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ALMOST AUTOMORPHIC SOLUTIONS OF SEMILINEAR STOCHASTIC HYPERBOLIC DIFFERENTIAL EQUATIONS IN INTERMEDIATE SPACE

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Abstract

In this paper, we investigate the existence, uniqueness of almost automorphic in one-dimensional distribution mild solution for semilinear stochastic differential equations driven by Lévy noise. The semigroup theory, fixed point theorem and stochastic analysis technique are the main tools in carrying out proof. Finally, we give one example to illustrate the main findings.

1. Introduction

Almost automorphy of deterministic differential equations in abstract space has been extensively investigated and many authors have made important contributions to this theory [7, 9, 14, 16, 26, 28]. For stochastic differential equations (SDEs), the concept of almost automorphy can be defined in square-mean and in distribution, respectively. Square-mean almost automorphy for stochastic processes is first introduced in [22] with the applications to the SDEs. For almost automorphy in distribution sense, Fu [20] systematically explore the properties, and existence, uniqueness of almost automorphic in distribution solutions to nonautonomous SDEs are studied.

Stochastic differential equations are used widely in many fields, such as nonlinear vibration, engineering, population dynamics, neural networks, control theory and so on. The asymptotic properties of solutions for SDEs have been studied from different points, such as square-mean almost periodicity [4, 6], almost periodicity in distribution [2, 15, 30, 24], square-mean almost automorphy [5, 10], almost autommorphy in distribution [3, 21], ergodicity [11, 17, 12], stability [8, 13, 23] and so on. Note that most studies are concerned with differential

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equations perturbed by Gaussian noise. It is worthwhile to mention that many models involve jump perturbations, or more general Lévy noise. Almost automorphy of SDEs with Lévy noise is first investigated by Liu and Sun [27], where Poisson square-mean almost automorphy is introduced and studied. For SDEs driven by Lévy noise, one can see [25, 32, 33] for more details.

In this paper, we investigate the existence, unqueness of almost automorphic in distribution mild solution for SDEs driven by Lévy noise. Intuitively, a large jump may destroy almost automorphy, but we will see that the almost automorphy property may persist under some suitable conditions. Different form [25, 27, 33], here the state space we consider is intermediate space, and the operator is the infinitesimal generator of an analytic semigroup.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. Sections 3 is devoted to the existence, uniqueness of almost automorphic in one-dimensional distribution mild solution for semilinear SDEs with Lévy noise. In Section 4, to illustrate the main findings, we consider one example with Lévy noise.

2. Preliminaries and basic results

Throughout the paper, $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ stand for the set of natural numbers, integers, real numbers and complex numbers, respectively. For A being a linear operator, $D(A)$, $\rho(A)$, $R(\lambda, A)$, $\sigma(A)$ stand for the domain, the resolvent set, the resolvent and spectrum of A. We assume that $(H, \|\cdot\|), (V, |\cdot|_V)$ are real separable Hilbert spaces. $L(V, H)$ denotes the space of all bounded linear operators from V to H. We assume that $(\Omega, \mathscr{F}, P)$ is a probability space, and $\mathscr{L}^2(P, H)$ stands for the space of all H-valued random variables Y such that

$$\mathbf{E}\|Y\|^2 = \int_\Omega \|Y\|^2 \, dP < \infty.$$

For $Y \in \mathscr{L}^2(P, H)$, let

$$\|Y\|_2 := \left(\int_\Omega \|Y\|^2 \, dP\right)^{1/2},$$

then $\mathscr{L}^2(P, H)$ is a Hilbert space equipped with the norm $\|\cdot\|_2$.

2.1. Sectorial operators and intermediate space

Definition 2.1 ([19]). A linear operator $A : D(A) \subset H \to H$ is said to be ω-sectorial of angle θ if the following hold: there exist constants $\omega \in \mathbf{R}$, $\theta \in (\pi/2, \pi)$ and $\tilde{M} > 0$ such that

$$\rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbf{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}, \tag{2.1}$$

$$\|R(\lambda, A)\| \le \frac{\tilde{M}}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega},$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ for each $\lambda \in S_{\theta,\omega}$.

It is well know that [29] if A is ω-sectorial of angle θ, then it generates an analytic semigroup $(T(t))_{t\geq 0}$ in the sector $S_{\theta-\pi/2,0}$, which maps $(0,+\infty)$ to $L(H,H)$ such that there exist $M_0, M_1 > 0$ with

$$\|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0,$$
$$\|t(A-\omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0.$$

Definition 2.2 ([19]). A semigroup $(T(t))_{t\geq 0}$ is said to be hyperbolic, if there exist projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, $Ker\,P$ is invariant with respect to $T(t)$, $T(t) : Im\,J \to Im\,J$ is invertible and

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\| \quad \text{for } t \geq 0,$$
$$\|T(t)Jx\| \leq Me^{\delta t}\|x\| \quad \text{for } t \leq 0,$$

where $J := I - P$ and $T(t) := (T(-t))^{-1}$ for $t \leq 0$.

Recall that if a semigroup $(T(t))_{t\geq 0}$ is analytic, then $(T(t))_{t\geq 0}$ is hyperbolic if and only if

$$\sigma(A) \cap i\mathbf{R} = \emptyset,$$

see for instance [19].

Next, we recall the definition of intermediate space.

Definition 2.3 ([29]). Let $\alpha \in (0,1)$. A Banach space $(H_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(A)$ and H, if $D(A) \subset H_\alpha \subset H$ and there exists a constant $\tilde{c} > 0$ such that

$$\|x\|_\alpha \leq \tilde{c}\|x\|^{1-\alpha}\|x\|_A^\alpha, \quad x \in D(A),$$

where $\|\cdot\|_A$ is the graph norm of A. Here $\|x\|_A^\alpha = \|x\| + \|Ax\|$ for each $x \in D(A)$.

Concrete examples of H_α include $D((-A)^\alpha)$ for $\alpha \in (0,1)$, the domains of the fraction power of $-A$, the real interpolation spaces $D_A(\alpha,\infty)$, $\alpha \in (0,1)$, defined as follows:

$$\begin{cases} D_A(\alpha,\infty) := \left\{ x \in X : [x]_\alpha := \sup\limits_{0<t\leq 1} \|t^{1-\alpha}(A-\omega)e^{-\omega t}T(t)x\| < +\infty \right\} \\ \|x\|_\alpha = \|x\| + [x]_\alpha, \end{cases}$$

and the abstract Hölder space $D_A(\alpha) := \overline{D(A)}^{\|\cdot\|_\alpha}$ as well as the complex interpolation space $[X, D(A)]_\alpha$, see [29] for more details.

Similar as $\mathscr{L}^2(P,H)$, for $0<\alpha<1$, we can define $\mathscr{L}^2(P,H_\alpha)$, where the norm is defined by

$$\|Y\|_{\alpha,2} := \left(\int_\Omega \|Y\|_\alpha^2 \, dP\right)^{1/2}.$$

LEMMA 2.1 ([29]). *For the hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, there exist constants $c>0$, $M>0$, $\delta>0$ and $\gamma>0$ such that*

$$\|T(t)Jx\|_\alpha \leq ce^{\delta t}\|x\| \quad \textit{for } t\leq 0, \tag{2.2}$$

$$\|T(t)Px\|_\alpha \leq Mt^{-\alpha}e^{-\gamma t}\|x\| \quad \textit{for } t>0. \tag{2.3}$$

2.2. Lévy process

DEFINITION 2.4 ([1, 31]). A V-valued stochastic process $L=(L(t),t\geq 0)$ is called Lévy process if
(i) $L(0)=0$ almost surely;
(ii) L has independent and stationary increments;
(iii) L is stochastically continuous, i.e., $\lim_{t\to s} P(|L(t)-L(s)|_V>\varepsilon)=0$ for all $\varepsilon>0$ and $s>0$.

Given a Lévy process L, we define the process of jumps of L by $\Delta L(t) = L(t)-L(t-)$, $t\geq 0$. For any Borel set B in $V-\{0\}$, define the random counting measure

$$N(t,B)(\omega) := \#\{0\leq s\leq t : \Delta L(s)(\omega)\in B\} = \sum_{0\leq s\leq t} \chi_B(\Delta L(s)(\omega)),$$

where # means the counting and χ_B is the indicator function. We write $\nu(\cdot) = \mathbf{E}(N(1,\cdot))$ and call it the intensity measure associated with L. If a Borel set B in $V-\{0\}$ is bounded below (that is $0\notin \bar{B}$, where $\bar{B}$ is the closure of B), then $N(t,B)<\infty$ almost surely for all $t\geq 0$ and $\{N(t,B),t\geq 0\}$ is a Poisson process with the intensity $\nu(B)$. So N is called Poisson random measure. For each $t\geq 0$ and B bounded below, the compensated Poisson random measure is defined by

$$\tilde{N}(t,B) = N(t,B) - t\nu(B).$$

PROPOSITION 2.1 (Lévy-Itô decomposition [31]). *If L is a V-valued Lévy process, then there exist $a\in V$, a V-valued Wiener process W with covariance operator Q, and an independent Poisson random measure N on $\mathbf{R}^+\times(V-\{0\})$ such that for each $t\geq 0$,*

$$L(t) = at + W(t) + \int_{|x|_V<1} x\tilde{N}(t,dx) + \int_{|x|_V\geq 1} xN(t,dx). \tag{2.4}$$

Here the Poisson random measure N has the intensity measure ν which satisfies

$$\int_V (|y|_V^2 \wedge 1)\nu(dy) < \infty \tag{2.5}$$

and $\tilde{N}$ is the compensated Poisson random measure of N.

Given two independent, identically distributed Lévy processes L_1 and L_2 with decompositions as in Proposition 2.1 with a, Q, W, N. Let

$$L(t) = \begin{cases} L_1(t), & \text{for } t \geq 0, \\ -L_2(-t), & \text{for } t \leq 0. \end{cases}$$

Then L is a two-sided Lévy process. In this paper, we need consider two-sided Lévy process. The two-sided Lévy process L is defined on the filtered probability space $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t \in \mathbf{R}})$. We assume that the covariance operator Q of W is of trace class, i.e., $Tr\, Q < \infty$.

Remark 2.1. It follows from that (2.5) that $\int_{|x|_V \geq 1} \nu(dx) < \infty$. For convenience, we denote

$$b := \int_{|x|_V \geq 1} \nu(dx)$$

throughout the paper.

Remark 2.2. Note that the stochastic process $\tilde{L} = (\tilde{L}(t), t \in \mathbf{R})$ given by $\tilde{L}(t) = L(t+s) - L(s)$ for some $s \in \mathbf{R}$ is also a two-sided Lévy process which share the same law as L. In particular, when $s \in \mathbf{R}^+$, the similar conclusions hold for one-sided Lévy process.

2.3. Square-mean almost automorphic process

Definition 2.5. A stochastic process $Y : \mathbf{R} \to \mathscr{L}^2(P, H)$ is said to be $\mathscr{L}^2$-bounded if there exists a constant $M > 0$ such that

$$\mathbf{E}\|Y(t)\|^2 = \int_\Omega \|Y(t)\|^2\, dP \leq M.$$

Definition 2.6. A stochastic process $Y : \mathbf{R} \to \mathscr{L}^2(P, H)$ is said to be $\mathscr{L}^2$-continuous if for any $s \in \mathbf{R}$,

$$\lim_{t \to s} \mathbf{E}\|Y(t) - Y(s)\|^2 = 0.$$

Note that if an H-valued process is $\mathscr{L}^2$-continuous, then it is necessarily stochastically continuous.

Denoted by $SBC(\mathbf{R}, \mathscr{L}^2(P,H))$ the collection of all the $\mathscr{L}^2$-bounded and $\mathscr{L}^2$-continuous processes. It is a Banach space equipped with the norm $\|Y\|_\infty := \sup_{t\in\mathbf{R}}\|Y(t)\|_2 = \sup_{t\in\mathbf{R}}(\mathbf{E}\|Y(t)\|^2)^{1/2}$.

Definition 2.7 ([22]). An $\mathscr{L}^2$-continuous process $Y : \mathbf{R} \to \mathscr{L}^2(P,H)$ is said to be square-mean almost automorphic if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and a stochastic process $\tilde{Y} : \mathbf{R} \to \mathscr{L}^2(P,H)$ such that

$$\lim_{n\to\infty} \mathbf{E}\|Y(t+s_n) - \tilde{Y}(t)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}\|\tilde{Y}(t-s_n) - Y(t)\|^2 = 0,$$

hold for each $t \in \mathbf{R}$. The collection of all square-mean almost automorphic processes $Y : \mathbf{R} \to \mathscr{L}^2(P,H)$ is denoted by $SAA(\mathbf{R}, \mathscr{L}^2(P,H))$.

Remark 2.3. (i) [22] Any square-mean almost automorphic process is $\mathscr{L}^2$-bounded.

(ii) [22] $SAA(\mathbf{R}, \mathscr{L}^2(P,H))$ is Banach space with the supremum norm $\|Y\|_\infty = \sup_{t\in\mathbf{R}}\|Y(t)\|_2$.

Definition 2.8 ([22]). A function $f : \mathbf{R} \times \mathscr{L}^2(P,H) \to L(V, \mathscr{L}^2(P,H))$, $(t, Y) \to f(t, Y)$ is said to be square-mean almost automorphic in $t \in \mathbf{R}$ for each $Y \in \mathscr{L}^2(P,H)$ if f is continuous in the following sense

$$\mathbf{E}\|f(t, Y) - f(t', Y')\|^2_{L(V,\mathscr{L}^2(P,H))} \to 0 \quad \text{as } (t, Y) \to (t', Y')$$

and that for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and an function $\tilde{f} : \mathbf{R} \times \mathscr{L}^2(P,H) \to L(V, \mathscr{L}^2(P,H))$ such that

$$\lim_{n\to\infty} \mathbf{E}\|f(t+s_n, Y) - \tilde{f}(t, Y)\|^2_{L(V,\mathscr{L}^2(P,H))} = 0$$

$$\lim_{n\to\infty} \mathbf{E}\|\tilde{f}(t-s_n, Y) - f(t, Y)\|^2_{L(V,\mathscr{L}^2(P,H))} = 0$$

hold for each $t \in \mathbf{R}$ and each $Y \in \mathscr{L}^2(P,H)$. Denote by $SAA(\mathbf{R} \times \mathscr{L}^2(P,H), L(V, \mathscr{L}^2(P,H)))$ the set of such functions.

Theorem 2.1 ([22]). *Assume that $f : \mathbf{R} \times \mathscr{L}^2(P,H) \to \mathscr{L}^2(P,H)$, $(t, Y) \to f(t, Y)$ is square-mean almost automorphic in $t \in \mathbf{R}$ for each $Y \in \mathscr{L}^2(P,H)$, and there exists a constant $L_f > 0$ such that*

$$\mathbf{E}\|f(t, Y) - f(t, Z)\|^2 \le L_f \cdot \mathbf{E}\|Y - Z\|^2, \quad \text{for all } Y, Z \in \mathscr{L}^2(P,H),\ t \in \mathbf{R}.$$

Then $f(\cdot, Y(\cdot)) \in SAA(\mathbf{R}, \mathscr{L}^2(P,H))$ if $Y(\cdot) \in SAA(\mathbf{R}, \mathscr{L}^2(P,H))$.

Now, we introduce the concept of almost automorphy in distribution. Let $\mathscr{P}(H)$ be the space of all Borel probability measures on H endowed with the β metric:

$$\beta(\mu, \nu) := \sup\left\{\left|\int f\, d\mu - \int f\, d\nu\right| : \|f\|_{BL} \le 1\right\}, \quad \mu, \nu \in \mathscr{P}(H),$$

where f are Lipschitz continuous real-valued functions on H with the norm

$$\|f\|_{BL} = \|f\|_L + \|f\|_\infty, \quad \|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}, \quad \|f\|_\infty = \sup_{x \in H} |f(x)|.$$

A sequence $\{\mu_n\} \subset \mathscr{P}(H)$ is said to weakly converge to μ if $\int f \, d\mu_n \to \int f \, d\mu$ for all $f \in BC(\mathbf{R}, H)$, the space of all bounded continuous real-valued functions on H. It is well know that the β metric is a complete metric on $\mathscr{P}(H)$ and a sequence $\{\mu_n\}$ weakly converges to μ if and only if $\beta(\mu_n, \mu) \to 0$ as $n \to \infty$, one can see [18, §11.3] for more details.

Definition 2.9 ([27]). An H-valued stochastic process f is said to be almost automorphic in one-dimensional distribution if its law $\mu(t)$ is a $\mathscr{P}(H)$-valued almost automorphic mapping, that is, for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$ and a $\mathscr{P}(H)$-valued mapping $\tilde{\mu}(t)$ such that

$$\lim_{n\to\infty} \beta(\mu(t + s_n), \tilde{\mu}(t)) = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta(\tilde{\mu}(t - s_n), \mu(t)) = 0$$

hold for each $t \in \mathbf{R}$.

Remark 2.4. Note that square-mean almost automorphic stochastic process is necessarily an almost automorphic in one-dimensional distribution, but the converse is not true, one can see [27] for more details.

2.4. Poisson square-mean almost automorphic process

Definition 2.10. A stochastic process $F(t, x) : \mathbf{R} \times V \to \mathscr{L}^2(P, H)$ is said to be Poisson stochastically bounded if there exists a constant $M > 0$ such that

$$\int_V \mathbf{E}\|F(t, x)\|^2 \nu(dx) \leq M \quad \text{for all } t \in \mathbf{R},\ x \in V.$$

Definition 2.11. A stochastic process $F(t, x) : \mathbf{R} \times V \to \mathscr{L}^2(P, H)$ is said to be Poisson stochastically continuous if

$$\lim_{t\to s} \int_V \mathbf{E}\|F(t, x) - F(s, x)\|^2 \nu(dx) = 0.$$

Denoted by $PSBC(\mathbf{R} \times V, \mathscr{L}^2(P, H))$ the collection of all the Poisson stochastically bounded and continuous processes.

Definition 2.12 ([27]). A stochastic process $F : \mathbf{R} \times V \to \mathscr{L}^2(P, H)$, $(t, x) \to F(t, x)$ is said to Poisson square-mean almost automorphic in $t \in \mathbf{R}$ if F is Poisson stochastically continuous and for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$ and a function $\tilde{F} : \mathbf{R} \times V \to \mathscr{L}^2(P, H)$ with

$\int_V \mathbf{E}\|\tilde{F}(t,x)\|^2 \nu(dx) < \infty$, such that

$$\lim_{n\to\infty} \int_V \mathbf{E}\|F(t+s_n,x) - \tilde{F}(t,x)\|^2 \nu(dx) = 0$$

and

$$\lim_{n\to\infty} \int_V \mathbf{E}\|\tilde{F}(t-s_n,x) - F(t,x)\|^2 \nu(dx) = 0$$

for each $t \in \mathbf{R}$. The collection of all Poisson square-mean almost automorphic stochastic processes $F : \mathbf{R} \times V \to \mathscr{L}^2(P,H)$ is denoted by $PSAA(\mathbf{R} \times V, \mathscr{L}^2(P,H))$.

Next, we give the concept of Poisson square-mean almost automorphic process with the parameter.

Definition 2.13. A stochastic process $F : \mathbf{R} \times \mathscr{L}^2(P,H) \times V \to \mathscr{L}^2(P,H)$, $(t, Y, x) \to F(t, Y, x)$ is said to be Poisson stochastically bounded if there exists a constant $M > 0$ such that

$$\int_V \mathbf{E}\|F(t,Y,x)\|^2 \nu(dx) \le M \quad \text{for all } t \in \mathbf{R},\ x \in V.$$

Definition 2.14. A stochastic process $F : \mathbf{R} \times \mathscr{L}^2(P,H) \times V \to \mathscr{L}^2(P,H)$, $(t, Y, x) \to F(t, Y, x)$ is said to be Poisson stochastically continuous if

$$\int_V \mathbf{E}\|F(t,Y,x) - F(t',Y',x)\|^2 \nu(dx) \to 0 \quad \text{as } (t,Y) \to (t',Y').$$

Denoted by $PSBC(\mathbf{R} \times \mathscr{L}^2(P,H) \times V, \mathscr{L}^2(P,H))$ the collection of all the Poisson stochastically bounded and continuous processes.

Definition 2.15 ([27]). A stochastic process $F : \mathbf{R} \times \mathscr{L}^2(P,H) \times V \to \mathscr{L}^2(P,H)$, $(t, Y, x) \to F(t, Y, x)$ is said to uniformly Poisson square-mean almost automorphic if F is Poisson stochastically continuous and for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$ and a function $\tilde{F} : \mathbf{R} \times \mathscr{L}^2(P,H) \times V \to \mathscr{L}^2(P,H)$ with $\int_V \mathbf{E}\|\tilde{F}(t,Y,x)\|^2 \nu(dx) < \infty$, such that

$$\lim_{n\to\infty} \int_V \mathbf{E}\|F(t+s_n,Y,x) - \tilde{F}(t,Y,x)\|^2 \nu(dx) = 0$$

and

$$\lim_{n\to\infty} \int_V \mathbf{E}\|\tilde{F}(t-s_n,Y,x) - F(t,Y,x)\|^2 \nu(dx) = 0$$

for each $t \in \mathbf{R}$ and each $Y \in \mathscr{L}^2(P,H)$. The collection of all uniformly Poisson square-mean almost automorphic stochastic processes $F : \mathbf{R} \times \mathscr{L}^2(P,H) \times V \to \mathscr{L}^2(P,H)$ is denoted by $PSAA(\mathbf{R} \times \mathscr{L}^2(P,H) \times V, \mathscr{L}^2(P,H))$.

THEOREM 2.2 ([27]). *Assume that* $F \in PSAA(\mathbf{R} \times \mathscr{L}^2(P,H) \times V, \mathscr{L}^2(P,H))$, *and there exists a constant* $L > 0$ *such that*

$$\int_V \mathbf{E}\|F(t,Y,x) - F(t,Z,x)\|^2 \nu(dx)$$
$$\leq L \cdot \mathbf{E}\|Y - Z\|^2, \quad \text{for all } Y, Z \in \mathscr{L}^2(P,H),\ t \in \mathbf{R}.$$

Then $F(\cdot, Y(\cdot), x) \in PSAA(\mathbf{R} \times V, \mathscr{L}^2(P,H))$ *if* $Y(\cdot) \in SAA(\mathbf{R}, \mathscr{L}^2(P,H))$.

3. SDEs driven by Lévy noise

We consider semilinear SDEs with Lévy noise:

$$\begin{aligned} dY(t) &= AY(t)\,dt + f(t,Y(t))\,dt + g(t,Y(t))\,dW(t) \\ &\quad + \int_{|x|_V<1} F(t,Y(t-),x)\tilde{N}(dt,dx) \\ &\quad + \int_{|x|_V\geq 1} G(t,Y(t-),x)N(dt,dx), \quad t \in \mathbf{R}, \end{aligned} \tag{3.1}$$

where $f : \mathbf{R} \times \mathscr{L}^2(P,H_\alpha) \to \mathscr{L}^2(P,H)$, $g : \mathbf{R} \times \mathscr{L}^2(P,H_\alpha) \to L(V, \mathscr{L}^2(P,H))$, $F, G : \mathbf{R} \times \mathscr{L}^2(P,H_\alpha) \times V \to \mathscr{L}^2(P,H)$, W and N are the Lévy-Itô decomposition components of the two-sided Lévy process L with assumptions stated in Subsection 2.2.

First, we make the following assumptions:

(H_1) The operator A is a sectorial and $\sigma(A) \cap i\mathbf{R} = \emptyset$.

(H_2) $f \in SAA(\mathbf{R} \times \mathscr{L}^2(P,H_\alpha), \mathscr{L}^2(P,H))$, $g \in SAA(\mathbf{R} \times \mathscr{L}^2(P,H_\alpha), L(V, \mathscr{L}^2(P,H)))$, $F, G \in PSAA(\mathbf{R} \times \mathscr{L}^2(P,H_\alpha) \times V, \mathscr{L}^2(P,H))$ and there exists a constant $L > 0$ such that

$$\mathbf{E}\|f(t,Y) - f(t,Z)\|^2 \leq L \cdot \mathbf{E}\|Y - Z\|_\alpha^2,$$
$$\mathbf{E}\|(g(t,Y) - g(t,Z))Q^{1/2}\|_{L(V,\mathscr{L}^2(P,H))}^2 \leq L \cdot \mathbf{E}\|Y - Z\|_\alpha^2,$$
$$\int_{|x|_V<1} \mathbf{E}\|F(t,Y,x) - F(t,Z,x)\|^2 \nu(dx) \leq L \cdot \mathbf{E}\|Y - Z\|_\alpha^2,$$
$$\int_{|x|_V\geq 1} \mathbf{E}\|G(t,Y,x) - G(t,Z,x)\|^2 \nu(dx) \leq L \cdot \mathbf{E}\|Y - Z\|_\alpha^2,$$

for all $t \in \mathbf{R}$, $Y, Z \in \mathscr{L}^2(P,H_\alpha)$.

Before starting our main results, we recall the definition of the mild solution to (3.1).

DEFINITION 3.1 ([33]). Let $\alpha \in (0,1)$. An $\mathscr{F}_t$-progressively measurable stochastic process $\{Y(t)\}_{t\in\mathbf{R}}$ is called a mild solution of (3.1) if it satisfies the corresponding stochastic integral equation:

$$
\begin{aligned}
(3.2) \qquad Y(t) = {} & T(t-a)Y(a) + \int_a^t T(t-s)f(s, Y(s))\, ds \\
& + \int_a^t T(t-s)g(s, Y(s))\, dW(s) \\
& + \int_a^t \int_{|x|_V<1} T(t-s)F(s, Y(s-), x)\tilde{N}(ds, dx) \\
& + \int_a^t \int_{|x|_V\geq 1} T(t-s)G(s, Y(s-), x)N(ds, dx),
\end{aligned}
$$

for all $t \geq a$ and each $a \in \mathbf{R}$.

Theorem 3.1. *Let* (H_1)–(H_2) *be satisfied, then* (3.1) *has a unique* $\mathscr{L}^2$-*bounded mild solution if*

$$
\begin{aligned}
\Theta := {} & 8L(1+2b)(M\gamma^{\alpha-1}\Gamma(1-\alpha))^2 + 32LM^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha) \\
& + 8L(1+2b)c^2\delta^{-2} + 16Lc^2\delta^{-1} < 1.
\end{aligned}
$$

Furthermore, this unique $\mathscr{L}^2$-*bounded mild solution is almost automorphic in one-dimensional distribution if*

$$
\begin{aligned}
\vartheta := {} & 16L(1+2b)(M\gamma^{\alpha-1}\Gamma(1-\alpha))^2 + 64LM^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha) \\
& + 16L(1+2b)c^2\delta^{-2} + 32Lc^2\delta^{-1} < 1.
\end{aligned}
$$

Proof. Note that if $Y(t)$ is $\mathscr{L}^2$-bounded, then $Y(t)$ is a mild solution of (3.1) if and only if it satisfies the following integral equation

$$
\begin{aligned}
(3.3) \qquad Y(t) = {} & \int_{-\infty}^t T(t-s)Pf(s, Y(s))\, ds - \int_t^{+\infty} T(t-s)Jf(s, Y(s))\, ds \\
& + \int_{-\infty}^t T(t-s)Pg(s, Y(s))\, dW(s) - \int_t^{+\infty} T(t-s)Jg(s, Y(s))\, dW(s) \\
& + \int_{-\infty}^t \int_{|x|_V<1} T(t-s)PF(s, Y(s-), x)\tilde{N}(ds, dx) \\
& - \int_t^{+\infty} \int_{|x|_V<1} T(t-s)JF(s, Y(s-), x)\tilde{N}(ds, dx) \\
& + \int_{-\infty}^t \int_{|x|_V\geq 1} T(t-s)PG(s, Y(s-), x)N(ds, dx) \\
& - \int_t^{\infty} \int_{|x|_V\geq 1} T(t-s)JG(s, Y(s-), x)N(ds, dx),
\end{aligned}
$$

where $P + J = I$. In fact, for $t \geq a$, $a \in \mathbf{R}$, by the properties of $T(t-a)T(a-s) = T(t-s)$, one has

$$\begin{aligned}
T(t-a)Y(a) &= \int_{-\infty}^{a} T(t-s)Pf(s, Y(s))\,ds - \int_{a}^{+\infty} T(t-s)Jf(s, Y(s))\,ds \\
&\quad + \int_{-\infty}^{a} T(t-s)Pg(s, Y(s))\,dW(s) \\
&\quad - \int_{a}^{+\infty} T(t-s)Jg(s, Y(s))\,dW(s) \\
&\quad + \int_{-\infty}^{a} \int_{|x|_V<1} T(t-s)PF(s, Y(s-), x)\tilde{N}(ds, dx) \\
&\quad - \int_{a}^{+\infty} \int_{|x|_V<1} T(t-s)JF(s, Y(s-), x)\tilde{N}(ds, dx) \\
&\quad + \int_{-\infty}^{a} \int_{|x|_V \geq 1} T(t-s)PG(s, Y(s-), x)N(ds, dx) \\
&\quad - \int_{a}^{\infty} \int_{|x|_V \geq 1} T(t-s)JG(s, Y(s-), x)N(ds, dx) \\
&= Y(t) - \int_{a}^{t} T(t-s)f(s, Y(s))\,ds - \int_{a}^{t} T(t-s)g(s, Y(s))\,dW(s) \\
&\quad - \int_{a}^{t} \int_{|x|_V<1} T(t-s)F(s, Y(s-), x)\tilde{N}(ds, dx) \\
&\quad - \int_{a}^{t} \int_{|x|_V \geq 1} T(t-s)G(s, Y(s-), x)N(ds, dx),
\end{aligned}$$

so $Y(t)$ is a mild solution of (3.1). On the other hand, if $Y(t)$ is a mild solution of (3.1), then

$$\begin{aligned}
PY(t) &= T(t-a)PY(a) + \int_{a}^{t} T(t-s)Pf(s, Y(s))\,ds \\
&\quad + \int_{a}^{t} T(t-s)Pg(s, Y(s))\,dW(s) \\
&\quad + \int_{a}^{t} \int_{|x|_V<1} T(t-s)PF(s, Y(s-), x)\tilde{N}(ds, dx) \\
&\quad + \int_{a}^{t} \int_{|x|_V \geq 1} T(t-s)PG(s, Y(s-), x)N(ds, dx), \\
&\quad \text{for all } t \geq a,\ a \in \mathbf{R},
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
(3.5)\qquad JY(t) = {} & T(t-a)JY(a) + \int_a^t T(t-s)Jf(s, Y(s))\, ds \\
& + \int_a^t T(t-s)Jg(s, Y(s))\, dW(s) \\
& + \int_a^t \int_{|x|_V<1} T(t-s)JF(s, Y(s-), x)\tilde{N}(ds, dx) \\
& + \int_a^t \int_{|x|_V\geq 1} T(t-s)JG(s, Y(s-), x)N(ds, dx), \\
& \text{for all } t \geq a,\ a \in \mathbf{R}.
\end{aligned}
$$

Since Y is $\mathscr{L}^2$-bounded, then letting $a \to -\infty$ in (3.4) and $a \to +\infty$ in (3.5), by using (2.2), (2.3) and $P + J = I$, we have (3.3) holds.

In the following, we will prove (3.3) is the unique almost automorphic in one-dimensional distribution mild solution of (3.1). We divide the proof into several steps.

STEP 1. $\mathscr{L}^2$-bounded mild solution is $\mathscr{L}^2$-continuous.

Similar as the proof of [33], it is not difficult to see the $\mathscr{L}^2$-continuity of $Y(t)$.

STEP 2. Existence and uniqueness of $\mathscr{L}^2$-bounded mild solution.

Let $\mathscr{S} : SBC(\mathbf{R}, \mathscr{L}^2(P, H_\alpha)) \to SBC(\mathbf{R}, \mathscr{L}^2(P, H_\alpha))$ be the operator defined by (3.3) and let

$$(\mathscr{S}Y)(t) := (\mathscr{S}_1 Y)(t) + (\mathscr{S}_2 Y)(t) + (\mathscr{S}_3 Y)(t) + (\mathscr{S}_4 Y)(t),$$

where

$$
\begin{aligned}
(\mathscr{S}_1 Y)(t) &= \int_{-\infty}^t T(t-s)Pf(s, Y(s))\, ds - \int_t^{+\infty} T(t-s)Jf(s, Y(s))\, ds \\
(\mathscr{S}_2 Y)(t) &= \int_{-\infty}^t T(t-s)Pg(s, Y(s))\, dW(s) - \int_t^{+\infty} T(t-s)Jg(s, Y(s))\, dW(s) \\
(\mathscr{S}_3 Y)(t) &= \int_{-\infty}^t \int_{|x|_V<1} T(t-s)PF(s, Y(s-), x)\tilde{N}(ds, dx) \\
&\quad - \int_t^{+\infty} \int_{|x|_V<1} T(t-s)JF(s, Y(s-), x)\tilde{N}(ds, dx) \\
(\mathscr{S}_4 Y)(t) &= \int_{-\infty}^t \int_{|x|_V\geq 1} T(t-s)PG(s, Y(s-), x)N(ds, dx) \\
&\quad - \int_t^{\infty} \int_{|x|_V\geq 1} T(t-s)JG(s, Y(s-), x)N(ds, dx).
\end{aligned}
$$

If $Y(t)$ is $\mathscr{L}^2$-bounded, it is not difficult to see that $(\mathscr{S}Y)(t)$ is $\mathscr{L}^2$-bounded. By the proof of Step 1, $(\mathscr{S}Y)(t)$ is $\mathscr{L}^2$-continuous if $Y(t)$ is $\mathscr{L}^2$-bounded. Hence $\mathscr{S}$ is well-defined. Next, we will show that $\mathscr{S}$ is a contraction mapping on $SBC(\mathbf{R}, \mathscr{L}^2(P, H_\alpha))$. For $Y_1, Y_2 \in SBC(\mathbf{R}, \mathscr{L}^2(P, H_\alpha))$, one has

$$\begin{aligned}
&\mathbf{E}\|(\mathscr{S}Y_1)(t) - (\mathscr{S}Y_2)(t)\|_\alpha^2 \\
&\quad \le 4\mathbf{E}\|(\mathscr{S}_1 Y_1)(t) - (\mathscr{S}_1 Y_2)(t)\|_\alpha^2 + 4\mathbf{E}\|(\mathscr{S}_2 Y_1)(t) - (\mathscr{S}_2 Y_2)(t)\|_\alpha^2 \\
&\qquad + 4\mathbf{E}\|(\mathscr{S}_3 Y_1)(t) - (\mathscr{S}_3 Y_2)(t)\|_\alpha^2 + 4\mathbf{E}\|(\mathscr{S}_4 Y_1)(t) - (\mathscr{S}_4 Y_2)(t)\|_\alpha^2.
\end{aligned}$$

Similar as the proof of [4], the first term of the right-hand side of the above inequality can be estimated as follows:

$$\begin{aligned}
&\mathbf{E}\|(\mathscr{S}_1 Y_1)(t) - (\mathscr{S}_1 Y_2)(t)\|_\alpha^2 \\
&\quad \le 2\mathbf{E}\left\|\int_{-\infty}^t T(t-s)P[f(s, Y_1(s)) - f(s, Y_2(s))]\,ds\right\|_\alpha^2 \\
&\qquad + 2\mathbf{E}\left\|\int_t^{+\infty} T(t-s)J[f(s, Y_1(s)) - f(s, Y_2(s))]\,ds\right\|_\alpha^2 \\
&\quad \le 2M^2\mathbf{E}\left(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}\|f(s, Y_1(s)) - f(s, Y_2(s))\|\,ds\right)^2 \\
&\qquad + 2c^2\mathbf{E}\left(\int_t^{+\infty} e^{\delta(t-s)}\|f(s, Y_1(s)) - f(s, Y_2(s))\|\,ds\right)^2,
\end{aligned}$$

Note that

$$\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}\,ds = \gamma^{\alpha-1}\Gamma(1-\alpha),$$

where Γ is the Gamma function. Hence by Cauchy-Schwarz inequality, one has

$$\begin{aligned}
&\mathbf{E}\|(\mathscr{S}_1 Y_1)(t) - (\mathscr{S}_1 Y_2)(t)\|_\alpha^2 \\
&\quad \le 2M^2\gamma^{\alpha-1}\Gamma(1-\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}\mathbf{E}\|f(s, Y_1(s)) - f(s, Y_2(s))\|^2\,ds \\
&\qquad + 2c^2\delta^{-1}\int_t^{+\infty} e^{\delta(t-s)}\mathbf{E}\|f(s, Y_1(s)) - f(s, Y_2(s))\|^2\,ds \\
&\quad \le 2L[(M\gamma^{\alpha-1}\Gamma(1-\alpha))^2 + c^2\delta^{-2}] \cdot \sup_{s\in\mathbf{R}} \mathbf{E}\|Y_1(s) - Y_2(s)\|_\alpha^2.
\end{aligned}$$

By Itô isometry, the second term as follows:

$$
\begin{aligned}
&\mathbf{E}\|(\mathscr{S}_2 Y_1)(t) - (\mathscr{S}_2 Y_2)(t)\|_\alpha^2 \\
&\quad \le 2\mathbf{E}\left\|\int_{-\infty}^{t} T(t-s)P[g(s, Y_1(s)) - g(s, Y_2(s))]\, dW(s)\right\|_\alpha^2 \\
&\qquad + 2\mathbf{E}\left\|\int_{t}^{+\infty} T(t-s)J[g(s, Y_1(s)) - g(s, Y_2(s))]\, dW(s)\right\|_\alpha^2 \\
&\quad \le 2M^2 \int_{-\infty}^{t} (t-s)^{-2\alpha} e^{-2\gamma(t-s)} \mathbf{E}\|(g(s, Y_1(s)) - g(s, Y_2(s)))Q^{1/2}\|_{L(V,\mathscr{L}^2(P,H))}^2\, ds \\
&\qquad + 2c^2 \int_{t}^{+\infty} e^{2\delta(t-s)} \mathbf{E}\|(g(s, Y_1(s)) - g(s, Y_2(s)))Q^{1/2}\|_{L(V,\mathscr{L}^2(P,H))}^2\, ds,
\end{aligned}
$$

Note that

$$
\int_{-\infty}^{t} (t-s)^{-2\alpha} e^{-2\gamma(t-s)}\, ds = \int_{0}^{+\infty} s^{-2\alpha} e^{-2\gamma s}\, ds = (2\gamma)^{2\alpha-1}\Gamma(1-2\alpha),
$$

Hence

$$
\begin{aligned}
&\mathbf{E}\|(\mathscr{S}_2 Y_1)(t) - (\mathscr{S}_2 Y_2)(t)\|_\alpha^2 \\
&\qquad \le 2L\left[M^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha) + \frac{1}{2}c^2\delta^{-1}\right] \cdot \sup_{s\in\mathbf{R}} \mathbf{E}\| Y_1(s) - Y_2(s)\|_\alpha^2.
\end{aligned}
$$

By the properties of the integral for the Poisson random measure, we have

$$
\begin{aligned}
&\mathbf{E}\|(\mathscr{S}_3 Y_1)(t) - (\mathscr{S}_3 Y_2)(t)\|_\alpha^2 \\
&\quad \le 2\mathbf{E}\left\|\int_{-\infty}^{t}\int_{|x|_V<1} T(t-s)P[F(s, Y_1(s-), x) - F(s, Y_2(s-), x)]\tilde{N}(ds, dx)\right\|_\alpha^2 \\
&\qquad + 2\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V<1} T(t-s)J[F(s, Y_1(s-), x) - F(s, Y_2(s-), x)]\tilde{N}(ds, dx)\right\|_\alpha^2 \\
&\quad \le 2M^2 \int_{-\infty}^{t}\int_{|x|_V<1} (t-s)^{-2\alpha} e^{-2\gamma(t-s)} \\
&\qquad \times \mathbf{E}\|F(s, Y_1(s-), x) - F(s, Y_2(s-), x)\|^2 \nu(dx)\, ds \\
&\qquad + 2c^2 \int_{t}^{+\infty}\int_{|x|_V<1} e^{2\delta(t-s)} \mathbf{E}\|F(s, Y_1(s-), x) - F(s, Y_2(s-), x)\|^2 \nu(dx)\, ds \\
&\quad \le 2L\left[M^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha) + \frac{1}{2}c^2\delta^{-1}\right] \cdot \sup_{s\in\mathbf{R}} \mathbf{E}\| Y_1(s) - Y_2(s)\|_\alpha^2,
\end{aligned}
$$

Note that

$$\mathbf{E}\left\|\int_{-\infty}^{t}\int_{|x|_V\ge 1} T(t-s)P[G(s,Y_1(s-),x)-G(s,Y_2(s-),x)]\nu(dx)\,ds\right\|_\alpha^2$$
$$+\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V\ge 1} T(t-s)J[G(s,Y_1(s-),x)-G(s,Y_2(s-),x)]\nu(dx)\,ds\right\|_\alpha^2$$
$$\le M^2\int_{-\infty}^{t}\int_{|x|_V\ge 1}(t-s)^{-\alpha}e^{-\gamma(t-s)}\nu(dx)\,ds$$
$$\times\int_{-\infty}^{t}\int_{|x|_V\ge 1}(t-s)^{-\alpha}e^{-\gamma(t-s)}\mathbf{E}\|G(s,Y_1(s-),x)-G(s,Y_2(s-),x)\|^2\nu(dx)\,ds$$
$$+c^2\int_{t}^{+\infty}\int_{|x|_V\ge 1}e^{2\delta(t-s)}\mathbf{E}\|G(s,Y_1(s-),x)-G(s,Y_2(s-),x)\|^2\nu(dx)\,ds$$
$$\times\int_{t}^{+\infty}\int_{|x|_V\ge 1}e^{\delta(t-s)}\mathbf{E}\|G(s,Y_1(s-),x)-G(s,Y_2(s-),x)\|^2\nu(dx)\,ds$$
$$\le L[b(M\gamma^{\alpha-1}\Gamma(1-\alpha))^2+bc^2\delta^{-2}]\cdot\sup_{s\in\mathbf{R}}\mathbf{E}\|Y_1(s)-Y_2(s)\|_\alpha^2,$$

then similarly as the proof of $\mathscr{S}_3$, one has

$$\mathbf{E}\|(\mathscr{S}_4Y_1)(t)-(\mathscr{S}_4Y_2)(t)\|_\alpha^2$$
$$\le 4L\left[M^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha)+b(M\gamma^{\alpha-1}\Gamma(1-\alpha))^2+\frac{1}{2}c^2\delta^{-1}+bc^2\delta^{-2}\right]$$
$$\cdot\sup_{s\in\mathbf{R}}\mathbf{E}\|Y_1(s)-Y_2(s)\|_\alpha^2.$$

Hence, it follow that, for each $t\in\mathbf{R}$

$$\mathbf{E}\|(\mathscr{S}Y_1)(t)-(\mathscr{S}Y_2)(t)\|_\alpha^2\le\Theta\cdot\sup_{s\in\mathbf{R}}\mathbf{E}\|Y_1(s)-Y_2(s)\|_\alpha^2.$$

Note that

$$\sup_{s\in\mathbf{R}}\|Y_1(s)-Y_2(s)\|_{\alpha,2}^2\le\left(\sup_{s\in\mathbf{R}}\|Y_1(s)-Y_2(s)\|_{\alpha,2}\right)^2,$$

then

$$\|(\mathscr{S}Y_1)(t)-(\mathscr{S}Y_2)(t)\|_{\alpha,2}\le\sqrt{\Theta}\|Y_1-Y_2\|_{\alpha,\infty}.$$

Hence

$$\|\mathscr{S}Y_1-\mathscr{S}Y_2\|_{\alpha,\infty}:=\sup_{t\in\mathbf{R}}\|(\mathscr{S}Y_1)(t)-(\mathscr{S}Y_2)(t)\|_{\alpha,2}\le\sqrt{\Theta}\|Y_1-Y_2\|_{\alpha,\infty}.$$

Since $\Theta<1$, it follow that $\mathscr{S}$ is a contraction mapping on $SBC(\mathbf{R},\mathscr{L}^2(P,H_\alpha))$. Therefore, by the Banach contraction mapping principle, $\mathscr{S}$ has a unique fixed point in $SBC(\mathbf{R},\mathscr{L}^2(P,H_\alpha))$, which is the unique $\mathscr{L}^2$-bounded mild solution to (3.1).

STEP 3. Almost automorphy in one-dimensional distribution of $\mathscr{L}^2$-bounded mild solution.

Since $f \in SAA(\mathbf{R} \times \mathscr{L}^2(P, H_\alpha), \mathscr{L}^2(P, H))$, $g \in SAA(\mathbf{R} \times \mathscr{L}^2(P, H_\alpha), L(V, \mathscr{L}^2(P, H)))$, $F, G \in PSAA(\mathbf{R} \times \mathscr{L}^2(P, H_\alpha) \times V, \mathscr{L}^2(P, H))$, thus for every sequence of real numbers $\{s_n'\}$, there exists a subsequence $\{s_n\}$ and some functions $\tilde{f}$, $\tilde{g}$, $\tilde{F}$, $\tilde{G}$, such that

$$\lim_{n\to\infty} \mathbf{E}\|f(t+s_n, Y) - \tilde{f}(t, Y)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}\|\tilde{f}(t-s_n, Y) - f(t, Y)\|^2 = 0;$$

$$\lim_{n\to\infty} \mathbf{E}\|(g(t+s_n, Y) - \tilde{g}(t, Y))Q^{1/2}\|^2_{L(V, \mathscr{L}^2(P,H))} = 0,$$

$$\lim_{n\to\infty} \mathbf{E}\|(\tilde{g}(t-s_n, Y) - g(t, Y))Q^{1/2}\|^2_{L(V, \mathscr{L}^2(P,H))} = 0;$$

$$\lim_{n\to\infty} \int_{|x|<1} \mathbf{E}\|F(t+s_n, Y, x) - \tilde{F}(t, Y, x)\|^2 \nu(dx) = 0,$$

$$\lim_{n\to\infty} \int_{|x|<1} \mathbf{E}\|\tilde{F}(t-s_n, Y, x) - F(t, Y, x)\|^2 \nu(dx) = 0;$$

and

$$\lim_{n\to\infty} \int_{|x|\geq 1} \mathbf{E}\|G(t+s_n, Y, x) - \tilde{G}(t, Y, x)\|^2 \nu(dx) = 0,$$

$$\lim_{n\to\infty} \int_{|x|\geq 1} \mathbf{E}\|\tilde{G}(t-s_n, Y, x) - G(t, Y, x)\|^2 \nu(dx) = 0;$$

for each $t \in \mathbf{R}$, $Y \in \mathscr{L}^2(P, H_\alpha)$.

Let $\tilde{Y}(t)$ satisfy the integral equation

$$\begin{aligned}
\tilde{Y}(t) = &\int_{-\infty}^{t} T(t-s)P\tilde{f}(s, \tilde{Y}(s))\, ds - \int_{t}^{+\infty} T(t-s)J\tilde{f}(s, \tilde{Y}(s))\, ds \\
&+ \int_{-\infty}^{t} T(t-s)P\tilde{g}(s, \tilde{Y}(s))\, dW(s) - \int_{t}^{+\infty} T(t-s)J\tilde{g}(s, \tilde{Y}(s))\, dW(s) \\
&+ \int_{-\infty}^{t} \int_{|x|_V<1} T(t-s)P\tilde{F}(s, \tilde{Y}(s-), x)\tilde{N}(ds, dx) \\
&- \int_{t}^{+\infty} \int_{|x|_V<1} T(t-s)J\tilde{F}(s, \tilde{Y}(s-), x)\tilde{N}(ds, dx) \\
&+ \int_{-\infty}^{t} \int_{|x|_V\geq 1} T(t-s)P\tilde{G}(s, \tilde{Y}(s-), x)N(ds, dx) \\
&- \int_{t}^{\infty} \int_{|x|_V\geq 1} T(t-s)J\tilde{G}(s, \tilde{Y}(s-), x)N(ds, dx),
\end{aligned}$$

by the proof of Step 1 and Step 2, it follows that $\tilde{Y}$ is unique and $\mathscr{L}^2$-bounded.

Let $W_n(\tau) := W(\tau + s_n) - W(s_n)$, $N_n(\tau, x) := N(\tau + s_n, x) - N(s_n, x)$ and $\tilde{N}_n(\tau, x) := \tilde{N}(\tau + s_n, x) - \tilde{N}(s_n, x)$ for each $\tau \in \mathbf{R}$. It is easy to show that W_n is a Q-Wiener process with the same law as W, N_n is also a Poisson random measure and has the same law as N, so are $\tilde{N}_n$ and $\tilde{N}$, moreover, $\tilde{N}_n$ is the compensated Poisson measure of N_n. Let $\tau = s - s_n$, one has

$$
\begin{aligned}
Y(t+s_n) = & \int_{-\infty}^{t} T(t-\tau)Pf(\tau+s_n, Y(\tau+s_n))\,d\tau \\
& - \int_{t}^{+\infty} T(t-\tau)Jf(\tau+s_n, Y(\tau+s_n))\,d\tau \\
& + \int_{-\infty}^{t} T(t-\tau)Pg(\tau+s_n, Y(\tau+s_n))\,dW_n(\tau) \\
& - \int_{t}^{+\infty} T(t-\tau)Jg(\tau+s_n, Y(\tau+s_n))\,dW_n(\tau) \\
& + \int_{-\infty}^{t}\int_{|x|_V<1} T(t-\tau)PF(\tau+s_n, Y(\tau+s_n-), x)\tilde{N}_n(d\tau, dx) \\
& - \int_{t}^{+\infty}\int_{|x|_V<1} T(t-\tau)JF(\tau+s_n, Y(\tau+s_n-), x)\tilde{N}_n(d\tau, dx) \\
& + \int_{-\infty}^{t}\int_{|x|_V\geq 1} T(t-\tau)PG(\tau+s_n, Y(\tau+s_n-), x)N_n(d\tau, dx) \\
& - \int_{t}^{\infty}\int_{|x|_V\geq 1} T(t-\tau)JG(\tau+s_n, Y(\tau+s_n-), x)N_n(d\tau, dx).
\end{aligned} \tag{3.6}
$$

Consider the process

$$
\begin{aligned}
Y_n(t) = & \int_{-\infty}^{t} T(t-\tau)Pf(\tau+s_n, Y_n(\tau))\,d\tau - \int_{t}^{+\infty} T(t-\tau)Jf(\tau+s_n, Y_n(\tau))\,d\tau \\
& + \int_{-\infty}^{t} T(t-\tau)Pg(\tau+s_n, Y_n(\tau))\,dW(\tau) \\
& - \int_{t}^{+\infty} T(t-\tau)Jg(\tau+s_n, Y_n(\tau))\,dW(\tau) \\
& + \int_{-\infty}^{t}\int_{|x|_V<1} T(t-\tau)PF(\tau+s_n, Y_n(\tau-), x)\tilde{N}(d\tau, dx) \\
& - \int_{t}^{+\infty}\int_{|x|_V<1} T(t-\tau)JF(\tau+s_n, Y_n(\tau-), x)\tilde{N}(d\tau, dx)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-\infty}^{t}\int_{|x|_V\geq 1} T(t-\tau)PG(\tau+s_n, Y_n(\tau-), x)N(d\tau, dx) \\
&-\int_{t}^{\infty}\int_{|x|_V\geq 1} T(t-\tau)JG(\tau+s_n, Y_n(\tau-), x)N(d\tau, dx),
\end{aligned}
$$

It is easy to see that $Y(t+s_n)$ has the same distribution as $Y_n(t)$ for each $t\in\mathbf{R}$ and $Y_n(t)$ is unique and $\mathscr{L}^2$-bounded.

Note that

$$
\begin{aligned}
&\mathbf{E}\|Y_n(t)-\tilde{Y}(t)\|_\alpha^2 \\
&\quad\leq 8\mathbf{E}\left\|\int_{-\infty}^{t} T(t-\tau)P[f(\tau+s_n, Y_n(\tau))-\tilde{f}(\tau, \tilde{Y}(\tau))]\,d\tau\right\|_\alpha^2 \\
&\quad+8\mathbf{E}\left\|\int_{t}^{+\infty} T(t-\tau)J[f(\tau+s_n, Y_n(\tau))-\tilde{f}(\tau, \tilde{Y}(\tau))]\,d\tau\right\|_\alpha^2 \\
&\quad+8\mathbf{E}\left\|\int_{-\infty}^{t} T(t-\tau)P[g(\tau+s_n, Y_n(\tau))-\tilde{g}(\tau, \tilde{Y}(\tau))]\,dW(\tau)\right\|_\alpha^2 \\
&\quad+8\mathbf{E}\left\|\int_{t}^{+\infty} T(t-\tau)J[g(\tau+s_n, Y_n(\tau))-\tilde{g}(\tau, \tilde{Y}(\tau))]\,dW(\tau)\right\|_\alpha^2 \\
&\quad+8\mathbf{E}\left\|\int_{-\infty}^{t}\int_{|x|_V<1} T(t-\tau)P[F(\tau+s_n, Y_n(\tau-), x)-\tilde{F}(\tau, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx)\right\|_\alpha^2 \\
&\quad+8\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V<1} T(t-\tau)J[F(\tau+s_n, Y_n(\tau-), x)-\tilde{F}(\tau, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx)\right\|_\alpha^2 \\
&\quad+8\mathbf{E}\left\|\int_{-\infty}^{t}\int_{|x|_V\geq 1} T(t-\tau)P[G(\tau+s_n, Y_n(\tau-), x)-\tilde{G}(\tau, \tilde{Y}(\tau-), x)]N(d\tau, dx)\right\|_\alpha^2 \\
&\quad+8\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V\geq 1} T(t-\tau)J[G(\tau+s_n, Y_n(\tau-), x)-\tilde{G}(\tau, \tilde{Y}(\tau-), x)]N(d\tau, dx)\right\|_\alpha^2 \\
&\quad:= I_1+I_2+I_3+I_4.
\end{aligned}
$$

It follows from the Cauchy-Schwarz inequality that

$$
\begin{aligned}
I_1 &\leq 16\mathbf{E}\left\|\int_{-\infty}^{t} T(t-\tau)P[f(\tau+s_n, Y_n(\tau))-f(\tau+s_n, \tilde{Y}(\tau))]\,d\tau\right\|_\alpha^2 \\
&\quad+16\mathbf{E}\left\|\int_{-\infty}^{t} T(t-\tau)P[f(\tau+s_n, \tilde{Y}(\tau))-\tilde{f}(\tau, \tilde{Y}(\tau))]\,d\tau\right\|_\alpha^2
\end{aligned}
$$

$$
\begin{aligned}
&+16\mathbf{E}\left\|\int_t^{+\infty} T(t-\tau)J[f(\tau+s_n, Y_n(\tau)) - f(\tau+s_n, \tilde{Y}(\tau))]\,d\tau\right\|_\alpha^2 \\
&+16\mathbf{E}\left\|\int_t^{+\infty} T(t-\tau)J[f(\tau+s_n, \tilde{Y}(\tau)) - \tilde{f}(\tau, \tilde{Y}(\tau))]\,d\tau\right\|_\alpha^2 \\
\le\ & 16M^2\gamma^{\alpha-1}\Gamma(1-\alpha)\int_{-\infty}^t (t-\tau)^{-\alpha}e^{-\gamma(t-\tau)} \\
&\times \mathbf{E}\|f(\tau+s_n, Y_n(\tau)) - f(\tau+s_n, \tilde{Y}(\tau))\|^2\,d\tau \\
&+16M^2\gamma^{\alpha-1}\Gamma(1-\alpha)\int_{-\infty}^t (t-\tau)^{-\alpha}e^{-\gamma(t-\tau)}\mathbf{E}\|f(\tau+s_n, \tilde{Y}(\tau)) - \tilde{f}(\tau, \tilde{Y}(\tau))\|^2\,d\tau \\
&+16c^2\delta^{-1}\int_t^{+\infty} e^{\delta(t-\tau)}\mathbf{E}\|f(\tau+s_n, Y_n(\tau)) - f(\tau+s_n, \tilde{Y}(\tau))\|^2\,d\tau \\
&+16c^2\delta^{-1}\int_t^{+\infty} e^{\delta(t-\tau)}\mathbf{E}\|f(\tau+s_n, \tilde{Y}(\tau)) - \tilde{f}(\tau, \tilde{Y}(\tau))\|^2\,d\tau \\
\le\ & 16L[(M\gamma^{\alpha-1}\Gamma(1-\alpha))^2 + c^2\delta^{-2}]\cdot\sup_{\tau\in\mathbf{R}}\mathbf{E}\|Y_n(\tau) - \tilde{Y}(\tau)\|_\alpha^2 + \mathscr{E}_1^n(t),
\end{aligned}
$$

with

$$
\begin{aligned}
\mathscr{E}_1^n(t) = {} & 16M^2\gamma^{\alpha-1}\Gamma(1-\alpha)\int_{-\infty}^t (t-\tau)^{-\alpha}e^{-\gamma(t-\tau)}\mathbf{E}\|f(\tau+s_n, \tilde{Y}(\tau)) - \tilde{f}(\tau, \tilde{Y}(\tau))\|^2\,d\tau \\
&+16c^2\delta^{-1}\int_t^{+\infty} e^{\delta(t-\tau)}\mathbf{E}\|f(\tau+s_n, \tilde{Y}(\tau)) - \tilde{f}(\tau, \tilde{Y}(\tau))\|^2\,d\tau.
\end{aligned}
$$

By Lebesgue dominated convergence theorem, one has $\mathscr{E}_1^n(t) \to 0$ as $n \to +\infty$.

By Itô isometry, we have

$$
\begin{aligned}
I_2 \le\ & 16\mathbf{E}\left\|\int_{-\infty}^t T(t-\tau)P[g(\tau+s_n, Y_n(\tau)) - g(\tau+s_n, \tilde{Y}(\tau))]\,dW(\tau)\right\|_\alpha^2 \\
&+16\mathbf{E}\left\|\int_{-\infty}^t T(t-\tau)P[g(\tau+s_n, \tilde{Y}(\tau)) - \tilde{g}(\tau, \tilde{Y}(\tau))]\,dW(\tau)\right\|_\alpha^2 \\
&+16\mathbf{E}\left\|\int_t^{+\infty} T(t-\tau)J[g(\tau+s_n, Y_n(\tau)) - g(\tau+s_n, \tilde{Y}(\tau))]\,dW(\tau)\right\|_\alpha^2 \\
&+16\mathbf{E}\left\|\int_t^{+\infty} T(t-\tau)J[g(\tau+s_n, \tilde{Y}(\tau)) - \tilde{g}(\tau, \tilde{Y}(\tau))]\,dW(\tau)\right\|_\alpha^2 \\
\le\ & 16M^2\int_{-\infty}^t (t-\tau)^{-2\alpha}e^{-2\gamma(t-\tau)} \\
&\times \mathbf{E}\|(g(\tau+s_n, Y_n(\tau)) - g(\tau+s_n, \tilde{Y}(\tau)))Q^{1/2}\|_{L(V,\mathscr{L}^2(P,H))}^2\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ 16M^2 \int_{-\infty}^{t} (t-\tau)^{-2\alpha} e^{-2\gamma(t-\tau)} \\
&\times \mathbf{E}\|(g(\tau+s_n, \tilde{Y}(\tau)) - \tilde{g}(\tau, \tilde{Y}(\tau)))Q^{1/2}\|^2_{L(V,\mathscr{L}^2(P,H))}\, d\tau \\
&+ 16c^2 \int_{t}^{+\infty} e^{2\delta(t-\tau)}\mathbf{E}\|(g(\tau+s_n, Y_n(\tau)) - g(\tau+s_n, \tilde{Y}(\tau)))Q^{1/2}\|^2_{L(V,\mathscr{L}^2(P,H))}\, d\tau \\
&+ 16c^2 \int_{t}^{+\infty} e^{2\delta(t-\tau)}\mathbf{E}\|(g(\tau+s_n, \tilde{Y}(\tau)) - \tilde{g}(\tau, \tilde{Y}(\tau)))Q^{1/2}\|^2_{L(V,\mathscr{L}^2(P,H))}\, d\tau \\
\le{}& 16L\left[M^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha) + \frac{1}{2}c^2\delta^{-1}\right]\cdot \sup_{\tau\in\mathbf{R}} \mathbf{E}\|Y_n(\tau) - \tilde{Y}(\tau)\|^2_\alpha + \mathscr{E}_2^n(t),
\end{aligned}
$$

with

$$
\begin{aligned}
\mathscr{E}_2^n(t) := 16M^2 &\int_{-\infty}^{t} (t-\tau)^{-2\alpha} e^{-2\gamma(t-\tau)} \\
&\times \mathbf{E}\|(g(\tau+s_n, \tilde{Y}(\tau)) - \tilde{g}(\tau, \tilde{Y}(\tau)))Q^{1/2}\|^2_{L(V,\mathscr{L}^2(P,H))}\, d\tau \\
&+ 16c^2 \int_{t}^{+\infty} e^{2\delta(t-\tau)}\mathbf{E}\|(g(\tau+s_n, \tilde{Y}(\tau)) - \tilde{g}(\tau, \tilde{Y}(\tau)))Q^{1/2}\|^2_{L(V,\mathscr{L}^2(P,H))}\, d\tau.
\end{aligned}
$$

By Lebesgue dominated convergence theorem, one has $\mathscr{E}_2^n(t) \to 0$ as $n \to +\infty$.

By the properties of the integral for the Poisson random measure, one has

$$
\begin{aligned}
I_3 \le{}& 16\mathbf{E}\left\| \int_{-\infty}^{t}\int_{|x|_V<1} T(t-\tau)P[F(\tau+s_n, Y_n(\tau-), x) \right. \\
&\left. - F(\tau+s_n, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx) \right\|^2_\alpha \\
&+ 16\mathbf{E}\left\| \int_{-\infty}^{t}\int_{|x|_V<1} T(t-\tau)P[F(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{F}(\tau, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx) \right\|^2_\alpha \\
&+ 16\mathbf{E}\left\| \int_{t}^{+\infty}\int_{|x|_V<1} T(t-\tau)J[F(\tau+s_n, Y_n(\tau-), x) \right. \\
&\left. - F(\tau+s_n, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx) \right\|^2_\alpha \\
&+ 16\mathbf{E}\left\| \int_{t}^{+\infty}\int_{|x|_V<1} T(t-\tau)J[F(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{F}(\tau, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx) \right\|^2_\alpha
\end{aligned}
$$

$$
\begin{aligned}
&\leq 16M^2 \int_{-\infty}^{t} \int_{|x|_V<1} (t-\tau)^{-2\alpha} e^{-2\gamma(t-\tau)} \\
&\quad \times \mathbf{E}\|F(\tau+s_n, Y_n(\tau-), x) - F(\tau+s_n, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau \\
&\quad + 16M^2 \int_{-\infty}^{t} \int_{|x|_V<1} (t-\tau)^{-2\alpha} e^{-2\gamma(t-\tau)} \\
&\quad \times \mathbf{E}\|F(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{F}(\tau, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau \\
&\quad + 16c^2 \int_{t}^{+\infty} \int_{|x|_V<1} e^{2\delta(t-\tau)} \\
&\quad \times \mathbf{E}\|F(\tau+s_n, Y_n(\tau-), x) - F(\tau+s_n, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau \\
&\quad + 16c^2 \int_{t}^{+\infty} \int_{|x|_V<1} e^{2\delta(t-\tau)} \mathbf{E}\|F(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{F}(\tau, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau \\
&\leq 16L\left[M^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha) + \frac{1}{2}c^2\delta^{-1}\right] \cdot \sup_{s\in\mathbf{R}} \mathbf{E}\|Y_n(\tau) - \tilde{Y}(\tau)\|_\alpha^2 + \mathscr{E}_3^n(t),
\end{aligned}
$$

with

$$
\begin{aligned}
\mathscr{E}_3^n(t) := 16M^2 &\int_{-\infty}^{t} \int_{|x|_V<1} (t-\tau)^{-2\alpha} e^{-2\gamma(t-\tau)} \\
&\times \mathbf{E}\|F(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{F}(\tau, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau \\
&+ 16c^2 \int_{t}^{+\infty} \int_{|x|_V<1} e^{2\delta(t-\tau)} \\
&\times \mathbf{E}\|F(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{F}(\tau, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau.
\end{aligned}
$$

By Lebesgue dominated convergence theorem, one has $\mathscr{E}_3^n(t) \to 0$ as $n \to +\infty$.

For I_4, similarly as the proof of I_3, one has

$$
\begin{aligned}
I_4 \leq 32\mathbf{E}\Bigg\| &\int_{-\infty}^{t} \int_{|x|_V\geq 1} T(t-\tau)P[G(\tau+s_n, Y_n(\tau-), x) \\
&- G(\tau+s_n, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx)\Bigg\|_\alpha^2 \\
+ 32\mathbf{E}\Bigg\| &\int_{-\infty}^{t} \int_{|x|_V\geq 1} T(t-\tau)P[G(\tau+s_n, Y_n(\tau-), x) \\
&- G(\tau+s_n, \tilde{Y}(\tau-), x)]\nu(dx)\, d\tau\Bigg\|_\alpha^2
\end{aligned}
$$

$$+ 32\mathbf{E}\left\|\int_{-\infty}^{t}\int_{|x|_V\geq 1} T(t-\tau)P[G(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx)\right\|_\alpha^2$$

$$+ 32\mathbf{E}\left\|\int_{-\infty}^{t}\int_{|x|_V\geq 1} T(t-\tau)P[G(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)]\nu(dx)\ d\tau\right\|_\alpha^2$$

$$+ 32\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V\geq 1} T(t-\tau)J[G(\tau+s_n, Y_n(\tau-), x) - G(\tau+s_n, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx)\right\|_\alpha^2$$

$$+ 32\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V\geq 1} T(t-\tau)J[G(\tau+s_n, Y_n(\tau-), x) - G(\tau+s_n, \tilde{Y}(\tau-), x)]\nu(dx)\ d\tau\right\|_\alpha^2$$

$$+ 32\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V\geq 1} T(t-\tau)J[G(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)]\tilde{N}(d\tau, dx)\right\|_\alpha^2$$

$$+ 32\mathbf{E}\left\|\int_{t}^{+\infty}\int_{|x|_V\geq 1} T(t-\tau)J[G(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)]\nu(dx)\ d\tau\right\|_\alpha^2$$

$$\leq 32L\left[M^2(2\gamma)^{2\alpha-1}\Gamma(1-2\alpha) + b(M\gamma^{\alpha-1}\Gamma(1-\alpha))^2 + \frac{1}{2}c^2\delta^{-1} + bc^2\delta^{-2}\right] \cdot \sup_{s\in\mathbf{R}} \mathbf{E}\|Y_n(\tau) - \tilde{Y}(\tau)\|_\alpha^2 + \mathscr{E}_4^n(t),$$

with

$$\begin{aligned}\mathscr{E}_4^n(t) := 32M^2 &\int_{-\infty}^{t}\int_{|x|_V\geq 1}(t-\tau)^{-2\alpha}e^{-2\gamma(t-\tau)} \\ &\times \mathbf{E}\|G(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)\|^2\nu(dx)\ d\tau \\ &+ 32bM^2\gamma^{\alpha-1}\Gamma(1-\alpha) \\ &\times \int_{-\infty}^{t}\int_{|x|_V\geq 1}(t-\tau)^{-\alpha}e^{-\gamma(t-\tau)} \\ &\times \mathbf{E}\|G(\tau+s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)\|^2\nu(dx)\ d\tau\end{aligned}$$

$$+ 32c^2 \int_t^{+\infty} \int_{|x|_V \geq 1} e^{2\delta(t-\tau)}$$
$$\times \mathbf{E}\|G(\tau + s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau$$
$$+ 32b\delta^{-1}c^2 \int_t^{+\infty} \int_{|x|_V \geq 1} e^{\delta(t-\tau)}$$
$$\times \mathbf{E}\|G(\tau + s_n, \tilde{Y}(\tau-), x) - \tilde{G}(\tau, \tilde{Y}(\tau-), x)\|^2 \nu(dx)\, d\tau.$$

By Lebesgue dominated convergence theorem, one has $\mathscr{E}_4^n(t) \to 0$ as $n \to +\infty$.

By the estimates of I_1–I_4, one has

$$\mathbf{E}\| Y_n(t) - \tilde{Y}(t)\|_\alpha^2 \leq \mathscr{E}^n(t) + \vartheta \cdot \sup_{\tau \in \mathbf{R}} \mathbf{E}\| Y_n(\tau) - \tilde{Y}(\tau)\|_\alpha^2,$$

where $\mathscr{E}^n(t) = \sum_{i=1}^4 \mathscr{E}_i^n(t)$. Hence

$$\sup_{t \in \mathbf{R}} \mathbf{E}\| Y_n(t) - \tilde{Y}(t)\|_\alpha^2 \leq \sup_{t \in \mathbf{R}} \mathscr{E}^n(t) + \vartheta \cdot \sup_{\tau \in \mathbf{R}} \mathbf{E}\| Y_n(\tau) - \tilde{Y}(\tau)\|_\alpha^2,$$

By $\vartheta < 1$ and $\lim_{n \to +\infty} \sup_{t \in \mathbf{R}} \mathscr{E}^n(t) = 0$, it follows that

$$\sup_{t \in \mathbf{R}} \mathbf{E}\| Y_n(t) - \tilde{Y}(t)\|_\alpha^2 \to 0 \quad \text{as } n \to +\infty.$$

Since $Y(t + s_n)$ has the same distribution as $Y_n(t)$, by [27], one has $Y(t + s_n) \to \tilde{Y}(t)$ in distribution as $n \to +\infty$. Similarly, we have $\tilde{Y}(t - s_n) \to Y(t)$ in distribution as $n \to +\infty$. Hence Y is almost automorphic in one-dimensional distribution. The proof is complete. □

4. Example

Consider the stochastic heat differential equations with Dirichlet boundary condition:

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,\xi) = \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + au(t,\xi) + f(t,u(t,\xi)) + g(t,u(t,\xi))\frac{\partial W}{\partial t}(t,\xi) \\ &\qquad + k(t,u(t,\xi))\frac{\partial Z}{\partial t}(t,\xi), \quad t > 0,\ \xi \in (0,1), \\ &u(t,0) = u(t,1) = 0, \quad t > 0, \end{aligned} \tag{4.1}$$

where $a > 0$, f, g are square-mean almost automorphic with respect to t, k is Poisson square-mean almost automorphic with respect to t, W is a Q-Wiener process with $TrQ < \infty$, and Z is a Lévy pure jump process which is independent of W. Denote $H = V := C([0,1],\mathbf{R})$ equipped with the sup norm and defined the operator A by

$$Au := u'' + au, \quad u \in D(A),$$

where $D(A) := \{u \in C^2([0,1], \mathbf{R}), u(0) = u(1) = 0\}$. The operator A is sectorial, and the resolvent and spectrum of A are respectively given by [29]

$$\rho(A) = \mathbf{C} - \{-n^2\pi^2 + a : a \in \mathbf{N}\}, \quad \sigma(A) = \{-n^2\pi^2 + a : a \in \mathbf{N}\},$$

if $a \neq n^2\pi^2$, one has $\sigma(A) \cap i\mathbf{R} = \emptyset$. Hence, the analytic semigroup generated by A is hyperbolic. For $\alpha \in (0,1)$, the intermediate space H_α take the domains of the fraction power of $-A$, i.e., $H_\alpha = D((-A)^\alpha)$. Then the stochastic heat equation can be written as

$$dY = (AY + F(t, Y))\,dt + G(t, Y)\,dW + \int_{|Z|_V<1} K(t, Y, z)\tilde{N}(dt, dz) + \int_{|Z|_V \geq 1} K(t, Y, z)N(dt, dz)$$

on the Hilbert space H, where

$$F(t, Y) := f(t,u), \quad G(t, Y)\,dW := g(t,u)\,dW$$

$$k(t,u)\,dZ := \int_{|z|_V<1} K(t, Y, z)\tilde{N}(dt, dz) + \int_{|z|_V \geq 1} K(t, Y, z)N(dt, dz)$$

with

$$Z(t, \xi) = \int_{|z|_V<1} z\tilde{N}(t, dz) + \int_{|z|_V \geq 1} zN(t, dz), \quad K(t, Y, z) = k(t,u)z.$$

Here we assume for simplicity that the Lévy pure process Z on $L^2(0,1)$ is decomposed as above by the Lévy-Itô decomposition.

Note that if $f(t,u)$, $g(t,u)$ are Lipschitz with respect to u, then $F(t, Y)$, $G(t, Y)$ are Lipschitz with respect to Y. When $k(t,u)$ is Lipschitz with respect to u, and v is a finite measure, then the Lipschitz condition holds for K. Hence, assume that $f(t,u)$, $g(t,u)$ are Lipschitz with respect to u, and the finite intensity measure v of the Poisson process Z on $L^2(0,1)$ satisfies the Lipschitz condition for K in the sense of (H_2) with Lipschitz constant L. By Theorem 3.1, (4.1) admits a unique almost automorphic in one-dimensional distribution mild solution if L is small enough.

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