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KODAI MATH. J.
35 (2012), 425–438

ON FANO MANIFOLDS WITH AN UNSPLIT DOMINATING FAMILY OF RATIONAL CURVES

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Abstract

We study Fano manifolds X admitting an unsplit dominating family of rational curves and we prove that the Generalized Mukai Conjecture holds if X has pseudoindex $i_X = (\dim X)/3$ or dimension $\dim X = 6$. We also show that this conjecture is true for all Fano manifolds with $i_X > (\dim X)/3$.

1. Introduction

Let X be a Fano manifold, *i.e.* a smooth complex projective variety whose anticanonical bundle $-K_X$ is ample. A Fano manifold is associated with two invariants, namely the *index*, r_X, defined as the largest integer dividing $-K_X$ in the Picard group of X, and the *pseudoindex*, i_X, defined as the minimum anticanonical degree of rational curves on X.

In 1988 Mukai proposed the following conjecture, involving the index and the Picard number of a Fano manifold:

Conjecture 1.1 [10]. *Let X be a Fano manifold of dimension n. Then $\rho_X(r_X - 1) \le n$, with equality if and only if $X = (\mathbf{P}^{r_X-1})^{\rho_X}$.*

In 1990, in [15], where the notion of pseudoindex was introduced, the first step towards the conjecture was made and it was proved that if $i_X > (\dim X + 2)/2$ then $\rho_X = 1$; moreover, if $r_X = (\dim X + 2)/2$ then either $\rho_X = 1$ or $X = (\mathbf{P}^{r_X-1})^2$.

In 2002 Bonavero, Casagrande, Debarre and Druel reconsidered this problem and proposed the following more general conjecture:

Conjecture 1.2 [2]. *Let X be a Fano manifold of dimension n. Then $\rho_X(i_X - 1) \le n$, with equality if and only if $X = (\mathbf{P}^{i_X-1})^{\rho_X}$.*

2010 *Mathematics Subject Classification*. 14J45, 14E30.
Key words and phrases. Fano manifolds, rational curves.
Received September 15, 2011; revised February 1, 2012.

Moreover, in [2], they proved Conjecture (1.2) for Fano manifolds of dimension 4 (in lower dimension the result can be read off from the classification), for homogeneous manifolds, and for toric Fano manifolds of pseudoindex $i_X \geq (\dim X + 3)/3$ or dimension ≤ 7. In 2006, in [5], the toric case was completely settled.

In 2004, in [1], Conjecture (1.2) was proved for Fano manifolds of dimension 5 and for Fano manifolds of pseudoindex $i_X \geq (\dim X + 3)/3$ admitting an unsplit dominating family of rational curves (see Definition (2.1)).

In 2010, in [12], Conjecture (1.2) was proved for Fano manifolds of pseudoindex $i_X \geq (\dim X + 3)/3$, and simplified proofs of this conjecture for Fano manifolds of dimension 4 and 5 were provided.

In this paper we reconsider Fano manifolds X admitting an unsplit dominating family of rational curves, and we prove Conjecture (1.2) if X has dimension 6 (Theorem (6.3)), or X has pseudoindex $i_X \geq \dim X/3$ (Theorem (6.4)).

The paper is organized as follows: in Sections (2) and (3) we recall definitions and results on families of rational curves and on chains of rational curves on projective manifolds, while in Section (4) we consider families of rational curves on Fano manifolds; in Section (5) we prove Conjecture (1.2) for Fano manifolds X of pseudoindex $i_X > (\dim X)/3$; in Section (6) we consider Fano manifolds X admitting an unsplit dominating family of rational curves and we prove Conjecture (1.2) if $\dim X = 6$, or $i_X \geq (\dim X)/3$.

2. Families of rational curves

Let X be a smooth complex projective variety.

Definition 2.1. A *family of rational curves* V on X is an irreducible component of the scheme $\mathrm{Ratcurves}^n(X)$ (see [7, Definition II.2.11]).

Given a rational curve we will call a *family of deformations* of that curve any irreducible component of $\mathrm{Ratcurves}^n(X)$ containing the point parameterizing that curve.

We define $\mathrm{Locus}(V)$ to be the set of points of X through which there is a curve among those parameterized by V; we say that V is a *covering family* if $\mathrm{Locus}(V) = X$ and that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.

By abuse of notation, given a line bundle $L \in \mathrm{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C$, with C any curve among those parameterized by V.

We will say that V is *unsplit* if it is proper; clearly, an unsplit dominating family is covering.

We denote by V_x the subscheme of V parameterizing rational curves passing through a point x and by $\mathrm{Locus}(V_x)$ the set of points of X through which there is a curve among those parameterized by V_x. If, for a general point $x \in \mathrm{Locus}(V)$, V_x is proper, then we will say that the family is *locally unsplit*; by Mori's Bend and Break arguments, if V is a locally unsplit family, then $-K_X \cdot V \leq \dim X + 1$.

If X admits dominating families, we can choose among them one with minimal degree with respect to a fixed ample line bundle A, and we call it a *minimal dominating family*. Such a family is locally unsplit. Indeed, if for a general point $x \in \mathrm{Locus}(V)$ the family V_x were not unsplit, then there would be a rational curve Γ through x such that $A \cdot \Gamma < A \cdot V$; then there would exist a dominating family of rational curves of degree $< A \cdot V$, and this would be a contradiction with the minimality of V.

Definition 2.2. Let U be an open dense subset of X and $\pi : U \to Z$ a proper surjective morphism to a quasi-projective variety; we say that a family of rational curves V is a *horizontal dominating family with respect to* π if $\mathrm{Locus}(V)$ dominates Z and curves parameterized by V are not contracted by π. If such families exist, we can choose among them one with minimal degree with respect to a fixed ample line bundle and we call it a *minimal horizontal dominating family* with respect to π; such a family is locally unsplit.

Remark 2.3. By fundamental results in [9], a Fano manifold admits dominating families of rational curves; also horizontal dominating families with respect to proper morphisms defined on an open set exist, as proved in [8]. In the case of Fano manifolds with "minimal" we will mean minimal with respect to $-K_X$, unless otherwise stated.

Definition 2.4. We define a *Chow family of rational* 1*-cycles* $\mathscr{W}$ to be an irreducible component of $\mathrm{Chow}(X)$ parameterizing rational and connected 1-cycles.

We define $\mathrm{Locus}(\mathscr{W})$ to be the set of points of X through which there is a cycle among those parameterized by $\mathscr{W}$; notice that $\mathrm{Locus}(\mathscr{W})$ is a closed subset of X ([7, II.2.3]). We say that $\mathscr{W}$ is a *covering family* if $\mathrm{Locus}(\mathscr{W}) = X$.

If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$, denoted by $\mathscr{V}$, is called the *Chow family associated to* V.

Remark 2.5. If V is proper, *i.e.* if the family is unsplit, then V corresponds to the normalization of the associated Chow family $\mathscr{V}$.

Definition 2.6. Let V be a family of rational curves and let $\mathscr{V}$ be the associated Chow family. We say that V (and also $\mathscr{V}$) is *quasi-unsplit* if every component of any reducible cycle parameterized by $\mathscr{V}$ has numerical class proportional to the numerical class of a curve parameterized by V.

Definition 2.7. Let $V^1, \dots, V^k$ be families of rational curves on X and $Y \subset X$.

We define $\mathrm{Locus}(V^1)_Y$ to be the set of points $x \in X$ such that there exists a curve C among those parameterized by V^1 with $C \cap Y \neq \emptyset$ and $x \in C$. We inductively define $\mathrm{Locus}(V^1, \dots, V^k)_Y := \mathrm{Locus}(V^2, \dots, V^k)_{\mathrm{Locus}(V^1)_Y}$. Notice that, by this definition, we have $\mathrm{Locus}(V)_x = \mathrm{Locus}(V_x)$. Analogously we define $\mathrm{Locus}(\mathscr{W}^1, \dots, \mathscr{W}^k)_Y$ for Chow families $\mathscr{W}^1, \dots, \mathscr{W}^k$ of rational 1-cycles.

NOTATION. We denote by ρ_X the Picard number of X, *i.e.* the dimension of the **R**-vector space $N_1(X)$ of 1-cycles modulo numerical equivalence. If Γ is a 1-cycle, then we will denote by $[\Gamma]$ its numerical equivalence class in $N_1(X)$; if V is a family of rational curves, we will denote by $[V]$ the numerical equivalence class of any curve among those parameterized by V.

If $Y \subset X$, we will denote by $N_1(Y, X) \subseteq N_1(X)$ the vector subspace generated by numerical classes of curves of X contained in Y; moreover, we will denote by $NE(Y, X) \subseteq NE(X)$ the subcone generated by numerical classes of curves of X contained in Y.

We will make frequent use of the following dimensional estimates:

PROPOSITION 2.8 ([7, IV.2.6]). *Let V be a family of rational curves on X and $x \in \mathrm{Locus}(V)$ a point such that every component of V_x is proper. Then*

(a) $\dim V - 1 = \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq \dim X - K_X \cdot V - 1$;

(b) $\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1$.

DEFINITION 2.9. We say that k quasi-unsplit families $V^1, \ldots, V^k$ are numerically independent if in $N_1(X)$ we have $\dim\langle [V^1], \ldots, [V^k] \rangle = k$.

LEMMA 2.10 (Cf. [1, Lemma 5.4]). *Let $Y \subset X$ be a closed subset and $V^1, \ldots, V^k$ numerically independent unsplit families of rational curves such that $\langle [V^1], \ldots, [V^k] \rangle \cap NE(Y, X) = \underline{0}$. Then either* $\mathrm{Locus}(V^1, \ldots, V^k)_Y = \emptyset$ *or*

$$\dim \mathrm{Locus}(V^1, \ldots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k.$$

A key fact underlying our strategy to obtain bounds on the Picard number, based on [7, Proposition II.4.19], is the following:

LEMMA 2.11 ([1, Lemma 4.1]). *Let $Y \subset X$ be a closed subset, $\mathscr{V}$ a Chow family of rational 1-cycles. Then every curve contained in $\mathrm{Locus}(\mathscr{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and of irreducible components of cycles parameterized by $\mathscr{V}$ which meet Y.*

COROLLARY 2.12. *Let V^1 be a locally unsplit family of rational curves, and $V^2, \ldots, V^k$ unsplit families of rational curves. Then, for a general $x \in \mathrm{Locus}(V^1)$,*

(a) $N_1(\mathrm{Locus}(V^1)_x, X) = \langle [V^1] \rangle$;

(b) *either* $\mathrm{Locus}(V^1, \ldots, V^k)_x = \emptyset$, *or* $N_1(\mathrm{Locus}(V^1, \ldots, V^k)_x, X) = \langle [V^1], \ldots, [V^k] \rangle$.

We end this section by recalling two results that we will use in the following.

THEOREM 2.13 [6, Theorem 1.2]. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$. Then* codim $N_1(D, X) \leq 1$ *for every prime divisor $D \subset X$ and one of the following holds*:

(a) $i_X = 2$ *and there exists a smooth morphism $X \to Y$ with fibers isomorphic to $\mathbf{P}^1$ onto a Fano manifold Y of pseudoindex $i_Y \geq 2$;*

(b) $N_1(D, X) = N_1(X)$ *for every prime divisor $D \subset X$.*

THEOREM 2.14 [12, Theorem 3]. *Let X be a Fano manifold of Picard number ρ_X and pseudoindex $i_X \geq (\dim X + 3)/3$. Then $\rho_X(i_X - 1) \leq \dim X$ and equality holds if and only if $X = (\mathbf{P}^{i_X-1})^{\rho_X}$.*

3. Chains of rational curves

Let X be a smooth complex projective variety. Let V be a dominating family of rational curves on X and denote by $\mathscr{V}$ the associated Chow family.

DEFINITION 3.1. Let $Y \subset X$ be a closed subset; define $\mathrm{ChLocus}_m(\mathscr{V})_Y$ to be the set of points $x \in X$ such that there exist cycles $\Gamma_1, \dots, \Gamma_m$ with the following properties:

- Γ_i belongs to the family $\mathscr{V}$;
- $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$;
- $\Gamma_1 \cap Y \neq \emptyset$ and $x \in \Gamma_m$,

i.e. $\mathrm{ChLocus}_m(\mathscr{V})_Y$ is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the family $\mathscr{V}$.

If we consider among cycles parameterized by $\mathscr{V}$ only irreducible ones, in the same way we can define $\mathrm{ChLocus}_m(V)_Y$.

Define a relation of *rational connectedness with respect to $\mathscr{V}$* on X in the following way: two points x and y of X are in rc($\mathscr{V}$)-relation if there exists a chain of cycles in $\mathscr{V}$ which joins x and y, *i.e.* if $y \in \mathrm{ChLocus}_m(\mathscr{V})_x$ for some m. In particular, X is *rc($\mathscr{V}$)-connected* if for some m we have $X = \mathrm{ChLocus}_m(\mathscr{V})_x$.

The family $\mathscr{V}$ defines a proper prerelation in the sense of [7, Definition IV.4.6]. This prerelation is associated with a fibration, which we will call the *rc($\mathscr{V}$)-fibration*:

THEOREM 3.2 ([7, IV.4.16], Cf. [3]). *Let X be a normal and proper variety and $\mathscr{V}$ a proper prerelation; then there exists an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that*

- *$\langle \mathscr{U} \rangle$ restricts to an equivalence relation on X^0;*
- *$\pi^{-1}(z)$ is a $\langle \mathscr{U} \rangle$-equivalence class for every $z \in Z^0$;*
- *$\forall z \in Z^0$ and $\forall x, y \in \pi^{-1}(z)$, $x \in \mathrm{ChLocus}_m(\mathscr{V})_y$ with $m \leq 2^{\dim X - \dim Z^0} - 1$.*

Clearly X is rc($\mathscr{V}$)-connected if and only if $\dim Z^0 = 0$.

Given $\mathcal{V}^1, \dots, \mathcal{V}^k$ Chow families of rational 1-cycles, it is possible to define a relation of $\mathrm{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$-connectedness, which is associated with a fibration, that we will call $\mathrm{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$-fibration. The variety X will be called $\mathrm{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$-*connected* if the target of the fibration is a point.

For such varieties we have the following application of Lemma (2.11):

Proposition 3.3 (Cf. [1, Corollary 4.4]). *If X is rationally connected with respect to some Chow families of rational* 1*-cycles $\mathcal{V}^1, \dots, \mathcal{V}^k$, then $\mathrm{N}_1(X)$ is generated by the classes of irreducible components of cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$.*

In particular, if $\mathcal{V}^1, \dots, \mathcal{V}^k$ are quasi-unsplit families, then $\rho_X \le k$ and equality holds if and only if $\mathcal{V}^1, \dots, \mathcal{V}^k$ are numerically independent.

A straightforward consequence of the above proposition is the following:

Corollary 3.4 ([12, Corollary 3]). *If X is rationally connected with respect to Chow families of rational* 1*-cycles $\mathcal{V}^1, \dots, \mathcal{V}^k$ and D is an effective divisor, then D cannot be trivial on every irreducible component of every cycle parameterized by $\mathcal{V}^1, \dots, \mathcal{V}^k$.*

We will also make use of the following

Lemma 3.5 ([12, Lemma 3]). *Let X be a Fano manifold of pseudoindex i_X, let $Y \subset X$ be a closed subset of dimension $\dim Y > \dim X - i_X$ and let W be an unsplit non dominating family of rational curves such that $[W] \notin \mathrm{NE}(Y, X)$. Then $\mathrm{Locus}(W) \cap Y = \emptyset$.*

4. Families of rational curves on Fano manifolds

We start this section by recalling the following

Construction 4.1 ([12, Construction 1]). Let X be a Fano manifold; let V^1 be a minimal dominating family of rational curves on X and consider the associated Chow family $\mathcal{V}^1$.

If X is not $\mathrm{rc}(\mathcal{V}^1)$-connected, let V^2 be a minimal horizontal dominating family with respect to the $\mathrm{rc}(\mathcal{V}^1)$-fibration, $\pi_1 : X \dashrightarrow Z^1$. If X is not $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-connected, we denote by V^3 a minimal horizontal dominating family with respect to the the $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-fibration, $\pi_2 : X \dashrightarrow Z^2$, and so on. Since $\dim Z^{i+1} < \dim Z^i$, for some integer k we have that X is $\mathrm{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$-connected.

Notice that, by construction, the families $V^1, \dots, V^k$ are numerically independent.

Examples 4.2. Consider $X = \mathbf{P}^m \times \mathbf{P}^m$, with $m \ge 2$. Clearly X is a Fano manifold of Picard number $\rho_X = 2$ and pseudoindex $i_X = m + 1$. This manifold admits two numerically independent unsplit dominating families of rational curves, say V^a and V^b, of anticanonical degree equal to i_X, which are the families of lines in the $\mathbf{P}^m$s (cf. [14, Theorem 1]). We can take $V^1 = V^a$ and so the rc($\mathscr{V}^1$)-fibration $\pi_1 : X \dashrightarrow Z^1$ corresponds to the projection of X onto one of its factors. Since $\rho_X > 1$, then X cannot be rc($\mathscr{V}^1$)-connected, so there exists a family V^2 which is a minimal horizontal dominating family with respect to π_1; note that we can take $V^2 = V^b$. Then X is rc($\mathscr{V}^1, \mathscr{V}^2$)-connected. In fact, if the target of the rc($\mathscr{V}^1, \mathscr{V}^2$)-fibration $\pi_2 : X \dashrightarrow Z^2$ were positive dimensional, there would be a minimal horizontal dominating family with respect to π_2, V^3; since at the i-th step the dimension of the quotient drops at least by $\dim \mathrm{Locus}(V^i)_{x_i}$, where x_i is a general point of Locus(V^i), we would have $2m = \dim X \geq \sum_{i=1}^{3} \dim \mathrm{Locus}(V^i)_{x_i} \geq \sum_{i=1}^{3}(-K_X \cdot V^i - 1) \geq 3(i_X - 1) = 3m,$ which gives $m \leq 0$, a contradiction.

Consider $X = \mathrm{Bl}_{\mathbf{P}^s} \mathbf{P}^{2s+3}$, with $s \geq 1$. This is a Fano manifold of Picard number $\rho_X = 2$ and pseudoindex $i_X = s + 2$ admitting a divisorial contraction $\sigma : X \to \mathbf{P}^{2s+3}$ and a fiber type contraction, say $\varphi : X \to Y$. Note that the minimal anticanonical degree of curves contracted by σ and by φ is equal to i_X and that $\dim Y = s + 2$ (*e.g.* see the proof of [13, Proposition 4.1]). Denote by C_φ (resp. C_σ) a curve contracted by φ (resp. σ) such that $-K_X \cdot C_\varphi = i_X$ (resp. $-K_X \cdot C_\sigma = i_X$). We can take V^1 to be a family of deformations of C_φ, and the rc($\mathscr{V}^1$)-fibration corresponds to φ. Now, a family of deformation of C_σ can be taken as V^2. Arguing as before we get that X is rc($\mathscr{V}^1, \mathscr{V}^2$)-connected.

LEMMA 4.3 ([12, Lemma 4]). *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and let $V^1, \ldots, V^k$ be families of rational curves as in Construction* (4.1). *Then*

$$\sum_{i=1}^{k}(-K_X \cdot V^i - 1) \leq \dim X.$$

In particular, $k(i_X - 1) \leq \dim X$, and equality holds if and only if $X = (\mathbf{P}^{i_X - 1})^k$.

LEMMA 4.4. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and let $V^1, \ldots, V^k$ be families of rational curves as in Construction* (4.1). *If $V^1, \ldots, V^{h-1}$, with $h \leq k$, are unsplit and* $\dim \mathrm{Locus}(V^h, \ldots, V^1)_{x_h} = \dim X - 1$ *for a general point $x_h \in$* Locus(V^h), *then either $\rho_X = h = k$, or $i_X = 2$, $\rho_X = h + 1$ and $k - h \leq 1$.*

Proof. Let D be an irreducible component of maximal dimension of $\mathrm{Locus}(V^h, \ldots, V^1)_{x_h}$. Then by part (b) of Corollary (2.12) $\mathrm{N}_1(D, X) \subseteq \langle [V^1], \ldots, [V^h] \rangle \subseteq \mathrm{N}_1(X)$. Clearly $\rho_X \geq k$, so the assertion follows by Theorem (2.13). □

LEMMA 4.5. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and let $V^1, \ldots, V^k$ be families of rational curves as in Construction* (4.1). *Assume that* *at least one of these families, say* V^j, *is not unsplit. Then* $k(i_X - 1) \leq \dim X - i_X$.

Moreover,

(a) *if* $j = \dfrac{\dim X - i_X}{i_X - 1}$, *then* $j = k$ *and* $\rho_X(i_X - 1) = \dim X - i_X$;

(b) *if* $j = \dfrac{\dim X - i_X - 1}{i_X - 1}$, *then* $j = k$ *and either* $\rho_X(i_X - 1) = \dim X - i_X - 1$, *or* $i_X = 2$ *and* $\rho_X = \dim X - 2$.

Proof. Since V^j is not unsplit, we have $-K_X \cdot V^j \geq 2i_X$. By Lemma (4.3) we get $(k-1)(i_X - 1) + (2i_X - 1) \leq \dim X$, hence $k \leq \dfrac{\dim X - i_X}{i_X - 1}$. Moreover, by part (b) of Proposition (2.8), we have $\dim \mathrm{Locus}(V^j)_{x_j} \geq 2i_X - 1$ for a general point $x_j \in \mathrm{Locus}(V^j)$.

If $j = \dfrac{\dim X - i_X}{i_X - 1}$, then $j = k$ and V^j is the only non unsplit family. Then, for a general point $x_k \in \mathrm{Locus}(V^k)$, we have $X = \mathrm{Locus}(V^k, \ldots, V^1)_{x_k}$ by Lemma (2.10). Therefore, by part (b) of Corollary (2.12), we obtain that $\mathrm{N}_1(X) = \langle [V^1], \ldots, [V^k] \rangle$, so $\rho_X = k$, and we obtain case (a) of the statement.

Assume now that $j = \dfrac{\dim X - i_X - 1}{i_X - 1}$. Then V^j is the only non unsplit family; moreover, $\dim \mathrm{Locus}(V^j, \ldots, V^1)_{x_j} \geq \dim X - 1$ by Lemma (2.10).

We claim that X is $\mathrm{rc}(V^1, \ldots, \mathscr{V}^j)$-connected.

In fact, a general fiber of the $\mathrm{rc}(V^1, \ldots, \mathscr{V}^j)$-fibration has dimension at least $\dim \mathrm{Locus}(V^j, \ldots, V^1)_{x_j} \geq \dim X - 1$ by Lemma (2.10). This implies $\dim Z^j \leq 1$, and thus, if X were not $\mathrm{rc}(V^1, \ldots, \mathscr{V}^j)$-connected, we would have $\dim \mathrm{Locus}(V^{j+1})_{x_{j+1}} = 1$ for a general point $x_{j+1} \in \mathrm{Locus}(V^{j+1})$. Hence, by part (b) of Proposition (2.8), $-K_X \cdot V^{j+1} = 2 = i_X$, so V^{j+1} would be unsplit and, by part (a) of the same proposition, covering, against the minimality of V^j. Therefore $j = k$.

Consider an irreducible component D of $\mathrm{Locus}(V^k, \ldots, V^1)_{x_k}$ of maximal dimension (which is at least $\dim X - 1$). Therefore, either $X = \mathrm{Locus}(V^k, \ldots, V^1)_{x_k}$ and $\rho_X = k$ by part (b) of Corollary (2.12), or D is a divisor in X. In this last case, either $\rho_X = k$, or $i_X = 2$ and $\rho_X = \dim X - 2$ by Lemma (4.4). $\square$

5. Bounds on the Picard number of Fano manifolds

In this section we show that Conjecture (1.2) holds for Fano manifolds of pseudoindex $i_X > \dim X/3$.

THEOREM 5.1. *Let* X *be a Fano manifold of Picard number* ρ_X *and pseudoindex* $i_X > \dim X/3$. *Then* $\rho_X(i_X - 1) \leq \dim X$ *and equality holds if and only if* $X = (\mathbf{P}^{i_X - 1})^{\rho_X}$.

Proof. Note that in view of Theorem (2.14) we can restrict to $i_X < (\dim X + 3)/3$. Moreover, since for $i_X = 1$ there is nothing to prove, we assume $i_X \geq 2$ (and so $\dim X > 3$).

Let $V^1, \ldots, V^k$ be families of rational curves as in Construction (4.1).

If all the families are unsplit, then Lemma (4.3) gives $k \leq 3$ unless either $i_X = 2$, $\dim X = 5$ and $k = 4$, or $X = (\mathbf{P}^1)^5$, or $X = (\mathbf{P}^1)^4$, or $X = (\mathbf{P}^2)^4$.

Since $\rho_X = k$ by Proposition (3.3), the assertion follows.

We can thus assume that at least one of these families, say V^j, is not unsplit. Then, by Lemma (4.3), $k \leq 3$ and exactly one of these families is not unsplit. Moreover, if $j = 3$, by computing $\dim \mathrm{Locus}(V^3, V^2, V^1)$ with Lemma (2.10), we get a contradiction unless $\dim X = 5$ and $i_X = 2$, so $\rho_X = 3$ by part (b) of Corollary (2.12). If $j = 2$ and $i_X = (\dim X + 2)/3$, then $\rho_X = 2$ by part (a) of Lemma (4.5). If $j = 2$ and $i_X = (\dim X + 1)/3$, denoted by T an irreducible component of maximal dimension of $\mathrm{Locus}(V^2, V^1)_{x_2}$, we have $\dim T \geq \dim X - 1$ by Lemma (2.10). So if $\dim T = \dim X$ then $\rho_X = 2$ by part (b) of Corollary (2.12), while if $\dim T = \dim X - 1$ then either $\rho_X = 2$ or $\dim X = 5$, $i_X = 2$ and $\rho_X = 3$ by Lemma (4.4).

Therefore we are left with $j = 1$. Then a general fiber of the $\mathrm{rc}(\mathscr{V}^1)$-fibration $X \dashrightarrow Z^1$ has dimension at least $\dim \mathrm{Locus}(V^1)_{x_1} \geq 2i_X - 1$.

Assume first that $\dim Z^1 \geq 1$. For a general point $x_2 \in \mathrm{Locus}(V^2)$ we know that $\dim \mathrm{Locus}(V^2)_{x_2} \leq \dim Z^1 \leq \dim X - (2i_X - 1) < i_X + 1$. By part (b) of Proposition (2.8) we have $\dim \mathrm{Locus}(V^2)_{x_2} \geq -K_X \cdot V^2 - 1$, and so we deduce that $-K_X \cdot V^2 \leq i_X + 1$. So V^2 is unsplit and V^2 is not dominating, since $-K_X \cdot V^2 < -K_X \cdot V^1$. Denote by D an irreducible component of maximal dimension of $\mathrm{Locus}(V^1, V^2)_{x_1}$. Then $\dim D = \dim X - 1$, so we are done by Lemma (4.4).

Finally we deal with the case in which $\dim Z^1 = 0$, so X is $\mathrm{rc}(\mathscr{V}^1)$-connected. Let x be a general point. Since x is general and V^1 is minimal we have $\overline{\mathrm{Locus}(V^1)_x} = \mathrm{Locus}(V^1)_x$ and $\mathrm{N}_1(\mathrm{Locus}(V^1)_x, X) = \langle [V^1] \rangle$ by part (a) of Corollary (2.12).

If $\mathrm{Locus}(V^1)_x = X$, then $\rho_X = 1$. So we can suppose that $\dim \mathrm{Locus}(V^1)_x < \dim X$ and thus, by part (b) of Proposition (2.8), $-K_X \cdot V^1 \leq \dim X$. In particular every reducible cycle parameterized by $\mathscr{V}^1$ has at most two irreducible components.

If every irreducible component of a $\mathscr{V}^1$-cycle in a connected m-chain through x is numerically proportional to V^1, then $\rho_X = 1$ by repeated applications of Lemma (2.11).

We can thus assume that there exist m-chains through x, $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$, with $x \in \Gamma_1$ and $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$, such that, for some $j \in \{1, \ldots, m\}$ the irreducible components Γ_j^1 and Γ_j^2 of Γ_j are not numerically proportional to $\mathscr{V}^1$.

Let $j_0 \in \{1, \ldots, m\}$ be the minimum integer for which such a chain exists; by the generality of x we have $j_0 \geq 2$. If $j_0 = 2$ set $x_1 = x$, otherwise let x_1 be a point in $\Gamma_{j_0-1} \cap \Gamma_{j_0-2}$. Since $\Gamma_{j_0-1} \subset \mathrm{Locus}(\mathscr{V}^1)_{x_1}$ there is an irreducible component Y of $\mathrm{Locus}(V^1)_{x_1}$ which meets Γ_{j_0}. By Lemma (2.11), $\mathrm{N}_1(Y, X) = \langle [V^1] \rangle$.

Let γ be a component of Γ_{j_0} meeting Y and denote by W a family of deformations of γ; then the family W is unsplit and it is not covering, by the minimality of V^1.

Then $\dim \mathrm{Locus}(W)_Y = \dim X - 1$, and so $\mathrm{Locus}(W) = \mathrm{Locus}(W)_Y$. Moreover, in this case, by part (b) of Corollary (2.12) we get $\mathrm{N}_1(\mathrm{Locus}(W)_Y, X) = \langle [V^1], [W] \rangle$. Therefore $\rho_X = 2$ by Theorem (2.13). $\square$

Now, in view of Theorem (5.1) it is straightforward to derive the following results.

PROPOSITION 5.2. *Let X be a Fano manifold of dimension ≥ 7, Picard number ρ_X and pseudoindex $i_X > (\dim X - 3)/2$. Then $\rho_X(i_X - 1) \leq \dim X$ and equality holds if and only if $X = (\mathbf{P}^{i_X-1})^{\rho_X}$.*

PROPOSITION 5.3. *Let X be a Fano manifold of Picard number ρ_X and pseudoindex $i_X > \dim X - 4$. Then $\rho_X(i_X - 1) \leq \dim X$ and equality holds if and only if $X = (\mathbf{P}^{i_X-1})^{\rho_X}$.*

Remark 5.4. All the previous results can be improved once the Generalized Mukai Conjecture is proved in the case of Fano manifolds of dimension 6. However, this seems to be much more difficult, so in the next section we prove the conjecture under some additional assumption.

6. Fano manifolds with an unsplit dominating family

Since the Generalized Mukai Conjecture holds for Fano manifolds of dimension lower than or equal to five, in the next theorems we deal with manifolds of dimension at least six: in Theorem (6.2) we consider Fano manifolds of dimension greater than six and pseudoindex $\dim X/3$, while in Theorem (6.3) we consider Fano sixfolds.

We start with the following

LEMMA 6.1. *Let X be a Fano manifold of Picard number ρ_X and pseudoindex $i_X = \dim X/3$. If X admits an unsplit dominating family V of rational curves such that $-K_X \cdot V > \dim X/3$, then $\rho_X(i_X - 1) < \dim X$.*

Proof. Note that for $i_X = 1$ there is nothing to prove, so we can assume $i_X \geq 2$ (and so $\dim X \geq 6$).

Since V is an unsplit dominating family of rational curves on X, then either X is $\mathrm{rc}(V)$-connected and so $\rho_X = 1$, or there exists a minimal horizontal dominating family V' with respect to the $\mathrm{rc}(V)$-fibration.

In this last case, if V' is not unsplit, we get that an irreducible component D of $\mathrm{Locus}(V', V)_{x'}$, for a general point $x' \in \mathrm{Locus}(V')$, has dimension at least $\dim X - 1$ by Lemma (2.10). If $D = X$, then $\rho_X = 2$ by part (b) of Corollary (2.12); if D is a divisor, then $\rho_X = 2$ unless $\dim X = 6$ and $\rho_X = 3$ by Lemma (4.4).

We can thus assume that V' is unplit. Now, either X is rc(V, V')-connected and so $\rho_X = 2$, or there exists a minimal horizontal dominating family V'' with respect to the rc(V, V')-fibration. If V'' is not unsplit, then by Lemma (2.10) we can compute $\dim \mathrm{Locus}(V'', V', V)_{x''}$ for a general point $x'' \in \mathrm{Locus}(V'')$; then we reach a contradiction unless $\dim X = 6$ and $\rho_X = 3$ by part (b) of Corollary (2.12). If otherwise V'' is unsplit, then either X is rc(V, V', V'')-connected and so $\rho_X = 3$, or there exists a minimal horizontal dominating family V''' with respect to the rc(V, V', V'')-fibration. Then, for a general point $x''' \in \mathrm{Locus}(V''')$, computing the dimension of $\mathrm{Locus}(V''', V'', V', V)_{x'''}$, we find that either $\dim X = 6$ or 9, $X = \mathrm{Locus}(V''', V'', V', V)_{x'''}$ and $\rho_X = 4$, or $\dim X = 6$, an irreducible component of maximal dimension of $\mathrm{Locus}(V''', V'', V', V)_{x'''}$ is a divisor and $\rho_X = 4$, or 5 by Lemma (4.4). $\square$

THEOREM 6.2. *Let X be a Fano manifold of Picard number ρ_X, dimension $\dim X > 6$ and pseudoindex $i_X = \dim X/3$. If X admits an unsplit dominating family of rational curves, then $\rho_X(i_X - 1) \leq \dim X$ and equality holds if and only if $X = (\mathbf{P}^3)^4$.*

Proof. Denote by V any unsplit dominating family of rational curves on X. We can assume that $-K_X \cdot V = \dim X/3$, since if there exists an unsplit dominating family such that $-K_X \cdot V > \dim X/3$, the assertion follows by Lemma (6.1). Let $V^1, \dots, V^k$ be families of rational curves as in Construction (4.1); then by Lemma (4.3) we get $k \leq 3$, unless $k = 4$, $\dim X = 9$ and $i_X = 3$, or $X = (\mathbf{P}^3)^4$.

If all the families V^i are unsplit, then $\rho_X = k$ by Proposition (3.3).

We can thus assume that at least one of these families, say V^j, is not unsplit. Since $-K_X \cdot V^j \geq 2 \dim X/3$, by Lemma (4.3) we can have only one non-unsplit family among $V^2, \dots, V^k$ and $k \leq 3$. Moreover, if $j = 3$, then $\dim X = 9$ by Lemma (4.3), so $\rho_X = 3$ by part (a) of Lemma (4.5).

So we are left to consider $j = 2$. We claim that in this case X is rc$(V^1, \mathcal{V}^2)$-connected. In fact, if this were not the case, there should be a family V^3 which is horizontal with respect to the rc$(V^1, \mathcal{V}^2)$-fibration. Then, by Lemma (4.3), we would have that $\dim X = 9$ and, by Proposition (2.8), that all the V^is are dominating with $-K_X \cdot V^2 > -K_X \cdot V^3$, which is a contradiction.

Consider an irreducible component G of $\mathrm{Locus}(V^2, V^1)_{x_2}$ of maximal dimension. Then $\dim G \geq \dim X - 2$ by Lemma (2.10) and $\mathrm{N}_1(G, X) \subseteq \langle [V^1], [V^2] \rangle$ by part (b) of Corollary (2.12). If $\dim G = \dim X$ then clearly $\rho_X = 2$, while if $\dim G = \dim X - 1$ then $\rho_X = 2$ by Lemma (4.4).

We can thus assume that $\dim G = \dim X - 2$. Since, if all the components of the reducible cycles are contained in $\langle [V^1], [V^2] \rangle$ then $\rho_X = 2$, we can assume that this is not the case. Let $\Gamma = \Gamma_1 + \Gamma_2$ be a reducible cycle of $\mathcal{V}^2$ which is not contained in $\langle [V^1], [V^2] \rangle$ and denote by W^i a family of deformations of Γ_i, $i = 1, 2$.

By Lemma (2.10) we get $-K_X \cdot V^1 = i_X$, $-K_X \cdot V^2 = 2i_X$ and $\dim \mathrm{Locus}(V^2)_{x_2} = 2i_X - 1$, so that V^2 is covering by Proposition (2.8).

We claim that there does not exist any W^i, among the families that are not contained in $\langle[V^1],[V^2]\rangle$, such that $\dim \operatorname{Locus}(W^i) = \dim X - 1$. In fact, if such a family W^i would exist, then it could not be trivial on both V^1 and V^2 by Corollary (3.4) and Lemma (3.5). Therefore $\operatorname{Locus}(W^i)$ would intersect $\operatorname{Locus}(V^2, V^1)_{x_2}$, so $\dim \operatorname{Locus}(V^2, V^1, W^i)_{x_2} \geq \dim X$, which is a contradiction since W^i is not covering.

It follows that $\dim \operatorname{Locus}(W^i) \leq \dim X - 2$ for any family W^i that is not contained in $\langle[V^1],[V^2]\rangle$. Then $\operatorname{Locus}(W^1, W^2, V^1)_x$ has an irreducible component D of dimension at least $\dim X - 1$ by combining Lemma (2.10) and part (b) of Proposition (2.8). As $\mathrm{N}_1(D, X) \subseteq \langle[V^1],[W^1],[W^2]\rangle$ by part (b) of Corollary (2.12), we conclude that $\rho_X = 3$: this is clear if $\dim D = \dim X$, while it follows by Theorem (2.13) if $\dim D = \dim X - 1$. $\square$

Theorem 6.3. *Let X be a Fano manifold of Picard number ρ_X, pseudoindex i_X and dimension 6. If X admits an unsplit dominating family of rational curves, then $\rho_X(i_X - 1) \leq 6$. Moreover, equality holds if and only if $X = \mathbf{P}^6$, or $X = \mathbf{P}^3 \times \mathbf{P}^3$, or $X = \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$, or $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.*

Proof. Clearly we can assume $i_X \geq 2$. Moreover, we can restrict to $i_X = 2$, since otherwise we can apply Theorem (2.14). So we have to show that $\rho_X \leq 6$, with equality if and only if $X = (\mathbf{P}^1)^6$.

Denote by V any unsplit dominating family of rational curves on X. We can assume that $-K_X \cdot V = 2$, since if there exists an unsplit dominating family such that $-K_X \cdot V \geq 3$ then the assertion follows by Lemma (6.1). Let $V^1, \dots, V^k$ be families of rational curves as in Construction (4.1); then by Lemma (4.3) we get $k \leq 5$, unless $X = (\mathbf{P}^1)^6$.

If all the families V^i are unsplit, then $\rho_X = k$ by Proposition (3.3).

We can thus assume that at least one of these families, say V^j, is not unsplit. Since $-K_X \cdot V^j \geq 4$, by Lemma (4.3) we can have only one non-unsplit family among $V^2, \dots, V^k$ and $k \leq 4$. Moreover, if $j = 4$, then $\rho_X = 4$ by part (a) of Lemma (4.5), while, if $j = 3$, then we conclude by part (b) of the same lemma.

Therefore we are left with $j = 2$. In this case, a general fiber of the $\mathrm{rc}(V^1, \mathcal{V}^2)$-fibration $\pi_2 : X \dashrightarrow Z^2$ has dimension at least $\dim \operatorname{Locus}(V^2, V^1)_{x_2}$, which is at least four by combining Lemma (2.10) and part (b) of Proposition (2.8). Then $\dim Z^2 \leq 2$.

Assume first that $\dim Z^2 \geq 1$ and denote by V^3 a minimal horizontal dominating family with respect to π_2. Then $\dim \operatorname{Locus}(V^3)_{x_3} \leq 2$, so $-K_X \cdot V^3 \leq 3$, by part (b) of Proposition (2.8), and V^3 is unsplit. Moreover, if $-K_X \cdot V^3 = 3$, then V^3 would be covering by Proposition (2.8), contradicting the minimality of V^2. Therefore $-K_X \cdot V^3 = 2$; since V^3 cannot be covering, the same proposition implies that $\dim \operatorname{Locus}(V^3)_{x_3} = 2$. It follows that X is $\mathrm{rc}(V^1, \mathcal{V}^2, V^3)$-connected.

Let F be a general fiber of the $\mathrm{rc}(V^1, \mathcal{V}^2)$-fibration, whose dimension is equal to four. Consider an irreducible component D of $\operatorname{Locus}(V^3)_F$ of maximal dimension. By Lemma (2.10), D is a divisor, and so we are done by Lemma (4.4).

Assume now that $\dim Z^2 = 0$, so that X is $\mathrm{rc}(V^1, \mathcal{V}^2)$-connected.

If $-K_X \cdot V^2 \geq 6$, then Lemma (4.3) implies that $-K_X \cdot V^2 = 6$. It follows by Lemma (2.10) that $X = \mathrm{Locus}(V^2, V^1)_{x_2}$, for a general $x_2 \in \mathrm{Locus}(V^2)$ and $\rho_X = 2$ by part (b) of Corollary (2.12).

Therefore we can assume that $-K_X \cdot V^2 < 6$, so that the reducible cycles of $\mathcal{V}^2$ have exacly two irreducible components. Consider an irreducible component G of $\mathrm{Locus}(V^2, V^1)_{x_2}$ of maximal dimension. Then $\dim G \geq 4$ by Lemma (2.10).

Moreover, if $\dim G = 6$, then $\rho_X = 2$, so we need to consider $\dim G = 4$ or 5. Since, if all the components of these cycles are contained in $\langle [V^1], [V^2] \rangle$ then $\rho_X = 2$, we can assume that this is not the case. Let $\Gamma = \Gamma_1 + \Gamma_2$ be a reducible cycle of $\mathcal{V}^2$ not contained in $\langle [V^1], [V^2] \rangle$ and denote by W^i a family of deformations of Γ_i, $i = 1, 2$.

If $\dim G = 5$, then by Lemma (3.5) $G \cdot \Gamma_i = 0$, for $i = 1, 2$. It follows that $G \cdot V^2 = 0$, whence $G \cdot V^1 > 0$ by Corollary (3.4). Then $X = \mathrm{Locus}(V^1)_G$, so $\mathrm{N}_1(X) = \langle [V^1], [V^2] \rangle$, a contradiction.

Therefore we are left with $\dim G = 4$. By Proposition (2.8) we get $-K_X \cdot V^1 = 2$, $-K_X \cdot V^2 = 4$ and $\dim \mathrm{Locus}(V^2)_{x_2} = 3$, so V^2 is covering.

Assume that there exists a family W^i, among the families that are not contained in $\langle [V^1], [V^2] \rangle$, such that $\dim \mathrm{Locus}(W^i) = 5$. Then it cannot be trivial on both V^1 and V^2 by Corollary (3.4) and Lemma (3.5). Therefore $\mathrm{Locus}(W^i)$ intersects $\mathrm{Locus}(V^2, V^1)_{x_2}$, so $\dim \mathrm{Locus}(V^2, V^1, W^i)_{x_2} = 5$, so we conclude by Theorem (2.13).

We can thus assume that $\dim \mathrm{Locus}(W^i) = 4$ for any family that is not contained in $\langle [V^1], [V^2] \rangle$. Then $\mathrm{Locus}(W^1, W^2, V^1)_{y_1}$ has an irreducible component D of dimension at least five by Lemma (2.10). Then we conclude by part (b) of Corollary (2.12) if $\dim D = 6$ and by Theorem (2.13) if $\dim D = 5$. $\square$

By combining the results of this section we actually have the following

Theorem 6.4. *Let X be a Fano manifold of Picard number ρ_X and pseudo-index $i_X \geq \min\{\dim X - 4, (\dim X - 2)/2, \dim X/3\}$. If X admits an unsplit dominating family of rational curves, then $\rho_X(i_X - 1) \leq \dim X$ and equality holds if and only if $X = (\mathbf{P}^{i_X - 1})^{\rho_X}$.*

Acknowledgements. I would like to thank Gianluca Occhetta for helpful suggestions during the preparation of the paper.

References

[1] M. Andreatta, E. Chierici and G. Occhetta, Generalized Mukai conjecture for special Fano varieties, Cent. Eur. J. Math. **2** (2004), 272–293.

[2] L. Bonavero, C. Casagrande, O. Debarre and S. Druel, Sur une conjecture de Mukai, Comment. Math. Helv. **78** (2003), 601–626.

[3] F. Campana, Coréduction algébrique d'un espace analytique faiblement Kählérien compact, Invent. Math. **63** (1981), 187–223.

[4] F. Campana, Orbifolds, special varieties and classification theory: an appendix, Ann. Inst. Fourier (Grenoble) **54** (2004), 631–665.

[5] C. Casagrande, The number of vertices of a Fano polytope, Ann. Inst. Fourier (Grenoble) **56** (2006), 121–130.

[6] C. Casagrande, On the Picard number of divisors in Fano manifolds, to appear in Annales Scientifiques de l'Ecole Normale Superieure, arXiv:0905.3239v4.

[7] J. Kollár, Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete **32**, Springer-Verlag, Berlin, 1996.

[8] J. Kollár, Y. Miyaoka and S. Mori, Rational connectedness and boundedness of Fano manifolds, J. Differential Geom. **36** (1992), 765–779.

[9] S. Mori, Projective manifolds with ample tangent bundles, Ann. of Math. (2) **110** (1979), 593–606.

[10] S. Mukai, Open problems, Birational geometry of algebraic varieties (T. Taniguchi foundation, ed.), Katata, 1988.

[11] D. Mumford, The red book of varieties and schemes, Lecture notes in mathematics **1358**, Springer-Verlag, Berlin, 1999.

[12] C. Novelli and G. Occhetta, Rational curves and bounds on the Picard number of Fano manifolds, Geom. Dedicata **147** (2010), 207–217.

[13] C. Novelli and G. Occhetta, Projective manifolds containing a large linear subspace with nef normal bundle, Michigan Math. J. **60** (2011), 441–462.

[14] G. Occhetta, A characterization of products of projective spaces, Canad. Math. Bull. **49** (2006), 270–280.

[15] J. A. Wiśniewski, On a conjecture of Mukai, Manuscripta Math. **68** (1990), 135–141.

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