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MEROMORPHIC SOLUTIONS OF FUNCTIONAL EQUATION $P(f)P(g) = 1$

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Abstract

By utilizing Nevanlinna's value distribution theory, we find the meromorphic solutions of the functional equations of the type $P(f)P(g) = 1$, where P is a polynomial with three distinct zeros at least.

1. Introduction

Let **C** denote the complex plane and $f(z)$ a nonconstant function meromorphic on **C**. The value distribution theory was derived and developed by R. Nevanlinna in 1925, with the well-known Jensen formula as the starting point. The theory mainly consists of the so-called first and second fundamental theorems, expressed in terms of the quantities $T(r, f)$, $m(r, f)$, $N(r, f)$ and $\overline{N}(r, f)$; they are called characteristic function, proximate function, counting function and reduced counting function (see, e.g., [4]). We use $S(r, f)$ to denote the quantity $o(T(r, f))$, $(r \to \infty, r \notin E)$, here and in sequel, the letter E is a set of $r \in (0, \infty)$ with finite linear measure not necessarily the same at each occurrence. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function of $f(z)$ provided that $T(r, a) = S(r, f)$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and c a finite complex number. If $f(z) - c$ and $g(z) - c$ have the same zeros counting multiplicity, then we say that $f(z)$ and $g(z)$ share the value c CM. Let a, b be two constants. We recall the definition (see, e.g., [5]) on f and g which share a value a CM*, which means that

$$\overline{N}\left(r, \frac{1}{f - a}\right) - \overline{N}(r, f = a, g = a) = S(r, f),$$

and

$$\overline{N}\left(r, \frac{1}{g - a}\right) - \overline{N}(r, f = a, g = a) = S(r, g),$$

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where $\bar{N}(r, f = a, g = b)$ denote the reduced counting function of the common zeros of $f - a$ and $g - b$. It is obvious that f and g share a value a CM implies that f and g share a CM*.

Nevanlinna's value distribution theory has been used to study the Fermat type of equations of meromorphic functions since 1960s (see [2], [8]). And we refer the reader to [3] for some recent developments of value sharing and more general type equation $P(f) = Q(g)$ of meromorphic functions, where P, Q are two polynomials in $\mathbf{C}[z]$, see [3], [10].

In 1997, C.-C. Yang and X.-H. Hua [9] proved the following theorem.

Theorem A. *Suppose that f, g are two nonconstant meromorphic functions and $n \geq 6$ is an integer. If $f^n f' g^n g' = 1$, then $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

This theorem is also true for $n \geq 2$ (see [5]). It is nature to ask what will happen when f^n and g^n in Theorem A are replaced by general polynomials in f and g, respectively. In another paper [11], we have studied the existence or solvability of meromorphic solutions of the functional equations of the type $P(f)f'P(g)g' = 1$, where P is a polynomial with two distinct zeros at least, and obtain some results. In this paper, by using Nevanlinna's value distribution theory, we further study the existence or solvability of meromorphic solutions of the functional equations of the type $P(f)P(g) = 1$, where P is a polynomial with three distinct zeros at least, and prove the following results.

Theorem 1. *Suppose that $P(z)$ is a complex polynomial having at least three distinct zeros r_1, r_2 and r_3. Let $P(z) = (z - r_1)^{k_1}(z - r_2)^{k_2}(z - r_3)^{k_3} Q(z)$, where k_1, k_2, k_3 are three positive integers, $Q(z)$ is a polynomial of degree m and $Q(r_i) \neq 0$, $i = 1, 2, 3$, the constant term of $Q(z)$ is D. If (f, g) is a pair of nonconstant meromorphic solutions of the functional equation $P(f)P(g) = 1$, then (f, g) must satisfy one of the following three equations*:

(i) $(f - r_1)(f - r_2)(f - r_3)(g - r_1)(g - r_2)(g - r_3) = d$;

(ii) $(f - r_1)^2(f - r_2)(f - r_3)(g - r_1)^2(g - r_2)(g - r_3) = d$;

(iii) $(f - r_1)^3(f - r_2)^2(f - r_3)(g - r_1)^3(g - r_2)^2(g - r_3) = d$,

where d is a nonzero constant.

Obviously, we can assume that one of those three numbers is zero in Theorem 1 by parallel moving: $r_1 \to 0$, $r_2 \to r_2 - r_1$ (denoted by r_1), $r_3 \to r_3 - r_1$ (denoted by r_2), so we only need to solve the the following three equations,

$$f(f - r_1)(f - r_2)g(g - r_1)(g - r_2) = d, \tag{1.1}$$

$$f^2(f - r_1)(f - r_2)g^2(g - r_1)(g - r_2) = d, \tag{1.2}$$

$$f^3(f - r_1)^2(f - r_2)g^3(g - r_1)^2(g - r_2) = d. \tag{1.3}$$

Now we have the following theorem:

THEOREM 2. *Let r_1, r_2 and d be nonzero constants, and $r_1 \neq r_2$. Then the functional equations* (1.2) *and* (1.3) *have no nonconstant meromorphic solutions*; *the functional equation* (1.1) *has nonconstant meromorphic solutions, if and only if r_1 and r_2 satisfy*

$$r_1^2 - r_1 r_2 + r_2^2 = 0, \tag{1.4}$$

and when r_1, r_2 satisfy (1.4), *the pair of nonconstant meromorphic solution (f, g) of equation* (1.1) *must satisfy*

$$f = r - \frac{rc(\sqrt{3} - \wp'(W))}{2\wp(W)} \tag{1.5}$$

$$g = r - \frac{r^2 f'(f - r)}{\sqrt{3} c W' f (f - r_1)(f - r_2)}, \tag{1.6}$$

where W is an entire function of z, $\wp(z)$ is the Weierstrass elliptic function satisfying $(\wp')^2 = 4\wp^3 - 1$, $r = \dfrac{r_1 + r_2}{3}$ and c is a cube root of unity.

COROLLARY 1. *Suppose that f and g are two nonconstant meromorphic functions. Let m, n be two positive integers satisfying $m + n \geq 14$, and a, b, c three distinct constants. Let $H(z) = (z - a)(z - b)^m (z - c)^n$. If $H(f)$ and $H(g)$ share 1 CM, then $H(f) = H(g)$.*

2. Some lemmas

The following lemmas will be used in the proof of our theorems. Lemma 1 is obvious by the lemma of logarithmic derivative, *i.e.*, $m(r, f'/f) = S(r, f)$ (see e.g. [4]). Lemma 3 is well-known.

LEMMA 1. *Let $f(z)$ be a nonconstant meromorphic function, and let $P_l(f)$ be a polynomial in f of degree l, and a_i, $i = 1, 2 \ldots, n$ be distinct complex numbers in* $\mathbf{C}$, *and j be a natural number. Let*

$$g = \frac{P_l(f) f^{(j)}}{(f - a_1) \cdots (f - a_n)}.$$

If $l < n$, then $m(r, g) = S(r, f)$.

LEMMA 2 ([1, 2]). *Any functions $F(z)$, $G(z)$, which are meromorphic in the plane and satisfy*

$$F^3 + G^3 = 1, \tag{2.1}$$

have the form

$$F = f(W(z)), \quad G = cg(W(z)) = cf(-W(z)) = f(-c^2 W(z)), \tag{2.2}$$

where f and g are the following functions:

$$f(z) = \frac{3 + \sqrt{3}\wp'(z)}{6\wp(z)}, \quad g(z) = \frac{3 - \sqrt{3}\wp'(z)}{6\wp(z)}, \tag{2.3}$$

where W is an entire function of z, $\wp(z)$ is the Weierstrass elliptic function satisfying $(\wp')^2 = 4\wp^3 - 1$ and c is a cube-root of unity.

LEMMA 3 ([7]). *Let $f(z)$ be a nonconstant meromorphic function. If*

$$R(f) = \frac{P_1(f)}{Q_1(f)} = \frac{a_p f^p + a_{p-1} f^{p-1} + \cdots + a_0}{b_q f^q + b_{q-1} f^{q-1} + \cdots + b_0},$$

where $P_1(f)$ and $Q_1(f)$ are two relatively prime polynomials of degree p and q, respectively, and the coefficients $a_i(z)$ and $b_j(z)$ are all small functions of $f(z)$ with $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$, $i = 1, 2 \ldots, p$, $j = 1, 2 \ldots, q$, then we have

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f). \tag{2.4}$$

LEMMA 4 ([5] or [6]). *Suppose that f and g are two nonconstant meromorphic functions sharing the value 1 CM. If $f \neq g$ and $fg \neq 1$, then the following inequality holds*:

$$T(r, f) \leq N_2(r, f) + N_2(r, g) + N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + S(r, f) + S(r, g), \tag{2.5}$$

where the notation $N_2(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f)$.

3. Proof of Theorem 1

Suppose that (f, g) is a pair of nonconstant meromorphic solution of the functional equation $P(f)P(g) = 1$, where $P(z)$ is a polynomial having k $(k \geq 3)$ distinct roots $r_1, r_2, \ldots, r_k$. By Nevanlinna's first fundamental theorem and Lemma 3, we have $T(r, f) = T(r, g) + S(r)$, where $S(r) := S(r, f) = S(r, g)$. It is obvious that any r_j point of f is a pole of g. If $k \geq 4$, then by Nevanlinna's second fundamental theorem, we have

$$2T(r, f) \leq \sum_{j=0}^{k} \overline{N}\left(r, \frac{1}{f - r_j}\right) + S(r) \leq \overline{N}(r, g) + S(r) \leq T(r, g) + S(r),$$

which implies $T(r, f) \leq S(r)$, a contradiction. Hence equation $P(z) = 0$ only has three distinct roots r_1, r_2, r_3, and $Q(z)$ is constant. We write $Q(z)$ as D. Suppose that z is a r_i point of f with multiplicity n_i, and also a pole f with multiplicity p. Then $n_i k_i = p(k_1 + k_2 + k_3)$. Therefore, $n_i \geq m_i := (k_1 + k_2 + k_3)/k_i$. This means that the multiplicities of all r_i points of f are at least m_i, $i = 1, 2, 3$. Since

$$\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}=1, \tag{3.1}$$

by Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}T(r,f) &\le \bar{N}\left(r,\frac{1}{f-r_1}\right)+\bar{N}\left(r,\frac{1}{f-r_2}\right)+\bar{N}\left(r,\frac{1}{f-r_3}\right)+S(r)\\ &\le \frac{1}{m_1}N\left(r,\frac{1}{f-r_1}\right)+\frac{1}{m_2}N\left(r,\frac{1}{f-r_2}\right)+\frac{1}{m_3}N\left(r,\frac{1}{f-r_3}\right)+S(r)\\ &\le T(r,f)+S(r),\end{aligned}$$

which implies that

$$N\left(r,\frac{1}{f-r_i}\right)=m_i\bar{N}\left(r,\frac{1}{f-r_i}\right)+S(r)\neq S(r),\quad i=1,2,3.$$

Therefore, "almost all" r_i points of f have multiplicity m_i, and thus "almost all" poles of g are simple. Symmetrically, we see that "almost all" r_i points of g have multiplicity m_i, and "almost all" poles of f are simple. For convenience, we assume $m_1 \le m_2 \le m_3$. Since

$$k_1+k_2+k_3=m_ik_i,\quad i=1,2,3, \tag{3.2}$$

we have $3(k_1+k_2+k_3) \ge m_1(k_1+k_2+k_3)$, that is $m_1 \le 3$. By (3.1), we have $m_1 > 1$. Therefore, $m_1 = 2$ or $m_1 = 3$.

Now we distinguish two cases below.

If $m_1 = 3$, note $m_1 \le m_2 \le m_3$, by (3.1) we get $m_1 = m_2 = m_3 = 3$, obviously there exists a natural number k such that $k_i = k$, $i = 1,2,3$.

If $m_1 = 2$, then we have $k_1 = k_2 + k_3$, by (3.2) and $m_1 \le m_2 \le m_3$, we get $2(k_1+k_2+k_3) \ge m_2(k_2+k_3)$. Therefore, $4(k_2+k_3) \ge m_2(k_2+k_3)$, and thus $m_2 \le 4$. Obviously by (3.1), we have $m_2 \neq 2$. Hence we have $m_2 = 3$ or $m_2 = 4$. If $m_2 = 3$, then we have $m_3 = 6$. Thus there exists a natural number k such that $k_1 = 3k$, $k_2 = 2k$, $k_3 = k$. If $m_2 = 4$, then we have $m_3 = 4$, thus there exists a natural number k such that $k_1 = 2k$, $k_2 = k$, $k_3 = k$. Therefore, the equation $P(f)P(g) = 1$ can be reduced to the following three equations:

(i) $(f-r_1)^k(f-r_2)^k(f-r_3)^k(g-r_1)^k(g-r_2)^k(g-r_3)^k = 1/D$,

(ii) $(f-r_1)^{2k}(f-r_2)^k(f-r_3)^k(g-r_1)^{2k}(g-r_2)^k(g-r_3)^k = 1/D$,

(iii) $(f-r_1)^{3k}(f-r_2)^{2k}(f-r_3)^k(g-r_1)^{3k}(g-r_2)^{2k}(g-r_3)^k = 1/D$.

The conclusion of Theorem 1 follows.

4. Proof of Theorem 2

Suppose that f and g are nonconstant meromorphic functions satisfying one of the equations (1.1), (1.2), and (1.3). Then the 0 points, r_1 points, and r_2 points of f are poles of g. By Nevanlinna's second fundamental theorem, we have $T(r,f) \le T(r,g) + S(r,f)$. Symmetrically, we have $T(r,g) \le T(r,f) + S(r,g)$.

Hence $T(r,f)=T(r,g)+S(r)$, where $S(r):=S(r,f)=S(r,g)$. We shall consider the three functional equations (1.1), (1.2) and (1.3), respectively.

4.1. Solution of equation (1.1)

Suppose that f and g are nonconstant meromorphic functions satisfying equation (1.1). In this case, by the arguments in the proof of Theorem 1, we see that the multiplicities of 0 points, r_1 points and r_2 points of f or g are almost all 3, the poles of f or g are almost all simple. Let

$$\varphi_1=\frac{(f')^3}{f^2(f-r_1)^2(f-r_2)^2},\quad \varphi_2=\frac{(g')^3}{g^2(f-g_1)^2(g-r_2)^2}. \tag{4.1}$$

Then we have $\varphi_i\not\equiv 0$ and $N(r,\varphi_i)=S(r)$, $i=1,2$. From the expression

$$\varphi_1=\frac{f'}{f(f-r_1)}\cdot\frac{f'}{f(f-r_2)}\cdot\frac{f'}{(f-r_1)(f-r_2)},$$

and by Lemma 1, we get $m(r,\varphi_1)=S(r)$. Therefore, $T(r,\varphi_1)=S(r)$. Similarly, we have $T(r,\varphi_2)=S(r)$. By the first equation in (4.1), we get

$$f=\frac{1}{\varphi_1}\frac{f'}{f}\left(\frac{f'}{(f-r_1)(f-r_2)}\right)^2,$$

then we have $m(r,f)=S(r)$, similarly we have $m(r,g)=S(r)$.

Suppose that z_1 is a zero of g of multiplicity of 3. Then it is a simple pole of f, we have the following Laurent expansions in a neighborhood of z_1,

$$f(z)=\frac{A_1}{z-z_1}+O(1),\quad g(z)=A_2(z-z_1)^3+O((z-z_1)^4),$$

where A_1 and A_2 are nonzero constant. Then we get

$$f'(z)=\frac{-A_1}{(z-z_1)^2}+O(1),\quad g'(z)=3A_2(z-z_1)^2+O((z-z_1)^3).$$

Substitute the above two equations into (1.1) and (4.1), respectively, we get $d=A_1^3A_2r_1r_2$, $\varphi_1(z_1)=-1/A_1^3$, $\varphi_2(z_1)=27A_2/(r_1^2r_2^2)$. And thus

$$\frac{\varphi_1(z_1)}{\varphi_2(z_1)}=-\frac{r_1^3r_2^3}{27d}.$$

Note that $N(r,1/g)\neq S(r,g)$. We deduce that

$$\frac{\varphi_1}{\varphi_2}=-\frac{r_1^3r_2^3}{27d}. \tag{4.2}$$

Similarly we have

$$\frac{\varphi_2}{\varphi_1} = -\frac{r_1^3 r_2^3}{27d}. \tag{4.3}$$

Therefore, we get $\varphi_1 \equiv \varphi_2$ or $\varphi_1 \equiv -\varphi_2$.

Taking the derivative in equation (1.1) gives

$$\frac{f' L(f)}{f(f-r_1)(f-r_2)} + \frac{g' L(g)}{g(g-r_1)(g-r_2)} = 0, \tag{4.4}$$

where $L(z)$ is a polynomial defined by

$$L(z) = 3\left(z^2 - \frac{2}{3}(r_1+r_2)z + \frac{r_1 r_2}{3}\right). \tag{4.5}$$

Note that the zeros of f' are the poles of g and the zeros of g' are the poles of f, hence by (4.4), we see that $L(f)$ and $L(g)$ share 0 CM.

We divide our argument into two cases below:

Case (a): $r_1^2 - r_1 r_2 + r_2^2 = 0$.

In this case, the equation $L(z) = 0$ has a multiple root $r = (r_1 + r_2)/3$, hence we have $L(z) = 3(z-r)^2$. So, r is a shared value of f and g. By (1.1), we get

$$d = (r(r-r_1)(r-r_2))^2 = \left(\frac{r_1 r_2}{3}\right)^3 = r^6. \tag{4.6}$$

Combine this with (4.2), we obtain $\varphi_2 = -\varphi_1$. By (4.1) and (4.4), we get

$$\frac{f'}{(L(f))^2} = -\frac{g'}{(L(g))^2}.$$

Let

$$A = \frac{(g-r)^2}{(f-r)^2} f'. \tag{4.7}$$

Note that $d = r^6$ and $z(z-r_1)(z-r_2) = (z-r)^3 + r^3$. (1.1) can be rewritten as

$$\frac{1}{(g-r)^3} = -\frac{1}{r^3}\frac{f(f-r_1)(f-r_2)}{(f-r)^3}. \tag{4.8}$$

Hence we get $A^3 = r^6 \varphi_1$. From (4.7) and (4.8), we get

$$g = r - \frac{r^3 f'(f-r)}{Af(f-r_1)(f-r_2)}, \tag{4.9}$$

By the first equation in (4.1), $A^3 = r^6\varphi_1$ and $z(z-r_1)(z-r_2) = (z-r)^3 + r^3$, we get

$$\left(\frac{f' r^2}{A}\right)^3 = ((f-r)^3 + r^3)^2. \tag{4.10}$$

Let $G = f'r^2/A$ and $F = (f - r)^3 + r^3$. Then the above equation yields $G = h^2$ and $F = h^3$, where $h = F/G$. Therefore,

$$\left(\frac{h}{r}\right)^3 + \left(1 - \frac{f}{r}\right)^3 = 1, \tag{4.11}$$

By Lemma 2, we get

$$\frac{h}{r} = \frac{3 + \sqrt{3}\wp'(W)}{6\wp(W)}, \quad \frac{r - f}{r} = c\frac{3 - \sqrt{3}\wp'(W)}{6\wp(W)}, \tag{4.12}$$

where W is an entire function of z, $\wp(z)$ is the Weierstrass elliptic function satisfying $(\wp')^2 = 4\wp^3 - 1$ and c is a cube root of unity.

Taking the derivative of the both sides in the second equation of (4.12) and deducing, we get

$$\frac{f'}{r} = \frac{cW'}{2}\frac{\frac{2}{\sqrt{3}}\wp^3(W) + \wp'(W) + \frac{1}{\sqrt{3}}}{\wp^2(W)}. \tag{4.13}$$

By the first equation of (4.12), combined with $G = f'r^2/A$ and $G = h^2$, we get

$$f' = A\left(\frac{1 + \frac{1}{\sqrt{3}}\wp'(W)}{2\wp(W)}\right)^2. \tag{4.14}$$

The above two equations yield

$$\left(\frac{2}{\sqrt{3}} - \frac{2rcW'}{A}\right)\wp'(W) = \left(\frac{4rcW'}{\sqrt{3}A} - \frac{4}{3}\right)\wp^3(W) + \frac{2rcW'}{\sqrt{3}A} - \frac{2}{3}.$$

Note that $(\wp')^2 = 4\wp^3 - 1$, we know $\dfrac{2}{\sqrt{3}} - \dfrac{2rcW'}{A} \equiv 0$, hence $A = \sqrt{3}rcW'$, combined with (4.9) and the second equation in (4.12), we have

$$f = r - \frac{rc(\sqrt{3} - \wp'(W))}{2\wp(W)}$$

$$g = r - \frac{r^2 f'(f - r)}{\sqrt{3}cW'f(f - r_1)(f - r_2)}.$$

Hence we proved the result of Theorem (2) in this case.

Case (b): $r_1^2 - r_1 r_2 + r_2^2 \neq 0$.

In this case, the equation $L(z) = 0$ has two distinct roots denoted by a_1, a_2. (4.5) can be rewritten as

$$L(z) = 3(z - a_1)(z - a_2).$$

By (4.4), we know that f and g share the set $\{a_1, a_2\}$ CM. (4.4) can be rewritten as

$$\frac{f'(f-a_1)(f-a_2)}{f(f-r_1)(f-r_2)} + \frac{g'(g-a_1)(g-a_2)}{g(g-r_1)(g-r_2)} = 0. \tag{4.15}$$

Note that

$$\begin{aligned} a_1(a_1-r_1)(a_1-r_2) &= a_1(a_1^2 - (r_1+r_2)a_1 + r_1r_2) \\ &= a_1\left(\frac{2}{3}(r_1+r_2)a_1 - \frac{r_1r_2}{3} - (r_1+r_2)a_1 + r_1r_2\right) \\ &= \frac{2}{3}r_1r_2a_1 - \frac{1}{3}(r_1+r_2)a_1^2 \\ &= \frac{2}{3}r_1r_2a_1 - \frac{1}{3}(r_1+r_2)\left(\frac{2}{3}(r_1+r_2)a_1 - \frac{r_1r_2}{3}\right), \end{aligned}$$

thus we obtain

$$a_1(a_1-r_1)(a_1-r_2) = -\frac{2}{9}(r_1^2 - r_1r_2 + r_2^2)a_1 + \frac{r_1r_2(r_1+r_2)}{9}. \tag{4.16}$$

Similarly we can get

$$a_2(a_2-r_1)(a_2-r_2) = -\frac{2}{9}(r_1^2 - r_1r_2 + r_2^2)a_2 + \frac{r_1r_2(r_1+r_2)}{9}. \tag{4.17}$$

When $\overline{N}(r, f = a_1, g = a_1) \neq S(r)$ and $\overline{N}(r, f = a_1, g = a_2) \neq S(r)$ occur at the same time, by (1.1), we have

$$d = (a_1(a_1-r_1)(a_1-r_2))^2 \quad \text{and} \quad d = a_1(a_1-r_1)(a_1-r_2)a_2(a_2-r_1)(a_2-r_2),$$

thus we get

$$a_1(a_1-r_1)(a_1-r_2) = a_2(a_2-r_1)(a_2-r_2).$$

From (4.16) and (4.17), we get $a_1 = a_2$, a contradiction. Similarly $\overline{N}(r, f = a_1, g = a_1) \neq S(r)$ and $\overline{N}(r, f = a_2, g = a_1) \neq S(r)$ cannot occur at the same time. Hence when $\overline{N}(r, f = a_1, g = a_1) \neq S(r)$, we have $\overline{N}(r, f = a_1, g = a_2) = S(r)$ and $\overline{N}(r, f = a_2, g = a_1) = S(r)$, thus f and g share a_1, a_2 CM*. Let

$$\alpha = \frac{f-a_1}{g-a_1}\frac{g-a_2}{f-a_2}.$$

Obviously, we have $T(r, \alpha) = S(r)$ and $\alpha \not\equiv 0$, thus we get

$$g = a_1 + \frac{(a_1-a_2)(f-a_1)}{(\alpha-1)f + a_1 - \alpha a_2} = \frac{(\alpha a_1 - a_2)f + (1-\alpha)a_1a_2}{(\alpha-1)f + a_1 - \alpha a_2}.$$

Hence

$$g - r_i = \frac{(\alpha a_1 - a_2 - (\alpha - 1)r_i)f + (1 - \alpha)a_1 a_2 - r_i(a_1 - \alpha a_2)}{(\alpha - 1)f + a_1 - \alpha a_2}, \quad i = 1, 2.$$

From (1.1) and the above equation, we get

$$\begin{aligned} d = f(f - r_1)(f - r_2) &\frac{(\alpha a_1 - a_2)f + (1 - \alpha)a_1 a_2}{(\alpha - 1)f + a_1 - \alpha a_2} \\ &\cdot \frac{(\alpha a_1 - a_2 - (\alpha - 1)r_1)f + (1 - \alpha)a_1 a_2 - r_1(a_1 - \alpha a_2)}{(\alpha - 1)f + a_1 - \alpha a_2} \\ &\cdot \frac{(\alpha a_1 - a_2 - (\alpha - 1)r_2)f + (1 - \alpha)a_1 a_2 - r_2(a_1 - \alpha a_2)}{(\alpha - 1)f + a_1 - \alpha a_2}. \end{aligned}$$

By Lemma 3 and the above equation, we deduce that d is not a constant, a contradiction.

Similarly when $\overline{N}(r, f = a_2, g = a_2) \neq S(r)$, we can deduce that $f - a_1$ and $g - a_2$ share 0 CM*, and that $f - a_2$ and $g - a_1$ share 0 CM*. Let

$$\beta = \frac{f - a_1}{g - a_2} \frac{g - a_1}{f - a_2}.$$

Obviously, we have $T(r, \beta) = S(r)$ and $\beta \not\equiv 0$. By a similar argument as the above, we can also deduce a contradiction. Hence equation (1.1) has no nonconstant meromorphic solutions in Case (b), which completes the proof about solutions of equation (1.1).

4.2. Solution of equation (1.2)

Suppose that f and g are nonconstant meromorphic functions satisfying equation (1.2). In this case, the multiplicities of 0 points of f, g are almost all 2, the multiplicities of r_1 points, r_2 points of f, g are almost all 4, their poles are almost all simple. Let

$$(4.18) \qquad \phi_1 = \frac{(f')^4}{f^2(f - r_1)^3(f - r_2)^3}, \quad \phi_2 = \frac{(g')^4}{g^2(g - r_1)^3(g - r_2)^3}.$$

Obviously we have $T(r, \phi_i) = S(r)$ and $\phi_i \not\equiv 0$, $i = 1, 2$. By the first equation in (4.18), we get

$$f = \frac{1}{\phi_1} \frac{f'}{f} \left(\frac{f'}{(f - r_1)(f - r_2)} \right)^3,$$

and by Lemma 1, we have $m(r, f) = S(r)$, similarly we have $m(r, g) = S(r)$.

By considering the Laurent expansion in the neighborhood of a zero with multiplicity 2 of f and g, respectively, we can obtain $\phi_1 = \phi_2$ or $\phi_1 = -\phi_2$, and by (1.2)

$$(4.19) \qquad d = \frac{(r_1 r_2)^4}{16} \frac{\phi_1}{\phi_2}.$$

On the other hand, by considering the Laurent expansions in the neighborhood of a r_1 point with multiplicity 4 of f, we can get

$$\frac{\phi_1}{\phi_2} = \frac{256d}{(r_1(r_1 - r_2))^4}, \tag{4.20}$$

combined with (4.19) and by the symmetry of r_1 and r_2, we get $r_1 = -r_2$. Let $r = r_1 = -r_2$. Hence $d = r^8/16$ or $d = -r^8/16$. If $d = r^8/16$, then by (1.2), we get

$$f^2(f^2 - r^2)g^2(g^2 - r^2) = \frac{r^8}{16}. \tag{4.21}$$

Let $h_1 = fg$. Then by (4.21), we get

$$f^2 + g^2 = \frac{h_1^4 - r^4 h_1^2 - \dfrac{r^8}{16}}{r^2 h_1^2}.$$

Hence

$$(f + g)^2 = \frac{h_1^4 + 2r^2 h_1^3 - r^4 h_1^2 - \dfrac{r^8}{16}}{r^2 h_1^2}, \tag{4.22}$$

$$(f - g)^2 = \frac{h_1^4 - 2r^2 h_1^3 - r^4 h_1^2 - \dfrac{r^8}{16}}{r^2 h_1^2}. \tag{4.23}$$

Note $r \neq 0$, either the equation $z^4 + 2r^2z^3 - r^4z^2 - r^8/16 = 0$ or the equation $z^4 - 2r^2z^3 - r^4z^2 - r^8/16 = 0$ have no multiple roots. All of the roots of the two equations are pairwise distinct, thus by (4.22) and (4.23), we deduce that h_1 has eight multiple value points. By Nevanlinna's second fundamental theorem, we know that a nonconstant meromorphic function has four multiple value points at most, thus h_1 is a constant, hence $f + g$ and $f - g$ are also constants, which implies that f and g are constants, a contradiction. If $d = -r^8/16$, then we can also get a contradiction by using the similar argument as the above. Hence equation (1.2) has no nonconstant meromorphic solutions.

4.3. Solution of equation (1.3)

Suppose that f and g are nonconstant meromorphic functions satisfying equation (1.3). In this case, the multiplicities of 0 points of f, g are almost all 2, the multiplicities of r_1 points of f, g are almost all 3, the multiplicities of r_2 points of f, g are almost all 6, their poles are almost all simple. Let

$$\psi_1 = \frac{(f')^6}{f^3(f - r_1)^4(f - r_2)^5}, \quad \psi_2 = \frac{(g')^6}{g^3(g - r_1)^4(g - r_2)^5}. \tag{4.24}$$

Obviously we have $T(r,\psi_i)=S(r)$ and $\psi_i \not\equiv 0$, $i=1,2$. Simultaneously we have $m(r,f)=S(r)$ and $m(r,g)=S(r)$.

Suppose that z_2 is a zero point of f of multiplicity 2. Then it is the simple pole of g. By considering the Laurent expansions of f and g in a neighborhood of z_2, we can prove

$$d=\left(\frac{r_1r_2}{2}\right)^6\frac{\psi_1}{\psi_2}, \tag{4.25}$$

and $\psi_1=\psi_2$ or $\psi_1=-\psi_2$.

Suppose that z_3 is a r_1-point of f of multiplicity 3. Then it is the simple pole of g. By considering the Laurent expansions of f and g in a neighborhood of z_3, we can obtain

$$d=\left(\frac{r_1(r_1-r_2)}{3}\right)^6\frac{\psi_1}{\psi_2}. \tag{4.26}$$

Combined with (4.25), we get

$$\left(\frac{r_2}{2}\right)^6=\left(\frac{r_1-r_2}{3}\right)^6. \tag{4.27}$$

Suppose that z_4 is a r_2-point of f of multiplicity 6. Then it is the simple pole of g. By considering the Laurent expansions of f and g in a neighborhood of z_4, we can get

$$d=\frac{6^6}{r_2^6(r_2-r_1)^6}\frac{\psi_1}{\psi_2}. \tag{4.28}$$

Combined with (4.25), we get

$$\left(\frac{r_1}{2}\right)^6=\left(\frac{r_2-r_1}{6}\right)^6. \tag{4.29}$$

From (4.27) and (4.29), we can get $r_2^3+8r_1^3=0$ or $r_2^3-8r_1^3=0$. When $r_2^3+8r_1^3=0$, we have $r_2=-2r_1$ or $r_2^2=2r_1r_2-4r_1^2$.

If $r_2=-2r_1$. Taking it into (4.27), we get $r_1^3=0$, hence $r_1=0$, a contradiction.

If $r_2^2=2r_1r_2-4r_1^2$, then combining with (4.27), we still get $r_1=0$, a contradiction. When $r_2^3-8r_1^3=0$, we can still get a contradiction by using a similar argument as the above. Hence equation (1.3) has no nonconstant solutions. Therefore, the proof of Theorem 2 is completed.

5. Proof of Corollary 1

Let

$$F=H(f)=(f-a)(f-b)^m(f-c)^n$$

and

$$G = H(g) = (g-a)(g-b)^m(g-c)^n.$$

Then we have

$$N_2(r,F) = 2\overline{N}(r,f) \le 2T(r,f), \quad N_2(r,G) = 2\overline{N}(r,g) \le 2T(r,g), \tag{5.1}$$

$$N_2\left(r,\frac{1}{F}\right) \le 5T(r,f) + O(1), \quad N_2\left(r,\frac{1}{G}\right) \le 5T(r,g) + O(1). \tag{5.2}$$

If $F \ne G$ and $FG \ne 1$, then by Lemma 4 and inequalities (5.1), (5.2), we have

$$T(r,F) \le 7T(r,f) + 7T(r,g) + S(r), \tag{5.3}$$

where $S(r) = S(r,F) + S(r,G) = S(r,f) + S(r,g)$. Since

$$T(r,F) = (m+n+1)T(r,f) + O(1)$$

and by (5.3), we get

$$(m+n-6)T(r,f) \le 7T(r,g) + S(r).$$

Symmetrically, we have

$$(m+n-6)T(r,g) \le 7T(r,f) + S(r).$$

These two inequalities yield

$$(m+n-13)(T(r,f)+T(r,g)) \le S(r),$$

which is impossible for $m+n \ge 14$ and nonconstant meromorphic functions f and g.

When $m+n \ge 14$, we can rule out the case $FG = 1$ by Theorem 2. Therefore, we have $F = G$, i.e., $H(f) = H(g)$, which completes the proof of Corollary 1.

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