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ABSOLUTE ZETA FUNCTIONS AND THE AUTOMORPHY

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Introduction

In this paper we study the absolute zeta function $\zeta_f(s)$ associated to a certain "absolute automorphic form" $f(x)$ on the group

$$\Gamma = \mathbf{R}_{>0} = \{x \in \mathbf{R} \mid x > 0\}.$$

We require $f(x)$ to satisfy the following automorphy:

$$f\left(\frac{1}{x}\right) = Cx^{-D}f(x),$$

where $D \in \mathbf{Z}$ with $C = \pm 1$.

To explain our problem we first recall the history of absolute zeta functions briefly. Soulé [17] (2004) introduced the absolute zeta function (the zeta functions over $\mathbf{F}_1$) of a suitable scheme X as the limit of the congruence zeta function

$$\zeta_{X/\mathbf{F}_1} = \lim_{p \to 1} \zeta_{X/\mathbf{F}_p}(s),$$

where

$$\zeta_{X/\mathbf{F}_p}(s) = \exp\left(\sum_{m=1}^{\infty} \frac{|X(\mathbf{F}_{p^m})|}{m} p^{-ms}\right);$$

see Kurokawa [12] (2005) and Deitmar [4] (2006). Later Connes-Consani [2] (2010) [3] (2011) interpreted it as

$$\zeta_{X/\mathbf{F}_1}(s) = \exp\left(\int_1^{\infty} \frac{f(x)x^{-s-1}}{\log x}\, dx\right)$$

when

$$f(x) = |X(\mathbf{F}_x)| \in \mathbf{Z}[x].$$

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At this point, it would be suggestive to explain this integral via the Jackson p-integral. The congruence zeta function

$$\zeta_{X/\mathbf{F}_p}(s) = \exp\left(\sum_{m=1}^{\infty} \frac{f(p^m)}{m} p^{-ms}\right)$$

is written as

$$\zeta_{X/\mathbf{F}_p}(s) = \exp\left(\frac{\log p}{1-p^{-1}} \int_1^{\infty} \frac{f(x)x^{-s-1}}{\log x} \, d_p x\right),$$

where the Jackson p-integral ($p > 1$)

$$\int_1^{\infty} g(x) \, d_p x = \sum_{m=1}^{\infty} g(p^m)(p^m - p^{m-1})$$

will have the property

$$\lim_{p \to 1} \int_1^{\infty} g(x) \, d_p x = \int_1^{\infty} g(x) \, dx$$

for a suitable class of functions $g(x)$. Hence, for

$$g(x) = \frac{\log p}{1-p^{-1}} \cdot \frac{f(x)x^{-s-1}}{\log x}$$

it would be reasonable to expect that

$$\lim_{p \to 1} \zeta_{X/\mathbf{F}_p}(s) = \exp\left(\int_1^{\infty} \frac{f(x)x^{-s-1}}{\log x} \, dx\right).$$

Unfortunately this integral has divergency in general. For example, let $X = \mathbf{P}^n$. Then

$$f(x) = |\mathbf{P}^n(\mathbf{F}_x)| = x^n + x^{n-1} + \cdots + 1 = \frac{x^{n+1}-1}{x-1},$$

so

$$\int_1^{\infty} \frac{f(x)x^{-s-1}}{\log x} \, dx = \infty$$

from the contribution around $x = 1$, where $f(1) = n+1$.

To remedy this difficulty, Kurokawa-Ochiai [9] (2013) and Deitmar-Koyama-Kurokawa [5] (2015) used the zeta-regularization process:

$$\zeta_{X/\mathbf{F}_1}(s) = \exp\left(\frac{\partial}{\partial w} Z_{X/\mathbf{F}_1}(w,s)\bigg|_{w=0}\right)$$

with

$$Z_{X/\mathbf{F}_1}(w,s) = \frac{1}{\Gamma(w)}\int_1^\infty f(x)x^{-s-1}(\log x)^{w-1}\,dx.$$

We notice that

$$Z_{X/\mathbf{F}_1}(w,s) = \frac{1}{\Gamma(w)}\int_0^\infty f(e^t)e^{-st}t^{w-1}\,dt$$

is the Mellin transform frequently used in the theory of zeta functions, where $f(e^t)$ is usually an automorphic form.

As an example of this procedure, we see that

$$Z_{\mathbf{P}^n/\mathbf{F}_1}(w,s) = (s-n)^{-w} + (s-(n-1))^{-w} + \cdots + s^{-w}$$

and

$$\zeta_{\mathbf{P}^n/\mathbf{F}_1}(s) = \frac{1}{(s-n)(s-(n-1))\cdots s},$$

which has a functional equation

$$\zeta_{\mathbf{P}^n/\mathbf{F}_1}(n-s) = (-1)^{n+1}\zeta_{\mathbf{P}^n/\mathbf{F}_1}(s).$$

Here

$$n = \dim \mathbf{P}^n$$

and

$$n+1 = \chi(\mathbf{P}^n) = f(1),$$

where $\chi(X)$ is the Euler-Poincaré characteristic. At the same time

$$f(x) = \frac{x^{n+1}-1}{x-1}$$

has the automorphy

$$f\left(\frac{1}{x}\right) = x^{-n}f(x)$$

corresponding to the functional equation of $\zeta_{\mathbf{P}^n/\mathbf{F}_1}(s)$ under $s \leftrightarrow n-s$.

The situation is exactly similar in the case of the Grassmannian scheme $Gr(n,m)$ classifying the m-dimensional linear subspaces in the n-dimensional linear space, where

$$\begin{aligned} f(x) &= |Gr(n,m)(\mathbf{F}_x)| \\ &= \frac{(x^n-1)\cdots(x^{n-m+1}-1)}{(x^m-1)\cdots(x-1)} \end{aligned}$$

with the automorphy

$$f\left(\frac{1}{x}\right) = x^{-m(n-m)} f(x).$$

Here

$$m(n-m) = \dim \, Gr(n,m).$$

The absolute zeta function $\zeta_{Gr(n,m)/\mathbf{F}_1}(s)$ is a rational function satisfying the functional equation

$$\zeta_{Gr(n,m)/\mathbf{F}_1}(m(n-m)-s) = (-1)^{\binom{n}{m}} \zeta_{Gr(n,m)/\mathbf{F}_1}(s),$$

where

$$\binom{n}{m} = \chi(Gr(n,m)) = f(1).$$

The case of $\mathbf{P}^n$ is a special case of the Grassmannian:

$$\mathbf{P}^n = Gr(n+1,1).$$

These are particular cases of the following Theorem 1, which is the first result of this paper.

THEOREM 1. *Let*

$$f(x) = x^l \frac{(x^{m(1)}-1)\cdots(x^{m(a)}-1)}{(x^{n(1)}-1)\cdots(x^{n(b)}-1)}$$

with an integer $l \geq 0$, *and positive integers* $m(i)$, $n(j)$. *Put*

$$\begin{aligned} \deg(f) &= l + |\underline{m}| - |\underline{n}|, \\ \widetilde{\deg}(f) &= \deg(f) + l \\ &= 2l + |\underline{m}| - |\underline{n}|, \end{aligned}$$

where

$$|\underline{m}| = \sum_{i=1}^{a} m(i),$$

$$|\underline{n}| = \sum_{j=1}^{b} n(j).$$

Define

$$\zeta_f(s) = \exp\left(\frac{\partial}{\partial w} Z_f(w,s)\bigg|_{w=0}\right)$$

with

$$Z_f(w,s) = \frac{1}{\Gamma(w)} \int_1^{\infty} f(x) x^{-s-1} (\log x)^{w-1} \, dx.$$

Then the following properties hold.

(1)

$$f\left(\frac{1}{x}\right) = Cx^{-D} f(x)$$

with $C = (-1)^{a-b}$ *and* $D = \widetilde{\deg}(f)$.

(2) $\zeta_f(s)$ *is a meromorphic function on* $\mathbf{C}$ *written explicitly by multiple gamma functions of order b. Moreover, zeros and poles of* $\zeta_f(s)$ *belong to* $\mathbf{Z}$.

(3) $\zeta_f(s)$ *is non-zero holomorphic in* $\mathrm{Re}(s) > \deg(f) - 1$ *except for the simple pole at* $s = \deg(f)$.

(4) $\zeta_f(s)$ *has a functional equation*

$$\zeta_f(D-s)^C = \varepsilon_f(s) \zeta_f(s),$$

where $\varepsilon_f(s)$ *is written explicitly by multiple sine functions of order b.*

(5) $\zeta_f(s)$ *is a rational function if and only if* $f(x) \in \mathbf{Z}[x]$.

(6) *When* $\zeta_f(s)$ *is a rational function* (*i.e.*, $f(x) \in \mathbf{Z}[x]$ *by* (5))

$$\zeta_f(D-s)^C = (-1)^{f(1)} \zeta_f(s).$$

We may refer to $f(x)$ and $\zeta_f(s)$ in Theorem 1 as "cyclotomic absolute automorphic forms" and "cyclotomic absolute zeta functions" respectively since

$$\left\{ f(x) = x^l \frac{\prod_{i=1}^a (x^{m(i)} - 1)}{\prod_{j=1}^b (x^{n(j)} - 1)} \;\middle|\; \begin{matrix} l \in \mathbf{Z}_{\geq 0}, \\ m(i), n(j) \in \mathbf{Z}_{>0}, \\ a, b \in \mathbf{Z}_{\geq 0} \end{matrix} \right\}$$

$$= \left\{ f(x) = x^l \prod_{m=1}^{M} \Phi_m(x)^{c(m)} \;\middle|\; \begin{matrix} l \in \mathbf{Z}_{\geq 0}, \\ c(m) \in \mathbf{Z}, \\ M \in \mathbf{Z}_{\geq 0} \end{matrix} \right\},$$

where $\Phi_m(x)$ is the m-th cyclotomic polynomial.

We explain general procedures of constructing absolute zeta functions with the theory of multiple gamma functions and multiple sine functions in §1. The detailed formulations and the proof of Theorem 1 are given in §2. We show that many examples of X such as $\mathbf{A}^n$, $\mathbf{P}^n$, $Gr(n,m)$, $GL(n)$, $SL(n)$, $Sp(n)$ are special cases of "cyclotomic" Theorem 1 in §3.

The next theme of this paper is to consider the (absolute) zeta function $\zeta_\rho(s)$ of a virtual representation ρ of Γ.

Theorem 2. *Let* $\rho = (\rho_+, \rho_-)$ *be a virtual representation of* Γ:

$$\rho_\pm : \Gamma \to GL(d_\pm, \mathbf{C}).$$

Define

$$\zeta_\rho(s) = \operatorname{sdet}(s - D_\rho)^{-1} = \frac{\det(s - D_{\rho_-})}{\det(s - D_{\rho_+})}$$

with

$$D_{\rho_\pm} = \lim_{x \to 1} \frac{\rho_\pm(x) - \rho_\pm(1)}{x - 1}.$$

Let

$$f(x) = \operatorname{str}(\rho(x)) = \operatorname{tr}(\rho_+(x)) - \operatorname{tr}(\rho_-(x)).$$

Then the following properties hold.

(1)

$$\zeta_\rho(s) = \zeta_f(s),$$

where

$$\zeta_f(s) = \exp\left(\frac{\partial}{\partial w} Z_f(w,s)\bigg|_{w=0}\right)$$

with

$$Z_f(w,s) = \frac{1}{\Gamma(w)} \int_1^\infty f(x) x^{-s-1} (\log x)^{w-1}\, dx.$$

(2) *When* ρ *is self-dual* (*self-contragradient*) *unitary*,

$$\zeta_\rho(-s) = (-1)^{\deg(\rho)} \zeta_\rho(s)$$

with

$$\deg(\rho) = \deg(\rho_+) - \deg(\rho_-) = d_+ - d_-$$

and

$$f\left(\frac{1}{x}\right) = f(x).$$

In this case $\zeta_\rho(s)$ (*and* $\zeta_f(s)$) *satisfies the analogue of the Riemann hypothesis*:

$$\zeta_\rho(s) = 0, \quad \infty \Rightarrow \operatorname{Re}(s) = 0.$$

The proof is given in §4. We notice that the correspondence from a representation ρ to an absolute automorphic form f may be regarded as the Langlands correspondence over $\mathbf{F}_1$.

The next theme of this paper is to consider the absolute zeta function of several valuables $\zeta_f(s_1, \dots, s_q)$.

THEOREM 3. *Let*

$$f(x_1, \dots, x_q) = x_1^{l(1)/2} \cdots x_q^{l(q)/2} \frac{\prod_{i=1}^{a}(x_1^{m(i,1)} \cdots x_q^{m(i,q)} - 1)}{\prod_{j=1}^{b}(x_1^{n(j,1)} \cdots x_q^{n(j,q)} - 1)}$$

with an integer $l(k) \geq 0$, and positive integers $m(i,k)$, $n(j,k)$.

Put

$$\widetilde{\deg}_k(f) = l(k) + \sum_{i=1}^{a} m(i,k) - \sum_{j=1}^{b} n(j,k).$$

Define

$$\zeta_f(s_1, \dots, s_q) = \exp\left(\frac{\partial}{\partial w} Z_f(w, (s_1, \dots, s_q))\bigg|_{w=0}\right)$$

with

$$Z_f(w, (s_1, \dots, s_q)) = \frac{1}{\Gamma(w)^q} \int_1^\infty \cdots \int_1^\infty f(x_1, \dots, x_q) x_1^{-s_1} \cdots x_q^{-s_q}$$
$$\times ((\log x_1) \cdots (\log x_q))^{w-1} \frac{dx_1}{x_1} \cdots \frac{dx_q}{x_q}.$$

Then the following properties hold.

(1)

$$f\left(\frac{1}{x_1}, \dots, \frac{1}{x_q}\right) = C x_1^{-D(1)} \cdots x_q^{-D(q)} f(x_1, \dots, x_q)$$

with $C = (-1)^{a-b}$ and $D(k) = \widetilde{\deg}_k(f)$.

(2) *$\zeta_f(s_1, \dots, s_q)$ is a meromorphic function on $\mathbf{C}^q$ written explicitly by generalized multiple gamma functions.*

(3) *$\zeta_f(s_1, \dots, s_q)$ has a functional equation*

$$\zeta_f(D(1) - s_1, \dots, D(q) - s_q)^C = \varepsilon_f(s_1, \dots, s_q) \zeta_f(s_1, \dots, s_q),$$

where $\varepsilon_f(s_1, \dots, s_q)$ is written explicitly by generalized multiple sine functions.

The proof is given in §5. We may refer to $f(x_1, \dots, x_q)$ and $\zeta_f(s_1, \dots, s_q)$ in Theorem 3 as "cyclotomic absolute automorphic forms of several valuables" and "cyclotomic absolute zeta functions of several variables" respectively.

THEOREM 4. *We assume $q \geq 2$ and $f(x_1, \dots, x_q) \in \mathbf{Z}[x_1, \dots, x_q]$. Then the following conditions* (1) *and* (2) *are equivalent.*

(1)

$$f(x_1, \dots, x_q) \in ((x_i - 1)(x_j - 1) \mid 1 \leq i < j \leq q).$$

(2)
$$\zeta_f(s_1, \dots, s_q) = 1.$$

The proof is given in §6. We remark that in the one variable case, $\zeta_f(s) = 1$ if and only if $f(x) = 0$.

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1. General constructions

Let
$$f : \Gamma - \{1\} \to \mathbf{C} \cup \{\infty\}$$
be a function. We define
$$Z_f(w,s) = \frac{1}{\Gamma(w)} \int_1^\infty f(x) x^{-s} (\log x)^{w-1} \frac{dx}{x}$$
and
$$\zeta_f(s) = \exp\left(\frac{\partial}{\partial w} Z_f(w,s) \Big|_{w=0} \right),$$
where we assume that $Z_f(w,s)$ has an analytic continuation in w to a region containing $w = 0$.

We define
$$f^*(x) = f\left(\frac{1}{x}\right)$$
and
$$\varepsilon_f(s) = \frac{\zeta_{f^*}(-s)}{\zeta_f(s)}.$$
For example, let
$$f(x) = x^\alpha \quad (\alpha \in \mathbf{C}).$$
Then
$$Z_f(w,s) = (s-\alpha)^{-w},$$
$$\zeta_f(s) = \frac{1}{s-\alpha},$$
$$f^*(x) = x^{-\alpha},$$
$$\zeta_{f^*}(s) = \frac{1}{s+\alpha},$$

and

$$\varepsilon_f(s) = -1.$$

It would be interesting to extend the above construction to the case of several variables, where

$$f : (\Gamma - \{1\})^q \to \mathbf{C} \cup \{\infty\},$$

$$Z_f(w, (s_1, \dots, s_q)) = \frac{1}{\Gamma(w)^q} \int_1^\infty \cdots \int_1^\infty f(x_1, \dots, x_q) x_1^{-s_1} \cdots x_q^{-s_q} \times ((\log x_1) \cdots (\log x_q))^{w-1} \frac{dx_1}{x_1} \cdots \frac{dx_q}{x_q},$$

$$\zeta_f(s_1, \dots, s_q) = \exp\left(\frac{\partial}{\partial w} Z_f(w, (s_1, \dots, s_q)) \bigg|_{w=0} \right),$$

$$f^*(x_1, \dots, x_q) = f\left(\frac{1}{x_1}, \dots, \frac{1}{x_q} \right),$$

and

$$\varepsilon_f(s_1, \dots, s_q) = \frac{\zeta_{f^*}(-s_1, \dots, -s_q)}{\zeta_f(s_1, \dots, s_q)}.$$

For example, let

$$f(x_1, \dots, x_q) = x_1^{\alpha_1} \cdots x_q^{\alpha_q} \quad (\alpha_1, \dots, \alpha_q \in \mathbf{C}).$$

Then

$$Z_f(w, (s_1, \dots, s_q)) = (s_1 - \alpha_1)^{-w} \cdots (s_q - \alpha_q)^{-w},$$

$$\zeta_f(s_1, \dots, s_q) = \frac{1}{(s_1 - \alpha_1) \cdots (s_q - \alpha_q)},$$

$$f^*(x_1, \dots, x_q) = x_1^{-\alpha_1} \cdots x_q^{-\alpha_q},$$

$$\zeta_{f^*}(s_1, \dots, s_q) = \frac{1}{(s_1 + \alpha_1) \cdots (s_q + \alpha_q)},$$

and

$$\varepsilon_f(s_1, \dots, s_q) = (-1)^q.$$

Now, since the theory of multiple gamma functions and multiple sine functions is essential in the proof of Theorem 1 we briefly review the construction.

For

$$\underline{\omega} = (\omega_1, \dots, \omega_r)$$

with $\omega_1, \dots, \omega_r > 0$ (actually this condition can be relaxed: see [1], [10], [11], [8], [18], [19]) we define the multiple Hurwitz zeta function as

$$\zeta_r(s, x, \underline{\omega}) = \sum_{n_1, \dots, n_r \geq 0} (n_1 \omega_1 + \cdots + n_r \omega_r + x)^{-s}.$$

This converges absolutely in $\mathrm{Re}(s) > r$ and it has an analytic continuation to all $s \in \mathbf{C}$. Moreover, $\zeta_r(s, x, \underline{\omega})$ is holomorphic at $s = 0$. The multiple gamma function $\Gamma_r(x, \underline{\omega})$ and the multiple sine function $S_r(x, \underline{\omega})$ are defined as

$$\Gamma_r(x, \underline{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, x, \underline{\omega})\bigg|_{s=0}\right)$$

and

$$S_r(x, \underline{\omega}) = \Gamma_r(x, \underline{\omega})^{-1}\Gamma_r(|\underline{\omega}| - x, \underline{\omega})^{(-1)^r},$$

where

$$|\underline{\omega}| = \omega_1 + \cdots + \omega_r.$$

Both functions $\Gamma_r(x, \underline{\omega})$ and $S_r(x, \underline{\omega})$ are meromorphic in $x \in \mathbf{C}$. When $r = 1$ we get classical functions:

$$\begin{aligned}\zeta_1(s, x, \omega) &= \sum_{n=0}^{\infty}(n\omega + x)^{-s} \\ &= \omega^{-s}\zeta\left(s, \frac{x}{\omega}\right)\end{aligned}$$

with the Hurwitz zeta function

$$\zeta(s, x) = \sum_{n=0}^{\infty}(n + x)^{-s},$$

$$\Gamma_1(x, \omega) = \frac{\Gamma\left(\frac{x}{\omega}\right)}{\sqrt{2\pi}}\omega^{x/\omega - 1/2}$$

with the usual gamma function $\Gamma(x)$, and

$$S_1(x, \omega) = 2\sin\left(\frac{\pi x}{\omega}\right).$$

The analytic continuation of $\zeta_r(s, x, \underline{\omega})$ from $\mathrm{Re}(s) > r$ to all $s \in \mathbf{C}$ is obtained by using the integral expression

$$\begin{aligned}\zeta_r(s, x, \underline{\omega}) &= \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{e^{-xt}t^{s-1}}{(1 - e^{-\omega_1 t})\cdots(1 - e^{-\omega_r t})}\,dt \\ &= \frac{1}{\Gamma(s)}\int_1^{\infty}\frac{u^{-x-1}(\log u)^{s-1}}{(1 - u^{-\omega_1})\cdots(1 - u^{-\omega_r})}\,du.\end{aligned}$$

Hence, we see that the theory of multiple gamma functions and multiple sine functions is obtained as a special case of the absolute zeta function starting from the absolute automorphic function

$$f(x) = \frac{1}{(1-x^{-\omega_1})\cdots(1-x^{-\omega_r})}$$

as follows:

$$f\left(\frac{1}{x}\right) = Cx^{-D}f(x)$$

with $C = (-1)^r$ and $D = |\underline{\omega}| = \omega_1 + \cdots + \omega_r$,

$$\begin{aligned} \zeta_r(w, s, \underline{\omega}) &= Z_f(w, s), \\ \Gamma_r(s, \underline{\omega}) &= \zeta_f(s), \\ \Gamma_r(|\underline{\omega}| + s, \underline{\omega})^{(-1)^r} &= \zeta_{f^*}(s), \end{aligned}$$

and

$$S_r(s, \underline{\omega}) = \varepsilon_f(s).$$

We describe the "multiple gamma function of negative order" $\Gamma_{-r}(x, \underline{\omega})$ for $\underline{\omega} = (\omega_1, \ldots, \omega_r)$ also. This is defined as

$$\Gamma_{-r}(x, \underline{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_{-r}(s, x, \underline{\omega})\bigg|_{s=0}\right)$$

for

$$\begin{aligned} \zeta_{-r}(s, x, \underline{\omega}) &= \sum_{n_1, \ldots, n_r = 0, 1} (-1)^{n_1 + \cdots + n_r}(n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s} \\ &= \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|}(\omega(I) + x)^{-s}, \end{aligned}$$

where

$$\omega(I) = \sum_{i \in I} \omega_i.$$

Hence, we have the explicit formula

$$\begin{aligned} \Gamma_{-r}(x, \underline{\omega}) &= \prod_{n_1, \ldots, n_r = 0, 1} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{(-1)^{n_1 + \cdots + n_r + 1}} \\ &= \prod_{I \subset \{1, \ldots, r\}} (\omega(I) + x)^{(-1)^{|I|+1}}. \end{aligned}$$

The multiple sine function of negative order $S_{-r}(x, \underline{\omega})$ is defined by

$$S_{-r}(x, \underline{\omega}) = \Gamma_{-r}(x, \underline{\omega})^{-1}\Gamma_{-r}(-|\underline{\omega}| - x, \underline{\omega})^{(-1)^r}.$$

We remark the simplest case $r=0$:

$$\zeta_0(s,x,\emptyset)=x^{-s},$$
$$\Gamma_0(x,\emptyset)=\frac{1}{x},$$

and

$$\begin{aligned}S_0(x,\emptyset)&=\Gamma_0(x,\emptyset)^{-1}\Gamma_0(-x,\emptyset)\\&=-1.\end{aligned}$$

We have

$$S_{-r}(x,\underline{\omega})=1$$

for $r\geq 1$ as shown below. Let

$$\begin{aligned}f(x)&=(1-x^{-\omega_1})\cdots(1-x^{-\omega_r})\\&=\sum_{I\subset\{1,\ldots,r\}}(-1)^{|I|}x^{-\omega(I)}.\end{aligned}$$

Then

$$f\left(\frac{1}{x}\right)=Cx^{-D}f(x)$$

with $C=(-1)^r$ and $D=-|\underline{\omega}|$. Moreover we have

$$\begin{aligned}\zeta_{-r}(w,s,\underline{\omega})&=Z_f(w,s),\\\Gamma_{-r}(s,\underline{\omega})&=\zeta_f(s),\\\Gamma_{-r}(-|\underline{\omega}|+s,\underline{\omega})^{(-1)^r}&=\zeta_{f^*}(s),\end{aligned}$$

and

$$S_{-r}(s,\underline{\omega})=\varepsilon_f(s).$$

Now, the fact

$$S_{-r}(s,\underline{\omega})=1$$

is proved exactly in the similar way as Theorem 1 (6), but we show it directly here from

$$\begin{aligned}S_{-r}(s,\underline{\omega})&=\Gamma_{-r}(s,\underline{\omega})^{-1}\Gamma_{-r}(-|\underline{\omega}|-s,\underline{\omega})^{(-1)^r}\\&=\prod_{I\subset\{1,\ldots,r\}}(s+\omega(I))^{(-1)^{|I|+1}}\times\prod_{J\subset\{1,\ldots,r\}}(-|\underline{\omega}|-s+\omega(J))^{(-1)^{r-|J|}}.\end{aligned}$$

In the second factor replace J by $I = \{1, \dots, r\} - J$. Then we have

$$\begin{aligned} S_{-r}(x, \underline{\omega}) &= \prod_{I \subset \{1,\dots,r\}} \{(s+\omega(I))^{(-1)^{|I|+1}} \cdot (-s-\omega(I))^{(-1)^{|I|}}\} \\ &= \prod_{I \subset \{1,\dots,r\}} (-1)^{(-1)^{|I|}} \\ &= (-1)^0 \\ &= 1 \end{aligned}$$

as expected above.

2. Proof of Theorem 1

(1) From

$$f(x) = x^l \frac{(x^{m(1)}-1)\cdots(x^{m(a)}-1)}{(x^{n(1)}-1)\cdots(x^{n(b)}-1)}$$

we obtain

$$\begin{aligned} f\left(\frac{1}{x}\right) &= x^{-l} \frac{(x^{-m(1)}-1)\cdots(x^{-m(a)}-1)}{(x^{-n(1)}-1)\cdots(x^{-n(b)}-1)} \\ &= (-1)^{a-b} x^{-\deg(f)} \frac{(x^{m(1)}-1)\cdots(x^{m(a)}-1)}{(x^{n(1)}-1)\cdots(x^{n(b)}-1)} \\ &= Cx^{-D} f(x) \end{aligned}$$

with

$$C = (-1)^{a-b}$$

and

$$D = \widetilde{\deg}(f) = \deg(f) + l.$$

(2) Since

$$\begin{aligned} f(x) &= x^{l-|\underline{n}|} \frac{(x^{m(1)}-1)\cdots(x^{m(a)}-1)}{(1-x^{-n(1)})\cdots(1-x^{-n(b)})} \\ &= \frac{1}{(1-x^{-n(1)})\cdots(1-x^{-n(b)})} \sum_{I \subset \{1,\dots,a\}} (-1)^{a-|I|} x^{l-|\underline{n}|+m(I)} \end{aligned}$$

with

$$m(I) = \sum_{i \in I} m(i),$$

we get

$$Z_f(w,s) = \sum_{I\subset\{1,\dots,a\}} \frac{(-1)^{a-|I|}}{\Gamma(w)} \int_1^\infty \frac{x^{-(s-l+|\underline{n}|-m(I))}(\log x)^{w-1}}{(1-x^{-n(1)})\cdots(1-x^{-n(b)})}\frac{dx}{x}$$
$$= \sum_{I\subset\{1,\dots,a\}} (-1)^{a-|I|}\zeta_b(w, s-l+|\underline{n}|-m(I), \underline{n}).$$

Hence

$$\zeta_f(s) = \prod_{I\subset\{1,\dots,a\}} \Gamma_b(s-l+|\underline{n}|-m(I),\underline{n})^{(-1)^{a-|I|}}.$$

Especially, $\zeta_f(s)$ is meromorphic on $\mathbf{C}$, and its zeros and poles belong to $\mathbf{Z}$.

(3) From the expression

$$\begin{aligned}\zeta_f(s) &= \Gamma_b(s-\deg(f),\underline{n}) \\ &\quad\times \prod_{I\subsetneq\{1,\dots,a\}} \Gamma_b(s-l+|\underline{n}|-m(I),\underline{n})^{(-1)^{a-|I|}} \\ &= \Gamma_b(s-\deg(f),\underline{n}) \\ &\quad\times \prod_{\substack{I\subset\{1,\dots,a\}\\ I\neq\emptyset}} \Gamma_b(s-\deg(f)+m(I),\underline{n})^{(-1)^{|I|}}\end{aligned}$$

we know that $\zeta_f(s)$ is non-zero holomorphic in $\mathrm{Re}(s) > \deg(f)-1$ except for the simple pole at $s=\deg(f)$ coming from the first factor.

Moreover

$$\mathrm{Res}_{s=\deg(f)}\,\zeta_f(s) = \rho_b(\underline{n})^{-1} \times \prod_{I\neq\emptyset} \Gamma_b(m(I),\underline{n})^{(-1)^{|I|}}$$

with

$$\rho_b(\underline{n}) = \prod_{\substack{k_1,\dots,k_b\geq 0\\ (k_1,\dots,k_b)\neq(0,\dots,0)}} (k_1 n(1)+\cdots+k_b n(b)),$$

which is the Stirling modular form of Barnes [1] (1904) and $\prod$ denotes the regularized product of Deninger [6] (1992):

$$\prod_{\lambda\in\Lambda}\lambda = \exp\left(-\frac{d}{ds}\sum_{\lambda\in\Lambda}\lambda^{-s}\bigg|_{s=0}\right).$$

In [20] (2012), Tanaka investigated the Stirling modular form.

(4) From (1) we see that

$$f^*(x) = Cx^{-D}f(x)$$

with $C=(-1)^{a-b}$ and $D=\widetilde{\deg}(f)$.

Hence, we know exactly as in (2) that

$$\zeta_{f^*}(s) = \zeta_f(D+s)^C$$

is a meromorphic function. Thus

$$\begin{aligned}\varepsilon_f(s) &= \frac{\zeta_{f^*}(-s)}{\zeta_f(s)} \\ &= \frac{\zeta_f(D-s)^C}{\zeta_f(s)}\end{aligned}$$

is also a meromorphic function and

$$\zeta_f(D-s)^C = \zeta_f(s)\varepsilon_f(s).$$

Moreover, from the explicit formula for $\zeta_f(s)$ using the multiple gamma function proved in (2) (3) we obtain the following explicit formula for $\varepsilon_f(s)$ using the multiple sine function:

$$\varepsilon_f(s) = \prod_{I\subset\{1,\ldots,a\}} S_b(s-\deg(f)+m(I),\underline{n})^{(-1)^{|I|}}.$$

In fact,

$$\zeta_f(s) = \prod_I \Gamma_b(s-\deg(f)+m(I),\underline{n})^{(-1)^{|I|}}$$

and

$$\begin{aligned}\zeta_f(D-s)^C &= \prod_I \Gamma_b(D-s-\deg(f)+m(I),\underline{n})^{C(-1)^{|I|}} \\ &= \prod_I \Gamma_b(D-s-\deg(f)+|\underline{m}|-m(I),\underline{n})^{C(-1)^{|I|-a}} \\ &= \prod_I \Gamma_b(l+|\underline{m}|-m(I)-s,\underline{n})^{(-1)^{|I|+b}} \\ &= \prod_I \Gamma_b(|\underline{n}|-(s-\deg(f)+m(I)),\underline{n})^{(-1)^{|I|+b}}\end{aligned}$$

give

$$\begin{aligned}\varepsilon_f(s) &= \zeta_f(s)^{-1}\zeta_f(D-s)^C \\ &= \prod_I (\Gamma_b(s-\deg(f)+m(I),\underline{n})^{-1}\Gamma_b(|\underline{n}|-(s-\deg(f)+m(I)),\underline{n})^{(-1)^b})^{(-1)^{|I|}} \\ &= \prod_I S_b(s-\deg(f)+m(I),\underline{n})^{(-1)^{|I|}}.\end{aligned}$$

(5)
(a) proof of $\Leftarrow$):
Let

$$f(x) = \sum_k a(k)x^k \in \mathbf{Z}[x].$$

Then

$$Z_f(w,s) = \sum_k a(k)(s-k)^{-w},$$

so

$$\zeta_f(s) = \prod_k (s-k)^{-a(k)}$$

is a rational function.
(b) proof of $\Rightarrow$):
From

$$f(x) = \frac{x^{l-|\underline{n}|}}{(1-x^{-n(1)})\cdots(1-x^{-n(b)})} \sum_{I\subset\{1,\dots,a\}} (-1)^{a-|I|} x^{m(I)}$$

we obtain the following expression in $x > 1$:

$$f(x) = \sum_{I\subset\{1,\dots,a\}} (-1)^{a-|I|} x^{l-|\underline{n}|+m(I)} \sum_{k_1,\dots,k_b\geq 0} x^{-(k_1 n(1)+\cdots+k_b n(b))}.$$

Let

$$\sum_{k_1,\dots,k_b\geq 0} x^{-(k_1 n(1)+\cdots+k_b n(b))} = \sum_{\mu\geq 0} v_{\underline{n}}(\mu) x^{-\mu}$$

with

$$v_{\underline{n}}(\mu) = |\{(k_1,\dots,k_b) \mid k_1,\dots,k_b \geq 0, k_1 n(1)+\cdots+k_b n(b) = \mu\}|.$$

Then we get

$$\begin{aligned} f(x) &= \sum_{I\subset\{1,\dots,a\}} (-1)^{a-|I|} x^{l-|\underline{n}|+m(I)} \left(\sum_{\mu\geq 0} v_{\underline{n}}(\mu) x^{-\mu} \right) \\ &= \sum_{\mu} \left(\sum_{I\subset\{1,\dots,a\}} (-1)^{a-|I|} v_{\underline{n}}(\mu + l - |\underline{n}| + m(I)) \right) x^{-\mu} \\ &= \sum_{\mu} \left(\sum_{I\subset\{1,\dots,a\}} (-1)^{|I|} v_{\underline{n}}(\mu + \deg(f) - m(I)) \right) x^{-\mu}. \end{aligned}$$

Now, from the expression

$$\zeta_f(s) = \prod_{I \subset \{1,\dots,a\}} \Gamma_b(s - \deg(f) + m(I), \underline{n})^{(-1)^{|I|}}$$

we see that

$$\mathrm{ord}_{s=-\mu}\, \zeta_f(s) = \sum_{I \subset \{1,\dots,a\}} (-1)^{|I|} v_{\underline{n}}(\mu + \deg(f) - m(I)),$$

where the order is considered as the order of a pole (so, negative order for a zero). Thus we obtain the expression

$$f(x) = \sum_{\mu} (\mathrm{ord}_{s=-\mu}\, \zeta_f(s)) x^{-\mu}.$$

When $\zeta_f(s)$ is a rational function,

$$\mathrm{ord}_{s=-\mu}\, \zeta_f(s) = 0$$

for $|\mu|$ sufficiently large.

Hence, $f(x)$ is a Laurent polynomial with $\mathbf{Z}$-coefficients. Since

$$f(x) = x^l \frac{\prod_i (x^{m(i)} - 1)}{\prod_j (x^{n(j)} - 1)}$$

shows that $f(0)$ is finite, $f(x) \in \mathbf{Z}[x]$.

(6) From (5), put

$$f(x) = \sum_k a(k) x^k \in \mathbf{Z}[x].$$

Then

$$f^*(x) = \sum_k a(k) x^{-k}.$$

Since

$$Z_f(w,s) = \sum_k a(k)(s-k)^{-w}$$

and

$$Z_{f^*}(w,s) = \sum_k a(k)(s+k)^{-w}$$

we get

$$\zeta_f(s) = \prod_k (s-k)^{-a(k)}$$

and

$$\zeta_{f^*}(s) = \prod_k (s+k)^{-a(k)}.$$

Hence

$$\begin{aligned}\varepsilon_f(s) &= \frac{\zeta_{f^*}(-s)}{\zeta_f(s)} \\ &= \frac{\prod_k (-s+k)^{-a(k)}}{\prod_k (s-k)^{-a(k)}} \\ &= (-1)^{\sum_k a(k)} \\ &= (-1)^{f(1)}.\end{aligned}$$

Thus, by (4) we obtain the functional equation

$$\zeta_f(\widetilde{\deg}(f)-s)^{(-1)^{a-b}} = (-1)^{f(1)}\zeta_f(s).$$

Otherwise we may argue as follows. The equation

$$f\left(\frac{1}{x}\right) = Cx^{-D}f(x)$$

of (1) ($C = \pm 1$, $D \in \mathbf{Z}$) for

$$f(x) = \sum_k a(k)x^k \in \mathbf{Z}[x]$$

implies

$$\begin{aligned}\sum_k a(k)x^{-k} &= \sum_k C \cdot a(k)x^{k-D} \\ &= \sum_k C \cdot a(D-k)x^{-k}.\end{aligned}$$

Hence, we get

$$a(k) = C \cdot a(D-k).$$

By the way

$$\zeta_f(s) = \prod_k (s-k)^{-a(k)}$$

gives

$$\begin{aligned}\zeta_f(D-s)^C &= \prod_k ((D-k)-s)^{-C\cdot a(k)} \\ &= \prod_k (k-s)^{-C\cdot a(D-k)} \\ &= \prod_k (k-s)^{-a(k)},\end{aligned}$$

where we used

$$a(k) = C \cdot a(D-k).$$

Thus

$$\begin{aligned}\varepsilon_f(s) &= \frac{\zeta_f(D-s)^C}{\zeta_f(s)} \\ &= \frac{\prod_k (k-s)^{-a(k)}}{\prod_k (s-k)^{-a(k)}} \\ &= (-1)^{f(1)}.\end{aligned}$$

[QED of Theorem 1]

3. Examples

In this section we illustrate by examples that the absolute zeta functions $\zeta_{X/\mathbf{F}_1}(s)$ of some typical $\mathbf{Z}$-schemes over $\mathbf{F}_1$ are identified with the zeta functions $\zeta_f(s)$ of certain cyclotomic absolute automorphic forms f such that $f(x) = |X(\mathbf{F}_x)|$ for primes powers x; and that in each subsection we list the $f(x)$, $\zeta_{X/\mathbf{F}_1}(s) = \zeta_f(s)$, the invariants D, C, l, a, b, $f(1)$ as Theorem 1, and the functional equation for $\zeta_{X/\mathbf{F}_1}(s)$.

3.1. Affine space $\mathbf{A}^n$.

$$f(x) = x^n,$$

$$\zeta_{\mathbf{A}^n/\mathbf{F}_1}(s) = \zeta_f(s) = \frac{1}{s-n}.$$

$[D = 2n,\ C = +1;\ l = n,\ a = b = 0;\ f(1) = 1]$

$$\zeta_{\mathbf{A}^n/\mathbf{F}_1}(2n-s) = -\zeta_{\mathbf{A}^n/\mathbf{F}_1}(s).$$

3.2. Projective space $\mathbf{P}^n$.

$$f(x) = x^n + x^{n-1} + \cdots + 1 = \frac{x^{n+1}-1}{x-1},$$

$$\zeta_{\mathbf{P}^n/\mathbf{F}_1}(s) = \frac{1}{(s-n)\cdots s}.$$

$[D=n,\ C=+1;\ l=0,\ a=b=1;\ m(1)=n+1,\ n(1)=1;\ f(1)=n+1]$

$$\zeta_{\mathbf{P}^n/\mathbf{F}_1}(n-s)=(-1)^{n+1}\zeta_{\mathbf{P}^n/\mathbf{F}_1}(s).$$

3.3. Grassmannian space $Gr(n,m)$.

$$f(x)=\frac{(x^n-1)\cdots(x^{n-m+1}-1)}{(x^m-1)\cdots(x-1)},$$

$\zeta_{Gr(n,m)/\mathbf{F}_1}(s)$ is a rational function.
$[D=m(n-m),\ C=+1;\ l=0,\ a=b=m;\ f(1)=\binom{n}{m}.]$

$$\zeta_{Gr(n,m)/\mathbf{F}_1}(m(n-m)-s)=(-1)^{\binom{n}{m}}\zeta_{Gr(n,m)/\mathbf{F}_1}(s).$$

3.4. General linear group $GL(n)$.

$$f(x)=x^{n(n-1)/2}(x-1)(x^2-1)\cdots(x^n-1),$$

$\zeta_{GL(n)/\mathbf{F}_1}(s)$ is the rational function $\Gamma_{-n}(s-n^2,(1,2,\ldots,n))$.
$[D=\frac{n(3n-1)}{2},\ C=(-1)^n;\ l=\frac{n(n-1)}{2},\ a=n,\ b=0;\ f(1)=0.]$

$$\zeta_{GL(n)/\mathbf{F}_1}\left(\frac{n(3n-1)}{2}-s\right)^{(-1)^n}=\zeta_{GL(n)/\mathbf{F}_1}(s).$$

3.5. Special linear group $SL(n)$.

$$f(x)=x^{n(n-1)/2}(x^2-1)\cdots(x^n-1),$$

$\zeta_{SL(n)/\mathbf{F}_1}(s)$ is the rational function $\Gamma_{-(n-1)}(s-(n^2-1),(2,3,\ldots,n))$.
$[D=\frac{n(3n-1)}{2}-1,\ C=(-1)^{n-1};\ l=\frac{n(n-1)}{2},\ a=n-1,\ b=0;\ f(1)=0.]$

$$\zeta_{SL(n)/\mathbf{F}_1}\left(\frac{n(3n-1)}{2}-1-s\right)^{(-1)^{n-1}}=\zeta_{SL(n)/\mathbf{F}_1}(s).$$

3.6. Symplectic group $Sp(n)$ (size $2n$).

$$f(x)=x^{n^2}(x^2-1)(x^4-1)\cdots(x^{2n}-1),$$

$\zeta_{Sp(n)/\mathbf{F}_1}(s)$ is the rational function $\Gamma_{-n}(s-(2n^2+n),(2,4,\ldots,2n))$.
$[D=n(3n+1),\ C=(-1)^n;\ l=n^2,\ a=n,\ b=0;\ f(1)=0.]$

$$\zeta_{Sp(n)/\mathbf{F}_1}(n(3n+1)-s)^{(-1)^n}=\zeta_{Sp(n)/\mathbf{F}_1}(s).$$

Remarks.

$$(1)\ \ \zeta_{Sp(n)/\mathbf{F}_1}(s)=\zeta_{GL(n)/\mathbf{F}_1}\left(\frac{s-n}{2}\right),$$

$$(2)\ \ \zeta_{Sp(n)/\mathbf{F}_q}(s)=\zeta_{GL(n)/\mathbf{F}_{q^2}}\left(\frac{s-n}{2}\right).$$

3.7. Multiple gamma functions of order r.

$$f(x) = \frac{1}{(x^{n(1)} - 1)\cdots(x^{n(r)} - 1)},$$

$$\zeta_f(s) = \Gamma_r(s + |\underline{n}|, \underline{n}).$$

$[D = -|\underline{n}|,\ C = (-1)^r;\ l = 0,\ a = 0,\ b = r.]$

$$\zeta_f(-|\underline{n}| - s)^{(-1)^r} = \zeta_f(s)\varepsilon_f(s),$$

$$\varepsilon_f(s) = S_r(s + |\underline{n}|, \underline{n}).$$

3.8. Multiple gamma functions of oder $-r$.

$$f(x) = (x^{m(1)} - 1)\cdots(x^{m(r)} - 1),$$

$$\zeta_f(s) = \prod_{I\subset\{1,\dots,r\}} (s - m(I))^{(-1)^{r-|I|+1}},$$

$$m(I) = \sum_{i\in I} m(i).$$

$[D = |\underline{m}|,\ C = (-1)^r;\ l = 0,\ a = r,\ b = 0;\ f(1) = 0]$

$$\zeta_f(|\underline{m}| - s)^{(-1)^r} = \zeta_f(s).$$

4. Proof of Theorem 2

(1) Let

$$\rho_+ \cong \bigoplus_{j=1}^{d_+} \chi_{\alpha(j)},$$

$$\rho_- \cong \bigoplus_{k=1}^{d_-} \chi_{\beta(k)}$$

with

$$\chi_\alpha : \Gamma \to GL(1, \mathbf{C})$$

defined by

$$\chi_\alpha(x) = x^\alpha.$$

Then

$$D_{\rho_+} \cong \lim_{x\to 1} \bigoplus_{j=1}^{d_+} \left(\frac{\chi_{\alpha(j)}(x) - 1}{x - 1}\right)$$

$$= \begin{pmatrix} \alpha(1) & & O \\ & \ddots & \\ O & & \alpha(d_+) \end{pmatrix},$$

and

$$D_{\rho_-} \cong \lim_{x \to 1} \bigoplus_{k=1}^{d_-} \left(\frac{\chi_{\beta(k)}(x) - 1}{x - 1} \right)$$
$$= \begin{pmatrix} \beta(1) & & O \\ & \ddots & \\ O & & \beta(d_-) \end{pmatrix}.$$

Hence

$$\zeta_\rho(s) = \frac{\det(s - D_{\rho_-})}{\det(s - D_{\rho_+})}$$
$$= \frac{\prod_{k=1}^{d_-}(s - \beta(k))}{\prod_{j=1}^{d_+}(s - \alpha(j))}.$$

On the other hand we see that

$$f(x) = \mathrm{tr}(\rho_+(x)) - \mathrm{tr}(\rho_-(x))$$
$$= \sum_{j=1}^{d_+} x^{\alpha(j)} - \sum_{k=1}^{d_-} x^{\beta(k)}.$$

Hence

$$Z_f(w, s) = \sum_{j=1}^{d_+} (s - \alpha(j))^{-w} - \sum_{k=1}^{d_-} (s - \beta(k))^{-w}$$

and

$$\zeta_f(s) = \frac{\prod_{k=1}^{d_-}(s - \beta(k))}{\prod_{j=1}^{d_+}(s - \alpha(j))}.$$

Thus, we get the identity

$$\zeta_\rho(s) = \zeta_f(s).$$

(2) From the unitariness of $\rho_\pm$, we have

$$f(x) = \sum_{j=1}^{d_+} x^{\alpha(j)} - \sum_{k=1}^{d_-} x^{\beta(k)}$$

with $\alpha(j), \beta(k) \in \sqrt{-1}\mathbf{R}$.

By the way, the expression

$$\zeta_\rho(s) = \frac{\prod_{k=1}^{d_-}(s - \beta(k))}{\prod_{j=1}^{d_+}(s - \alpha(j))}$$

obtained in (1) gives

$$\begin{aligned}
\zeta_\rho(-s) &= \frac{\prod_{k=1}^{d_-}(-s-\beta(k))}{\prod_{j=1}^{d_+}(-s-\alpha(j))} \\
&= (-1)^{d_+-d_-}\frac{\prod_{k=1}^{d_-}(s+\beta(k))}{\prod_{j=1}^{d_+}(s+\alpha(j))} \\
&= (-1)^{\deg(\rho)}\zeta_{f^*}(s),
\end{aligned}$$

where we used

$$f^*(x) = \sum_{j=1}^{d_+} x^{-\alpha(j)} - \sum_{k=1}^{d_-} x^{-\beta(k)}.$$

Now, the self-duality of $\rho_\pm$ means

$$\check{\rho}_\pm \cong \rho_\pm,$$

where

$$\check{\rho}_\pm(x) = {}^t\rho_\pm(x^{-1}).$$

In particular, we get

$$\begin{aligned}
f^*(x) &= f(x^{-1}) \\
&= \mathrm{tr}({}^t\rho_+(x^{-1})) - \mathrm{tr}({}^t\rho_-(x^{-1})) \\
&= \mathrm{tr}(\check{\rho}_+(x)) - \mathrm{tr}(\check{\rho}_-(x)) \\
&= \mathrm{tr}(\rho_+(x)) - \mathrm{tr}(\rho_-(x)) \\
&= f(x).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\zeta_\rho(-s) &= (-1)^{\deg(\rho)}\zeta_f(s) \\
&\overset{(1)}{=} (-1)^{\deg(\rho)}\zeta_\rho(s).
\end{aligned}$$

[QED of Theorem 2]

5. Proof of Theorem 3

Now, we introduce the generalized multiple gamma function $\Gamma(x_1, \dots, x_q; \Omega)$ and the generalized multiple sine function $S(x_1, \dots, x_q; \Omega)$ respectively.

For

$$\Omega = \begin{pmatrix} \omega(1,1) & \cdots & \omega(1,q) \\ \vdots & & \vdots \\ \omega(r,1) & \cdots & \omega(r,q) \end{pmatrix}$$

with $\omega(1,1),\dots,\omega(r,q)>0$ (actually this condition can be relaxed: see [13], [14], [15], [16], [7]) we define Shintani zeta function as

$$\zeta(s,(x_1,\dots,x_q),\Omega)=\sum_{n_1,\dots,n_r\geq 0}\prod_{k=1}^{q}\left(\sum_{i=1}^{r}n_i\omega(i,k)+x_k\right)^{-s}.$$

We remark that Shintani zeta function was introduced by Shintani [13], [14] (1976) but the above definition is a modified version by Friedman-Ruijsenaars [7] (2004). This converges absolutely in $\mathrm{Re}(s)>\dfrac{r}{q}$ and it has an analytic continuation to all $s\in\mathbf{C}$. Moreover, $\zeta(s,(x_1,\dots,x_q),\Omega)$ is holomorphic at $s=0$. The generalized multiple gamma function $\Gamma(x_1,\dots,x_q;\Omega)$ and the generalized multiple sine function $S(x_1,\dots,x_q;\Omega)$ are defined as

$$\Gamma(x_1,\dots,x_q;\Omega)=\exp\left(\frac{\partial}{\partial s}\zeta(s,(x_1,\dots,x_q),\Omega)\bigg|_{s=0}\right)$$

and

$$\begin{aligned}&S(x_1,\dots,x_q;\Omega)\\&\quad=\Gamma(x_1,\dots,x_q;\Omega)^{-1}\Gamma\left(\sum_{i=1}^{r}\omega(i,1)-x_1,\dots,\sum_{i=1}^{r}\omega(i,q)-x_q;\Omega\right)^{(-1)^r}.\end{aligned}$$

Both functions $\Gamma(x_1,\dots,x_q;\Omega)$ and $S(x_1,\dots,x_q;\Omega)$ are meromorphic in $(x_1,\dots,x_q)\in\mathbf{C}^q$.

(1) From

$$f(x_1,\dots,x_q)=x_1^{l(1)/2}\cdots x_q^{l(q)/2}\frac{\prod_{i=1}^{a}(x_1^{m(i,1)}\cdots x_q^{m(i,q)}-1)}{\prod_{j=1}^{b}(x_1^{n(j,1)}\cdots x_q^{n(j,q)}-1)}$$

we obtain

$$f\left(\frac{1}{x_1},\dots,\frac{1}{x_q}\right)=Cx_1^{-D(1)}\cdots x_q^{-D(q)}f(x_1,\dots,x_q)$$

with

$$C=(-1)^{a-b}$$

and

$$D(k)=\widetilde{\deg}_k(f)=l(k)+\sum_{i=1}^{a}m(i,k)-\sum_{j=1}^{b}n(j,k).$$

(2) Since

$$f(x_1,\dots,x_q) = x_1^{l(1)/2-\sum_j n(j,1)} \cdots x_q^{l(q)/2-\sum_j n(j,q)}$$
$$\times \left(\sum_{I\subset\{1,\dots,a\}} (-1)^{a-|I|} x_1^{m(I,1)} \cdots x_q^{m(I,q)} \right)$$
$$\times \left(\sum_{v_1,\dots,v_q \geq 0} (x_1^{-n(1,1)} \cdots x_q^{-n(1,q)})^{v_1} \cdots (x_1^{-n(b,1)} \cdots x_q^{-n(b,q)})^{v_q} \right)$$

with

$$m(I,k) = \sum_{i\in I} m(i,k),$$

we get

$$Z_f(w,(s_1,\dots,s_q))$$
$$= \sum_{v_1,\dots,v_q\geq 0} \sum_{I\subset\{1,\dots,a\}} (-1)^{a-|I|} \prod_{k=1}^{q} \left(s_k + \sum_{j=1}^{b} (v_j+1)n(j,k) - m(I,k) - \frac{l(k)}{2} \right)^{-w}.$$

Hence

$$\zeta_f(s_1,\dots,s_q)$$
$$= \prod_{I\subset\{1,\dots,a\}} \left(\prod_{v_1,\dots,v_q} \prod_{k=1}^{q} \left(s_k + \sum_{j=1}^{b} (v_j+1)n(j,k) - m(I,k) - \frac{l(k)}{2} \right) \right)^{(-1)^{a-|I|+1}}$$
$$= \prod_{I\subset\{1,\dots,a\}} \left(\prod_{v_1,\dots,v_q} \prod_{k=1}^{q} \left(s_k - \deg_k(f) + m(I,k) + \sum_{j=1}^{b} v_j n(j,k) \right) \right)^{(-1)^{|I|+1}},$$

where $\deg_k(f) = \dfrac{l(k)}{2} + \sum_{i=1}^{a} m(i,k) - \sum_{j=1}^{b} n(j,k)$. Thus we obtain

$$\zeta_f(s_1,\dots,s_q) = \prod_{I\subset\{1,\dots,a\}} \Gamma(s_1 - \deg_1(f) + m(I,1),\dots,$$
$$s_q - \deg_q(f) + m(I,q); N)^{(-1)^{|I|}},$$

where

$$N = \begin{pmatrix} n(1,1) & \cdots & n(1,q) \\ \vdots & & \vdots \\ n(b,1) & \cdots & n(b,q) \end{pmatrix}$$

and

$$\Gamma(x_1,\dots,x_q;N)=\Gamma_{b,q}(x_1,\dots,x_q;N)$$

is the generalized multiple gamma function. Hence $\zeta_f(s_1,\dots,s_q)$ is a meromorphic function on $\mathbf{C}^q$.

(3) From (1) we see that

$$f^*(x_1,\dots,x_q)=Cx_1^{-D(1)}\cdots x_q^{-D(q)}f(x_1,\dots,x_q)$$

with $C=(-1)^{a-b}$ and $D(k)=\widetilde{\deg}_k(f)$. Hence, we have

$$\begin{aligned}
&\zeta_{f^*}(-s_1,\dots,-s_q)\\
&\quad=\zeta_f(D(1)-s_1,\dots,D(q)-s_q)^C\\
&\quad=\prod_{I\subset\{1,\dots,a\}}\Gamma\left(\frac{l(1)}{2}-s_1+m(I,1),\dots,\frac{l(q)}{2}-s_q+m(I,q);N\right)^{(-1)^{a-b+|I|}}\\
&\quad=\prod_{I\subset\{1,\dots,a\}}\Gamma\left(\sum_{j=1}^{b}n(j,1)-s_1+\deg_1(f)-m(I,1),\dots,\right.\\
&\qquad\qquad\left.\sum_{j=1}^{b}n(j,q)-s_q+\deg_q(f)-m(I,q);N\right)^{(-1)^{b+|I|}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\varepsilon_f(s_1,\dots,s_q)=\prod_{I\subset\{1,\dots,a\}}&S(s_1-\deg_1(f)+m(I,1),\dots,\\
&s_q-\deg_q(f)+m(I,q);N)^{(-1)^{|I|}},
\end{aligned}$$

where

$$\begin{aligned}
&S(x_1,\dots,x_q;N)\\
&\quad=\Gamma(x_1,\dots,x_q;N)^{-1}\Gamma\left(\sum_{j=1}^{b}n(j,1)-x_1,\dots,\sum_{j=1}^{b}n(j,q)-x_q;N\right)^{(-1)^b}
\end{aligned}$$

is the generalized multiple sine function. [QED of Theorem 3]

6. Proof of Theorem 4

(a) Proof of (1) $\Rightarrow$ (2):
Let

$$f(x_1,\dots,x_q)=\sum_{1\le i<j\le n}(x_i-1)(x_j-1)g_{ij}(x_1,\dots,x_q)$$

with

$$g_{ij}(x_1, \dots, x_q) = \sum_{k_1, \dots, k_q} b_{ij}(k_1, \dots, k_q) x_1^{k_1} \cdots x_q^{k_q} \in \mathbf{Z}[x_1, \dots, x_q].$$

Then we have

$$\begin{aligned} Z_f(w, (s_1, \dots, s_q)) = \sum_{1 \le i < j \le n} \sum_{k_1, \dots, k_q} & b_{ij}(k_1, \dots, k_q) \frac{1}{(s_1 - k_1)^w} \cdots \frac{1}{(s_{i-1} - k_{i-1})^w} \\ & \times \left(\frac{1}{(s_i - k_i - 1)^w} - \frac{1}{(s_i - k_i)^w} \right) \frac{1}{(s_{i+1} - k_{i+1})^w} \cdots \frac{1}{(s_{j-1} - k_{j-1})^w} \\ & \times \left(\frac{1}{(s_j - k_j - 1)^w} - \frac{1}{(s_j - k_j)^w} \right) \frac{1}{(s_{j+1} - k_{j+1})^w} \cdots \frac{1}{(s_q - k_q)^w}. \end{aligned}$$

Since

$$\frac{\partial}{\partial w} Z_f(w, (s_1, \dots, s_q)) \bigg|_{w=0} = 0,$$

we have

$$\begin{aligned} \zeta_f(s_1, \dots, s_q) &= \exp\left(\frac{\partial}{\partial w} Z_f(w, (s_1, \dots, s_q)) \bigg|_{w=0} \right) \\ &= 1. \end{aligned}$$

Thus, we obtain (1) $\Rightarrow$ (2) of Theorem 4.

(b) Proof of (2) $\Rightarrow$ (1):

Let

$$f(x_1, \dots, x_q) = \sum_{k_1, \dots, k_q} a(k_1, \dots, k_q) x_1^{k_1} \cdots x_q^{k_q} \in \mathbf{Z}[x_1, \dots, x_q].$$

Since

$$\zeta_f(s_1, \dots, s_q) = \prod_{k_1, \dots, k_q} \left(\frac{1}{(s_1 - k_1) \cdots (s_q - k_q)} \right)^{a(k_1, \dots, k_q)} = 1,$$

we have

$$\prod_{k_1} (s_1 - k_1)^{-\sum_{k_2, \dots, k_q} a(k_1, \dots, k_q)} = 1.$$

This gives

$$\sum_{k_2, \dots, k_q} a(k_1, \dots, k_q) = 0$$

for every k_1. Similarly, we have

$$\sum_{k_1,k_3,\dots,k_q} a(k_1,\dots,k_q) = 0, \dots, \sum_{k_1,k_2,\dots,k_{q-2},k_q} a(k_1,\dots,k_q) = 0$$

and

$$\sum_{k_1,k_2,\dots,k_{q-1}} a(k_1,\dots,k_q) = 0$$

for every $k_2,\dots,k_{q-1}$ and k_q respectively. Now $f(x_1,\dots,x_q)$ divided by $x_1 - 1$ can be written as

$$f(x_1,\dots,x_q) = (x_1 - 1)\phi_1(x_1,\dots,x_q) + r_1(x_2,\dots,x_q).$$

Since $r_1(x_2,\dots,x_q)$ divided by $x_2 - 1$ can be written as

$$r_1(x_2,\dots,x_q) = (x_2 - 1)\phi_2(x_2,\dots,x_q) + r_2(x_3,\dots,x_q),$$

we have

$$f(x_1,\dots,x_q) = (x_1 - 1)\phi_1(x_1,\dots,x_q) + (x_2 - 1)\phi_2(x_2,\dots,x_q) + r_2(x_3,\dots,x_q).$$

When it's repeated, we have

$$\begin{aligned} f(x_1,\dots,x_q) = {} & (x_1 - 1)\phi_1(x_1,\dots,x_q) + (x_2 - 1)\phi_2(x_2,\dots,x_q) \\ & + \cdots + (x_{q-1} - 1)\phi_{q-1}(x_{q-1},x_q) + r_{q-1}(x_q). \end{aligned}$$

Since

$$\sum_{k_1,k_2,\dots,k_{q-1}} a(k_1,\dots,k_q) = 0,$$

we have

$$r_{q-1}(x_q) = 0.$$

Thus, we have

$$f(x_1,\dots,x_q) \in (x_1 - 1, x_2 - 1, \dots, x_{q-1} - 1).$$

Similarly, we have

$$\begin{aligned} f(x_1,\dots,x_q) & \in (x_1 - 1, x_2 - 1, \dots, x_{q-2} - 1, x_q - 1), \dots, f(x_1,\dots,x_q) \\ & \in (x_2 - 1, x_3 - 1, \dots, x_q - 1). \end{aligned}$$

Hence, we have

$$f(x_1, \dots, x_q) \in \bigcap_{i=1}^{q}(x_1 - 1, \dots, x_{i-1} - 1, x_{i+1} - 1, \dots, x_q - 1).$$

Finally, we show

$$\bigcap_{i=1}^{q}(x_1 - 1, \dots, x_{i-1} - 1, x_{i+1} - 1, \dots, x_q - 1) = ((x_i - 1)(x_j - 1) \mid 1 \le i < j \le q).$$

Obviously, we have

$$((x_i - 1)(x_j - 1) \mid 1 \le i < j \le q) \subset \bigcap_{i=1}^{q}(x_1 - 1, \dots, x_{i-1} - 1, x_{i+1} - 1, \dots, x_q - 1).$$

Now we show

$$\bigcap_{i=1}^{q}(x_1 - 1, \dots, x_{i-1} - 1, x_{i+1} - 1, \dots, x_q - 1) \subset ((x_i - 1)(x_j - 1) \mid 1 \le i < j \le q).$$

If $f(x_1, \dots, x_q) \in \bigcap_{i=1}^{q}(x_1 - 1, \dots, x_{i-1} - 1, x_{i+1} - 1, \dots, x_q - 1)$, then we have

$$\begin{aligned} f(x_1, \dots, x_q) &= (x_2 - 1)h_{12}(x_1, \dots, x_q) + (x_3 - 1)h_{13}(x_1, \dots, x_q) \\ &\quad + \cdots + (x_q - 1)h_{1q}(x_1, \dots, x_q), \\ f(x_1, \dots, x_q) &= (x_1 - 1)h_{21}(x_1, \dots, x_q) + (x_3 - 1)h_{23}(x_1, \dots, x_q) \\ &\quad + \cdots + (x_q - 1)h_{2q}(x_1, \dots, x_q), \\ &\cdots \\ f(x_1, \dots, x_q) &= (x_1 - 1)h_{q1}(x_1, \dots, x_q) + \cdots + (x_{q-1} - 1)h_{qq-1}(x_1, \dots, x_q). \end{aligned}$$

Using

$$\begin{aligned} f(x_1, \dots, x_q) = (x_2 - 1)h_{12}(x_1, \dots, x_q) + (x_3 - 1)h_{13}(x_1, \dots, x_q) \\ + \cdots + (x_q - 1)h_{1q}(x_1, \dots, x_q) \end{aligned}$$

and

$$\begin{aligned} f(x_1, \dots, x_q) = (x_1 - 1)h_{21}(x_1, \dots, x_q) + (x_3 - 1)h_{23}(x_1, \dots, x_q) \\ + \cdots + (x_q - 1)h_{2q}(x_1, \dots, x_q), \end{aligned}$$

we have $h_{12}(x_1, \dots, x_q) \in (x_1 - 1, x_3 - 1, \dots, x_q - 1)$. Similarly, we have

$$\begin{aligned} h_{13}(x_1, \dots, x_q) \in (x_1 - 1, x_2 - 1, x_4 - 1, \dots, x_q - 1), \dots, h_{1q-1}(x_1, \dots, x_q) \\ \in (x_1 - 1, x_2 - 1, \dots, x_{q-2} - 1, x_q - 1) \end{aligned}$$

and $h_{1q}(x_1, \ldots, x_q) \in (x_1 - 1, x_2 - 1, \ldots, x_{q-1} - 1)$. Hence, we obtain

$$f(x_1, \ldots, x_q) \in ((x_i - 1)(x_j - 1) \mid 1 \le i < j \le q).$$

Thus, we obtain (2) $\Rightarrow$ (1) of Theorem 4. [QED of Theorem 4]

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