 and 2, and $P(I(\mathbf{x}) = \nu, m)$

$$w_{\pi^*}^{(\nu+1)}(\mathbf{x}; m+1 = \nu+1) = \sum_{x_\ell \geq K_\ell} x_\ell w_{\pi^*}^{(\nu)}(0_\ell \mathbf{x}; m = \nu) = (\nu+1)! \varpi(\mathbf{x})$$

(d) If $P(I(\mathbf{x}) = \nu, m)$ and $P(I(\mathbf{x}) = \nu+1, m)$ are true, then $P(I(\mathbf{x}) = \nu + 1, m+1)$ is true, since by relation (7), Propositions 1 and 2, $P(I(\mathbf{x}) = \nu, m)$ and $P(I(\mathbf{x}) = \nu+1, m)$

$$\begin{aligned} w_{\pi^*}^{(\nu+1)}(\mathbf{x}; m+1) &= (1-\lambda) w_{\pi^*}^{(\nu+1)}(\mathbf{x}; m) + \sum_{x_\ell \geq K_\ell} x_\ell w_{\pi^*}^{(\nu)}(0_\ell \mathbf{x}; m) \\ &= (\nu+1)!(1-\lambda)^{m-\nu} \binom{m+1}{\nu+1} \varpi(\mathbf{x}) \end{aligned}$$

The computational details are omitted.

Case 1.2: $P(\mathbf{x}) = Q(\mathbf{x}) = \emptyset$. Then, all subsystems belong in $R(\mathbf{x})$. Hence,

$$w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x}; m+1) = w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x}; m) + \sum_{\ell=1}^{N} x_\ell \left(w_{\pi^*}^{(I(\mathbf{x})-1)}(0_\ell \mathbf{x}; m) - w_{\pi^*}^{(I(\mathbf{x})-1)}(\mathbf{x}; m) \right)$$

Induction on m will then prove (11). $w_{\pi^*}^{(I(\mathbf{x})-1)}(\mathbf{x}; m) = 0$ (by Proposition 1), $w_{\pi^*}^{(I(\mathbf{x})-1)}(0_\ell \mathbf{x}; m)$ is given by case 1.1, and $w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x}; m)$ is given by induction on m. The details are algebraic operations which are omitted.

Case 2: $P(\mathbf{x}) = \emptyset$ and $Q(\mathbf{x}) \neq \emptyset$.

Case 2.1: $I(\mathbf{x}) = 0$. Then the proof is by induction on m.

(a) $P(0,0)$ is true, since $w_{\pi^*}^{(0)}(\mathbf{x};0) = 1$ (boundary condition 7. Notice that $n(\mathbf{x}) > 1$).

(b) if $P(0,m)$ is true, then $P(0,m+1)$ is true. To see this notice that from (7)

$$w_{\pi^*}^{(0)}(\mathbf{x};m+1) = (1-\lambda)w_{\pi^*}^{(0)}(\mathbf{x};m) + \lambda w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m)$$

$w_{\pi^*}^{(0)}(\mathbf{x};m)$ is given by $P(0,m)$. If $n(\mathbf{x}) = 2$ then $w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m)$ is given by case 1.1, otherwise it is given by $P(0,m)$. In both cases we have

(i) if $m+1 < n$, then $w_{\pi^*}^{(0)}(\mathbf{x};m+1) = (1-\lambda)w_{\pi^*}^{(0)}(\mathbf{x};m)+\lambda w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = (1-\lambda)+\lambda = 1$ (notice that $\sum_{z=0}^{m}\binom{m}{z}(1-\lambda)^{m-z}\lambda^z = 1$).

(ii) if $m+1 \geq n$, then $w_{\pi^*}^{(0)}(\mathbf{x};m+1) = (1-\lambda)\sum_{z=0}^{n-1}\binom{m}{z}(1-\lambda)^{m-z}\lambda^z + \lambda\sum_{z=0}^{n-2}\binom{m}{z}(1-\lambda)^{m-z}\lambda^z$ Simple algebra then gives

$$w_{\pi^*}^{(0)}(\mathbf{x};m+1) = \sum_{z=0}^{n-1}\binom{m+1}{z}(1-\lambda)^{m-z+1}\lambda^z$$

Therefore, from (i) and (ii), $P(0,m+1)$ is true.

Case 2.2: $I(\mathbf{x}) > 0$.

(a) Since $I(\mathbf{x}) > 0$, this means $R(\mathbf{x}) \neq \emptyset$. Propositions 1 and 2 applied to (7) show that

$$w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m=I(\mathbf{x})) = \sum_{\ell\in R(\mathbf{x})} x_\ell w_{\pi^*}^{(I(\mathbf{x})-1)}(0_\ell\mathbf{x};I(\mathbf{x})-1)$$

Equation (11) for Case 1.1 and for $\ell \in R(\mathbf{x})$ gives

$$w_{\pi^*}^{(I(\mathbf{x})-1)}(0_\ell\mathbf{x};I(\mathbf{x})-1) = (I(\mathbf{x})-1)!\varpi(0_\ell\mathbf{x})$$

Hence

$$w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m=I(\mathbf{x})) = \sum_{\ell\in R(\mathbf{x})}(x_\ell - K_\ell + 1)(I(\mathbf{x})-1)!\varpi(\mathbf{x}) = I(\mathbf{x})!\varpi(\mathbf{x})$$

which establishes $P(I(\mathbf{x}), m = I(\mathbf{x}))$.

(b) If $P(I(\mathbf{x}),m)$ is true, then $P(I(\mathbf{x}),m+1)$ is true. By eq.(7) we have

$$\begin{aligned} w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m+1) = \; & (1-\lambda)w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m) + \lambda w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) \\ & + \sum_{\ell=1}^{N} x_\ell\left(w_{\pi^*}^{(I(\mathbf{x})-1)}(0_\ell\mathbf{x};m) - w_{\pi^*}^{(I(\mathbf{x})-1)}(\mathbf{x};m)\right) \end{aligned}$$

(i) If $n \leq m - I(\mathbf{x}) + 1$, then $w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m)$ is given by $P(I(\mathbf{x}),m)$ with $\alpha(\mathbf{x};m) = \min(n-1, m-I(\mathbf{x})) = n-1$. If $n(\mathbf{x}) = 2$, then $w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x};m)$

is given by eq. (11) (Case 1.1) with $\alpha(\mathbf{x};m) = 0 = n-2$ and if $n(\mathbf{x}) > 2$ then it is given by $P(I(\mathbf{x}), m)$ with $I(1_{\pi^*(\mathbf{x})}\mathbf{x}) = I(\mathbf{x})$ and $\alpha(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = \min(n(1_{\pi^*(\mathbf{x})}\mathbf{x})-1, m-I(\mathbf{x})) = n-2$. If $\ell : x_\ell < K_\ell - 1$, then $w_{\pi^*}^{(I(\mathbf{x})-1)}(0_\ell\mathbf{x};m)$ is 0 (Proposition 1), else it is given by eq. (11) (Case 1.1). Finally, $w_{\pi^*}^{(I(\mathbf{x})-1)}(\mathbf{x};m)$ is 0 (Proposition 1). Hence,

$$\begin{aligned} w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m+1) = \; & (1-\lambda)\frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n-1} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z \\ & +\lambda\frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n-2} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z \\ & + \sum_{\ell\in R(\mathbf{x})} x_\ell \frac{(I(\mathbf{x})-1)!}{\lambda^{I(\mathbf{x})-1}}\varpi(0_\ell\mathbf{x}) \sum_{z=I(\mathbf{x})-1}^{I(\mathbf{x})-1} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z \end{aligned}$$

Then it is a matter of operations to check that

$$w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m+1) = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n-1} \binom{m+1}{z}(1-\lambda)^{m-z+1}\lambda^z$$

(ii) if $n > m - I(\mathbf{x}) + 1$, then similar substitutions in (7) give

$$\begin{aligned} w_{\pi^*}^{(I(\mathbf{x}))}(\mathbf{x};m+1) = \; & (1-\lambda)\frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{m} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z \\ & +\lambda\frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{m} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z \\ & + \sum_{\ell\in R(\mathbf{x})} x_\ell \frac{(I(\mathbf{x})-1)!}{\lambda^{I(\mathbf{x})-1}}\varpi(0_\ell\mathbf{x}) \sum_{z=I(\mathbf{x})-1}^{I(\mathbf{x})-1} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z \\ & = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{m} \binom{m+1}{z}(1-\lambda)^{m-z+1}\lambda^z \end{aligned}$$

So, $P(I(\mathbf{x}), m+1)$ is true. ∎

Proposition 3 *For any state* $\mathbf{x}$ *such that* $P(\mathbf{x}) \neq \emptyset$, *the repairman should be allocated to a subsystem belonging to* $P(\mathbf{x})$.

Proof: The proof is by comparing the $I(\mathbf{x})$-th term of the power series expansion of $w_{\pi^*}(1_{\pi^*(\mathbf{x})}\mathbf{x};m)$ and $w_{\pi^*}(1_i\mathbf{x};m)$, $i \in A(\mathbf{x})$ and $i \neq \pi^*(\mathbf{x})$. So, we compare the policy which at state $\mathbf{x}$ assigns the repairman to a subsystem which belongs in $P(\mathbf{x})$ with any policy which at state $\mathbf{x}$ assigns the repairman to a subsystem in $Q(\mathbf{x})$. Then $w_{\pi^*}^{(\nu)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = 0 \; \forall \nu \leq I(\mathbf{x})$ and $\forall m \geq 0$

(Proposition 1), while $w_{\pi^*}^{(\nu)}(1_i\mathbf{x};m) = 0\ \forall\nu < I(\mathbf{x})$ and $\forall m \geq 0$ (Proposition 1) and $w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x};m)$ is strictly positive (Lemma 1) $\forall m \geq I(\mathbf{x})$. Therefore $w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) < w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x};m)\ \forall m \geq I(\mathbf{x})$. ■

Proposition 3 shows that the first step of π^* is optimal, i.e. we should first restrict our attention to $A_1(\mathbf{x})$. If this has a unique member, then we stop and repairing that subsystem is optimal. Otherwise, we have two cases: (A) $P(\mathbf{x}) = \emptyset$, (B) There are two or more members in $P(\mathbf{x})$.

Proposition 4 *If $P(\mathbf{x}) = \emptyset$, then it is optimal to assign the repairman to a subsystem i_0 with $n_{i_0}(\mathbf{x}) = n(\mathbf{x})$*

Proof: If $P(\mathbf{x}) = \emptyset$, then $A(\mathbf{x}) = Q(\mathbf{x})$. We will prove that there exists an $m' \geq 0$ such that

$$w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x};m) \quad \forall m < m'$$
$$w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) < w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x};m) \quad \forall m \geq m'$$

Suppose that $n = n(\mathbf{x}) = n_{\pi^*(\mathbf{x})}(\mathbf{x}) < n_i(\mathbf{x})$. Then $n(1_{\pi^*(\mathbf{x})}\mathbf{x}) = n(\mathbf{x}) - 1$ and $n(1_i\mathbf{x}) = n(\mathbf{x})$.

(I) If $R(\mathbf{x}) = \emptyset$, then $I(\mathbf{x}) = 0$. Then, Lemma 1 leads to the following results:

(a) If $m < n-1$, then $w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = 1 = w_{\pi^*}^{(0)}(1_i\mathbf{x};m)$. (b) If $m = n-1$, then $w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = \sum_{z=0}^{n-2}\binom{m}{z}\lambda^z(1-\lambda)^{m-z} = 1-\lambda^{n-1}$ while $w_{\pi^*}^{(0)}(1_i\mathbf{x};m) = 1$. Hence, $w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) < w_{\pi^*}^{(0)}(1_i\mathbf{x};m)$ for $m = n-1$. (c) If $m > n-1$, then $w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = \sum_{z=0}^{n-2}\binom{m}{z}(1-\lambda)^{m-z}\lambda^z$ and $w_{\pi^*}^{(0)}(1_i\mathbf{x};m) = \sum_{z=0}^{n-1}\binom{m}{z}(1-\lambda)^{m-z}\lambda^z$. Now, $\sum_{z=0}^{n-2}\binom{m}{z}(1-\lambda)^{m-z}\lambda^z < \sum_{z=0}^{n-1}\binom{m}{z}(1-\lambda)^{m-z}\lambda^z$ is true. Therefore, $w_{\pi^*}^{(0)}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) < w_{\pi^*}^{(0)}(1_i\mathbf{x};m)$ $\forall m > n-1$. (a), (b) and (c) lead to the following conclusion: If at state $\mathbf{x}$ no subsystem is functioning and no subsystem would start functioning after a single repair, then assign the repairman to the subsystem that needs the least number of repairs to become operational.

(II) $R(\mathbf{x}) \neq \emptyset$. Then we have the following cases:

(a) If $m < I(\mathbf{x})$, then $w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x};m) = 0$ (Proposition 2).

(b) If $m = I(\mathbf{x})$, then Lemma 1 leads to

$$w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x};m) = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\min(n-2,0)} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z = I(\mathbf{x})!\varpi(\mathbf{x})$$

and for $i : n_i(\mathbf{x}) > n(\mathbf{x})$,

$$w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x};m) = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\min(n-1,0)} \binom{m}{z}(1-\lambda)^{m-z}\lambda^z = I(\mathbf{x})!\varpi(\mathbf{x})$$

Therefore, for $m = I(\mathbf{x})$ $w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) = w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x}; m)$.
(c) If $m > I(\mathbf{x})$, then Lemma 1 leads to

$$w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\min(n-2,m-I(\mathbf{x}))} \binom{m}{z}(1-\lambda)^{m-z}\lambda^{z} \quad (12)$$

and for $i : n_i(\mathbf{x}) > n(\mathbf{x})$,

$$w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x}; m) = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}}\varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\min(n-1,m-I(\mathbf{x}))} \binom{m}{z}(1-\lambda)^{m-z}\lambda^{z} \quad (13)$$

If $m - I(\mathbf{x}) < n - 1$, then $\min(n-2, m-I(\mathbf{x})) = \min(n-1, m-I(\mathbf{x})) = m - I(\mathbf{x})$ and $w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) = w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x}; m)$. If $m - I(\mathbf{x}) \geq n - 1$, then $\min(n-2, m-I(\mathbf{x})) = n-2 < \min(n-1, m-I(\mathbf{x})) = n-1$ and $w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) < w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x}; m)$.

Hence, $w_{\pi^*}^{(I(\mathbf{x}))}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) < w_{\pi^*}^{(I(\mathbf{x}))}(1_i\mathbf{x}; m)$, $\forall m \geq I(\mathbf{x}) + n - 1$ ■

Proposition 4 establishes step 2 of the optimal policy when $P(\mathbf{x})$ is the empty set.

Proposition 5 *If there are two or more subsystems in $P(\mathbf{x})$, then the optimal policy assigns the repairman to a subsystem i_0 with $\ell_{i_0}(\mathbf{x}) = \min_{i\in P(\mathbf{x})}(\ell_i(\mathbf{x}))$.*

Proof: Consider the power series expansion of $w(1_{\pi^*(\mathbf{x})}\mathbf{x}; m)$ and $w(1_i\mathbf{x}; m)$ $i \in P(\mathbf{x})$, $i \neq \pi^*(\mathbf{x})$. Then, for all m and $\nu \leq I(\mathbf{x})$ $w_{\pi^*}^{(\nu)}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) = w_{\pi^*}^{(\nu)}(1_i\mathbf{x}; m) = 0$ (Proposition 1) So, we compare the $(I(\mathbf{x})+1)$-th term. We will prove that there exists an $m' \geq 0$ such that

$$w_{\pi^*}^{(I(\mathbf{x})+1)}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) = w_{\pi^*}^{(I(\mathbf{x})+1)}(1_i\mathbf{x}; m) \;\; \forall m < m'$$
$$w_{\pi^*}^{(I(\mathbf{x})+1)}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) < w_{\pi^*}^{(I(\mathbf{x})+1)}(1_i\mathbf{x}; m) \;\; \forall m \geq m'$$

For $m < I(\mathbf{x}) + 1$ we have $w_{\pi^*}^{(I(\mathbf{x})+1)}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) = w_{\pi^*}^{(I(\mathbf{x})+1)}(1_i\mathbf{x}; m) = 0$ (Proposition 2). For $m \geq I(\mathbf{x}) + 1$, Lemma 1 leads to

$$w_{\pi^*}^{(I(\mathbf{x})+1)}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) = \binom{m}{I(\mathbf{x})+1}(I(\mathbf{x})+1)!\varpi(1_{\pi^*(\mathbf{x})}\mathbf{x})(1-\lambda)^{m-I(\mathbf{x})-1} \quad (14)$$

$$w_{\pi^*}^{(I(\mathbf{x})+1)}(1_i\mathbf{x}; m) = \binom{m}{I(\mathbf{x})+1}(I(\mathbf{x})+1)!\varpi(1_i\mathbf{x})(1-\lambda)^{m-I(\mathbf{x})-1} \quad (15)$$

Therefore, using (10), if $\ell_{\pi^*(\mathbf{x})}(\mathbf{x}) < \ell_i(\mathbf{x})$, then $w_{\pi^*}^{(I(\mathbf{x})+1)}(1_{\pi^*(\mathbf{x})}\mathbf{x}; m) < w_{\pi^*}^{(I(\mathbf{x})+1)}(1_i\mathbf{x}; m)\ \forall m \geq I(\mathbf{x}) + 1$. ■

Propositions 4 and 5 establish step 2 of the optimal policy, i.e., the consideration of the minimum of $\ell_i(\mathbf{x})$ for all subsystems i such that $\delta_i(\mathbf{x})$ is minimum. Hence, if $A_1(\mathbf{x})$ has more than one members, and if $A_2(\mathbf{x})$ is nonempty and has a unique member, then, it is optimal to repair that subsystem.

Theorem 1 *If a repair allocation policy, say π^0, satisfying steps 1 and 2 is followed, and if $\mathbf{x}(0) = \mathbf{x}$, then, the probability that the system is down at any time instant t is given by*

$$
\begin{aligned}
&P_{\pi^0}(c(\mathbf{x}(t)) = 0) \\
&= \begin{cases} t^{I(\mathbf{x})} e^{-\lambda t} \varpi(\mathbf{x}) \mu^{I(\mathbf{x})} + o(\mu^{I(\mathbf{x})}) & \text{if} \quad P(\mathbf{x}) \neq \emptyset \\ \lambda^{-I(\mathbf{x})} I(\mathbf{x})! \varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n(\mathbf{x})-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!} \mu^{I(\mathbf{x})} + o(\mu^{I(\mathbf{x})}) & \text{if} \quad P(\mathbf{x}) = \emptyset \end{cases} \\
&= e^{-\lambda t} t^{I(\mathbf{x})} \sum_{z=0}^{s(\mathbf{x})} \frac{(\lambda t)^z}{z!} \binom{I(\mathbf{x}) + z}{z}^{-1} \varpi(\mathbf{x}) \mu^{I(\mathbf{x})} + o(\mu^{I(\mathbf{x})})
\end{aligned}
$$

Proof:

$$
P_{\pi^0}(c(\mathbf{x}(t)) = 0) = \sum_{m=I(\mathbf{x})}^{\infty} w_{\pi^0}(\mathbf{x}; m) e^{-t} \frac{t^m}{m!} = e^{-t} \sum_{\nu=I(\mathbf{x})}^{\infty} \left\{ \sum_{m=I(\mathbf{x})}^{\infty} w_{\pi^0}^{(\nu)}(\mathbf{x}; m) \frac{t^m}{m!} \right\} \mu^\nu
$$

Hence for $\nu = I(\mathbf{x})$ the coefficient of μ^ν will be

$$
e^{-t} \sum_{m=I(\mathbf{x})}^{\infty} w_{\pi^0}^{(I(\mathbf{x}))}(\mathbf{x}; m) \frac{t^m}{m!} = e^{-t} \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}} \varpi(\mathbf{x}) \sum_{m=I(\mathbf{x})}^{\infty} \frac{t^m}{m!} \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\alpha(\mathbf{x};m)} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \tag{16}
$$

<u>Case 1</u> $P(\mathbf{x}) \neq \emptyset$. Then, $\alpha(\mathbf{x}; m) = 0$ and eq. (16) gives

$$
e^{-t} \sum_{m=I(\mathbf{x})}^{\infty} w_{\pi^0}^{(I(\mathbf{x}))}(\mathbf{x}; m) \frac{t^m}{m!} = e^{-t} \varpi(\mathbf{x}) \sum_{m=0}^{\infty} t^{I(\mathbf{x})} \frac{[t(1-\lambda)]^m}{m!} = t^{I(\mathbf{x})} e^{-\lambda t} \varpi(\mathbf{x})
$$

That is, if π^0 is followed, the propability that the system is down at the time instant t is
$t^{I(\mathbf{x})} e^{-\lambda t} \varpi(\mathbf{x}) \mu^{I(\mathbf{x})} + o(\mu^{I(\mathbf{x})})$ and the difference with the optimal policy is $o(\mu^{I(\mathbf{x})})$.

Case 2 $P(\mathbf{x}) = \emptyset$. Then, Subcase 2.1: If $Q(\mathbf{x}) \neq \emptyset$.

$$
\begin{aligned}
&\sum_{m=I(\mathbf{x})}^{\infty} \frac{t^m}{m!} \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\alpha(\mathbf{x};m)} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \\
&= \sum_{m=I(\mathbf{x})}^{I(\mathbf{x})+n-1} \frac{t^m}{m!} \sum_{z=I(\mathbf{x})}^{m} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \\
&\quad + \sum_{m=I(\mathbf{x})+n}^{\infty} \frac{t^m}{m!} \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n-1} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \\
&= \sum_{m=I(\mathbf{x})}^{I(\mathbf{x})+n-1} \frac{t^m}{m!} \left(1 - \sum_{z=0}^{I(\mathbf{x})-1} \binom{m}{z} \lambda^z (1-\lambda)^{m-z}\right) \\
&\quad + \sum_{m=I(\mathbf{x})+n}^{\infty} \frac{t^m}{m!} \sum_{z=0}^{I(\mathbf{x})+n-1} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \\
&\quad - \sum_{m=I(\mathbf{x})+n}^{\infty} \frac{t^m}{m!} \sum_{z=0}^{I(\mathbf{x})-1} \binom{m}{z} \lambda^z (1-\lambda)^{m-z}
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\sum_{m=I(\mathbf{x})+n}^{\infty} \frac{t^m}{m!} \sum_{z=0}^{I(\mathbf{x})+n-1} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \\
&= \sum_{z=0}^{I(\mathbf{x})+n-1} \frac{(\lambda t)^z}{z!} \sum_{m=I(\mathbf{x})+n-z}^{\infty} \frac{((1-\lambda)t)^m}{m!} \\
&= e^t \sum_{z=0}^{I(\mathbf{x})+n-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!} - \sum_{z=0}^{I(\mathbf{x})+n-1} \frac{(\lambda t)^z}{z!} \sum_{m=0}^{I(\mathbf{x})+n-z-1} \frac{((1-\lambda)t)^m}{m!} \\
&= e^t \sum_{z=0}^{I(\mathbf{x})+n-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!} - \sum_{z=0}^{I(\mathbf{x})+n-1} \frac{t^m}{m!}
\end{aligned}
$$

where the last equality uses the fact that a convolution of two Poisson is Poisson. Hence,

$$
\begin{aligned}
&\sum_{m=I(\mathbf{x})}^{\infty} \frac{t^m}{m!} \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+\alpha(\mathbf{x};m)} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \\
&= \sum_{m=I(\mathbf{x})}^{I(\mathbf{x})+n-1} \frac{t^m}{m!} + e^t \sum_{z=0}^{I(\mathbf{x})+n-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!} - \sum_{m=0}^{I(\mathbf{x})+n-1} \frac{t^m}{m!}
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{m=I(\mathbf{x})}^{\infty} \frac{t^m}{m!} \sum_{z=0}^{I(\mathbf{x})-1} \binom{m}{z} \lambda^z (1-\lambda)^{m-z} \\
= \; & e^t \sum_{z=0}^{I(\mathbf{x})+n-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!} - e^t \sum_{z=0}^{I(\mathbf{x})-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!} \\
= \; & e^t \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!}
\end{aligned}
$$

Therefore, (16) is reduced to

$$
e^{-t} \sum_{m=I(\mathbf{x})}^{\infty} w_{\pi^0}^{(I(\mathbf{x}))}(\mathbf{x}; m) \frac{t^m}{m!} = \frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}} \varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!}
$$

and if π^0 is followed, the probability that the system is down at the time instant t is

$$
\frac{I(\mathbf{x})!}{\lambda^{I(\mathbf{x})}} \varpi(\mathbf{x}) \sum_{z=I(\mathbf{x})}^{I(\mathbf{x})+n-1} e^{-\lambda t} \frac{(\lambda t)^z}{z!} \mu^{I(\mathbf{x})} + o(\mu^{I(\mathbf{x})})
$$

and the difference with the optimal policy is $o(\mu^{I(\mathbf{x})})$.
Subcase 2.2: If $Q(\mathbf{x}) = \emptyset$, then a simple change in the summation order of (16) will give the result.

Now, it is a matter of operations to check that both Case 1 and 2 can be expressed as

$$
P_{\pi^0}(c(\mathbf{x}(t)) = 0) = e^{-\lambda t} t^{I(\mathbf{x})} \sum_{z=0}^{s(\mathbf{x})} \frac{(\lambda t)^z}{z!} \binom{I(\mathbf{x})+z}{z}^{-1} \varpi(\mathbf{x}) \mu^{I(\mathbf{x})} + o(\mu^{I(\mathbf{x})})
$$

■

Now, what happens if $A_2(\mathbf{x})$ has two members or more? Then, one has to examine the higher order coefficients of μ^ν. This is very tedious and moreover the continuation of π^0 one may derive is not strongly optimal anymore. As pointed out in the Introduction, we don't believe that a uniformly optimal (i.e. independent of the particular values of μ and λ) policy exists then, except in particular cases. This situation may be remedied if the optimization criterion is changed to the total discounted operation time of the system, but again this will be so only for discount factors sufficiently small or large. For the discounted problem, the optimal policy (which will necessarily coincide with π^0 for the first two steps) is quite complicated but easy to program and compute on a machine.

4 The Corresponding Routing and Scheduling Problems

Let us consider N parallel queues, each with finite buffer size (capacity). The buffer size of queue i is denoted by N_i, $i = 1, \ldots, N$. Associated to queue i is a control level K_i, $i = 1, \ldots, N$. There are two problems we consider, the "routing" and the "scheduling" problem.

In the routing problem jobs arrive according to a Poisson process with rate λ. There are N processors, one assigned to each queue. All processors are identical and serve all waiting customers (jobs) in the queue simultaneously. The time it takes for a job to be processed is exponentially distributed with rate μ. Processing times and arrival times are all assumed to be independent. A controller assigns new arrivals to queues and arriving jobs that find all buffers occupied are lost to the system. A job assigned to a processor may not change queue at a future time. In the scheduling problem, the time it takes for any empty position in any buffer to be occupied is exponentially distributed with rate μ, and these times are independent. There is a unique processor which is to be assigned to a queue and which may process one job at a time. The service times for each job are identical independent exponentials with rate λ (and are also independent of the arrival times). The controller assigns the processor to a queue and preemptions are allowed. Again, arriving jobs that find all buffers occupied are lost to the system.

Now, let x_i represent the number of jobs waiting in queue i, $i = 1, \ldots, N$ and let π^0 be any policy satisfying steps 1 and 2 and σ^0 be any policy that is derived by changing min with max in steps 1 and 2. Then, parts 2 and 3 of the of the present paper show that for the routing problem and for mean processing times sufficiently long (i.e. for time consuming jobs) σ^0 (π^0) maximizes (minimizes) the probability all queues have size less than K_i at any time instant t.

Finally, let $\widehat{x}_i$ be the number of empty spaces in the buffer of queue i, $i = 1, \ldots, N$ and let $\widehat{\pi}^0(\widehat{\sigma}^0)$ be any policy that satisfies steps 1 and 2 (that satisfies steps 1 and 2 with the min replaced by max) Then, for the scheduling problem and for mean arrival times sufficiently long (i.e. for rare jobs) $\widehat{\pi}^0$ ($\widehat{\sigma}^0$) maximizes (minimizes) the probability at least one queue has K_i or more empty spaces in its buffer at any time instant t.

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