

THE GROTHENDIECK CONJECTURE FOR THE MODULI SPACES OF HYPERBOLIC CURVES OF GENUS ONE

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Abstract

We study the Grothendieck conjecture for the moduli spaces of hyperbolic curves of genus one. A consequence of the main results is that the isomorphism class of a certain moduli space of hyperbolic curves of genus one over a sub- p -adic field is completely determined by the isomorphism class of the étale fundamental group of the moduli space over the absolute Galois group of the sub- p -adic field. We also prove related results in absolute anabelian geometry.

Introduction

A. Grothendieck has proposed that there is a class of varieties called anabelian varieties such that, roughly speaking, the isomorphism class of a variety that belongs to this class should be completely determined by the étale fundamental group (cf. [1], [2]). He also suggested examples of varieties which should be found in the class of anabelian varieties. This is now called the Grothendieck conjecture of anabelian geometry. One may find moduli spaces of hyperbolic curves among these examples suggested by Grothendieck.

The present paper focuses on the Grothendieck conjecture of anabelian geometry for certain moduli spaces of hyperbolic curves of genus one. Let us introduce some notational conventions as follows: Let k be a field of characteristic zero and

$$\mathcal{P} : (\text{Ell}/k) \rightarrow (\text{Sets})$$

a moduli problem for elliptic curves over k (cf. [5], (4.2), (4.13)), which thus determines a contravariant functor of [5], (4.3.1)

$$\mathcal{M}(\mathcal{P}) : (\text{Sch}/k) \rightarrow (\text{Sets}),$$

i.e., the functor obtained by considering elliptic curves equipped with level \mathcal{P} structures. We shall say that the moduli problem \mathcal{P} is *hyperbolic* (cf. Definition

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1.1) if \mathcal{P} is represented by a hyperbolic curve over k , which we denote by

$$\mathcal{M}_{\mathcal{P},1},$$

i.e., the modular curve with respect to \mathcal{P} , and, moreover, the natural morphism from $\mathcal{M}_{\mathcal{P},1}$ to the coarse moduli scheme “ \mathbf{A}_k^1 ” of elliptic curves over k is non-constant, i.e., dominant. For instance, if n is an integer greater than two, and k contains a primitive n -th root of unity, then the moduli problem $[\Gamma(n)]$ over k obtained by considering “ $\Gamma(n)$ -structures” (cf. [5], (3.1), (5.1)) is hyperbolic.

For a positive integer r , we shall write

$$\mathcal{M}(\mathcal{P}, r) : (\text{Sch}/k) \rightarrow (\text{Sets})$$

for the functor obtained by considering collections of data consisting of projective smooth curves of genus one equipped with ordered distinct r points and level \mathcal{P} structures on the elliptic curves determined by the projective smooth curves of genus one and the first marked points (cf. Definition 1.3). If \mathcal{P} is hyperbolic, then $\mathcal{M}(\mathcal{P}, r)$ is represented by a smooth variety over k (cf. Proposition 1.5, (iii)), which we denote by

$$\mathcal{M}_{\mathcal{P},r}.$$

A consequence of the main results of the present paper is as follows (cf. Theorem 2.2 in the case where both “ $g_{\alpha}^{\mathcal{M}}$ ” and “ $g_{\beta}^{\mathcal{M}}$ ” are positive):

THEOREM. *Let k be a sub- p -adic field for some prime number p ; \bar{k} an algebraic closure of k . For $\xi = \alpha, \beta$, let \mathcal{P}_{ξ} be a hyperbolic (cf. Definition 1.1) moduli problem for elliptic curves over k ; $r_{\xi} \geq 1$ an integer. Suppose that, for $\xi = \alpha, \beta$, the modular curve $\mathcal{M}_{\mathcal{P}_{\xi},1}$ with respect to \mathcal{P}_{ξ} is of positive genus. Then the natural map*

$$\text{Isom}_k(\mathcal{M}_{\mathcal{P}_{\alpha},r_{\alpha}}, \mathcal{M}_{\mathcal{P}_{\beta},r_{\beta}}) \rightarrow \text{Isom}_{G_k}(\pi_1(\mathcal{M}_{\mathcal{P}_{\alpha},r_{\alpha}}), \pi_1(\mathcal{M}_{\mathcal{P}_{\beta},r_{\beta}})) / \text{Inn}(\pi_1(\mathcal{M}_{\mathcal{P}_{\beta},r_{\beta}} \otimes_k \bar{k}))$$

is bijective.

The proof is a combination of rather standard arguments in anabelian geometry. A part of it is similar to some arguments in [3], which implies Theorem in the case where either r_{α} or r_{β} is less than five.

After reviewing some definitions in section 0, we introduce, in section 1, certain moduli spaces of hyperbolic curves of genus one, that are the main objects of the present paper. We state the main results in section 2. In section 3, we prove the main results. Finally, in section 4, we prove some variants in absolute anabelian geometry.

Most of the results of the present paper are contained in the master thesis [6] of the second author.

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0. Preliminaries

0.1. Let p be a prime number. A p -adic local field is defined as a finite extension of \mathbf{Q}_p . A sub - p -adic field is defined as a field isomorphic to a subfield of a finitely generated extension of a p -adic local field (cf. [8], Definition 15.4, (i)). A $generalized$ sub - p -adic field is defined as a field isomorphic to a subfield of a finitely generated extension of the p -adic completion of the maximal unramified extension of a p -adic local field (cf. [9], Definition 4.11).

0.2. For a connected noetherian scheme X , we denote by

$$\pi_1(X)$$

the étale fundamental group (well-defined up to conjugation) for some base point.

For a morphism $X \rightarrow Y$ of connected noetherian schemes, we denote by

$$\Delta_{X/Y} \subseteq \pi_1(X)$$

the kernel of the induced outer homomorphism $\pi_1(X) \rightarrow \pi_1(Y)$.

0.3. Let S be a scheme. For a pair of nonnegative integers (g, r) with $2g - 2 + r > 0$, a *hyperbolic curve of type (g, r)* over S is defined as a scheme X over S such that there are a scheme X^{cpt} which is smooth, proper, geometrically connected, and of relative dimension one over S and a closed subscheme $D \subset X^{\text{cpt}}$ of X^{cpt} which is finite and étale over S satisfying the following conditions:

- (i) Any geometric fiber of $X^{\text{cpt}} \rightarrow S$ is of genus g .
- (ii) The finite étale covering $D \hookrightarrow X^{\text{cpt}} \rightarrow S$ is of degree r .
- (iii) X is isomorphic to $X^{\text{cpt}} \setminus D$ over S .

A *hyperbolic curve* over S is defined as a hyperbolic curve of type (g, r) over S for some pair of nonnegative integers (g, r) with $2g - 2 + r > 0$.

0.4. For a hyperbolic curve X/S and a positive integer n , the n -th *relative configuration space* X_n associated to X/S is defined as the complement in the fiber product X^n of n copies of X over S of the (i, j) -weak diagonals, where (i, j) ranges over the pairs such that $1 \leq i < j \leq n$. Here, the (i, j) -weak diagonal is the inverse image via the projection $X^n \rightarrow X^2$ to the i -th and j -th factors of the diagonal of X^2 . The 0 -th *relative configuration space* X_0 associated to X/S is defined as S .

For $1 \leq i \leq n$, by forgetting the i -th factor, we obtain a projection morphism $p_i : X_n \rightarrow X_{n-1}$. One verifies easily that the scheme X_n is, via the morphism p_i , a hyperbolic curve over X_{n-1} of type $(g, r + n - 1)$. Therefore, X_n is a *hyperbolic polycurve* over S in the sense of [3], Definition 2.1, (ii).

1. Moduli spaces of hyperbolic curves of genus one

In this section, we introduce certain moduli spaces of hyperbolic curves of genus one. Let k be a field of characteristic zero and

$$\mathcal{P} : (\text{Ell}/k) \rightarrow (\text{Sets})$$

a moduli problem for elliptic curves over k (cf. [5], (4.2), (4.13)). Hence we have a contravariant functor of [5], (4.3.1)

$$\mathcal{M}(\mathcal{P}) : (\text{Sch}/k) \rightarrow (\text{Sets}),$$

i.e., the functor obtained by considering elliptic curves equipped with level \mathcal{P} structures.

DEFINITION 1.1. We shall say that \mathcal{P} is *hyperbolic* if $\mathcal{M}(\mathcal{P})$ is represented by a hyperbolic curve over k , and, moreover, the natural morphism from the hyperbolic curve that represents $\mathcal{M}(\mathcal{P})$ to the coarse moduli scheme “ \mathbf{A}_k^1 ” of elliptic curves over k is nonconstant, i.e., dominant. If \mathcal{P} is hyperbolic, then we shall denote by

$$\mathcal{M}_{\mathcal{P},1}$$

the hyperbolic curve that represents $\mathcal{M}(\mathcal{P})$.

Remark 1.2. An example of a hyperbolic moduli problem is as follows: If n is an integer greater than two, and k contains a primitive n -th root of unity, then the moduli problem $[\Gamma(n)]$ over k obtained by considering “ $\Gamma(n)$ -structures” (cf. [5], (3.1), (5.1)) is hyperbolic.

DEFINITION 1.3. Let r be a positive integer. Then we shall write

$$\mathcal{M}(\mathcal{P}, r) : (\text{Sch}/k) \rightarrow (\text{Sets})$$

for the contravariant functor such that, for each k -scheme S , the set $\mathcal{M}(\mathcal{P}, r)(S)$ is defined as the set of isomorphism classes of collections of data as follows:

- (i) The elliptic curve E over S equipped with a level \mathcal{P} structure (i.e., an element of $\mathcal{P}(E)$).
- (ii) Ordered r sections $x_1, \dots, x_r : S \rightarrow E$ of the structure morphism $E \rightarrow S$ such that $\text{Im}(x_i) \cap \text{Im}(x_j) = \emptyset$ for $i \neq j$, and, moreover, x_1 is the identity section of $E \rightarrow S$.

Let r be a positive integer. Then it follows from the definition of $\mathcal{M}(\mathcal{P}, r)$ that we have a cartesian diagram of functors over k

$$\begin{array}{ccc} \mathcal{M}(\mathcal{P}, r) & \longrightarrow & \mathcal{M}_{1,r} \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathcal{P}) = \mathcal{M}(\mathcal{P}, 1) & \longrightarrow & \mathcal{M}_{1,1}, \end{array}$$

where we write $\mathcal{M}_{1,r}$ for the (functor represented by the) moduli stack of projective smooth curves of genus one over k equipped with ordered distinct r points (cf. [7]), the vertical arrows are the morphisms obtained by forgetting the last $r - 1$ marked points, and the horizontal arrows are the morphisms obtained by forgetting the data related to \mathcal{P} . Since the right-hand vertical arrow is representable (cf., e.g., the proof of [7], Theorem 2.7), we conclude that if \mathcal{P} is hyperbolic, then $\mathcal{M}(\mathcal{P}, r)$ is represented by a scheme over k .

DEFINITION 1.4. Suppose that \mathcal{P} is hyperbolic. We shall denote by

$$\mathcal{M}_{\mathcal{P},r}$$

the scheme that represents $\mathcal{M}(\mathcal{P}, r)$. Hence we have a cartesian diagram of stacks over k

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{P},r} & \longrightarrow & \mathcal{M}_{1,r} \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathcal{P},1} & \longrightarrow & \mathcal{M}_{1,1}. \end{array}$$

Note that $\mathcal{M}_{\mathcal{P},r}$ is a smooth variety over k (cf. Proposition 1.5, (iii), below).

PROPOSITION 1.5. Suppose that \mathcal{P} is hyperbolic. Then the following hold:

- (i) $\mathcal{M}_{\mathcal{P},2}$ is, via the left-hand vertical arrow of the diagram of Definition 1.4, a hyperbolic curve of type $(1,1)$ over $\mathcal{M}_{\mathcal{P},1}$.
- (ii) If $r \geq 2$, then $\mathcal{M}_{\mathcal{P},r}$ is, via the left-hand vertical arrow of the diagram of Definition 1.4, the $(r - 1)$ -st relative configuration space associated to the hyperbolic curve $\mathcal{M}_{\mathcal{P},2}/\mathcal{M}_{\mathcal{P},1}$.
- (iii) $\mathcal{M}_{\mathcal{P},r}$ is a hyperbolic polycurve over k in the sense of [3], Definition 2.1, (ii). In particular, $\mathcal{M}_{\mathcal{P},r}$ is a smooth variety over k (cf. [3], Remark 2.1.1).

Proof. Assertions (i) and (ii) follow from the definition of $\mathcal{M}(\mathcal{P}, r)$. Assertion (iii) follows from assertions (i), (ii). \square

2. Main results

The main results of the present paper are as follows:

THEOREM 2.1. Let k be a sub- p -adic field for some prime number p and \bar{k} an algebraic closure of k . For $\xi = \alpha, \beta$, let C^ξ be a hyperbolic curve of type (g_ξ^C, r_ξ^C) over k ; X_1^ξ a hyperbolic curve over C^ξ of type (g_ξ, r_ξ) ; $n_\xi \geq 1$ an integer; X^ξ the n_ξ -th relative configuration space associated to X_1^ξ/C^ξ ; $\Pi^\xi \stackrel{\text{def}}{=} \pi_1(X^\xi)$; $\bar{\Pi}^\xi \stackrel{\text{def}}{=} \pi_1(X^\xi \otimes_k \bar{k})$; $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$. Thus, we have an exact sequence of pro-finite groups

$$1 \rightarrow \bar{\Pi}^\xi \rightarrow \Pi^\xi \rightarrow G_k \rightarrow 1.$$

Assume that the following two conditions (1) and (2) hold:

- (1) $g_\alpha < g_\beta^C$ or $r_\alpha + n_\alpha - 1 < r_\beta^C$.
- (2) $g_\beta < g_\alpha^C$ or $r_\beta + n_\beta - 1 < r_\alpha^C$.

Then the natural map

$$\mathrm{Isom}_k(X^\alpha, X^\beta) \rightarrow \mathrm{Isom}_{G_k}(\Pi^\alpha, \Pi^\beta) / \mathrm{Inn}(\bar{\Pi}^\beta)$$

is bijective. If, moreover, $\mathrm{Isom}_k(X^\alpha, X^\beta) \neq \emptyset$, then

$$(g_\alpha^C, r_\alpha^C, g_\alpha, r_\alpha, n_\alpha) = (g_\beta^C, r_\beta^C, g_\beta, r_\beta, n_\beta).$$

THEOREM 2.2. Let k be a sub- p -adic field for some prime number p ; \bar{k} an algebraic closure of k . For $\xi = \alpha, \beta$, let \mathcal{P}_ξ be a hyperbolic (cf. Definition 1.1) moduli problem for elliptic curves over k ; $r_\xi \geq 1$ an integer. Write

$$\mathcal{M}_{\mathcal{P}_\xi, r_\xi}$$

for the smooth variety over k representing the functor $\mathcal{M}(\mathcal{P}_\xi, r_\xi)$ (cf. Definitions 1.3, 1.4); $(g_\xi^\mathcal{M}, r_\xi^\mathcal{M})$ for the type of the hyperbolic curve $\mathcal{M}_{\mathcal{P}_\xi, 1}$ over k . Assume that the following two conditions (1) and (2) hold:

- (1) $1 \leq g_\beta^\mathcal{M}$ or $r_\alpha \leq r_\beta^\mathcal{M}$.
- (2) $1 \leq g_\alpha^\mathcal{M}$ or $r_\beta \leq r_\alpha^\mathcal{M}$.

Then the natural map

$$\mathrm{Isom}_k(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}, \mathcal{M}_{\mathcal{P}_\beta, r_\beta}) \rightarrow \mathrm{Isom}_{G_k}(\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}), \pi_1(\mathcal{M}_{\mathcal{P}_\beta, r_\beta})) / \mathrm{Inn}(\pi_1(\mathcal{M}_{\mathcal{P}_\beta, r_\beta} \otimes_k \bar{k}))$$

is bijective. If, moreover, $\mathrm{Isom}_k(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}, \mathcal{M}_{\mathcal{P}_\beta, r_\beta}) \neq \emptyset$, then

$$(r_\alpha, g_\alpha^\mathcal{M}, r_\alpha^\mathcal{M}) = (r_\beta, g_\beta^\mathcal{M}, r_\beta^\mathcal{M}).$$

Remark 2.3. Note that, in the situation of Theorem 2.1 (resp. Theorem 2.2), if either $n_\alpha \leq 3$ or $n_\beta \leq 3$ (resp. $r_\alpha \leq 4$ or $r_\beta \leq 4$), then Theorem 2.1 (resp. Theorem 2.2) follows from [3], Theorem B, without the conditions (1) and (2).

3. Proofs of Theorems 2.1 and 2.2

In this section, we prove Theorems 2.1 and 2.2. First, we prove the following two lemmas:

LEMMA 3.1. For $\xi = \alpha, \beta$, let k_ξ be an algebraically closed field of characteristic zero; Y^ξ a hyperbolic curve over k_ξ of type (g_ξ, r_ξ) ; $n_\xi \geq 1$ an integer; $Y_{n_\xi}^\xi$ the n_ξ -th relative configuration space associated to Y^ξ/k_ξ . Let

$$\varphi : \pi_1(Y_{n_\alpha}^\alpha) \rightarrow \pi_1(Y_{n_\beta}^\beta)$$

be an isomorphism of profinite groups. Then the following hold:

- (i) $n_\alpha = n_\beta$. If, moreover, $n_\alpha = n_\beta \neq 1$, then $(g_\alpha, r_\alpha) = (g_\beta, r_\beta)$.
- (ii) There exist integers $1 \leq i_\alpha \leq n_\alpha$, $1 \leq i_\beta \leq n_\beta$ such that

$$\varphi(\Delta_{Y_{n_\alpha}^\alpha / Y_{n_\alpha-1}^\alpha}) = \Delta_{Y_{n_\beta}^\beta / Y_{n_\beta-1}^\beta},$$

where, for $\xi = \alpha, \beta$, we write $Y_{n_\xi-1}^\xi$ for the $(n_\xi - 1)$ -st relative configuration space associated to Y^ξ/k_ξ , and the morphism $Y_{n_\xi}^\xi \rightarrow Y_{n_\xi-1}^\xi$ implicit in the kernel $\Delta_{Y_{n_\xi}^\xi/Y_{n_\xi-1}^\xi}$ is the composite of an automorphism of $Y_{n_\xi}^\xi$ over k_ξ and the projection morphism obtained by forgetting the i_ξ -th factor.

Proof. These assertions follow immediately from [4], Theorem 2.5. \square

Remark 3.2. Lemma 3.1, (i) (resp. (ii)), in the case where the isomorphism φ is compatible with the respective weight filtrations of $\pi_1(Y_{n_\alpha}^\alpha)$, $\pi_1(Y_{n_\beta}^\beta)$ follows from [6], Theorem 2.1 (resp. 2.2).

LEMMA 3.3. *Let k be a generalized sub- p -adic field for some prime number p and \bar{k} an algebraic closure of k . For $\xi = \alpha, \beta$, let C^ξ be a hyperbolic curve of type (g_ξ^C, r_ξ^C) over k ; X_1^ξ a hyperbolic curve over C^ξ of type (g_ξ, r_ξ) ; $n_\xi \geq 1$ an integer; X^ξ the n_ξ -th relative configuration space associated to X_1^ξ/C^ξ ; $\Pi^\xi \stackrel{\text{def}}{=} \pi_1(X^\xi)$; $\bar{\Pi}^\xi \stackrel{\text{def}}{=} \pi_1(X^\xi \otimes_k \bar{k})$; $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$. Thus, we have an exact sequence of profinite groups*

$$1 \rightarrow \bar{\Pi}^\xi \rightarrow \Pi^\xi \rightarrow G_k \rightarrow 1.$$

Let

$$\varphi : \Pi^\alpha \rightarrow \Pi^\beta$$

be an isomorphism of profinite groups over G_k . Suppose that

$$\varphi(\Delta_{X^\alpha/C^\alpha}) = \Delta_{X^\beta/C^\beta}.$$

Then φ arises from an isomorphism $X^\alpha \xrightarrow{\sim} X^\beta$ over k . In other words, the element of $\text{Isom}_{G_k}(\Pi^\alpha, \Pi^\beta)/\text{Inn}(\bar{\Pi}^\beta)$ determined by φ is contained in the image of the natural map

$$\text{Isom}_k(X^\alpha, X^\beta) \rightarrow \text{Isom}_{G_k}(\Pi^\alpha, \Pi^\beta)/\text{Inn}(\bar{\Pi}^\beta).$$

Moreover,

$$(g_\alpha^C, r_\alpha^C, g_\alpha, r_\alpha, n_\alpha) = (g_\beta^C, r_\beta^C, g_\beta, r_\beta, n_\beta).$$

Proof. Since $\varphi(\Delta_{X^\alpha/C^\alpha}) = \Delta_{X^\beta/C^\beta}$, the isomorphism φ induces an isomorphism $\pi_1(C^\alpha) \xrightarrow{\sim} \pi_1(C^\beta)$ over G_k . Write $\varphi_0 : \pi_1(C^\alpha) \xrightarrow{\sim} \pi_1(C^\beta)$ for the isomorphism over G_k induced by φ . Then it follows from [9], Theorem 4.12, that φ_0 arises from an isomorphism $f_0 : C^\alpha \xrightarrow{\sim} C^\beta$ over k , which thus implies that $(g_\alpha^C, r_\alpha^C) = (g_\beta^C, r_\beta^C)$. Let \bar{x} be a geometric point of C^α , $Y^\alpha \stackrel{\text{def}}{=} X^\alpha \times_{C^\alpha} \bar{x}$, and Y^β the base change of the projection morphism $X^\beta \rightarrow C^\beta$ by the composite $\bar{x} \rightarrow C^\alpha \xrightarrow{f_0} C^\beta$. Here, observe that, by [3], Proposition 2.4, (i), Δ_{X^ξ/C^ξ} is naturally isomorphic to $\pi_1(Y^\xi)$. Moreover, observe that Y_1^ξ is the n_ξ -th relative configuration space associated to the hyperbolic curve $Y_1^\xi \stackrel{\text{def}}{=} X_1^\xi \times_{C^\xi} \bar{x}$ over \bar{x} . Thus, it follows from Lemma 3.1, (i), that $n_\alpha = n_\beta$, which we denote by n . Moreover, again by Lemma 3.1, (i), if $n \neq 1$, then $(g_\alpha, r_\alpha) = (g_\beta, r_\beta)$. Further-

more, by Lemma 3.1, (ii), there exist integers $1 \leq i_\alpha, i_\beta \leq n$ such that

$$\varphi(\Delta_{Y^\alpha/Y_{n-1}^\alpha}) = \Delta_{Y^\beta/Y_{n-1}^\beta},$$

where, for $\xi = \alpha, \beta$, we write Y_{n-1}^ξ for the $(n-1)$ -st relative configuration space associated to Y_1^ξ/\bar{x} , and the morphism $Y^\xi \rightarrow Y_{n-1}^\xi$ implicit in the kernel $\Delta_{Y^\xi/Y_{n-1}^\xi}$ is the composite of an automorphism of Y^ξ over \bar{x} and the projection morphism obtained by forgetting the i_ξ -th factor. In particular, since, for $\xi = \alpha, \beta$, the homomorphism $\Delta_{Y^\xi/Y_{n-1}^\xi} \rightarrow \Delta_{X^\xi/X_{n-1}^\xi}$ induced by the natural morphism $Y^\xi \rightarrow X^\xi$ is an isomorphism (cf. [3], Proposition 2.4, (ii)), we conclude that

$$\varphi(\Delta_{X^\alpha/X_{n-1}^\alpha}) = \Delta_{X^\beta/X_{n-1}^\beta},$$

where, for $\xi = \alpha, \beta$, we write X_{n-1}^ξ for the $(n-1)$ -st relative configuration space associated to X_1^ξ/C^ξ , and the morphism $X^\xi \rightarrow X_{n-1}^\xi$ implicit in the kernel $\Delta_{X^\xi/X_{n-1}^\xi}$ is the composite of an automorphism of X^ξ over C^ξ and the projection morphism obtained by forgetting the i_ξ -th factor. Thus, the isomorphism φ induces an isomorphism $\pi_1(X_{n-1}^\alpha) \xrightarrow{\sim} \pi_1(X_{n-1}^\beta)$ over G_k . Write $\varphi_{n-1} : \pi_1(X_{n-1}^\alpha) \xrightarrow{\sim} \pi_1(X_{n-1}^\beta)$ for the isomorphism over G_k induced by φ . Note that it follows from $\varphi(\Delta_{X^\alpha/C^\alpha}) = \Delta_{X^\beta/C^\beta}$ that

$$\varphi_{n-1}(\Delta_{X_{n-1}^\alpha/C^\alpha}) = \Delta_{X_{n-1}^\beta/C^\beta}.$$

In the remainder of the proof, let us prove Lemma 3.3 by induction on n . Suppose that φ_{n-1} arises from an isomorphism $f_{n-1} : X_{n-1}^\alpha \xrightarrow{\sim} X_{n-1}^\beta$ over k . Write $\eta \rightarrow X_{n-1}^\alpha$ for the generic point of X_{n-1}^α , $X_\eta^\alpha \stackrel{\text{def}}{=} X^\alpha \times_{X_{n-1}^\alpha} \eta$, and X_η^β for the base change of the projection morphism $X^\beta \rightarrow X_{n-1}^\beta$ by the composite $\eta \rightarrow X_{n-1}^\alpha \xrightarrow{f_{n-1}} X_{n-1}^\beta$. Then it follows from [3], Proposition 2.4, (ii), that φ induces an isomorphism $\pi_1(X_\eta^\alpha) \xrightarrow{\sim} \pi_1(X_\eta^\beta)$ over $\pi_1(\eta)$. In particular, by [9], Theorem 4.12, there exists an isomorphism $X_\eta^\alpha \xrightarrow{\sim} X_\eta^\beta$ over η from which the isomorphism $\pi_1(X_\eta^\alpha) \xrightarrow{\sim} \pi_1(X_\eta^\beta)$ arises, which thus implies that if $n = 1$, then $(g_\alpha, r_\alpha) = (g_\beta, r_\beta)$. Hence it follows from [3], Lemma 2.10, that φ arises from a morphism $X^\alpha \rightarrow X^\beta$ over k . Therefore, by applying a similar argument to φ^{-1} , it follows from [3], Proposition 3.2, (i), that φ arises from an isomorphism $X^\alpha \xrightarrow{\sim} X^\beta$ over k . This completes the proof of Lemma 3.3. \square

3.4. Let us prove Theorem 2.1. First, recall that the injectivity of the map $\text{Isom}_k(X^\alpha, X^\beta) \rightarrow \text{Isom}_{G_k}(\Pi^\alpha, \Pi^\beta)/\text{Inn}(\overline{\Pi}^\beta)$ follows from [3], Proposition 3.2, (ii). Let φ be an element of $\text{Isom}_{G_k}(\Pi^\alpha, \Pi^\beta)$. It follows from Lemma 3.3 that, to prove Theorem 2.1, it suffices to show that

$$\varphi(\Delta_{X^\alpha/C^\alpha}) = \Delta_{X^\beta/C^\beta}.$$

To this end, let us consider the composite

$$\Delta_{X^\alpha/X_{n_\alpha-1}^\alpha} \hookrightarrow \Delta_{X^\alpha/k} \xrightarrow{\varphi} \Delta_{X^\beta/k} \rightarrow \Delta_{C^\beta/k},$$

where we denote by $X_{n_\alpha-1}^\alpha$ the $(n_\alpha-1)$ -st relative configuration space associated to X_1^α/C^α , and the morphism $X^\alpha \rightarrow X_{n_\alpha-1}^\alpha$ implicit in the kernel $\Delta_{X^\alpha/X_{n_\alpha-1}^\alpha}$ is the

projection morphism obtained by forgetting the n_α -th factor. It follows from [3], Proposition 2.4, (iii), (iv), that $\Delta_{X^\alpha/X_{n_\alpha-1}^\alpha}$ is topologically finitely generated, and $\Delta_{C^\beta/k}$ is elastic. (Here, let us recall that a profinite group G is *elastic* if the following condition holds: Let H be a closed subgroup of G . Suppose that H is topologically finitely generated and normal in an open subgroup of G that contains H . Then H is either open in G or trivial.) Hence the image of the above composite is either open or trivial. Suppose that the image is open. Then, by the following discussion, we obtain a *contradiction*: Let x be a closed point of $X_{n_\alpha-1}^\alpha$ and $\bar{x} \rightarrow X_{n_\alpha-1}^\alpha$ a geometric point of $X_{n_\alpha-1}^\alpha$ lying over x . Then it follows from [3], Proposition 2.4, (i), that $\Delta_{X^\alpha/X_{n_\alpha-1}^\alpha}$ is isomorphic to $\pi_1(X^\alpha \times_{X_{n_\alpha-1}^\alpha} \bar{x})$. Hence, by [8], Theorem A, the above composite arises from a dominant morphism $X^\alpha \times_{X_{n_\alpha-1}^\alpha} x \rightarrow C^\beta$ over k . On the other hand, such a dominant morphism does not exist by the condition (1), together with the fact that the hyperbolic curve $X^\alpha \times_{X_{n_\alpha-1}^\alpha} \bar{x}$ is of type $(g_\alpha, r_\alpha + n_\alpha - 1)$. Therefore, we conclude that the image of the above composite is trivial, which thus implies that the morphism $\Pi^\alpha \rightarrow \pi_1(C^\beta)$ induced by φ factors through the projection $\Pi^\alpha \rightarrow \pi_1(X_{n_\alpha-1}^\alpha)$.

Moreover, by applying a similar argument to the argument of the preceding paragraph *inductively*, we conclude that the morphism $\Pi^\alpha \rightarrow \pi_1(C^\beta)$ induced by φ factors through the projection $\Pi^\alpha \rightarrow \pi_1(C^\alpha)$, which thus implies that

$$\varphi(\Delta_{X^\alpha/C^\alpha}) \subseteq \Delta_{X^\beta/C^\beta}.$$

Furthermore, by applying a similar argument to φ^{-1} , we obtain that

$$\varphi(\Delta_{X^\alpha/C^\alpha}) = \Delta_{X^\beta/C^\beta}.$$

This completes the proof of Theorem 2.1.

3.5. Let us prove Theorem 2.2. If either r_α or r_β is equal to one, then Theorem 2.2 follows immediately from [3], Theorem B. Suppose that both r_α and r_β are greater than one. Then $\mathcal{M}_{\mathcal{P}_\xi, r_\xi}$ is the $(r_\xi - 1)$ -st relative configuration space associated to the hyperbolic curve $\mathcal{M}_{\mathcal{P}_\xi, 2}/\mathcal{M}_{\mathcal{P}_\xi, 1}$ of type $(1, 1)$ (cf. Proposition 1.5, (i), (ii)). Hence it follows immediately from a similar argument to the argument applied in the proof of Theorem 2.1 that, to prove Theorem 2.2, it suffices to show the following assertion: If $\varphi: \pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}) \xrightarrow{\sim} \pi_1(\mathcal{M}_{\mathcal{P}_\beta, r_\beta})$ is an isomorphism over G_k , then the image of the composite

$$\Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}} \hookrightarrow \Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/k} \xrightarrow{\varphi} \Delta_{\mathcal{M}_{\mathcal{P}_\beta, r_\beta}/k} \rightarrow \Delta_{\mathcal{M}_{\mathcal{P}_\beta, 1}/k}$$

is not open. To this end, suppose that the image of the above composite is open. Then, by [8], Theorem A, for any geometric point \bar{s} of $\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}$ which lies over a closed point, there exists a dominant morphism

$$\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha} \times_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}} \bar{s} \rightarrow \mathcal{M}_{\mathcal{P}_\beta, 1} \times_k \bar{s}$$

from the hyperbolic curve $\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha} \times_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}} \bar{s}$ over \bar{s} of type $(1, r_\alpha - 1)$ to the hyperbolic curve $\mathcal{M}_{\mathcal{P}_\beta, 1} \times_k \bar{s}$ over \bar{s} of type $(g_\beta^{\mathcal{M}}, r_\beta^{\mathcal{M}})$ (cf. the discussion of the second paragraph of the proof of Theorem 2.1).

Suppose that $g_\beta^{\mathcal{M}} > 1$. Then it is immediate such a dominant morphism does not exist. Thus, we obtain a contradiction.

Suppose that $g_\beta^{\mathcal{M}} = 0$. Then it follows from the condition (1) that such a dominant morphism does not exist. Thus, we obtain a contradiction.

Suppose that $g_\beta^{\mathcal{M}} = 1$. Let us recall that, by the definition of a hyperbolic moduli problem, the natural morphism from $\mathcal{M}_{\mathcal{P}_\alpha, 1}$ to the coarse moduli scheme of elliptic curves over k is dominant. Thus, by varying \bar{s} , for all but finitely many $j \in \bar{k}$, one may take an elliptic curve over \bar{s} (i.e., over \bar{k}) whose j -invariant is $j \in \bar{k}$ as the smooth compactification of the source of the above dominant morphism

$$\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha} \times_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}} \bar{s} \rightarrow \mathcal{M}_{\mathcal{P}_\beta, 1} \times_k \bar{s}.$$

Hence one may conclude that, for all but finitely many $j \in \bar{k}$, the smooth compactification of the hyperbolic curve $\mathcal{M}_{\mathcal{P}_\beta, 1} \times_k \bar{s}$ is isogenous to an elliptic curve over \bar{s} whose j -invariant is $j \in \bar{k}$. Thus, we obtain a contradiction. This completes the proof of Theorem 2.2.

4. Absolute variants

In this section, we prove some variants of the results of section 2 in absolute anabelian geometry.

THEOREM 4.1. *For $\xi = \alpha, \beta$, let k_ξ be a finitely generated extension of \mathbf{Q} ; \bar{k}_ξ an algebraic closure of k_ξ ; C^ξ a hyperbolic curve of type (g_ξ^C, r_ξ^C) over k_ξ ; X_1^ξ a hyperbolic curve over C^ξ of type (g_ξ, r_ξ) ; $n_\xi \geq 1$ an integer; X^ξ the n_ξ -th relative configuration space associated to X_1^ξ/C^ξ ; $\Pi^\xi \stackrel{\text{def}}{=} \pi_1(X^\xi)$. Assume that the following two conditions hold:*

- (1) $g_\beta < g_\alpha^C$ or $r_\beta + n_\beta - 1 < r_\alpha^C$.
- (2) $g_\alpha < g_\beta^C$ or $r_\alpha + n_\alpha - 1 < r_\beta^C$.

Then the natural map

$$\text{Isom}(X^\alpha, X^\beta) \rightarrow \text{Isom}(\Pi^\alpha, \Pi^\beta) / \text{Inn}(\Pi^\beta)$$

is bijective. If, moreover, $\text{Isom}(X^\alpha, X^\beta) \neq \emptyset$, then

$$(g_\alpha^C, r_\alpha^C, g_\alpha, r_\alpha, n_\alpha) = (g_\beta^C, r_\beta^C, g_\beta, r_\beta, n_\beta).$$

Proof. The injectivity follows from [3], Proposition 3.2, (ii), and [12], Lemma 4.2, together with the well-known fact that the absolute Galois group of a finitely generated extension of \mathbf{Q} is center-free. Let φ be an element of $\text{Isom}(\Pi^\alpha, \Pi^\beta)$. We consider the composite $\Delta_{X^\alpha/k_\alpha} \hookrightarrow \Pi^\alpha \xrightarrow{\varphi} \Pi^\beta \rightarrow G_{k_\beta}$. Since $\Delta_{X^\alpha/k_\alpha}$ is topologically finitely generated (cf. [3], Lemmas 1.5, 1.7), it follows from [3], Proposition 3.19, (i), that the composite is trivial. By applying a similar argument, it follows that the composite $\Delta_{X^\beta/k_\beta} \hookrightarrow \Pi^\beta \xrightarrow{\varphi^{-1}} \Pi^\alpha \rightarrow G_{k_\alpha}$ is trivial. Hence it follows that φ lies over an isomorphism $G_{k_\alpha} \xrightarrow{\sim} G_{k_\beta}$. It follows from [3], Proposition 3.19, (ii), that the isomorphism $G_{k_\alpha} \xrightarrow{\sim} G_{k_\beta}$ arises from an isomor-

phism $\bar{k}_\beta \xrightarrow{\sim} \bar{k}_\alpha$ that determines an isomorphism $k_\beta \xrightarrow{\sim} k_\alpha$. Then, by taking the base change of X^β by the isomorphism $\text{Spec } k_\alpha \xrightarrow{\sim} \text{Spec } k_\beta$ induced by $k_\beta \xrightarrow{\sim} k_\alpha$, it follows from Theorem 2.1 that φ arises from an isomorphism $X^\alpha \xrightarrow{\sim} X^\beta$, and $(g_\alpha^C, r_\alpha^C, g_\alpha, r_\alpha, n_\alpha) = (g_\beta^C, r_\beta^C, g_\beta, r_\beta, n_\beta)$. This completes the proof of Theorem 4.1. \square

DEFINITION 4.2. Let k be a p -adic local field (for some prime number p) and \mathcal{P} a moduli problem for elliptic curves over k . Then we shall say that \mathcal{P} is of *quasi-Belyi type* if \mathcal{P} is hyperbolic, and, moreover, the hyperbolic curve $\mathcal{M}_{\mathcal{P},1}$ over k is of quasi-Belyi type (cf. [10], Definition 2.3, (iii)), i.e., the hyperbolic curve $\mathcal{M}_{\mathcal{P},1}$ over k is defined over a finite extension of \mathbf{Q} , and, moreover, there exist a connected finite étale covering $Z \rightarrow \mathcal{M}_{\mathcal{P},1}$ and a dominant morphism from Z to the complement in the projective line of three rational points over a finite extension of k .

Remark 4.3. In the situation of Definition 4.2, the example of a hyperbolic moduli problem of Remark 1.2 is of quasi-Belyi type.

THEOREM 4.4. For $\xi = \alpha, \beta$, let k_ξ be a field of characteristic zero; \bar{k}_ξ an algebraic closure of k_ξ ; \mathcal{P}_ξ a hyperbolic (cf. Definition 1.1) moduli problem for elliptic curves over k_ξ ; $r_\xi \geq 1$ an integer. Write

$$\mathcal{M}_{\mathcal{P}_\xi, r_\xi}$$

for the smooth variety over k_ξ representing the functor $\mathcal{M}(\mathcal{P}_\xi, r_\xi)$ (cf. Definitions 1.3, 1.4); $(g_\xi^{\mathcal{M}}, r_\xi^{\mathcal{M}})$ for the type of the hyperbolic curve $\mathcal{M}_{\mathcal{P}_\xi,1}$ over k_ξ . Assume that one of the following two conditions (i) and (ii) holds:

- (i) The following conditions hold:
 - (1) For $\xi = \alpha, \beta$, the field k_ξ is finitely generated over \mathbf{Q} .
 - (2) $1 \leq g_\beta^{\mathcal{M}}$ or $r_\alpha \leq r_\beta^{\mathcal{M}}$.
 - (3) $1 \leq g_\alpha^{\mathcal{M}}$ or $r_\beta \leq r_\alpha^{\mathcal{M}}$.
- (ii) The following conditions hold:
 - (1) For $\xi = \alpha, \beta$, the field k_ξ is p_ξ -adic local for some prime number p_ξ .
 - (2) $r_\alpha < 2g_\beta^{\mathcal{M}} + r_\beta^{\mathcal{M}} - 1$.
 - (3) $r_\beta < 2g_\alpha^{\mathcal{M}} + r_\alpha^{\mathcal{M}} - 1$.
 - (4) For $\xi = \alpha, \beta$, the moduli problem \mathcal{P}_ξ is of quasi-Belyi type (cf. Definition 4.2).

Then the natural map

$$\text{Isom}(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}, \mathcal{M}_{\mathcal{P}_\beta, r_\beta}) \rightarrow \text{Isom}(\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}), \pi_1(\mathcal{M}_{\mathcal{P}_\beta, r_\beta})) / \text{Inn}(\pi_1(\mathcal{M}_{\mathcal{P}_\beta, r_\beta}))$$

is bijective. If, moreover, $\text{Isom}(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}, \mathcal{M}_{\mathcal{P}_\beta, r_\beta}) \neq \emptyset$, then

$$(r_\alpha, g_\alpha^{\mathcal{M}}, r_\alpha^{\mathcal{M}}) = (r_\beta, g_\beta^{\mathcal{M}}, r_\beta^{\mathcal{M}}).$$

Proof. Suppose that the condition (i) holds. Then Theorem 4.4 follows from a similar argument as in the proof of Theorem 4.1, together with Theorem 2.2.

Suppose that the condition (ii) holds. The injectivity follows from [3], Proposition 3.2, (ii), and [12], Lemma 4.2, together with the well-known fact that the absolute Galois group of a p -adic local field is center-free. Let φ be an element of $\text{Isom}(\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}), \pi_1(\mathcal{M}_{\mathcal{P}_\beta, r_\beta}))$. It follows from [11], Corollary 2.8, (ii), that $\varphi(\Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/k_\alpha}) = \Delta_{\mathcal{M}_{\mathcal{P}_\beta, r_\beta}/k_\beta}$ and $p_\alpha = p_\beta$.

Let us consider the composite

$$\Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}} \hookrightarrow \Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/k_\alpha} \xrightarrow{\varphi} \Delta_{\mathcal{M}_{\mathcal{P}_\beta, r_\beta}/k_\beta} \rightarrow \Delta_{\mathcal{M}_{\mathcal{P}_\beta, 1}/k_\beta}.$$

It follows immediately from [3], Proposition 2.4, (iii), (iv), that $\Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}}$ is topologically finitely generated, and $\Delta_{\mathcal{M}_{\mathcal{P}_\beta, 1}/k_\beta}$ is elastic. Hence the image of the above composite is either open or trivial. Recall that the abelianization of $\Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1}}$ (resp. $\Delta_{\mathcal{M}_{\mathcal{P}_\beta, 1}/k_\beta}$) is a free $\hat{\mathbf{Z}}$ -module of rank r_α (resp. $2g_\beta^{\mathcal{M}} + r_\beta^{\mathcal{M}} - 1$) (cf. [3], Proposition 2.4, (v)). In particular, it follows from the condition (2) that the above composite is trivial. Hence the morphism $\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}) \rightarrow \pi_1(\mathcal{M}_{\mathcal{P}_\beta, 1})$ factors through the projection $\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}) \rightarrow \pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha-1})$.

Moreover, by applying a similar argument to the argument of the preceding paragraph *inductively*, we conclude that the morphism $\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}) \rightarrow \pi_1(\mathcal{M}_{\mathcal{P}_\beta, 1})$ induced by φ factors through the projection $\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}) \rightarrow \pi_1(\mathcal{M}_{\mathcal{P}_\alpha, 1})$, which thus implies that

$$\varphi(\Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/\mathcal{M}_{\mathcal{P}_\alpha, 1}}) \subseteq \Delta_{\mathcal{M}_{\mathcal{P}_\beta, r_\beta}/\mathcal{M}_{\mathcal{P}_\beta, 1}}.$$

Furthermore, by applying a similar argument to φ^{-1} , we obtain that

$$\varphi(\Delta_{\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha}/\mathcal{M}_{\mathcal{P}_\alpha, 1}}) = \Delta_{\mathcal{M}_{\mathcal{P}_\beta, r_\beta}/\mathcal{M}_{\mathcal{P}_\beta, 1}}.$$

Hence we conclude that the isomorphism φ induces an isomorphism $\pi_1(\mathcal{M}_{\mathcal{P}_\alpha, 1}) \xrightarrow{\sim} \pi_1(\mathcal{M}_{\mathcal{P}_\beta, 1})$. In particular, by the condition (4), it follows from [10], Corollary 2.3, that this isomorphism arises from an isomorphism $\mathcal{M}_{\mathcal{P}_\alpha, 1} \xrightarrow{\sim} \mathcal{M}_{\mathcal{P}_\beta, 1}$. Then, by applying a similar argument to the argument applied in the proof of Lemma 3.3, it follows that φ arises from an isomorphism $\mathcal{M}_{\mathcal{P}_\alpha, r_\alpha} \xrightarrow{\sim} \mathcal{M}_{\mathcal{P}_\beta, r_\beta}$, and $(r_\alpha, g_\alpha^{\mathcal{M}}, r_\alpha^{\mathcal{M}}) = (r_\beta, g_\beta^{\mathcal{M}}, r_\beta^{\mathcal{M}})$. This completes the proof of Theorem 4.4. \square

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