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CLASSIFICATION RESULTS OF QUASI EINSTEIN SOLITONS

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Abstract

We classify (ρ,τ)-quasi Einstein solitons with (a,τ)-concurrent vector fields. We also give a necessary and sufficient condition for a submanifold to be a (ρ,τ)-quasi Einstein soliton in a Riemannian manifold equipped with an (a,τ)-concurrent vector field.

1. Introduction

Let M be an m-dimensional Riemannian manifold with metric g. We call g (ρ,τ)-quasi Einstein if

$$\frac{1}{2}\mathscr{L}_v \mathrm{g} + \mathrm{Ric} - \frac{1}{\tau} v^* \otimes v^* = \rho R \mathrm{g} + \lambda \mathrm{g} \tag{1.1}$$

holds for some potential field v and some constant λ, where Ric is the Ricci curvature tensor of M, $\mathscr{L}_v \mathrm{g}$ is the Lie derivative of g with respect to v, ρ and $\tau > 0$ are two given constants and v^* is defined by $v^*(X) = \mathrm{g}(v, X)$ for any vector X tangent to M. A (ρ,τ)-quasi Einstein soliton is a manifold M whose metric satisfies (1.1). Recall that a Ricci soliton is a manifold M whose metric satisfies

$$\frac{1}{2}\mathscr{L}_v \mathrm{g} + \mathrm{Ric} = \lambda \mathrm{g} \tag{1.2}$$

for some potential field v. Hence the (ρ,τ)-quasi Einstein soliton is a generalization of the Ricci soliton.

Classification question for the (ρ,τ)-quasi Einstein soliton is complicated. Now some additional conditions (for example, the pinching of the Ricci curvature, the harmonicity of the Weyl tensor, the flatness of the Bach tensor, the locally conformally flatness, the conditions at infinity of the scalar curvature or the integral conditions of the potential function, etc) should be added. For works in this direction, we can refer to [1, 2, 3, 9, 10] and the references therein.

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A vector field v is called (a, τ)-concurrent if it satisfies

$$\nabla_X v = aX + \frac{1}{\tau} g(v, X)v \tag{1.3}$$

for any vector X tangent to M, where ∇ denotes the Levi-Civita connection of M. Then an $(a, 1)$-concurrent vector field is a closed torse forming [7] and a $(1, \infty)$-concurrent vector field is a concurrent vector field [4]. A closed torse forming is the generator of a biconcircular gradient vector field X, which satisfies [6]

$$\nabla_Y X = g(U, Y)X + g(X, Y)U$$

for any vector Y tangent to M. It was proved in [11] that the concurrent vector field is existed if the holonomy group of M leaves a point invariant. We can refer to [4, 5, 7, 8] to see the study of concurrent vector fields.

In [4] Chen and Deshmukh classified Ricci solitons with concurrent potential fields. They also derived a necessary and sufficient condition for a submanifold to be a Ricci soliton in a Riemannian manifold equipped with a concurrent vector field. Inspired by Chen-Deshmukh's work, we will classify (ρ, τ)-quasi Einstein solitons with (a, τ)-concurrent vector fields. We will also give a necessary and sufficient condition for a submanifold to be a (ρ, τ)-quasi Einstein soliton in a Riemannian manifold equipped with an (a, τ)-concurrent vector field.

2. Classification results via the (a, τ)-concurrent vector field

In this section we will classify (ρ, τ)-quasi Einstein solitons with (a, τ)-concurrent vector fields for some constant $a \neq 0$ and $\tau > 0$. We firstly state a lemma.

LEMMA 2.1. *Let v be an (a, τ)-concurrent vector field on an m-dimensional Riemannian manifold M with metric* g. *Then for any unit vector X orthogonal to v, the sectional curvature of M satisfies*

$$K(X, v) = \frac{a}{\tau}. \tag{2.1}$$

Proof. By (1.3) and the definition of the Riemannian curvature, we have

$$\begin{aligned} R(X, v, X, v) &= g(\nabla_X \nabla_v v - \nabla_v \nabla_X v - \nabla_{[X,v]} v, X) \\ &= g\left(\nabla_X\left(av + \frac{1}{\tau}|v|^2 v\right), X\right) - g\left(\nabla_v\left(aX + \frac{1}{\tau}g(v, X)v\right), X\right) \\ &\quad - g\left(a[X, v] - \frac{1}{\tau}g([X, v], v)v, X\right). \end{aligned}$$

Since $g(X, v) = 0$ and

$$\nabla_X v - \nabla_v X - [X, v] = 0,$$

we have

$$\begin{aligned} R(X, v, X, v) &= g\left(\nabla_X\left(\frac{1}{\tau}|v|^2 v\right), X\right) - g\left(\nabla_v\left(\frac{1}{\tau}g(v, X)v\right), X\right) \\ &= \frac{1}{\tau}|v|^2 g(\nabla_X v, X) = \frac{1}{\tau}|v|^2 g\left(aX + \frac{1}{\tau}(X, v)v, X\right) = \frac{a}{\tau}|v|^2. \end{aligned}$$

Hence (2.1) holds. □

Lemma 2.1 tells us that the Ricci curvature of M satisfies

$$\mathrm{Ric}(v, v) = \frac{m-1}{\tau} a|v|^2. \tag{2.2}$$

Now let's state the main result in this section.

Theorem 2.2. *Let $\tau > 0, a \neq 0$ and ρ be given constants. Then the metric* g *of an m-dimensional Riemannian manifold M is (ρ, τ)-quasi Einstein, i.e.,* g *satisfies* (1.1), *for some constant λ and some (a, τ)-concurrent vector field v, if and only if the following three conditions hold*:

(a)

$$\lambda = a + \frac{m-1}{\tau} a(1 - m\rho), \tag{2.3}$$

(b) *M^m is an open part of a warped product manifold $I \times_{f(s)} F$, where I is an open interval with arclength s and F is an $(m-1)$-dimensional Einstein manifold whose Ricci tensor satisfies $\mathrm{Ric}_F = (m-2)g_F$, g_F is the metric tensor of F,*

(c)

$$v(s, \cdot) = \mu(s)\frac{\partial}{\partial s}.$$

Moreover, after applying a suitable translation and dilation,

$$f(s) = \begin{cases} \sqrt{\dfrac{\tau}{a}} \sin \sqrt{\dfrac{a}{\tau}} s, & a > 0 \\ \sqrt{-\dfrac{\tau}{a}} \sinh \sqrt{-\dfrac{a}{\tau}} s, & a < 0, \end{cases} \tag{2.4}$$

$$\mu(s) = \begin{cases} \sqrt{a\tau} \tan \sqrt{\dfrac{a}{\tau}} s, & a > 0 \\ -\sqrt{-a\tau} \tanh \sqrt{-\dfrac{a}{\tau}} s, & a < 0. \end{cases} \tag{2.5}$$

Proof. The main idea comes from [4]. We firstly assume that g is (ρ, τ)-quasi Einstein for some (a, τ)-concurrent vector field v. Let us put $v = \mu e_1$, where e_1 is a unit vector field tangent to M^m. Also let us extend e_1 to a local orthonormal frame $\{e_1, \dots, e_m\}$ on M^m. Denote by $\{\omega_1, \dots, \omega_m\}$ the dual frame of 1-forms of $\{e_1, \dots, e_m\}$. Define the connection forms ω_i^j $(i, j = 1, \dots, m)$ on M^m by

$$\nabla_X e_i = \sum_{j=1}^{m} \omega_i^j(X) e_j, \quad i = 1, \dots, m. \tag{2.6}$$

Letting $X = e_1$ in (1.3) we have

$$e_1(\mu)e_1 + \mu \nabla_{e_1} e_1 = a e_1 + \frac{\mu^2}{\tau} e_1.$$

Since $(\nabla_{e_1} e_1, e_1) = 0$, we have

$$e_1(\mu) = a + \frac{\mu^2}{\tau} \tag{2.7}$$

and

$$\nabla_{e_1} e_1 = 0. \tag{2.8}$$

Put $D_1 = Span\{e_1\}$ and $D_2 = Span\{e_2, \dots, e_m\}$. It follows from (2.8) that D_1 is a totally geodesic distribution so that the leaves of D_1 are geodesics of M^m. Also, we may derive from (1.3) with $X = e_i$ $(i = 2, \dots, m)$ that

$$e_i(\mu)e_1 + \mu \nabla_{e_i} e_1 = a e_i, \quad 2 \le i \le m.$$

Hence

$$e_2(\mu) = \cdots = e_m(\mu) = 0 \tag{2.9}$$

and

$$\mu \nabla_{e_i} e_1 = a e_i, \quad 2 \le i \le m. \tag{2.10}$$

Due to (2.6), we have

$$\mu \omega_1^i(e_i) = a, \tag{2.11}$$

$$\omega_1^j(e_i) = 0, \quad j \ne i. \tag{2.12}$$

From Cartans structure equations, we have

$$\mathrm{d}\omega^i = -\sum_{j=1}^{m} \omega_j^i \wedge \omega^j, \quad i = 1, \dots, m. \tag{2.13}$$

Thus, after applying (2.12) and (2.13), we obtain $\mathrm{d}\omega^1 = 0$. Hence we have locally $\omega^1 = \mathrm{d}s$ for some function s on M^m. It follows from (2.12) that

$$\mathrm{g}([e_i, e_j], e_1) = \omega_j^1(e_i) - \omega_i^1(e_j) = 0, \quad 2 \le i \ne j \le m.$$

Therefore D_2 is an integrable distribution. Moreover, from (2.11) we know that the second fundamental form $\hat{h}$ of each leaf L of D_2 in M^m satisfies

$$\hat{h}(e_i, e_j) = -\frac{a\delta_{ij}}{\mu} e_1, \quad 2 \le i, j \le m, \tag{2.14}$$

which shows that the mean curvature of each leaf L is given by $-a\mu^{-1}$. Equation (2.14) implies that each leaf of D_2 is a totally umbilical hypersurface of M^m whose mean curvature vector is $\hat{H} = -\dfrac{ae_1}{\mu}$. Furthermore, by applying (2.9) we conclude that D_2 is a spherical distribution, i.e., the mean curvature vector of each totally umbilical leaf is parallel in the normal bundle. Consequently, as in [4] we know that M^m is locally a warped product manifold $I \times_{f(s)} F$ whose warped metric is given by

$$\mathrm{g} = \mathrm{d}s^2 + f^2(s)\mathrm{g}_F, \tag{2.15}$$

such that $e_1 = \dfrac{\partial}{\partial s}$. Hence (2.7) becomes

$$\mu'(s) = a + \frac{\mu^2}{\tau}. \tag{2.16}$$

Solving this equation and applying a suitable translation, we get (2.5).

Due to (2.15), a standard calculation shows that the sectional curvature of M^m satisfies

$$K(X, v) = -\frac{f''(s)}{f(s)}, \tag{2.17}$$

for each unit vector X orthogonal to v. Now, after comparing (2.1) with (2.17) we obtain

$$f''(s) = -\frac{a}{\tau} f(s). \tag{2.18}$$

Solving this differential equation yields

$$f(s) = \begin{cases} C_1 \sin\sqrt{\dfrac{a}{\tau}}s + C_2 \cos\sqrt{\dfrac{a}{\tau}}s, & a > 0 \\ C_1 \sinh\sqrt{-\dfrac{a}{\tau}}s + C_2 \cosh\sqrt{-\dfrac{a}{\tau}}s, & a < 0. \end{cases}$$

From (2.11) we have

$$\frac{f'(s)}{f(s)} = \frac{a}{\mu(s)}. \tag{2.19}$$

Hence

$$\begin{cases} \dfrac{C_1 \cos\sqrt{\dfrac{a}{\tau}}s - C_2 \sin\sqrt{\dfrac{a}{\tau}}s}{C_1 \sin\sqrt{\dfrac{a}{\tau}}s + C_2 \cos\sqrt{\dfrac{a}{\tau}}s} = \dfrac{\cos\sqrt{\dfrac{a}{\tau}}s}{\sin\sqrt{\dfrac{a}{\tau}}s}, & a > 0 \\ \dfrac{C_1 \cosh\sqrt{\dfrac{a}{\tau}}s + C_2 \sinh\sqrt{\dfrac{a}{\tau}}s}{C_1 \sinh\sqrt{\dfrac{a}{\tau}}s + C_2 \cosh\sqrt{\dfrac{a}{\tau}}s} = \dfrac{\cosh\sqrt{\dfrac{a}{\tau}}s}{\sinh\sqrt{\dfrac{a}{\tau}}s}, & a < 0. \end{cases}$$

Hence $C_2 = 0$ and

$$f(s) = \begin{cases} C_1 \sin\sqrt{\dfrac{a}{\tau}}s, & a > 0 \\ C_1 \sinh\sqrt{-\dfrac{a}{\tau}}s, & a < 0. \end{cases}$$

Now after applying a suitable dilation of the metric tensor of F, we can choose $f(s)$ as in (2.4).

On the other hand, by (1.3) and the definition of Lie-derivative we have

$$\begin{aligned} \mathscr{L}_v \mathrm{g}(X,Y) &= \mathrm{g}(\nabla_X v, Y) + \mathrm{g}(\nabla_Y v, X) \\ &= \mathrm{g}\left(aX + \frac{1}{\tau}\mathrm{g}(X,v)v, Y\right) + \mathrm{g}\left(aY + \frac{1}{\tau}\mathrm{g}(Y,v)v, X\right) \\ &= 2a\mathrm{g}(X,Y) + \frac{2}{\tau}\mathrm{g}(X,v)\mathrm{g}(Y,v) \end{aligned} \tag{2.20}$$

for any X, Y tangent to M^m. Combining (2.20) with (1.1) gives

$$\mathrm{Ric}(X,Y) = (\rho R + \lambda - a)\mathrm{g}(X,Y). \tag{2.21}$$

Choosing $X = Y = v$ and comparing (2.2) and (2.21) we get

$$\rho R + \lambda - a = \frac{m-1}{\tau}a. \tag{2.22}$$

Tracing (2.21) leads to

$$R = m(\rho R + \lambda - a). \tag{2.23}$$

Solving (2.22) and (2.23) we get that

$$\lambda = a + \frac{m-1}{\tau}a(1 - m\rho), \tag{2.24}$$

$$R = \frac{m(m-1)}{\tau}a. \tag{2.25}$$

Due to (2.15), a standard calculation shows that (see [10])

$$\mathrm{Ric}(e_i, e_i) = f^{-2}(s)\ \mathrm{Ric}_F(e_i, e_i) - [(\log f(s))'' + (m-1)((\log f(s))')^2],$$
$$i = 2, \dots, m,$$

where Ric_F denotes the Ricci curvature tensor of F. Hence

$$\mathrm{Ric}_F = (m-2)\mathrm{g}_F.$$

Conversely, we assume that (M^m, g) satisfies conditions a), b) and c) in Theorem 2.2. Let $e_1 = \dfrac{\partial}{\partial s}$, $e_2, \dots, e_m$ be a local orthonormal frame tangent to M^m. Note $f(s)$, $\mu(s)$ satisfy (2.16), (2.18), (2.19). Then

$$\nabla_{e_i}(v) = \mu(s)\nabla_{e_i}e_1 = \mu(s)\frac{f'(s)}{f(s)}e_i = ae_i = ae_i + \frac{1}{\tau}\mathrm{g}(v, e_i)v, \quad 2 \le i \le m,$$
$$\nabla_{e_1}(v) = \mu'(s)e_1 + \mu(s)\nabla_{e_1}e_1 = \mu'(s)e_1 = \left(a + \frac{\mu^2}{\tau}\right)e_1 = ae_1 + \frac{1}{\tau}\mathrm{g}(v, e_1)v.$$

Hence v is an (a, τ)-concurrent vector field. On the other hand, due to (2.15), a standard calculation shows that (see [10])

$$R_{1i} = -(m-1)[(\log f(t))'' + ((\log f(t))')^2]\delta_{1i} \tag{2.26}$$

and

$$R_{ij} = f^{-2}(t)R_{F,ij} - [(\log f(t))'' + (m-1)((\log f(t))')^2]\delta_{ij}, \quad 2 \le i, j \le m, \tag{2.27}$$

where R_{ij} and $R_{F,ij}$ denotes the Ricci curvature of M^m and F respectively in the local orthonormal frame. Since F is an Einstein $(m-1)$-manifold whose Ricci tensor satisfies $\mathrm{Ric}_F = (m-2)\mathrm{g}_F$, we can verify (1.1) directly by using (2.3), (2.20), (2.26) and (2.27). $\square$

When we choose $a = 1$, $\rho = 0$ and let $\tau \to \infty$ in Theorem 2.2, we will get the following classification result, which was derived in [4].

Corollary 2.3. *An m-dimensional Riemannian manifold (M, g) is a Ricci soliton, i.e.*, g *satisfies* (1.2), *for some constant λ and some concurrent vector field v, if and only if the following three conditions hold*:

(a) $\lambda = 1$,

(b) *M^m is an open part of a warped product manifold $I \times_s F$, where I is an open interval with arclength s and F is an Einstein $(m-1)$-manifold whose Ricci tensor satisfies* $\mathrm{Ric}_F = (m-2)\mathrm{g}_F$, g_F *is the metric tensor of F*,

(c) $v(s, \cdot) = s\dfrac{\partial}{\partial s}$.

3. Submanifolds with (ρ, τ)-quasi Einstein metric

Let $\phi : M^m \to N^n$ be an isometric immersion from an m-dimensional Riemannian manifold (M^m, g_M) into an n-dimensional Riemannian manifold (N^n, g_N). We use ∇^M, ∇^N to denote the Levi-Civita connections on M^m and N^n respectively. For vector fields X, Y tangent to M^m and η normal to M^m, the Gauss formula and Weingarten formula are given respectively by [4]

$$\nabla^N_X Y = \nabla^M_X Y + h(X, Y), \tag{3.1}$$

$$\nabla^N_X \eta = -A_\eta X + D_X \eta, \tag{3.2}$$

where $h(X, Y)$ is the normal components of $\nabla^N_X Y$, $-A_\eta X$ and $D_X \eta$ are the tangential and normal components of $\nabla^N_X \eta$. These two formulas define the second fundamental form h, the shape operator A, and the normal connection D of M^m in the ambient space N^n. For any vector field v on N^n, we use v^T and $v^\perp$ to denote the tangential and normal components of v on M^m, respectively.

The following result gives a necessary and sufficient condition, under which the metric of a submanifold in a Riemannian manifold equipped with an (a, τ)-concurrent vector field is (ρ, τ)-quasi Einstein.

Theorem 3.1. *Let (N^n, g_N) be a Riemannian manifold endowed with an (a, τ)-concurrent vector field v. Then the metric of a submanifold M^m in N^n is (ρ, τ)-quasi Einstein for the potential field v^T and some constant λ, if and only if the Ricci curvature of (M, g) satisfies*

$$\mathrm{Ric}_M(X, Y) = (\lambda + \rho R_M - 2a) g_M(X, Y) - g_M(h(X, Y), v^\perp) \tag{3.3}$$

for any X, Y tangent to M, where R_M denotes the scalar curvature of M.

Proof. Since $v = v^T + v^\perp$, from (1.3), (3.1) and (3.2) we have

$$\begin{aligned} aX &= \nabla^N_X v^T + \nabla^N_X v^\perp - \frac{1}{\tau} g_N(v, X) v \\ &= \nabla^M_X v^T + h(X, v^T) - A_{v^\perp} X + D_X v^\perp - \frac{1}{\tau} g_N(v, X)(v^T + v^\perp). \end{aligned} \tag{3.4}$$

By comparing the tangential and normal components of (3.4) we obtain

$$\nabla^M_X v^T = A_{v^\perp} X + aX + \frac{1}{\tau} g_M(v^T, X) v^T$$

and

$$h(X, v^T) = -D_X v^\perp + \frac{1}{\tau} g_M(v^T, X) v^\perp.$$

Then for vector fields X, Y tangent to M,

$$\begin{aligned}(3.5)\qquad (\mathscr{L}_{v^T}g_M)(X,Y) &= g_M(\nabla^M_X v^T, Y) + g_M(\nabla^M_Y v^T, X)\\ &= g_M\left(A_{v^\perp}X + aX + \frac{1}{\tau}g_M(v^T, X)v^T, Y\right)\\ &\quad + g_M\left(A_{v^\perp}Y + aY + \frac{1}{\tau}g_M(v^T, Y)v^T, X\right)\\ &= 2ag_M(X,Y) + 2g_M(A_{v^\perp}X, Y) + \frac{2}{\tau}g_M(v^T, X)g_M(v^T, Y).\end{aligned}$$

From (1.1) we have

$$\begin{aligned}(3.6)\qquad \frac{1}{2}\mathscr{L}_{v^T}g_M(X,Y) + \mathrm{Ric}_M(X,Y) - \frac{1}{\tau}g_M(v^T, X)g_M(v^T, Y)&\\ = \rho R g_M(X,Y) + \lambda g_M(X,Y).&\end{aligned}$$

Plugging (3.5) into (3.6) leads to

$$\mathrm{Ric}_M(X,Y) = (\lambda + \rho R - 2a)g_M(X,Y) - g_M(A_{v^\perp}X, Y).$$

Since the shape operator and the second fundamental form are related by

$$g_M(A_{v^\perp}X, Y) = g_M(h(X,Y), v^\perp).$$

We arrive at (3.3). □

Note that τ does not appear in (3.3). If we let $\rho = 0$, $a = 1$ and $\tau \to \infty$ in Theorem 3.1, we can get the following result, which was proved in [4].

Corollary 3.2. *Let (N^n, g_N) be a Riemannian manifold endowed with a concurrent vector field v. Then a submanifold M^m in N^n is a gradient soliton for the potential field v^T and some constant λ if and only if the Ricci curvature of (M, g) satisfies*

$$\mathrm{Ric}_M(X,Y) = (\lambda - 2)g_M(X,Y) - g_M(h(X,Y), v^\perp)$$

for any X, Y tangent to M.

Recall that a Riemannian submanifold M^m is called η-umbilical (with respect to a normal vector field η) if its shape operated satisfies $A_\eta = \varphi I$, where φ is a function on M^m and I is the identity map [4]. The following result can be deduced from Theorem 3.1 easily.

Theorem 3.3. *Let (N^n, g_N) be a Riemannian manifold endowed with an (a,τ)-concurrent vector field v. We assume that the metric of a submanifold M^m in N^n is (ρ,τ)-quasi Einstein for the potential field v^T and some constant λ. Then M^m is trivial in the sense that g_M is Einstein if and only if M^m is $v^\perp$-umbilical.*

Proof. M^m is $v^\perp$-umbilical is equivalent to the fact that for all X, Y tangent to M,

$$g_M(h(X,Y),v^\perp) = \varphi g_M(X,Y).$$

Hence (3.3) is equivalent to

$$\mathrm{Ric}_M(X,Y) = (\lambda + \rho R_M - 2a - \varphi) g_M(X,Y).$$

Hence this theorem holds. □

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