DOMAINS OF VARIABILITY OF LAURENT COEFFICIENTS AND THE CONVEX HULL FOR THE FAMILY OF CONCAVE UNIVALENT FUNCTIONS

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Abstract

Let $\mathbf{D}$ denote the open unit disc and let $p \in (0, 1)$. We consider the family $Co(p)$ of functions $f : \mathbf{D} \to \overline{\mathbf{C}}$ that satisfy the following conditions:
  (i) f is meromorphic in $\mathbf{D}$ and has a simple pole at the point p.
  (ii) $f(0) = f'(0) - 1 = 0$.
  (iii) f maps $\mathbf{D}$ conformally onto a set whose complement with respect to $\overline{\mathbf{C}}$ is convex.
We determine the exact domains of variability of some coefficients $a_n(f)$ of the Laurent expansion

$$f(z) = \sum_{n=-1}^{\infty} a_n(f)(z - p)^n, \quad |z - p| < 1 - p,$$

for $f \in Co(p)$ and certain values of p. Knowledge on these Laurent coefficients is used to disprove a conjecture of the third author on the closed convex hull of $Co(p)$ for certain values of p.

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In [7] the third author of the present article proved the following theorem.

THEOREM A. *Let $p \in (0, 1)$, $f \in Co(p)$, and let*

$$f(z) = \sum_{n=-1}^{\infty} a_n(f)(z - p)^n, \quad |z - p| < 1 - p,$$

be the Laurent expansion of f at the point p. Then the domain of variability of the residuum $a_{-1}(f)$ is determined by the inequality

$$(1) \qquad \left| a_{-1}(f) + \frac{p^2}{1 - p^4} \right| \le \frac{p^4}{1 - p^4}.$$

Equality is attained in (1) *if and only if*

$$(2) \qquad f(z) = \frac{z - \dfrac{p}{1 + p^2}(1 + e^{i\theta})z^2}{\left(1 - \dfrac{z}{p}\right)(1 - zp)}, \quad z \in \mathbf{D},$$

for some $\theta \in [0, 2\pi]$.

Theorem A follows without difficulty from the following representation theorem proved by Avkhadiev and Wirths in [2].

THEOREM B. *Let $p \in (0, 1)$. For any $f \in Co(p)$ there exists a function $\omega : \mathbf{D} \to \overline{\mathbf{D}}$ holomorphic in $\mathbf{D}$ such that*

$$(3) \qquad f(z) = \frac{z - \dfrac{p}{1 + p^2}(1 + \omega(z))z^2}{\left(1 - \dfrac{z}{p}\right)(1 - zp)}, \quad z \in \mathbf{D}.$$

On one hand, the present article originated in discussions among the authors whether it is possible to derive the domains of variability of Laurent coefficients $a_n(f)$, $n \ge 0$, for $f \in Co(p)$ from Theorem B. On the other hand, the third author hoped that it would be possible to prove that the family of functions defined by (3) represents the closed convex hull of $Co(p)$ in the topology of uniform convergence on compact subsets of $\mathbf{D}$.

In the sequel, we will determine the above domains of variability for $n = 0$ and $n = 1$ for certain values of p. For the remaining values of p we will use these considerations and some results of Livingston in [4] to show that the above mentioned hope was in vain.

Our first result is an application of Theorem A and Theorem 4 in [4].

THEOREM 1. *Let $p \in (0, 1)$ and $f \in Co(p)$. Then*

$$(4) \qquad \operatorname{Re} a_0(f) \ge -\frac{p}{(1 - p^2)^2}.$$

Equality is attained in (4) *if and only if*

$$(5) \qquad f(z) = \frac{z}{\left(1 - \dfrac{z}{p}\right)(1 - zp)}, \quad z \in \mathbf{D}.$$

Proof. In [4, Theorem 4], Livingston proved that for $f \in Co(p)$ the inequality

$$\left| p + \frac{a_0(f)(1 - p^2)}{a_{-1}(f)} \right| \leq \frac{1 + p^2}{p}$$

is valid. Hence, for any $f \in Co(p)$ there exists a number $\tau \in \overline{\mathbf{D}}$ such that

$$(6) \qquad a_0(f) = \frac{a_{-1}(f)}{1 - p^2} \left(-p + \tau \frac{1 + p^2}{p} \right).$$

To prove (4) we have to determine the minimal real part of the product at the right side of (6), where $a_{-1}(f)$ varies in the disc described by (1). To that end it is sufficient to consider the points $\tau = e^{i\varphi}$, $\varphi \in [0, 2\pi]$, and to compute the minimum of the quantity

$$\frac{-p}{(1 - p^4)(1 - p^2)} ((1 + p^2) \cos \varphi - p^2)$$

$$- \frac{p^3}{(1 - p^4)(1 - p^2)} ((1 + p^2)^2 \sin^2 \varphi + ((1 + p^2) \cos \varphi - p^2)^2)^{1/2},$$

where $\varphi \in [0, 2\pi]$. Letting $x = \cos \varphi \in [-1, 1]$ in this expression and differentiating with respect to x reveals there is no local extremum in the interval $(-1, 1)$. Therefore, it is easy to see that the minimum is attained for $\tau = 1$ and $a_{-1}(f) = -p^2/(1 - p^2)$. According to Theorem A, this residuum occurs only for the function (5) and for this function equality is attained in (4). This concludes the proof of Theorem 1. $\qquad \square$

For poles near the origin much more can be proved.

THEOREM 2. *Let $p \in (0, \sqrt{3} - 1]$ and $f \in Co(p)$. Then the domain of variability of $a_0(f)$ is determined by the inequality*

$$(7) \qquad \left| \frac{1 - p^2}{p} a_0(f) + \frac{1 - p^2 + p^4}{1 - p^4} \right| \leq \frac{p^2(2 - p^2)}{1 - p^4}.$$

Equality is attained in (7) if and only if f is one of the functions given in (2).

Proof. We multiply (3) by the denominator of the right side and expand both side in power series with expansion point at p. In the resulting equation, letting

$$\omega(z) = \sum_{n=0}^{\infty} c_n (z - p)^n, \quad z \in \mathbf{D},$$

and comparing the constant terms and the coefficients of $(z - p)$, we get

$$(8) \qquad a_{-1}(f) = \frac{-p^2}{1 - p^4} + \frac{p^4}{1 - p^4} c_0$$

and

$$(9) \qquad a_{-1}(f) - \frac{1 - p^2}{p} a_0(f) = \frac{1 - p^2}{1 + p^2} - \frac{p^2}{1 + p^2} (2c_0 + pc_1).$$

It may be mentioned at this place that (8) and the inequality $|c_0| \leq 1$ immediately prove Theorem A.

Further, we derive from (8) and (9) together the representation

$$(10) \qquad \frac{1 - p^2}{p} a_0(f) + \frac{1 - p^2 + p^4}{1 - p^4} = \frac{2p^2 - p^4}{1 - p^4} c_0 + \frac{p^3}{1 + p^2} c_1.$$

Using the inequalities

$$|c_0| \leq 1 \quad \text{and} \quad |c_1| \leq \frac{1 - |c_0|^2}{1 - p^2},$$

we get from (10) the inequality

$$\left| \frac{1 - p^2}{p} a_0(f) + \frac{1 - p^2 + p^4}{1 - p^4} \right| \leq \frac{p^2}{1 - p^4} ((2 - p^2)|c_0| + p(1 - |c_0|^2)).$$

The function

$$g(x) = (2 - p^2)x + p(1 - x^2)$$

has its local maximum at $x_M(p) = (2 - p^2)/2p$. Since $x_M(p) \geq 1$ for $p \in (0, \sqrt{3} - 1]$, we get that

$$\max\{g(x) \mid x \in [0, 1]\} = g(1) = 2 - p^2$$

for those p. This proves the inequality (7) for $f \in Co(p)$. Obviously, $|c_0| = 1$ implies that the only functions $f \in Co(p)$, for which equality can occur there, are the functions (2).

The points in the disc described by (7) are attained for the functions (3) with $\omega(z) \equiv c_0$, $|c_0| \leq 1$. The fact that they belong to the class $Co(p)$ has been proved in [1] and [7]. The proof of Theorem 2 is finished. □

Now, we turn to the values of p in the interval $(\sqrt{3} - 1, 1)$ and for them we get

THEOREM 3. *Let* $p \in (\sqrt{3} - 1, 1)$. *Then the closed convex hull of the class* $Co(p)$ *is a proper subset of the class of functions defined by* (3).

Proof. It is a direct consequence of Theorem 1 that the coefficients $a_0(f)$ of the functions in the closed convex hull of $Co(p)$ satisfy the inequality (4), likewise.

On the other hand, let us insert into (3) the functions

$$
(11) \qquad \omega_x(z) = \frac{-\left(\dfrac{z-p}{1-pz}\right) - x}{1 + x\left(\dfrac{z-p}{1-pz}\right)}, \quad z \in \mathbf{D},
$$

$x \in (0,1)$ fixed. A computation of the coefficients $a_0(f)$ for the resulting functions using (10) delivers

$$
a_0(f) = \frac{-p}{(1-p^2)^2}\left(1 + \frac{(1-x)p^2}{1+p^2}(p(1+x) - (2-p^2))\right).
$$

The right side is less than $-p/(1-p^2)^2$ for $x > (2-p^2-p)/p$ and $(2-p^2-p)/p < 1$ for $p \in (\sqrt{3}-1, 1)$. Hence, the functions f got by inserting (11) into (3) do not belong to the closed convex hull of $Co(p)$ for the values of p indicated in Theorem 3 and $x \in ((2-p^2-p)/p, 1)$.   □

In the sequel, we shall prove similar theorems as above concerning the coefficient $a_1(f)$. During this program Theorem 1 may be replaced by the following theorem.

THEOREM C (see [4, Theorem 3]). *Let $p \in (0,1)$ and $f \in Co(p)$. Then the inequality*

$$
|a_1(f)| \le \frac{p^2}{(1-p^2)^3}
$$

is valid.

Concerning the analogue to Theorem 2, much more effort than before is needed because of the appearance of c_0, c_1, and c_2 in the formulas. To get a sharp result nevertheless, we apply the theory of extremum problems for linear functionals on H^p, $1 \le p \le \infty$, due to Macintyre and Rogosinski [5], and Rogosinski and Shapiro [6] (see also Duren's Book [3] on H^p spaces, Ch. 8). This discussion enables us to prove

THEOREM 4. *Let $p \in \left(0, 1 - \dfrac{\sqrt{2}}{2}\right]$ and $f \in Co(p)$. Then the domain of variability of $a_1(f)$ is determined by the inequality*

$$
(12) \qquad \left| a_1(f)\left(\frac{1-p^2}{p}\right)^2 + \frac{p^2}{1-p^4} \right| \le \frac{1}{1-p^4}.
$$

Equality is attained in (12) if and only if f is one of the functions given in (2).

Proof. By the same procedure as in the proof of Theorem 2 we get in addition to (8) and (9) comparing the coefficients of $(z-p)^2$

$$a_0(f) - \frac{1-p^2}{p} a_1(f) = -\frac{p}{1+p^2}(1 + c_0 + 2pc_1 + p^2 c_2).$$

If we insert (10) into this equation, we get the following representation formula

$$(13) \quad a_1(f)\left(\frac{1-p^2}{p}\right)^2 + \frac{p^2}{1-p^4} = \frac{c_0}{1-p^4} + \frac{2p-p^3}{1+p^2}c_1 + \frac{p^2-p^4}{1+p^2}c_2 =: \Phi_p(\omega).$$

Our aim is to prove the inequality

$$(14) \qquad\qquad\qquad |\Phi_p(\omega)| \le \frac{1}{1-p^4},$$

where ω is as above. Obviously, it is sufficient to consider functions ω holomorphic on $\overline{\mathbf{D}}$. For them, we can represent the functional Φ_p in the form

$$(15) \qquad\qquad \Phi_p(\omega) = \frac{1}{2\pi i}\int_{\partial \mathbf{D}} \kappa_p(z)\omega(z)\,dz,$$

where

$$\kappa_p(z) = \frac{1}{(1-p^4)(z-p)} + \frac{2p-p^3}{(1+p^2)(z-p)^2} + \frac{p^2-p^4}{(1+p^2)(z-p)^3}.$$

The functional Φ_p remains unchanged, if we replace in (15) the kernel κ_p by a rational function K_p that has the same singular part at the point p as κ_p and is holomorphic elsewhere in $\overline{\mathbf{D}}$. Let

$$K_p(z) = \frac{1}{1-p^4}\left(\frac{1}{z-p} + \frac{p}{1-pz}\right) + \frac{2p-p^3}{1+p^2}\left(\frac{1}{(z-p)^2} + \frac{1}{(1-pz)^2}\right)$$

$$+ \frac{p^2-p^4}{1+p^2}\left(\frac{1}{(z-p)^3} + \frac{z}{(1-pz)^3}\right).$$

A lengthy but straightforward evaluation of K_p on the unit circle results in the following identity

$$e^{i\theta}K_p(e^{i\theta})(1+p^2)|1-pe^{i\theta}|^6$$

$$= (1 - 2p\cos\theta + p^2)^2$$

$$\quad + (2p - p^3)(-4p + 2(1+p^2)\cos\theta)(1 - 2p\cos\theta + p^2)$$

$$\quad + (p^2 - p^4)(4(\cos\theta)^2 - (2p^3 + 6p)\cos\theta - 2 + 6p^2)$$

$$= 4p^4(-2 + p^2)(\cos\theta)^2 + 4p^3(3 - p^2)\cos\theta$$

$$\quad + 1 - 8p^2 + 5p^4 - 2p^6$$

$$:= Q_p(\cos\theta),$$

where $\theta \in [0, 2\pi]$. The function $Q_p(x)$ has its local maximum at the point

$$x_M(p) = \frac{3 - p^2}{2p(2 - p^2)}.$$

Since $x_M(p) > 1$ for $p \in (0, 1)$, we get

$$Q_p(\cos \theta) \geq Q_p(-1) = 1 - 8p^2 - 12p^3 - 3p^4 + 4p^5 + 2p^6 := S(p), \quad \theta \in [0, 2\pi].$$

From $S'(p) < 0$ for $p \in (0, 1]$ and $S(1 - \sqrt{2}/2) = 0$ we conclude that

$$e^{i\theta} K_p(e^{i\theta}) \geq 0, \quad \theta \in [0, 2\pi] \text{ and } p \in \left(0, 1 - \frac{\sqrt{2}}{2}\right].$$

Hence the desired inequality (14) results from the following chain of relations

$$\left| \frac{1}{2\pi i} \int_{\partial \mathbf{D}} K_p(z)\omega(z) \, dz \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} K_p(e^{i\theta})| \, d\theta \|\omega\|_\infty$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} K_p(e^{i\theta}) \, d\theta \|\omega\|_\infty$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} K_p(e^{i\theta}) \, d\theta$$

$$= \frac{1}{2\pi i} \int_{\partial \mathbf{D}} K_p(z) \, dz$$

$$= \frac{1}{1 - p^4}.$$

This proves the inequality (14) and therefore (12).

For the proof that any point in the disc described by (12) occurs as the Laurent coefficient $a_1(f)$ of a function $f \in Co(p)$ we may use the same functions as in the analogous situation in the proof of Theorem 2.

To prove the second assertion of Theorem 4 we observe that in the above chain equality is attained everywhere if $\omega(z) \equiv 1$. If we apply the theory of extremum problems for linear functionals on H^∞ to the linear functional Φ_p (compare in particular [3, Theorem 8.1]), we see that there is a unique extremal function ω_E such that

$$\max\{|\Phi_p(\omega)| \, | \, \omega \in H^\infty, \|\omega\|_\infty \leq 1\} = \Phi_p(\omega_E).$$

The above considerations show that in our case $\omega_E(z) \equiv 1$. This implies that equality in (14) is attained if and only if $\omega(z) \equiv e^{i\theta}$ for some $\theta \in [0, 2\pi]$. This concludes the proof of Theorem 4. □

For the remaining values of p we can show that an improved version of Theorem 3 is valid.

THEOREM 5. *Let $p \in \left(1 - \dfrac{\sqrt{2}}{2}, 1 \right)$. Then the closed convex hull of the class $Co(p)$ is a proper subset of the class of functions defined by* (3).

Proof. For the proof, we use the same functions as in the proof of Theorem 3. For the coefficients of the Taylor expansion of ω_x, defined by (11), at the point p, we compute

$$c_0 = -x, \quad c_1 = -\frac{1 - x^2}{1 - p^2}, \quad \text{and} \quad c_2 = -\frac{1 - x^2}{(1 - p^2)^2}(p - x).$$

If we insert these identities into (13), we derive the following expression for the Laurent coefficients $a_1(f)$

$$a_1(f) = -\frac{p^2}{(1 - p^2)^3} \left(1 + \frac{1 - x}{1 + p^2}(-1 + (1 + x)(2p - p^2 x)) \right).$$

Let $R_p(x) = -1 + 2p + x(2p - p^2) - p^2 x^2$. Because of

$$R_p(1) = -1 + 4p - 2p^2 > 0 \quad \text{for} \quad p \in \left(1 - \frac{\sqrt{2}}{2}, 1 \right),$$

we see that there exist $x \in (0, 1)$ such that for the corresponding functions f the inequalities

$$a_1(f) < -\frac{p^2}{(1 - p^2)^3}$$

are valid. Hence, according to Theorem C, these functions f do not belong to the closed convex hull of $Co(p)$. □

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