C. PLÉNAT AND P. POPESCU-PAMPU
KODAI MATH. J.
**31** (2008), 199–218

# FAMILIES OF HIGHER DIMENSIONAL GERMS WITH BIJECTIVE NASH MAP

Camille Plénat and Patrick Popescu-Pampu

**Abstract**

Let $(X,0)$ be a germ of complex analytic normal variety, non-singular outside 0. An essential divisor over $(X,0)$ is a divisorial valuation of the field of meromorphic functions on $(X,0)$, whose center on any resolution of the germ is an irreducible component of the exceptional locus. The Nash map associates to each irreducible component of the space of arcs through 0 on $X$ the unique essential divisor intersected by the strict transform of the generic arc in the component. Nash proved its injectivity and asked if it was bijective. We prove that this is the case if there exists a divisorial resolution $\pi$ of $(X,0)$ such that its reduced exceptional divisor carries sufficiently many $\pi$-ample divisors (in a sense we define). Then we apply this criterion to construct an infinite number of families of 3-dimensional examples, which are not analytically isomorphic to germs of toric 3-folds (the only class of normal 3-fold germs with bijective Nash map known before).

## 1. Introduction

Let $X$ be a reduced complex algebraic variety. An *arc* contained in $X$ is a germ of formal map:

$$(\mathbf{C},0) \to X.$$

If $t$ denotes the local parameter of $\mathbf{C}$ at 0, notice that each arc comes equipped with a canonical parametrization: thought algebraically, it is a morphism of $\mathbf{C}$-algebras $\mathcal{O}_{X,0} \to \mathbf{C}[[t]]$.

In a preprint written around 1966, published later as [17], Nash defined the associated *arc space* $X_\infty$ of $X$, whose points represent the arcs contained in $X$. By looking at the Taylor expansions of the functions on $X$ with respect to the parameter $t$ and to their truncations at all the orders, Nash constructed this space as a projective limit of algebraic varieties of finite type over $X$.

If one associates to a formal arc the point of $X$ where it is based, that is the image of $0 \in \mathbf{C}$, one gets a natural map:

2000 *Mathematics Subject Classification*. 14B05, 32S25, 32S45.
*Key words and phrases*. Space of arcs, essential components, Nash map, Nash problem.
Received August 30, 2007; revised December 6, 2007.

$$\alpha : X_\infty \to X$$

If $Y$ is a closed subvariety of $X$, denote by:

$$(X, Y)_\infty := \alpha^{-1}(Y)$$

the *space of arcs on $X$ based at $Y$*.

Nash was thinking of the spaces $X_\infty$ and $(X, Y)_\infty$ for varying $Y \subset \mathrm{Sing}(X)$ as tools for studying the structure of $X$ in the neighborhood of its singular set. Indeed, the main object of his paper was to state a program for comparing the various resolutions of the singularities of $X$. Such resolutions always exist, as had recently been proven by Hironaka, but unlike in the case of surfaces, minimal ones do not necessarily exist. We quote from the introduction of [17] the two main problems formulated by Nash in this direction:

i) *For surfaces it seems possible that there are exactly as many families of arcs associated with a point as there are components of the image of the point in the minimal resolution of the singularities of the surface.*

ii) *In higher dimensions, the arc families associated with the singular set correspond to "essential components" which must appear in the image of the singular set in all resolutions. We do not know how complete is the representation of essential components by arc families.*

The first question is a local one, as it deals with the structure of $X$ only in a neighborhood of one of its (closed) points. The second one is more global, as it deals with the structure of $X$ in the neighborhood of its entire singular set.

Following Nash's paper, the foundations for his program were worked with more detail by Lejeune-Jalabert [15], Nobile [18] and Ishii & Kollár [12]. They also extended the program to other categories of spaces. For example, Ishii & Kóllar [12] considered schemes over arbitrary fields, Lejeune-Jalabert [15] and Nobile [18] considered formal germs of varieties. Their treatment extends readily to germs of complex analytic varieties.

For such germs, the space $(X, \mathrm{Sing}(X))_\infty$ of arcs based at the singular locus of $X$ can be canonically given the structure of a relative scheme over $X$, as the projective limit of relative schemes of finite type obtained by truncating arcs at each finite order.

*In the sequel we will restrict to the case where $(X,0)$ is a germ of a complex analytic variety and* $\mathrm{Sing}(X) = \{0\}$.

The space $(X,0)_\infty$ of arcs on $X$ based at 0 is a relative subscheme over $X$ of $X_\infty$. As it projects onto 0, we see that it is in fact a true scheme (but not of finite type over $\mathbf{C}$). This implies that it makes sense to speak about the set $\mathscr{C}(X,0)_\infty$ of its irreducible components.

Denote by

$$\pi : \tilde{X} \to X$$

a resolution of $X$. The *exceptional set* $\mathrm{Exc}(\pi) := \pi^{-1}(0)$ is not assumed to be of pure codimension 1, that is, the resolution is not necessarily divisorial.

An irreducible component of $\mathrm{Exc}(\pi)$ is called *an essential component of $\pi$* if it corresponds to an irreducible component of the exceptional set of any other resolution of $X$. In other words, if its birational transform is an irreducible component of the exceptional set in any resolution. An equivalence class of such essential components over all the resolutions of $X$ is called an *essential divisor over* $(X, 0)$. If we denote by $\mathscr{E}(X, 0)$ the set of essential divisors over $(X, 0)$, the essential components of the given resolution morphism $\pi$ are in a canonical bijective correspondence with the elements of $\mathscr{E}(X, 0)$.

Let $\mathscr{K}$ be an element of $\mathscr{C}(X, 0)_\infty$. For each arc represented by a point of $\mathscr{K}$, one can consider the intersection point with $\mathrm{Exc}(\pi)$ of its strict transform on $\tilde{X}$. For an arc generic with respect to the Zariski topology of $\mathscr{K}$, this intersection point is situated on a unique irreducible component of $\mathrm{Exc}(\pi)$; moreover, this component is essential (Nash [17], see also [15]). In this manner one defines a map:

$$\mathscr{N}_{X,0} : \mathscr{C}(X, 0)_\infty \to \mathscr{E}(X, 0)$$

which is called *the Nash map* associated to $(X, 0)$. Nash proved that the map $\mathscr{N}_{X,0}$ is always *injective* (which shows in particular that $\mathscr{C}(X, 0)_\infty$ is a finite set). In our context, one can reformulate question ii) above:

*When is the map* $\mathscr{N}_{X,0}$ *bijective*?

This question is also known as *the Nash problem on arcs*.

In [23] we listed the classes of isolated *surface* singularities for which the Nash map was proved to be bijective. In higher dimensions, the bijectivity of $\mathscr{N}$ was proved till now for the following classes of germs with not necessarily isolated singularities:

- for the germs which have resolutions with irreducible exceptional set, for trivial reasons;
- for germs of normal toric varieties by Ishii and Kollár in [12]; in this case, one can distinguish two types of Nash problems, as was done by Ishii [11]; Ishii [10] solved the Nash problem also for not-necessarily normal toric varieties;
- for various classes of not necessarily irreducible germs whose normalizations are disjoint unions of normal toric germs by Ishii [10], [11], Petrov [20] and González Pérez [7].

No surface or 3-fold is known for which the Nash map is not bijective. But Ishii and Kollár proved in [12] that it is not always bijective for algebraic varieties of dimension at least 4. Indeed, they gave a counterexample in dimension 4, which can be immediately transformed into a counterexample (with non-isolated singularity) in any larger dimension.

In this article we construct *a class of normal isolated singularities of arbitrary dimension* $(X, 0)$ for which the Nash map $\mathscr{N}_{X,0}$ is bijective (Corollary 4.3). The definition of the class uses a criterion ensuring that a divisorial component of the exceptional set of a given resolution is in the image of the Nash map (Theorem 4.1). In fact, we use that theorem in the following less general form (a reformulation of Corollary 4.2), which allows us to apply Kleiman's ampleness criterion:

Theorem. *Let $\pi : \tilde{X} \to X$ be a divisorial projective resolution of $(X, 0)$. Consider an irreducible component $E_i$ of* $\mathrm{Exc}(\pi)$. *Suppose that for any other component $E_j$, there exists an effective integral divisor $F_{ij}$ on $\tilde{X}$ whose support coincides with* $\mathrm{Exc}(\pi)$, *in which the coefficient of $E_i$ is strictly less than the coefficient of $E_j$ and such that the line bundle $\mathcal{O}_{\mathrm{Exc}(\pi)}(-F_{ij})$ is ample. Then $E_i$ is an essential component contained in the image of the Nash map.*

Using the previous criterion, we construct an infinite family of examples of 3-dimensional singularities with bijective Nash map (see Section 5). In Section 6, we distinguish some of the singularities constructed before using suitable analytical invariants. Moreover, we determine those which are isomorphic to germs of toric varieties, establishing like this the intersection of our class of examples with the classes known before.

*Acknowledgements*. We are grateful to Monique Lejeune-Jalabert and to the referee of [23] for having remarked that the main theorem of [23] could be obtained by replacing Laufer's vanishing theorem with a convenient use of the ampleness of adequate line bundles. This is the method we have extended in this paper to higher dimensions. The second author is also grateful to Sébastien Boucksom, David Eisenbud, Charles Favre and Christophe Mourougane for their explanations about ampleness and positivity. He addresses special thanks to C. Mourougane for his many helpful remarks on previous versions of this paper.

## 2. Essential divisors and essential components

In the sequel, if $A$ is a complex analytic space or a relative scheme over an analytic space, we denote by $\mathscr{C}A$ the set of its irreducible components.

Let $(X, 0)$ be an irreducible germ of complex analytic variety. We suppose that $\mathrm{Sing}(X) = \{0\}$, that is, the germ is smooth outside the origin (with a slight abuse of vocabulary due to the fact that $X$ is also allowed to be smooth, we say that the germ *has an isolated singularity*). Denote by $\mathfrak{m}$ the maximal ideal of its local ring $\mathcal{O}_{X,0}$. We also write $\mathscr{N}$ instead of $\mathscr{N}_{X,0}$, as we do not consider various Nash maps at the same time.

Consider a *resolution* $\pi : \tilde{X} \to X$. This means that $\pi$ is a proper bimeromorphic map with $\tilde{X}$ smooth, restricting to an isomorphism over the complement of 0 in $X$. The *exceptional set* $\mathrm{Exc}(\pi)$ *of* $\pi$ is by definition the subset of $\tilde{X}$ where $\pi$ is not a local isomorphism. If 0 is a singular point of $X$, it coincides with the preimage $\pi^{-1}(0)$. If each irreducible component of $\mathrm{Exc}(\pi)$ is of pure codimension 1 in $\tilde{X}$, we say that $\pi$ is *divisorial*. In the sequel, we do not suppose that this is the case. We do neither suppose that the morphism $\pi$ is projective.

*Remark* 2.1. In dimension 2, all the resolutions of a normal surface are divisorial. This is no longer true in higher dimensions: the simplest example of a normal germ with isolated singularity which has non-divisorial resolutions is the 3-fold hypersurface germ defined by the affine equation $xy - zt = 0$. Nevertheless, all the resolutions of a **Q**-factorial germ are divisorial (see Debarre [4, Section 1.40]).

Consider a closed irreducible subvariety $E$ of $\mathrm{Exc}(\pi)$ (not necessarily one of its irreducible components). Take the preimage $D$ of $E$ on $B_E(\tilde{X})$, the variety obtained by blowing-up $E$ in $\tilde{X}$. As $\tilde{X}$ is smooth, this preimage is an irreducible hypersurface of $B_E(\tilde{X})$. Therefore, it induces a discrete valuation $v_E$ of rank 1 on the field of meromorphic functions on $(X,0)$ (which associates to any such function the order of vanishing along $D$ of its total transform on $B_E(\tilde{X})$).

If $\psi : \overline{X} \to X$ is another resolution of $X$, *the birational transform* $E^{\psi}$ *of* $E$ on $\overline{X}$ is the center of the valuation $v_E$ on $\overline{X}$. We have obviously $v_E = v_{E^{\psi}}$. This allows to identify the valuation $v_E$ with the set whose elements are $E$ and its birational transforms on all the resolutions of $X$. Following [12], we say that $v_E$ (or the class of its centers on all the resolutions) is an *exceptional divisor over* $(X,0)$. The name is motivated by the fact that any resolution is dominated by another one on which the center of $v_E$ is a divisor (as above, just blow-up $E$, then resolve the singularities of the new space).

Conversely, if $v$ is an exceptional divisor over $(X,0)$ and $\pi : \tilde{X} \to X$ is a resolution, we denote by $E_v^{\pi}$ (or $E_v$, if $\pi$ is clear from the context) the center of $v$ on $\tilde{X}$. Among the exceptional divisors over $(X,0)$, Nash distinguished those whose centers are not only subvarieties, but irreducible *components* of the exceptional locus of any resolution of $(X,0)$ (in fact he considered this in the global case of an algebraic variety; Ishii [11, Definition 2.08] considers the same localized situation as ours):

Definition 2.2. An **essential divisor over** $(X,0)$ is an exceptional divisor over $(X,0)$ whose center on $\tilde{X}$ is an irreducible component of $\mathrm{Exc}(\pi)$, this for any resolution $\pi : \tilde{X} \to X$. We also say that the centers of the essential divisors on $\tilde{X}$ are the **essential components** of $\pi$.

In [2], Bouvier considered another definition of *essential divisors*. She called a component of codimension 1 of the exceptional set *essential* if its center on any resolution was a divisor. Her definition is strictly more restrictive than ours, as shown by the germs which admit resolutions without exceptional components of codimension 1 (see Remark 2.1). Ishii and Kollár introduced a third notion in [12], that of *divisorially essential divisors*. Namely, an exceptional divisor is of this type, if its center in any divisorial resolution is an irreducible component of the exceptional set. It follows directly from the Definition 2.2 that an essential divisor is a divisorially essential divisor, but it seems to be an open question if the converse is true.

*In the sequel we consider only the notion of essential components and essential divisors introduced in Definition 2.2.*

If $(X,0)$ is a normal surface singularity, then the essential divisors over $(X,0)$ are precisely the divisorial valuations generated by the irreducible components of the exceptional set of the minimal resolution of $(X,0)$. In higher dimensions it is much more difficult to determine them, as no minimal resolution (in the sense that it is dominated by all the other resolutions) exists in general.

The only class of normal singularities for which the essential divisors are completely known is that of germs of normal toric varieties. Indeed, Bouvier [2] determined combinatorially the essential divisors of all normal toric germs. Her work was based on preliminary results of Bouvier & González-Sprinberg [3]. Ishii [10] characterized the essential divisors with respect to the definition given in [12] also in the case of not necessarily normal toric varieties.

Two general criteria are known, ensuring that a 1-codimensional component of the exceptional locus of a given resolution is essential (see Ishii & Kollár [12, Examples 2.4, 2.5, 2.6]):

Proposition 2.3. *Let $E_i$ be an irreducible component of* $\mathrm{Exc}(\pi)$, *which is of codimension* 1 *in* $\tilde{X}$.

1) (*Nash* [17]) *If $E_i$ is not birationally ruled, then $E_i$ is essential.*

2) *If $(X,0)$ is a canonical singularity and $E_i$ is crepant, then $E_i$ is essential.*

*Moreover, in both cases the birational transform of $E_i$ on any other resolution has again codimension* 1.

One of the results of our work is to give a new criterion of essentiality for exceptional divisors, using the space of arcs on $X$ based at 0 (Theorem 4.1).

For each irreducible component $E$ of $\mathrm{Exc}(\pi)$, consider the smooth arcs on $\tilde{X}$ whose closed points are on $E - \bigcup_{F \neq E} F$, where $F$ varies among the elements of $\mathscr{C}\,\mathrm{Exc}(\pi)$, and which intersect $E$ transversely (that is, such that their tangent line and the tangent space to $E$ at their intersection point are direct summands). Consider the set of their images in $(X,0)_\infty$ and denote the closure of this set by $V(E)$.

*Remark* 2.4. In fact $V(E)$ only depends on the exceptional divisor $v_E$ over $(X,0)$ determined by $E$ (see Ishii [10, Example 2.14]). For this reason, in the sequel we also write $V(v)$ instead of $V(E)$, if $E = E_v$.

Nash [17] proved:

Proposition 2.5. 1) *The sets $V(E)$ are irreducible subvarieties of $(X,0)_\infty$* (*but not necessarily components*).

2)

$$(X,0)_\infty = \bigcup_{E \in \mathscr{C}\,\mathrm{Exc}(\pi)} V(E).$$

The next lemma gives a criterion to show that an exceptional divisor $v$ over $(X,0)$ is essential, using its image $V(v)$ in the space of arcs based at 0.

Lemma 2.6. *Let $v$ be an exceptional divisor over* 0. *If $V(v)$ is an irreducible component of $(X,0)_\infty$, then $v$ is essential.*

*Proof.* Suppose by contradiction that $v$ is inessential. This means that there exists a resolution $\pi : \tilde{X} \to X$ such that the center $E_v$ of the valuation $v$ on $\tilde{X}$ is strictly included in an irreducible component $E$ of $\mathrm{Exc}(\pi)$. We deduce that $V(v)$ is strictly included in $V(E)$. But this last variety is irreducible, by Proposition 2.5. This contradicts the fact that $V(v)$ is an irreducible component of $(X,0)_\infty$. □

The next proposition gives a criterion to prove that some components of the exceptional locus of a resolution of $(X,0)$ are essential, and in particular to prove that Nash's map $\mathcal{N}$ is bijective.

Proposition 2.7. *Let $\pi : \tilde{X} \to X$ be a resolution. Consider the irreducible components $(E_i)_{i \in I}$ of $\mathrm{Exc}(\pi)$. Suppose that one can write the index set $I$ as a disjoint union $I = J \sqcup K$ such that $V(E_j) \not\subset V(E_i)$, $\forall j \in J$, $\forall i \in I - \{j\}$. Then:*

1) *The varieties $(V(E_j))_{j \in J}$ are irreducible components of $(X,0)_\infty$. In particular, $(E_j)_{j \in J}$ are essential components of $\pi$.*

2) *If $(E_k)_{k \in K}$ are all inessential components of $\pi$, then $\mathcal{N}$ is bijective.*

*Proof.* 1) By Proposition 2.5, the irreducible components of $(X,0)_\infty$ are among the varieties $V(E)$ with $E \in \mathscr{C}E_\pi(X,0)$. Moreover, by definition, the irreducible components of $(X,0)_\infty$ are those which are not included in other irreducible subsets. Then the varieties $(V(E_j))_{j \in J}$ are irreducible components of $(X,0)_\infty$. By the previous lemma, it follows that $(E_j)_{j \in J}$ are essential components of $\pi$.

2) If the components $(E_k)_{k \in K}$ are all inessential, then, by Lemma 2.6, the varieties $(V(E_k))_{k \in K}$ are not irreducible components of $(X,0)_\infty$. The irreducible components of $(X,0)_\infty$ are exactly $(V(E_j))_{j \in J}$ and they do correspond bijectively to the essential components of the resolution. This implies that $\mathcal{N}$ is bijective. □

The following proposition was proven by the first author in [21, 2.2], as a generalization of Reguera [25, Theorem 1.10], who considered only the class of rational surface singularities. It appears also implicitely in the toric case in Ishii [9, Proposition 4.8]. It is an essential ingredient of all the criteria we prove in this paper. It was also the basis of our work [23].

Proposition 2.8. *Let $v_1$ and $v_2$ be exceptional divisors over $(X,0)$. If there exists a function $f \in \mathfrak{m}$ such that $v_1(f) < v_2(f)$, then $V(v_1) \not\subseteq V(v_2)$.*

In Section 4, we combine the propositions 2.7 and 2.8 in order to give criteria of essentiality for exceptional divisors in terms of global generation and ampleness of suitable line bundles. Before that, we need some background about ampleness and exceptional sets.

### 3. Background about ampleness and exceptional analytic sets

In this section we recall Kleiman's criterion of ampleness and Grauert's criterion of contractibility.

Let $Y$ be a complete algebraic variety. Let $Z_1(Y)_{\mathbf{R}}$ be the **R**-vector space of real one-cycles on $X$, consisting of all finite **R**-linear combinations of irreducible algebraic curves on $Y$. Two elements $\gamma_1$ and $\gamma_2$ of $Z_1(Y)_{\mathbf{R}}$ are *numerically equivalent* if one has the equality of intersection numbers

$$E \cdot \gamma_1 = E \cdot \gamma_2$$

for every $E \in \mathrm{Div}(Y) \otimes_{\mathbf{Z}} \mathbf{R}$, where $\mathrm{Div}(Y)$ denotes the group of Cartier divisors on $Y$. The corresponding vector space of numerical equivalence classes of one-cycles is written $N_1(Y)_{\mathbf{R}}$.

DEFINITION 3.1. Let $Y$ be a complete algebraic variety. The **cone of curves**

$$NE(Y) \subset N_1(Y)_{\mathbf{R}}$$

is the cone $\mathbf{R}_+$-spanned by the classes of all effective one-cycles on $Y$. Its closure $\overline{NE}(Y) \subset N_1(Y)_{\mathbf{R}}$ is the **closed cone of curves** or **Kleiman-Mori cone** of $Y$.

THEOREM 3.2 (Kleiman's criterion of ampleness). *Let $Y$ be a projective variety. A Cartier divisor $E$ on $Y$ is ample if and only if $E \cdot z > 0$ for all non zero $z \in \overline{NE}(Y)$.*

For details, we refer to Debarre [4] and Lazarsfeld [14].

Ampleness on a reducible variety can be tested on its irreducible components (see for example Lazarsfeld [14, proposition 1.2.16]):

PROPOSITION 3.3. *Let $Y$ be a projective variety and $L$ a line bundle on $Y$. Then $L$ is ample on $Y$ if and only if the restriction of $L$ to each irreducible component of $Y$ is ample.*

We took the following definition from Peternell [19, definition 2.8]:

DEFINITION 3.4. Let $Y$ be a reduced complex space and $E \subset Y$ a compact nowhere discrete and nowhere dense analytic set. $E$ is called **exceptional** (in $X$) if there is a complex space $Z$ and a proper surjective holomorphic map $\phi : Y \to Z$ such that:

(1) $\phi(E)$ is a finite set;

(2) $\phi : Y \backslash E \to Z \backslash \phi(E)$ is biholomorphic;

(3) $\phi_*(\mathcal{O}_Y) = \mathcal{O}_Z$.

Then one says that $\phi$ **contracts** $E$ (in $Y$).

One can show that a map $\phi$ which contracts $E$ in $Y$ is *unique* in the following sense: if $\phi_k : Y \to Z_k$, $k \in \{1, 2\}$ both contract $E$ in $Y$, then there exists a unique analytic isomorphism $u : Z_1 \to Z_2$ such that $\phi_2 = u \circ \phi_1$.

In the minimal model theory of algebraic varieties, one considers more general contractions, which are not necessarily birational maps.

The vocabulary is coherent with the one used in section 2. Indeed, if $\pi : \tilde{X} \to X$ is a resolution of a normal germ $X$ with isolated singularity, then its *exceptional set* $\mathrm{Exc}(\pi)$ in the sense of section 2 is *exceptional in* $\tilde{X}$ in the sense of Definition 3.4, and $\pi$ contracts $\mathrm{Exc}(\pi)$ in $\tilde{X}$.

The strategy we use for constructing examples of 3-dimensional singularities with bijective Nash maps (see Section 5) works thanks to Grauert's fundamental criterion of contractibility (Grauert [8], see also Peternell [19, theorem 2.12]). A particular case of it is sufficient for our purposes:

Theorem 3.5 (Grauert's criterion of contractibility). *Let $Y$ be a complex manifold and let $E$ be a reduced projective* (*not necessarily smooth or irreducible*) *hypersurface in $Y$. Suppose that there exists an effective divisor $A$ whose support is $E$, such that the restriction $\mathcal{O}_E(-A)$ of the line bundle $\mathcal{O}_Y(-A)$ to $E$ is ample. Then the analytic hypersurface $E$ is exceptional in $Y$.*

*Remark* 3.6. 1) If $Y$ is a surface, the converse of the theorem is also true. In this case, the hypothesis about the existence of $A$ is equivalent to the fact that the intersection form of $E$ is negative definite. For surfaces, the hypothesis of Grauert's criterion of contractibility is usually expressed in this last manner.

2) The converse of Theorem 3.5 is not true in a naive form if $\dim_{\mathbf{C}} Y \geq 3$, as shown by examples of Laufer [13] (see also Peternell [19, Example 2.14]). Nevertheless, there exists a converse if one replaces the search of an ample line bundle by that of a coherent sheaf $\mathcal{I}$ such that $\mathrm{supp}(\mathcal{O}_Y/\mathcal{I}) = E$ and $\mathcal{I}/\mathcal{I}^2$ is positive (see Peternell [19, Theorem 2.15]).

## 4. Criteria for an exceptional divisor to be essential

Recall that $(X,0)$ is supposed to be an irreducible germ with isolated singularity. *From now on, we suppose moreover that* $(X,0)$ *is normal.* We need this condition in order to be able to conclude that a bounded holomorphic function on $X\backslash 0$ extends to a function holomorphic over $X$.

Let $\pi : (\tilde{X}, \mathrm{Exc}(\pi)) \to (X,0)$ be a *divisorial* resolution of $(X,0)$. Denote by $(E_i)_{i\in I}$ the irreducible components of $\mathrm{Exc}(\pi)$.

Let

$$L(\pi) := \bigoplus_{i\in I} \mathbf{Z}E_i$$

be the lattice freely generated by the $E_i$'s, that is, the lattice of divisors on $\tilde{X}$ supported by $\mathrm{Exc}(\pi)$. Inside the associated real vector space $L_{\mathbf{R}}(\pi)$, consider the closed regular cone:

$$\sigma(\pi) := \bigoplus_{i\in I} \mathbf{R}_+E_i$$

of the effective $\mathbf{R}$-divisors on $\tilde{X}$ supported by $\mathrm{Exc}(\pi)$.

For each pair $(i, j) \in I^2$ with $i \neq j$, consider the closed convex sub-cone $\sigma_{ij}(\pi)$ of $\sigma(\pi)$ defined by:

$$\sigma_{ij}(\pi) := \left\{ \sum_{k \in I} a_k E_k \in \sigma(\pi) \mid a_i \leq a_j \right\}$$

THEOREM 4.1. *Fix $i \in I$. Suppose that for each $j \in I \backslash \{i\}$, the cone $\sigma_{ij}(\pi)$ contains in its interior an integral divisor $F_{ij}$ such that $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is generated by its global sections. Then $V(E_i)$ is in the image of the Nash map $\mathcal{N}$. In particular, $E_i$ is an essential component of $\pi$.*

*Proof.* Consider $F_{ij} \in \mathrm{int}(\sigma_{ij}(\pi))$ such that $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is generated by its global sections. Let us consider for a moment $\mathcal{O}_{\tilde{X}}(-F_{ij})$ not as a line bundle, but as the subsheaf of the structure sheaf $\mathcal{O}_{\tilde{X}}$ formed by the holomorphic functions vanishing along $\mathrm{Exc}(\pi)$ at least as much as indicated by the coefficients of $F_{ij}$.

If $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is generated by global sections, then there exists a function $f_{ij} \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-F_{ij}))$ whose divisor has a compact part coinciding with $F_{ij}$.

As $\pi$ realizes an isomorphism between $\tilde{X} \backslash \mathrm{Exc}(\pi)$ and $X \backslash 0$, there exists a function $g_{ij}$ on $X$ vanishing at 0, continuous on $X$, holomorphic on $X \backslash 0$ and such that $f_{ij} = \pi^*(g_{ij})$. As $X$ is supposed to be normal at 0 (see the beginning of the section), we deduce that $g_{ij} \in \mathfrak{m}$.

By construction, $v_{E_i}(g_{ij}) < v_{E_j}(g_{ij})$. Proposition 2.8 implies then that $V(E_i) \not\subseteq V(E_j)$. As this is true for any pair $(i, j) \in I^2$ with $i \neq j$, the proposition follows from Proposition 2.7. □

The following corollary is a direct consequence of the theorem. We state it as a separate result, in order to be able to use Kleiman's criterion of ampleness in combination with it.

COROLLARY 4.2. *Fix $i \in I$. Suppose that for each $j \in I \backslash \{i\}$, the cone $\sigma_{ij}(\pi)$ contains an integral divisor $F_{ij}$ such that $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is ample when restricted to each component of* Exc$(\pi)$. *Then $V(E_i)$ is in the image of $\mathcal{N}$ and $E_i$ is an essential component of $\pi$ relative to* 0.

*Proof.* As ampleness is an open condition with respect to the topology of $L(\pi)$, we see that the hypothesis implies that there exists an $F_{ij} \in \mathrm{int}(\sigma_{ij}(\pi))$ such that $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is ample when restricted to each component of $\mathrm{Exc}(\pi)$. By Proposition 3.3, it is also ample when restricted to $\mathrm{Exc}(\pi)$. This implies that $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is ample on a neighborhood of $E$ in $\tilde{X}$. But then there exists a multiple $-n_{ij}F_{ij}$ of the divisor $-F_{ij}$ (where $n_{ij} \in \mathbf{Z}_{>0}$) whose associated sheaf is very ample, which implies that $\mathcal{O}_{\tilde{X}}(-n_{ij}F_{ij})$ is generated by its global sections.

The divisor $n_{ij}F_{ij}$ is interior to the cone $\sigma_{ij}(\pi)$, as $F_{ij}$ was supposed to be so. This implies that the hypothesis of Theorem 4.1 are satisfied. The conclusion follows. □

A second corollary gives the criterion of bijectivity of the Nash map announced in the introduction:

COROLLARY 4.3. *Suppose that for each pair* $(i, j) \in I^2$ *with* $i \neq j$, *the cone* $\sigma_{ij}(\pi)$ *contains an integral divisor* $F_{ij}$ *such that* $\mathcal{O}_{\tilde{X}}(-F_{ij})$ *is ample when restricted to each component of* $E$. *Then the components of* $E$ *are precisely the essential components over* $0$ *and the Nash map* $\mathcal{N}$ *is bijective.*

*Proof.* This is an immediate consequence of Corollary 4.2. □

*Remark* 4.4. When $(X, 0)$ is a germ of normal surface and $\pi : \tilde{X} \to X$ is a resolution, the set of effective divisors $F \in \sigma(\pi)$ such that the line bundle $\mathcal{O}_{\tilde{X}}(-F)$ is ample is precisely what we called the strict Lipman semigroup in [23, Remark 4.4] (see Lipman [16, 10.4 and proof of 12.1 (iii)]). Then Corollary 4.3 restricted to germs of normal analytic surfaces gives exactly the class of singularities found in [23]. As explained in the Acknowledgements, the present work grew out from the wish to generalize the results of that article to higher dimensions.

## 5. An infinite number of families of examples in dimension 3

Corollary 4.3 gives a method to construct examples of singularities $(X, 0)$ for which the Nash map $\mathcal{N}$ is bijective. Namely, one starts from a divisorial resolution of a germ such that the components of the exceptional locus have closed cones of curves of finite type. The condition on an effective divisor supported by the exceptional set to have an ample opposite in restriction to the exceptional set translates then into a finite system of linear inequalities. If this system has solutions inside all the cones considered in the corollary then, using the corollary, one has an example with bijective Nash map.

One could try to start from germs defined explicitly by equations and to use one of the available algorithms of resolution. Nevertheless, those algorithms do not allow to compute the closed cone of curves of a component of the exceptional set.

For this reason we decided to work differently. The strategy we followed was to start from a finite collection $(E_i)_{i \in I}$ of smooth projective varieties, with cones of curves which are closed and of finite type. Then choose line bundles over the varieties $E_i$ with ample duals and glue analytically the total spaces of those line bundles along neighborhoods of suitable hypersurfaces of the $E_i$. Of course, the first thing to adjust in order to do such a gluing, is to make a pairing of the chosen hypersurfaces and to fix isomorphisms between the elements in each pair.

If the gluing succeeds, one gets a smooth analytic variety $X$ which contains a divisor $E$ obtained topologically by identifying the chosen pairs of hypersurfaces of the varieties $E_i$. The choices should be done in order to make $E$ *exceptional* in $X$, in the sense of Definition 3.4. Then try to construct the divisors $F_{ij}$ verifying the conditions of Corollary 4.3. The hypothesis on the finiteness of the cones of curves ensures, as explained before, that this search amounts to the resolution of a *finite* system of inequalities.

In this section we apply this strategy to construct an infinite number of families of 3-dimensional examples with bijective Nash map. All of them are defined by contracting (using Grauert's criterion 3.5) a divisor with two components inside a smooth algebraic threefold obtained by gluing algebraically along Zariski-open sets the total spaces of suitable line bundles over geometrically ruled surfaces. Both surfaces are obtained by compactifying total spaces of suitable line bundles over the same irreducible smooth projective curve. After the gluing, the two surfaces meet transversely along a curve which is isomorphic to the starting curve. We emphasize the fact that this starting curve is *any* irreducible smooth projective curve.

In the sequel we say that an algebraic surface $S$ is *geometrically ruled* over a curve $C$ if $S$ is the total space of an algebraicaly locally trivial bundle over $C$, with fibers projective lines. We say that $S$ is *birationally ruled* if it is birationally equivalent to a geometrically ruled surface.

Let $C$ be a smooth irreducible projective curve. Consider two algebraic line bundles $L_1$, $L_2$ over $C$ such that:

$$\deg_C L_i = -d_i, \quad \forall i \in \{1,2\}. \tag{1}$$

We suppose moreover that:

$$d_i > 0, \quad \forall i \in \{1,2\}. \tag{2}$$

Denote by $A_i$ the total space of the line bundle $L_i$ and by $C_i$ the image of the zero section of $L_i$ in $A_i$. The relations (1) and (2) imply:

$$C_i \cdot_{A_i} C_i = -d_i < 0, \quad \forall i \in \{1,2\} \tag{3}$$

(the notation $\cdot_Y$ means that one considers intersection numbers inside the smooth space $Y$).

One can compactify $A_i$ by adding a curve $\tilde{C}_i$ at infinity, getting like this a smooth projective surface $E_i$, which is geometrically ruled over $C$. Denote by $\pi_i$ the morphism:

$$\pi_i : E_i \to C$$

which extends the fibration morphism from $A_i$ to $C$.

By (3), one gets:

$$\tilde{C}_i \cdot_{E_i} \tilde{C}_i = d_i.$$

For $i \in \{1,2\}$, we consider the following line bundle on the geometrically ruled surface $E_i$:

$$H_i := \mathcal{O}_{E_i}(-x_i\tilde{C}_i) \otimes_{\mathcal{O}_{E_i}} \pi_i^*(L_j). \tag{4}$$

where $\{i,j\} = \{1,2\}$.

It is important to notice that, as an ingredient of the construction, we pull back one line bundle over $C$ to the total space of a compactification of the second line bundle. The important thing is that the total spaces of the restricted line bundles $\pi_1^*(L_2)|_{A_1}$ and $\pi_2^*(L_1)|_{A_2}$ are canonically isomorphic (see below the explanation of relation (8)), which allows to glue them. But $\pi_i^*(L_j)$ has not an ample inverse on $E_i$, as its degree on a fiber of the ruling is 0. This obliges us to twist the line bundle. We want to do this without changing the crucial property of the isomorphism of the total spaces of the rectrictions to $A_i$. That is why we twist with a line bundle having a meromorphic section whose divisor is supported by the curve at infinity $\tilde{C}_i$.

We pass now to the needed computations. In the definition (4), the integer $x_i$ is chosen such that the following condition is satisfied:

$$\check{H}_i \text{ is ample on } E_i, \quad \forall i \in \{1,2\}. \tag{5}$$

As $C_i \cdot_{E_i} C_i < 0$ (see relation (3)), one has (see Debarre [4, 1.35]):

$$\overline{NE}(E_i) = NE(E_i) = \mathbf{R}_+[C_i] \oplus \mathbf{R}_+[F_i]$$

where $[F_i]$ is the class of the fibers of the ruling $\pi_i$. By Kleiman's criterion of ampleness 3.2, condition (5) is equivalent to the system:

$$\begin{cases} \deg_{C_i} H_i < 0 \\ \deg_{F_i} H_i < 0 \end{cases}.$$

But:

$$\deg_{C_i} H_i = -x_i \tilde{C}_i \cdot_{E_i} C_i + \deg_{C_i} \pi_i^*(L_j) = 0 + \deg_C L_j = -d_j.$$

We have used the fact that the curves $C_i$ and $\tilde{C}_i$ are disjoint, the projection formula and relation (1). In the same manner, using the projection formula and the fact that the curves $\tilde{C}_i$ and $F_i$ meet transversely at one point of $E_i$, we get:

$$\deg_{F_i} H_i = -x_i.$$

As $d_j > 0$ by the hypothesis (2), we see that the condition (5) is equivalent to:

$$x_i > 0, \quad \forall i \in \{1,2\}. \tag{6}$$

The line bundle $\mathcal{O}_{E_i}(-x_i\tilde{C}_i)$ is equipped by construction with a meromorphic section $s_i$ whose divisor is exactly $-x_i\tilde{C}_i$. This implies that the restriction of $s_i$ to $A_i = E_i \backslash \tilde{C}_i$ is a regular and never vanishing section of the restricted line bundle $\mathcal{O}_{E_i}(-x_i\tilde{C}_i)|_{A_i}$. We deduce that this last line bundle is trivial. As a consequence:

$$H_i|_{A_i} \overset{(4)}{=} \mathcal{O}_{E_i}(-x_i\tilde{C}_i)|_{A_i} \otimes_{\mathcal{O}_{A_i}} \pi_i^*(L_j) \simeq \pi_i^*(L_j)|_{A_i}. \tag{7}$$

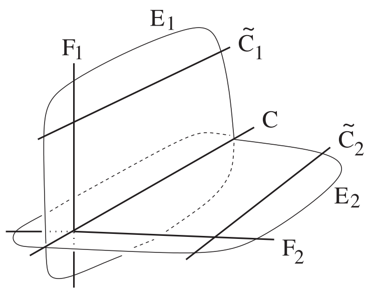


FIGURE 1. Construction of the 3-fold $M$

Denote by $M_i$ the total space of the line bundle $H_i$ over $E_i$ and by $N_i$ the total space of the line bundle $H_i$ over $A_i$. Consequently, $N_i$ is a Zariski open set of $M_i$. By relation (7), $N_i$ is isomorphic to the total space of the line bundle $\pi_i^*(L_j)$ over $A_i$, which in turn is isomorphic to the total space of the split vector bundle $L_1 \oplus L_2$ of rank 2 over $C$. This gives a canonical isomorphism:

$$N_1 \simeq N_2. \tag{8}$$

If we glue the algebraic manifolds $M_1$ and $M_2$ by identifying $N_1$ and $N_2$, we obtain a new 3-dimensional algebraic manifold $M := M_1 \cup M_2$ (with a slight abuse of notations), in which $E_1$ and $E_2$ are canonically embedded. We will consequently use the same notation for their images in $M$. Then:

$$E_1 \cap E_2 = C$$

where $C$ is the curve obtained by the identification under the preceding gluing of the curves $C_1$ and $C_2$ (see Figure 1), identified with the initial curve $C$.

By construction, one has the following identification of the algebraic normal bundles of $E_1$ and $E_2$ inside $M$:

$$N_{E_i|M} \simeq H_i. \tag{9}$$

Combining this with the relations (4) we get:

$$\begin{cases} C \cdot_M E_i = \deg_C N_{E_i|M} = \deg_{C_i} H_i = -d_j \\ F_i \cdot_M E_i = \deg_{F_i} N_{E_i|M} = \deg_{F_i} H_i = -x_i. \\ F_j \cdot_M E_i = \deg_{F_j} N_{E_i|M} = \deg_{F_j} H_i = 1 \end{cases} \tag{10}$$

In order to apply the criterion 4.3, we want to find under which conditions on the numbers $(d_1, d_2, x_1, x_2) \in \mathbf{Z}^4_{>0}$, there exist pairs $(\alpha_1, \alpha_2) \in \mathbf{Z}^2_{>0}$ such that the line bundle $\mathcal{O}_M(-(\alpha_1 E_1 + \alpha_2 E_2))$ is ample in restriction to $E_1 \cup E_2$ (remember that we have already imposed the restrictions (2) and (6)).

By Kleiman's ampleness criterion 3.2 and Proposition 3.5, this is equivalent to:

$$\begin{cases} C \cdot_M (\alpha_1 E_1 + \alpha_2 E_2) < 0 \\ F_1 \cdot_M (\alpha_1 E_1 + \alpha_2 E_2) < 0 \\ F_2 \cdot_M (\alpha_1 E_1 + \alpha_2 E_2) < 0 \end{cases} \overset{(10)}{\Longleftrightarrow} \begin{cases} \alpha_1 d_2 + \alpha_2 d_1 > 0 \\ \alpha_1 x_1 - \alpha_2 > 0 \\ \alpha_2 x_2 - \alpha_1 > 0 \end{cases}.$$

This in turn is equivalent to:

$$\frac{1}{x_1} < \frac{\alpha_1}{\alpha_2} < x_2. \tag{11}$$

The inequalities (11) have solutions $(\alpha_1, \alpha_2) \in \mathbf{Z}^2_{>0}$ if and only if at least one of the numbers $x_1$, $x_2$ is $\geq 2$. It has solutions in both half-planes $\alpha_1 \geq \alpha_2$ and $\alpha_2 \geq \alpha_1$ if and only if we have simultaneously $x_1 \geq 2$, $x_2 \geq 2$.

Combining this with Theorem 3.5 and Corollary 4.3, we get:

PROPOSITION 5.1. *Suppose that* $\deg_C L_1 < 0$, $\deg_C L_2 < 0$. *Consider* $x_1, x_2 \in \mathbf{Z}_{>0}$. *If* $x_1, x_2 \geq 1$ *and at least one of them is* $\geq 2$, *then* $E_1 \cup E_2$ *is* *exceptional in $M$. Let then $\pi : (M, E_1 \cup E_2) \to (X, 0)$ be the morphism which collapses $E_1 \cup E_2$ in $M$. If moreover both $x_1$ and $x_2$ are $\geq 2$, then $E_1$ and $E_2$ are both essential components over $(X, 0)$ and the Nash map $\mathcal{N}$ is bijective.*

*Remark* 5.2. We do not say that the Nash map is not bijective when one of the numbers $x_1$, $x_2$ is equal to 1. But the method of the present paper does not allow us to decide it in general.

Our construction shows that the analytic germ $(X, 0)$ defined in Proposition 5.1 is uniquely determined by the choice of the curve $C$, the line bundles $L_1$, $L_2$ and the numbers $x_1$, $x_2$. That is why, when we want to recall these ingredients, we denote it by

$$(X_{C, L_1, L_2, x_1, x_2}, 0).$$

In the same way, we denote by

$$M_{C, L_1, L_2, x_1, x_2}$$

the smooth algebraic manifold used to construct it.

## 6. Analytic invariants of our families of examples

In the introduction, we gave the list of the known examples of germs of dimension at least 3, which have a bijective Nash map. It is natural to ask if the examples constructed in the previous section are new, or cover partially the known ones. As the normal quasi-ordinary germs are isomorphic to germs of simplicial toric varieties (a result proved by the second author [24, Theorem 5.1], generalizing like this the hypersurface case treated by González Pérez [6, Prop. 14]), and as in all our examples there are exactly two essential divisors, this amounts to ask if some of them are analytically isomorphic to germs of normal toric varieties. Through the propositions 6.3 and 6.4, we show that this is the case only when the curve $C$ is rational.

The next proposition is a direct generalization of a result proved by Nash [17, page 35]. It compares from the viewpoint of birational algebraic geometry the centers on different resolutions of a given essential divisor over $(X, 0)$.

Proposition 6.1. *Let $(X, 0)$ be a germ of normal analytic variety of dimension $n \geq 2$, with isolated singularity. If $\pi_k : \tilde{X}_k \to X$, $k = 1, 2$ are two resolutions of $X$ and $A_k \subset \tilde{X}_k$ are essential components corresponding to the same essential divisor over $(X, 0)$, then $A_1 \times \mathbf{P}^{n-c_1-1}$ is birationally equivalent to $A_2 \times \mathbf{P}^{n-c_2-1}$, where $c_k := \operatorname{codim}_{\tilde{X}_k} A_k$.*

*Proof.* Denote by $v$ the essential divisor whose center on $\tilde{X}_k$ is $A_k$, for $k \in \{1, 2\}$. Consider the morphism $\beta_k : B_{A_k}(\tilde{X}_k) \to \tilde{X}_k$ obtained by blowing-up $A_k$ in $\tilde{X}_k$, and the exceptional divisor $D_k \subset B_{A_k}(\tilde{X}_k)$ of $\beta_k$. Then, $D_k$ is the center of $v$ in $B_{A_k}(\tilde{X}_k)$. It is birationally equivalent to $A_k \times \mathbf{P}^{n-c_k-1}$. Indeed, there exists a smooth Zariski open set $U_k \subset A_k$ whose normal bundle in $\tilde{X}_k$ is algebraically isomorphic to $U_k \times \mathbf{A}^{n-c_k}$, which shows that $\beta_k^{-1}(U_k) \simeq U_k \times \mathbf{P}^{n-c_k-1}$. But $D_k$ is birationally equivalent to $\beta_k^{-1}(U_k)$.

Now consider the bimeromorphic map $\rho := (\pi \circ \beta_2)^{-1} \circ (\pi \circ \beta_1) : B_{A_1}(\tilde{X}_1) \to B_{A_2}(\tilde{X}_2)$. As the center of the valuation $v$ on $B_{A_k}(\tilde{X}_k)$ is the irreducible hypersurface $D_k$, this shows that the closure of $\rho(D_1)$ in $B_{A_2}(\tilde{X}_2)$ is equal to $D_2$. This means that $\rho$ realizes a birational equivalence between $D_1$ and $D_2$. The conclusion of the proposition follows. □

In order to analyze the germs $(X_{C,L_1,L_2,x_1,x_2}, 0)$ constructed in the previous section, we will use Proposition 6.1 only through its Corollary 6.2. Before stating it, let us introduce some notations.

Suppose that $(X, 0)$ is a normal germ of 3-fold with isolated singularity. Consider a fixed resolution $\pi : \tilde{X} \to X$ of it. If $v$ is an essential divisor over $(X, 0)$, and $A_v$ is its center on $\tilde{X}$, define its *smooth representative* $R(v)$ to be:

- the unique minimal model of $A_v$, if $A_v$ is a curve or a surface which is not birationally ruled;
- the curve $C$, if $A_v$ is birationally equivalent to $C \times \mathbf{P}^1$.

Recall from the introduction that $\mathscr{E}(X, 0)$ denotes the set of essential divisors over $(X, 0)$.

Corollary 6.2. *The collection $(R(v))_{v \in \mathscr{E}(X,0)}$ of abstract smooth curves or minimal surfaces, parametrized by the set of essential divisors of $(X, 0)$, is independent of the choice of resolution.*

*Proof.* This is a direct consequence of the previous proposition and of the fact that a non-birationally ruled surface has a unique minimal model, whether if the smooth projective surfaces $C_1 \times \mathbf{P}^1$ and $C_2 \times \mathbf{P}^1$ are birationally equivalent, then the curves $C_1$ and $C_2$ are isomorphic (see Bădescu [1, Theorem 12.2 (c)]). □

An immediate consequence of the corollary is:

Proposition 6.3. 1) *If $(C, L_1, L_2, x_1, x_2)$ and $(C', L_1', L_2', x_1', x_2')$ are chosen such that $C$, $C'$ are non-isomorphic smooth projective curves, then the germs $(X_{C,L_1,L_2,x_1,x_2}, 0)$ and $(X_{C',L_1',L_2',x_1',x_2'}, 0)$ are not analytically isomorphic.*

2) *If $C$ is not rational, then $(X_{C,L_1,L_2,x_1,x_2}, 0)$ is not analytically isomorphic to a germ of toric variety.*

*Proof.* 1) The set of smooth representatives of the essential divisors of the germ $(X_{C,L_1,L_2,x_1,x_2}, 0)$, each representative being counted with its multiplicity, is equal to $2C$. Point 1) follows then from Corollary 6.2.

2) Any germ $(X, 0)$ of toric variety has toric resolutions. The irreducible components of the exceptional locus of such a resolution are orbit closures, and in particular are rational varieties. This shows that, when $X$ has dimension 3, all

the smooth representatives of the essential divisors are rational curves. Point 2) follows immediately. □

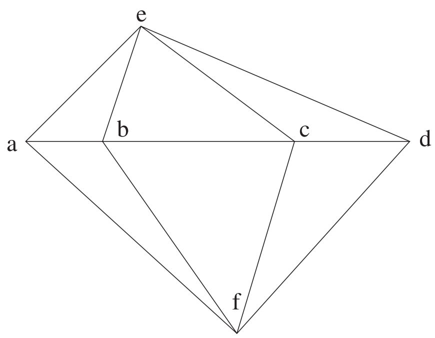


FIGURE 2. Figure illustrating the proof of Proposition 6.4

PROPOSITION 6.4. *If $C$ is rational curve, then $(X_{C,L_1,L_2,x_1,x_2}, 0)$ is analytically isomorphic to the germ of an affine normal toric variety at the unique 0-dimensional orbit.*

*Proof.* If $C$ is rational, it is isomorphic to a toric curve, and $E_1$, $E_2$ are isomorphic to toric surfaces, because the only geometrically ruled surfaces over $\mathbf{P}^1$ are the Hirzebruch surfaces. By recalling the shapes of the fans which define the Hirzebruch surfaces and the way one gets the fan defining an orbit closure from a given fan (see Fulton [5]), one sees that a candidate fan $\Delta$ for a toric 3-fold isomorphic to $M$ and such that $E_1$, $E_2$, $C$ are orbit closures should be as in Figure 2.

In it, we have represented the intersections of the edges $a$, $b$, $c$, $d$, $e$, $f$ of the fan (that is, its 1-dimensional cones) with a transversal plane. The fan $\Delta$ lives inside the 3-dimensional real vector space $N_{\mathbf{R}}$, where $N$ is the associated weight lattice. For each edge $l$, we denote by $v_l$ the unique primitive vector of $N$ situated on $l$. For each cone $\sigma$ of $\Delta$, we denote by $O_\sigma$ the associated orbit and by $V_\sigma := \bar{O}_\sigma$ the orbit closure inside the toric variety $\mathscr{X}(N,\Delta)$. If $\sigma$ is strictly convex with edges $l_1, \dots, l_n$, we write also $\sigma = \langle l_1, \dots, l_n \rangle$.

As we want $\mathscr{X}(N,\Delta)$ to be smooth, $\Delta$ must be a regular fan. Moreover, we would like to get $E_1 = V_b$, $E_2 = V_c$, $C = V_{\langle b,c \rangle}$ verifying the numerical constraints (10). Those equations are equivalent in our toric context with:

$$\begin{cases} V_{\langle b,c\rangle} \cdot V_b = -d_2 \\ V_{\langle b,c\rangle} \cdot V_c = -d_1 \\ V_{\langle b,e\rangle} \cdot V_b = -x_1 \\ V_{\langle c,e\rangle} \cdot V_c = -x_2 \end{cases} \tag{12}$$

where the intersection numbers are taken inside $\mathscr{X}(N,\Delta)$. The equalities $V_{\langle b,e\rangle} \cdot V_c = V_{\langle c,e\rangle} \cdot V_b = 1$ are automatically satisfied, as $\mathscr{X}(N,\Delta)$ is smooth.

We express the vectors $v_c$, $v_d$, $v_f$ in the basis $(v_a, v_b, v_e)$ of $N$. As we want $V_b$, $V_c$ to be Hirzebruch surfaces such that $V_{\langle b,e\rangle} = V_b \cap V_c$ has negative self-intersection in both of them, we require that $v_a$, $v_b$, $v_c$, $v_d$ be coplanar. It is the matter of a simple computation to see that this condition, combined with the requirement that $\Delta$ be regular, shows the existence of $\alpha_1, \dots, \alpha_4 \in \mathbf{Z}$ such that:

$$\tag{13} \begin{cases} v_c = -v_a + \alpha_1 v_b \\ v_d = -\alpha_2 v_a + (\alpha_1\alpha_2 - 1)v_b. \\ v_f = \alpha_3 v_a + \alpha_4 v_b - v_e \end{cases}$$

In order to compute the intersection numbers of the left-hand side of (12) from relations (13), we use the general formula (see Fulton [5, Section 3.3]):

$$\tag{14} \operatorname{div}(\chi^m) = \sum_{l \in \Delta^{(1)}} (m, v_l) V_l, \quad \forall m \in M$$

where $\chi^m$ is the monomial corresponding to $m \in M$ (a character of the associated algebraic torus) seen as a rational function on $\mathscr{X}(N, \Delta)$, and $\Delta^{(1)}$ is the set of edges of $\Delta$. Here $M := \operatorname{Hom}(N, \mathbf{Z})$ denotes the lattice of exponents of monomials.

In our case, if $(v_a^*, v_b^*, v_e^*)$ denotes the basis of $M$ dual of $(v_a, v_b, v_e)$, we get the following formulae by applying (14) to $m \in \{v_a^*, v_b^*\}$:

$$\begin{cases} \operatorname{div}(\chi^{v_a^*}) = V_a - V_c - \alpha_2 V_d + \alpha_3 V_f \\ \operatorname{div}(\chi^{v_b^*}) = V_b + \alpha_1 V_c + (\alpha_1\alpha_2 - 1) V_d + \alpha_4 V_f \end{cases}.$$

This implies:

$$\begin{cases} 0 = V_{\langle b,c\rangle} \cdot \operatorname{div}(\chi^{v_a^*}) = -V_{\langle b,c\rangle} \cdot V_c + \alpha_3 \\ 0 = V_{\langle b,c\rangle} \cdot \operatorname{div}(\chi^{v_b^*}) = V_{\langle b,c\rangle} \cdot V_b + \alpha_1 V_{\langle b,c\rangle} \cdot V_c + \alpha_4 \\ 0 = V_{\langle b,e\rangle} \cdot \operatorname{div}(\chi^{v_b^*}) = V_{\langle b,e\rangle} \cdot V_b + \alpha_1 \\ 0 = V_{\langle c,e\rangle} \cdot \operatorname{div}(\chi^{v_a^*}) = -V_{\langle c,e\rangle} \cdot V_c - \alpha_2 \end{cases}$$

By combining this with the equalities (12), we get:

$$\alpha_1 = x_1, \quad \alpha_2 = x_2, \quad \alpha_3 = -d_1, \quad \alpha_4 = d_2 + d_1 x_1.$$

We have obtained the required decomposition of the vectors $v_c$, $v_d$, $v_f$ in the basis $(v_a, v_b, v_c)$ of $N$:

$$\tag{15} \begin{cases} v_c = -v_a + x_1 v_b \\ v_d = -x_2 v_a + (x_1 x_2 - 1)v_b \\ v_f = -d_1 v_a + (d_2 + d_1 x_1)v_b - v_e \end{cases}.$$

We want to see now if this fan is a subdivision of a strictly convex cone $\gamma$ in $N_{\mathbf{R}}$. This is equivalent to the fact that $a$, $e$, $d$, $f$ are the edges of a strictly convex cone. After some routine computations, one sees that the only non-trivial requirement is that a linear form on $N_{\mathbf{R}}$ which vanishes on $v_d$ and $v_e$

takes non-vanishing values of the same sign on $v_a$ and $v_f$. Such a linear form is $m := x_2 v_b^* + (x_1 x_2 - 1) v_a^* \in M$. Then:

$$\begin{cases} (m, v_a) = x_1 x_2 - 1 \\ (m, v_f) = d_1 + x_2 d_2 \end{cases}.$$

As $d_1 + x_2 d_2 > 0$, we have to require that $x_1 x_2 > 1$. This is precisely the condition we had found at the end of Section 5, ensuring that $E_1 \cup E_2$ is exceptional in $M$ (see Proposition 5.1). We deduce that, if $x_1 x_2 > 1$, then $\gamma_{d_1, d_2, x_1, x_2} := \langle a, e, d, f \rangle$ is a strictly convex cone whose edges are $a$, $e$, $d$, $f$.

Now, as $\mathscr{Z}(N, \Delta)$ is toric, we can easily show that it is isomorphic to the manifold $M_{\mathbf{P}^1, L_1, L_2, x_1, x_2}$, where $\deg_{\mathbf{P}^1} L_1 = -d_1$, $\deg_{\mathbf{P}^1} L_2 = -d_2$. We deduce:

$$(X_{\mathbf{P}^1, L_1, L_2, x_1, x_2}, 0) \simeq (\mathscr{Z}(N, \gamma_{d_1, d_2, x_1, x_2}), 0).$$

The proposition is proved. $\square$

## References


[1] L. Bădescu, Algebraic surfaces, Springer, 2001.

[2] C. Bouvier, Diviseurs essentiels, composantes essentielles des variétés toriques singulières, Duke Math. J. **91** (1998), 609–620.

[3] C. Bouvier and G. Gonzalez-Sprinberg, Système générateur minimal, diviseurs essentiels et G-désingularisations des variétés toriques, Tohoku Math. J. **47** (1995), 125–149.

[4] O. Debarre, Higher-dimensional algebraic geometry, Springer, 2001.

[5] W. Fulton, Introduction to toric varieties, Annals of math. studies **131**, Princeton Univ. Press, 1993.

[6] P. D. González Pérez, Toric embedded resolutions of quasi-ordinary hypersurface singularities, Ann. Inst. Fourier (Grenoble) **53** (2003), 1819–1881.

[7] P. D. González Pérez, Bijectiveness of the Nash map for quasi-ordinary hypersurface singularities, Intern. Math. Res. Notices **2007**, rnm076.

[8] H. Grauert, Über Modifikationen und exzeptionnelle analytische Mengen, Math. Ann. **146** (1962), 331–368.

[9] S. Ishii, The arc space of a toric variety, J. of Algebra **278** (2004), 666–683.

[10] S. Ishii, Arcs, valuations and the Nash map, J. Reine Angew. Math. **588** (2005), 71–92.

[11] S. Ishii, The local Nash problem on arc families of singularities, Ann. Inst. Fourier (Grenoble) **56** (2006), 1207–1224.

[12] S. Ishii and J. Kollár, The Nash problem on arc families of singularities, Duke Math. Journal **120**, (2003), 601–620.

[13] H. B. Laufer, On $\mathbf{CP}^1$ as an exceptional set, Recent developments in several complex variables, Proc. Conf., Princeton Univ., Princeton, N. J., 1979, Ann. of math. stud. **100**, Princeton Univ. Press, 1981, 261–275.

[14] R. Lazarsfeld, Positivity in algebraic geometry, **I**, Springer, 2004.

[15] M. Lejeune-Jalabert, Courbes tracées sur un germe d'hypersurface, Amer. J. Math. **112** (1990), 525–568.

[16] J. Lipman, Rational singularities with applications to algebraic surfaces and unique factorization, Publ. Math. IHES **36** (1969), 195–279.

[17] J. F. Nash, Arc structure of singularities, Duke Math. J. **81** (1995), 31–38.

[18] A. NOBILE, On Nash theory of arc structure of singularities, Annali di Matematica pura ed applicata Ser. 4. **160** (1991), 129–146.

[19] TH. PETERNELL, Pseudoconvexity, the Levi problem and vanishing theorems, Several Complex Variables VII (H. Grauert, Th. Peternell and R. Remmert, eds.), Encyclopaedia of Maths. Sciences **74**, Springer, 1994.

[20] P. PETROV, Nash problem for stable toric varieties, ArXiv: math.AG/0604432.

[21] C. PLÉNAT, A propos du problème des arcs de Nash, Ann. Inst. Fourier (Grenoble) **55** (2005), 805–823.

[22] C. PLÉNAT, Résolution du problème des arcs de Nash pour les points doubles rationnels $D_n$ $(n \geq 4)$, Thèse, Univ. Paul Sabatier, Toulouse, Septembre, 2004, Available at http://fermat.ups-tlse.fr/%7Ewebthesards/theses.htm

[23] C. PLÉNAT AND P. POPESCU-PAMPU, A class of non-rational surface singularities with bijective Nash map, Bulletin de la SMF **134** (2006), 383–394.

[24] P. POPESCU-PAMPU, On the analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity, Duke Math. Journal **124** (2004), 67–104.

[25] A. J. REGUERA, Families of arcs on rational surface singularities, Manuscripta Math. **88** (1995), 321–333.

Camille Plénat
UNIVERSITÉ DE PROVENCE
LATP UMR 6632
CENTRE DE MATHÉMATIQUES ET INFORMATIQUE
39 RUE JOLIOT-CURIE
13453 MARSEILLE CEDEX 13
FRANCE
E-mail: plenat@cmi.univ-mrs.fr

Patrick Popescu-Pampu
UNIV. PARIS 7 DENIS DIDEROT
INST. DE MATHS.-UMR CNRS 7586
ÉQUIPE "GÉOMÉTRIE ET DYNAMIQUE"
CASE 7012
75205 PARIS CEDEX 13
FRANCE
E-mail: ppopescu@math.jussieu.fr