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SEMI-INVARIANT LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN PRODUCT MANIFOLD

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Abstract

In this paper, we introduce a new class of lightlike submanifolds, namely, semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. We investigate totally umbilical, curvature invariant lightlike submanifolds in real space forms $M_1(c_1) \times M_2(c_2)$ and discuss integrabilities of distributions on semi-Riemannian product manifold.

1. Introduction

Let $(\overline{M}, \bar{g})$ be a semi (pseudo) Riemannian manifold and let M be a submanifold of $\overline{M}$. If the restriction $g = \bar{g}|M$ of $\bar{g}$ to M is still non-degenerate, then (M, g) becomes a semi-Riemannian manifold and it can be studied as the submanifold of semi-Riemannian manifolds. A different situation appears when g is degenerate, then (M, g) is said to be a lightlike (degenerate) submanifold of semi-Riemannian manifold $\overline{M}$. Lightlike submanifolds M of a manifold $(\overline{M}, \bar{g})$ were considered by many authors (see [1], [2], [3], [4] and [9]). On the other hand, the geometry of submanifolds of a Riemannian product manifold (semi-Riemannian Product manifold) has been extensively studied by many geometers (see [6], [7] and [10]). It is known that a submanifold of semi-Riemannian product manifold is defined according to behaviours of almost product structure. Recently, in [7] the authors defined semi-invariant submanifolds of a Riemannian product manifold and proved some properties of these submanifolds.

In this paper, we have defined and studied a new class of lightlike submanifolds of a semi-Riemannian product manifold, i.e., proper semi-invariant lightlike submanifolds. We have discussed integrabilities of distributions and researched totally-umbilical proper semi-invariant lightlike and semi curvature-invariant lightlike submanifold in any positively or negatively curved semi-Riemannian product manifolds. Moreover, we give two necessary and sufficient conditions for semi-invariant lightlike submanifolds to be locally lightlike product manifolds.

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2. Preliminaries

In this section, we use the same notations and terminologies as in [4].

Let $\overline{M}$ be a real $(m+n)$-dimensional semi-Riemannian manifold, $m, n > 1$ and $\bar{g}$ be a semi-Riemannian metric tensor on $\overline{M}$. We denote by q the constant index of $\bar{g}$ and suppose that $\overline{M}$ is not a Riemannian manifold.

Let $(\overline{M}, \bar{g})$ be a $(m+n)$-dimensional semi-Riemannian manifold with index $q > 0$ and M be a submanifold of n-codimension of $\overline{M}$. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike (degenerate) submanifold of $\overline{M}$. We denote by g the induced metric of $\bar{g}$ on M. If we suppose that g is degenerate, then for each tangent space T_xM, we know that

$$T_xM^\perp = \{U_x \in T_x\overline{M} : g_x(U_x, V_x) = 0, \forall V_x \in T_xM\},$$

for each $x \in M$, is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus both T_xM and $T_xM^\perp$ are degenerate orthonormal distributions. Set

$$\mathrm{Rad}(T_xM) = T_xM \cap T_xM^\perp$$

which is called Radical subspace. If the mapping

$$\mathrm{Rad}(TM) : x \in M \to \mathrm{Rad}(T_xM),$$

defines a smooth distribution on M of $\mathrm{rank}(\mathrm{Rad}(TM)) = r > 0$, then the submanifold M of $\overline{M}$ is called r-lightlike submanifold and $\mathrm{Rad}(TM)$ is called the radical distribution on M. Furthermore, there are four possible cases with respect to the dimensional and codimensional of M and rank of $\mathrm{Rad}(TM)$. We recall that;

Case 1) M is called r-lightlike submanifold, if $1 \le r < \min\{m, n\}$

Case 2) M is called co-isotropic submanifold, if $1 \le r = n < m$

Case 3) M is called isotropic submanifold, if $1 \le r = m < n$

Case 4) M is called totally lightlike submanifold, if $1 \le r = m = n$.

For the dependence of all the induced geometric objects of M on $\{S(TM), S(TM^\perp)\}$ we refer to [4].

We shall consider to only Case 1. In this case, consider a complementary distribution $S(TM)$ to $\mathrm{Rad}(TM)$ in TM. It is called a screen distribution on M which is nondegerate. Therefore, we can write

$$TM = \mathrm{Rad}(TM) \perp S(TM). \tag{2.1}$$

As $S(TM)$ is nondegenerate vector subbundle of $T\overline{M}|_M$, we put

$$T\overline{M}|_M = S(TM) \perp S(TM)^\perp, \tag{2.2}$$

and

$$TM^\perp = \mathrm{Rad}\; TM \perp S(TM^\perp), \tag{2.3}$$

where $S(TM^{\perp})$ is a complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$ which is called screen transversal vector bundle of M. We denote a r-lightlike submanifold by $(M, g, S(TM^{\perp}))$. Let $\mathrm{tr}(TM)$ and $\ell\ \mathrm{tr}(TM)$ be complementary (but, not orthogonal) vector bundles to TM in $T\overline{M}$. Then we have

$$\mathrm{tr}(TM) = \ell\ \mathrm{tr}(TM) \oplus S(TM^{\perp}) \tag{2.4}$$

and

$$\begin{aligned} T\overline{M} &= TM \oplus \mathrm{tr}(TM) \\ &= (\mathrm{Rad}(TM) \oplus \ell\ \mathrm{tr}(TM)) \oplus S(TM) \oplus S(TM^{\perp}). \end{aligned} \tag{2.5}$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\overline{M}|_M$. If we use the fact that $S(TM)$ and $S(TM)^{\perp}$ are non-degenerate, we have the following orthogonal direct decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}. \tag{2.6}$$

Theorem 2.1 (Duggal-Bejancu [4]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be a r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then there exists a complementary vector bundle $\ell\ \mathrm{tr}(TM)$ called a lightlike transversal bundle of $\mathrm{Rad}(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(\ell\ \mathrm{tr}(TM)|_U)$ consists of smooth sections $\{N_1, \dots, N_r\}$ of $S(TM^{\perp})^{\perp}|_U$ such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad i, j = 0, 1 \dots r,$$

where $\{\xi_1, \dots, \xi_r\}$ is a basis of $\Gamma(\mathrm{Rad}\ TM|_U)$.

We consider the vector bundle

$$\mathrm{tr}(TM) = \ell\ \mathrm{tr}(TM) \perp S(TM^{\perp}). \tag{2.7}$$

Thus we have

$$T\overline{M} = TM \oplus \mathrm{tr}(TM) = S(TM) \perp S(TM^{\perp}) \perp (\mathrm{Rad}(TM) \oplus \ell\ \mathrm{tr}(TM)). \tag{2.8}$$

Now, let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \tag{2.9}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \quad \forall X \in \Gamma(TM) \tag{2.10}$$

for any $V \in \Gamma(\mathrm{tr}(TM))$. Using the projectoins $L : \mathrm{tr}(TM) \to \ell\ \mathrm{tr}(TM)$ and $S : \mathrm{tr}(TM) \to S(TM^{\perp})$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^{\ell}(X, Y) + h^{s}(X, Y) \tag{2.11}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\ell} N + D^{s}(X, N) \tag{2.12}$$

and

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^\ell(X, W), \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(l\ \mathrm{tr}(TM))$ and $W \in \Gamma(S(TM^\perp))$, where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\nabla^\ell_X N, D^\ell(X, W) \in \Gamma(\ell\ \mathrm{tr}(TM))$, $\nabla^s_X W, D^s(X, N) \in \Gamma(S(TM^\perp))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

By using equations (2.9), (2.10) and (2.11) we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^\ell(X, W)) = g(A_W X, Y). \tag{2.14}$$

We denote the projection morphism of TM to the screen distribution $S(TM)$ by P and consider the decomposition

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) \tag{2.15}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*'}_X \xi \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$. Then we have the following equations

$$\bar{g}(h^\ell(X, PY), \xi) = g(A^*_\xi X, PY), \quad \bar{g}(h^*(X, PY), N) = g(A_N X, PY) \tag{2.17}$$

$$g(A^*_\xi PX, PY) = g(PX, A^*_\xi PY), \quad A^*_\xi \xi = 0 \tag{2.18}$$

$$\bar{g}(A_N X, PY) = g(N, \overline{\nabla}_X PY) \tag{2.19}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma(\ell\ \mathrm{tr}(TM))$ ([4]).

In general, the induced connection on lightlike submanifold M is not metric connection. Since $\overline{\nabla}$ is metric connection, ∇g is obtained from equations (2.11), (2.12) and (2.13) as

$$(\nabla_X g)(Y, Z) = \bar{g}(h^\ell(X, Y), Z) + \bar{g}(h^\ell(X, Z), Y) \tag{2.20}$$

for any $X, Y, Z \in \Gamma(TM)$ [4].

Now, we recall that the equation of Gauss for the lightlike immersion of M in $\overline{M}$ is given by

$$\begin{aligned} \overline{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^\ell(X,Z)} Y - A_{h^\ell(Y,Z)} X + (\nabla_X h^\ell)(Y, Z) \\ & - (\nabla_Y h^\ell)(X, Z) + A_{h^s(X,Z)} Y + D^\ell(X, h^s(Y, Z)) \\ & - A_{h^s(Y,Z)} X - D^\ell(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z) \\ & - \nabla_Y h^s(X, Z) + D^s(X, h^\ell(Y, Z)) - D^s(Y, h^\ell(X, Z)) \end{aligned} \tag{2.21}$$

for any $X, Y, Z \in \Gamma(TM)$.

3. Semi-Riemannian product manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2-dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ be the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$. We denote the product manifold by $\overline{M} = (M_1 \times M_2, \bar{g})$, where

$$\bar{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(T\overline{M})$, where $*$ means the differential mapping. Then we have

$$\pi_*^2 = \pi_*, \quad \sigma_*^2 = \sigma_*, \quad \pi_*\sigma_* = \sigma_*\pi_* = 0 \quad \text{and} \quad \pi_* + \sigma_* = I,$$

where I is the identity map of $T(M_1 \times M_2)$. Thus $(\overline{M}, \bar{g})$ is a $(m_1 + m_2)$-dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The Riemannian product manifold $\overline{M} = M_1 \times M_2$ is characterized by M_1 and M_2 which are totally geodesic submanifolds of $\overline{M}$.

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$ and

$$\bar{g}(FX, Y) = \bar{g}(X, FY), \tag{3.1}$$

for any $X, Y \in \Gamma(T\overline{M})$, where F is called almost Riemannian product structure on $M_1 \times M_2$. If we denote the Levi-Civita connection on $\overline{M}$ by $\overline{\nabla}$, then it can be seen in [6] that

$$(\overline{\nabla}_X F)Y = 0,$$

for any $X, Y \in \Gamma(T\overline{M})$, that is, F is parallel with respect to $\overline{\nabla}$.

Now, let M_1 and M_2 be real space forms with constant sectional curvatures c_1 and c_2, respectively. Then the Riemannian curvature tensor $\overline{R}$ of $\overline{M} = M_1(c_1) \times M_2(c_2)$ is given by

$$\begin{aligned} \overline{R}(X, Y)Z = {} & \frac{1}{16}(c_1 + c_2)\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y \\ & + \bar{g}(FY, Z)FX - \bar{g}(FX, Z)FY\} \\ & + \frac{1}{16}(c_1 - c_2)\{\bar{g}(FY, Z)X - \bar{g}(FX, Z)Y \\ & + \bar{g}(Y, Z)FX - \bar{g}(X, Z)FY\}, \end{aligned} \tag{3.2}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$ [9].

4. Semi-invariant lightlike submanifolds of a semi-Riemannian product manifold

In this section, we will give definition of semi-invariant lightlike submanifolds and study integrabilities of distributions on $M_1 \times M_2$.

Definition 4.1. Let $(\overline{M}, \bar{g})$ be a semi-Riemannian product manifold and M be a r-lightlike submanifold of $\overline{M}$. We say that M is a semi-invariant lightlike submanifold of $\overline{M}$, if the following conditions are satisfied:

1) $F(\mathrm{Rad}(TM))$ is a distribution on $S(TM)$.

2) $F(L_1 \perp L_2)$ is a distribution on $S(TM)$, where $L_1 = \ell\ \mathrm{tr}(TM)$ and L_2 is a vector subbundle of $S(TM^\perp)$.

From the Definition 4.1, we have

$$S(TM) = (F(\mathrm{Rad}(TM)) \oplus F(L_1 \perp L_2)) \perp D_o. \tag{4.1}$$

Thus we obtain that the tangent bundle of M is decomposed as follow

$$TM = D \oplus D', \tag{4.2}$$

where

$$D = \mathrm{Rad}(TM) \perp F(\mathrm{Rad}(TM)) \perp D_o, \quad D' = F(L_1 \perp L_2).$$

Hence, from equations (4.1) and (4.2) we can write the following decompositions:

$$\begin{aligned} T\overline{M} &= TM \oplus \mathrm{tr}(TM) \\ &= TM \oplus (\ell\ \mathrm{tr}(TM) \perp S(TM^{\perp})) \\ &= (\mathrm{Rad}(TM) \oplus \ell\ \mathrm{tr}(TM)) \perp (F(\mathrm{Rad}(TM)) \oplus D') \perp D_o \perp S(TM^{\perp}). \end{aligned} \tag{4.3}$$

Lemma 4.1. *Three dimensional semi-invariant lightlike submanifold is* 1-*lightlike.*

Moreover, for D_o we have the following Lemma.

Lemma 4.3. *Let M be a proper semi-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the distribution D_o is a F-invariant distribution.*

Proof. We take $X_o \in \Gamma(D_o)$. Then from equation (4.1), for any $\xi \in \Gamma(\mathrm{Rad}\ TM)$, $N \in \Gamma(\ell\ \mathrm{tr}(TM))$ and $U \in \Gamma(S(TM^{\perp}))$, we have

$$\begin{aligned} \bar{g}(FX_o, \xi) &= \bar{g}(X_o, F\xi) = 0 \\ \bar{g}(FX_o, N) &= \bar{g}(X_o, FN) = 0 \\ \bar{g}(FX_o, FN) &= \bar{g}(X_o, N) = 0 \\ \bar{g}(FX_o, F\xi) &= \bar{g}(X_o, \xi) = 0 \\ \bar{g}(FX_o, U) &= \bar{g}(X_o, FU) = 0 \\ \bar{g}(FX_o, FU) &= \bar{g}(X_o, U) = 0, \end{aligned}$$

which imply that $FX_o \in \Gamma(D_o)$, that is, D_o is an invariant distribution with respect to F. □

From Lemma 4.2 and definition of D, we conclude that D is also F-invariant distribution.

Now, we shall construct an example for semi-invariant lightlike submanifold.

Example 1. Let M_1 and M_2 be $\mathbf{R}^4_2$ and $\mathbf{R}^3_2$, respectively. Then $\overline{M} = M_1 \times M_2$ is a semi-Riemannian product manifold with metric tensor $\bar{g} = \pi^* g_1 +$

$\sigma^* g_2$, where g_1 and g_2 are the standart metric tensors of $\mathbf{R}_2^4$ and $\mathbf{R}_2^3$, π_* and σ_* are the projection maps of $\Gamma(T\overline{M})$ onto $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. Suppose that an immersed submanifold M of $\overline{M}$ is given by equations;

$$x_1 = t_1 + t_4 - \frac{1}{2}t_5, \quad x_2 = t_2 + t_4 + \frac{1}{2}t_5$$

$$x_3 = \frac{1}{\sqrt{2}}(t_1 + t_2 + 2t_4 + t_5), \quad x_4 = \frac{1}{2}\log(1 + (t_1 - t_2)^2)$$

$$x_5 = t_2 - t_4 + \frac{1}{2}t_5, \quad x_6 = t_3, \quad x_7 = t_2 - t_4 - \frac{1}{2}t_5,$$

where t_i, $1 \leq i \leq 5$, are real parameters.

We set

$$U_1 = \sqrt{2}(1 + (t_1 - t_2)^2)\frac{\partial}{\partial x_1} + (1 + (t_1 - t_2)^2)\frac{\partial}{\partial x_3} + \sqrt{2}(t_1 - t_2)\frac{\partial}{\partial x_4}$$

$$U_2 = \sqrt{2}(1 + (t_1 - t_2)^2)\frac{\partial}{\partial x_2} + (1 + (t_1 - t_2)^2)\frac{\partial}{\partial x_3} - \sqrt{2}(t_1 - t_2)\frac{\partial}{\partial x_4}$$
$$+ \sqrt{2}(1 + (t_1 - t_2)^2)\frac{\partial}{\partial x_5} + \sqrt{2}(1 + (t_1 - t_2)^2)\frac{\partial}{\partial x_7}$$

$$U_3 = \frac{\partial}{\partial x_6}$$

$$U_4 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}$$

$$U_5 = -\frac{1}{2}\frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{2}\frac{\partial}{\partial x_5} - \frac{1}{2}\frac{\partial}{\partial x_7}$$

$$H_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}$$

$$H_2 = -2(t_1 - t_2)\frac{\partial}{\partial x_2} - \sqrt{2}(t_1 - t_2)\frac{\partial}{\partial x_3} + (1 + (t_1 - t_2)^2)\frac{\partial}{\partial x_4}.$$

By direct calculations we check that $\mathrm{Rad}(TM)$ is a distribution on M of rank one and spanned by $\xi = H_1$. Hence M is a 5-dimensional and 1-lightlike submanifold of $\overline{M}$. $S(TM)$ and $S(TM^\perp)$ are spanned by vector fields $\{U_1, U_3, U_4, U_5\}$ and $\{H_2\}$, respectively. Then the lightlike transversal vector bundle $\ell\,\mathrm{tr}(TM)$ is spanned by vector field

$$N = -\frac{1}{4}\frac{\partial}{\partial x_1} + \frac{1}{4}\frac{\partial}{\partial x_2} + \frac{1}{2\sqrt{2}}\frac{\partial}{\partial x_3} - \frac{1}{4}\frac{\partial}{\partial x_5} + \frac{1}{4}\frac{\partial}{\partial x_7}.$$

Therefore, $F\xi = U_4$, $FN = \frac{1}{2}U_5$, that is, $F(\mathrm{Rad}(TM))$ and $F(\ell\,\mathrm{tr}(TM))$ are subbundles of $S(TM)$, where $D_o = Sp\{U_1, U_3\}$, $L_2 = \{0\}$ and $F(S(TM^\perp)) = S(TM^\perp)$. Thus, M is a proper semi-invariant lightlike submanifold of $\overline{M}$.

Now, for any $X \in \Gamma(TM)$, FX can be written as follow

$$FX = fX + \omega X, \tag{4.4}$$

where fX and ωX are tangential and normal parts of FX, respectively. Similarly, for $V \in \Gamma(\mathrm{tr}(TM))$ we set

$$FV = BV + CV, \tag{4.5}$$

where BV and CV are also tangential and normal parts of FV, respectively.

Next we will study integrabilities of distributions on $M_1 \times M_2$. Since F is parallel on $\overline{M}$, from the equations (2.9), (2.10), (4.4) and (4.5) we obtain

$$\begin{aligned} &\overline{\nabla}_X FY = F\overline{\nabla}_X Y \\ &\nabla_X FY + h(X, FY) = f\nabla_X Y + \omega\nabla_X Y + Bh(X, Y) + Ch(X, Y), \end{aligned} \tag{4.6}$$

for any $X, Y \in \Gamma(TM)$. Comparing tangential part with normal one of the both sides of (4.6) we have

$$\nabla_X FY = f\nabla_X Y + Bh(X, Y) \tag{4.7}$$

and

$$h(X, FY) = \omega\nabla_X Y + Ch(X, Y). \tag{4.8}$$

Thus we can give following theorems.

THEOREM 4.1. *Let M be a proper semi-invariant r-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the distribution D is integrable if and only if the second fundamental form of M satisfies*

$$h(X, FY) = h(FX, Y),$$

for any $X, Y \in \Gamma(D)$.

Proof. From (4.6) we have

$$\nabla_X fY + h(X, FY) = f\nabla_X Y + \omega\nabla_X Y + Bh(X, Y) + Ch(X, Y), \tag{4.9}$$

for any $X, Y \in \Gamma(D)$. By replacing X by Y in equation (4.9), we obtain

$$\nabla_Y fX + h(Y, FX) = f\nabla_Y X + \omega\nabla_Y X + Bh(Y, X) + Ch(Y, X). \tag{4.10}$$

Taking account of h being symmetric, from equations (4.9) and (4.10) we conclude

$$\omega[X, Y] = h(X, FY) - h(Y, FX),$$

which proves our assertion. □

THEOREM 4.2. *Let M be a proper semi-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the distribution D' is integrable if and only if the shape operator of M satisfies*

$$A_{FX}Y = A_{FY}X,$$

for any $X, Y \in \Gamma(D')$.

Proof. We take $X, Y \in \Gamma(D')$. Noting that F is parallel with respect to $\overline{\nabla}$ and using the equations (2.9), (2.10), (4.4) and (4.5) we obtain

$$\begin{aligned} \overline{\nabla}_X FY &= F\overline{\nabla}_X Y \\ -A_{FY}X + \nabla^{\perp}_X FY &= f\nabla_X Y + \omega\nabla_X Y + Bh(X, Y) + Ch(X, Y). \end{aligned} \tag{4.11}$$

Moreover, replacing X by Y in (4.11) and taking by a direct calculation we obtain

$$-A_{FY}X + A_{FX}Y + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX = f[X, Y] + \omega[X, Y]. \tag{4.12}$$

Considering the tangential part of the equation (4.12), we obtain

$$-A_{FY}X + A_{FX}Y = f[X, Y].$$

From the last equation $[X, Y] \in \Gamma(D')$ if and only if $A_{FY}X = A_{FX}Y$. This completes the proof of the Theorem. □

Now we suppose that $\{N_1, N_2, \dots, N_r\}$ is a basis of $\Gamma(\ell\ \mathrm{tr}(TM))$ with respect to the basis $\{\xi_1, \xi_2, \dots, \xi_r\}$ of $\Gamma(\mathrm{Rad}(TM))$ such that $\{N_1, N_2, \dots, N_p\}$ is a basis of $\Gamma(L_1)$. Also we consider an orthonormal basis $\{W_1, W_2, \dots, W_s\}$ of $\Gamma(L_2)$. Then we state

Corollary 4.1. *Let M be a semi-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then invariant distribution D is integrable if and only if the second fundamental form of M satisfies*

$$\bar{g}(h(X, FY) - h(Y, FX), \xi_i) = 0, \quad i \in \{1, 2, \dots, r\} \tag{4.13}$$

and

$$\bar{g}(h(X, FY) - h(Y, FX), W_a) = 0, \quad a \in \{1, 2, \dots, s\} \tag{4.14}$$

for any $X, Y \in \Gamma(D)$.

Proof. Taking account of h being symmetric, we derive

$$\begin{aligned} \bar{g}([X, Y], F\xi_i) &= \bar{g}(\nabla_X Y - \nabla_Y X, F\xi_i) = \bar{g}(\overline{\nabla}_X Y - \overline{\nabla}_Y X, F\xi_i) \\ &= \bar{g}(\overline{\nabla}_X FY - \overline{\nabla}_Y FX, \xi_i) = \bar{g}(h(X, FY) - h(Y, FX), \xi_i) \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \bar{g}([X, Y], FW_a) &= \bar{g}(\nabla_X Y - \nabla_Y X, FW_a) = \bar{g}(\overline{\nabla}_X Y - \overline{\nabla}_Y X, FW_a) \\ &= \bar{g}(\overline{\nabla}_X FY - \overline{\nabla}_Y FX, W_a) = \bar{g}(h(X, FY) - h(Y, FX), W_a), \end{aligned} \tag{4.16}$$

for any $X, Y \in \Gamma(D)$. Thus from equations (4.15) and (4.16), we derive $[X, Y] \in \Gamma(D)$ if and only if (4.13) and (4.14) are satisfied. □

THEOREM 4.3. *Let M be a semi-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the invariant distribution D defines a totally geodesic foliation on M if and only if $h(X, Y)$ has no component in $\Gamma(L_1 \perp L_2)$, for any $X, Y \in \Gamma(D)$.*

Proof. D defines a totally geodesic foliation on M if and only if $\nabla_X Y \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. By using the equations (4.2) and taking account of $D' = F(L_1 \perp L_2)$, we conclude that $\nabla_X Y \in \Gamma(D)$ if and only if

$$\bar{g}(\nabla_X Y, F\xi_i) = 0, \quad i \in \{1, 2, \ldots, r\}$$

and

$$\bar{g}(\nabla_X Y, FW_a) = 0, \quad a \in \{1, 2, \ldots, s\},$$

for any $X, Y \in \Gamma(D)$. Moreover, we have

$$\begin{aligned}\bar{g}(\nabla_X Y, F\xi_i) &= \bar{g}(\overline{\nabla}_X Y - h(X, Y), F\xi_i) = \bar{g}(\overline{\nabla}_X FY, \xi_i) - \bar{g}(h(X, Y), F\xi_i)\\ &= \bar{g}(h(X, FY), \xi_i).\end{aligned}$$

In the same way, we have

$$\begin{aligned}\bar{g}(\nabla_X Y, FW_a) &= \bar{g}(\overline{\nabla}_X Y - h(X, Y), FW_a) = \bar{g}(\overline{\nabla}_X FY, W_a) - \bar{g}(h(X, Y), FW_a)\\ &= \bar{g}(h(X, FY), W_a),\end{aligned}$$

which proves our assertion. □

THEOREM 4.4. *Let M be a proper semi-invariant r-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is a locally lightlike Riemannian product if and only if $\nabla f = 0$.*

Proof. Let M be a locally lightlike Riemannian product. Then the leaves of distributions D and D' are both totally geodesics in M. By applying Gauss and Weingarten formulas, we infer

$$\nabla_U fX + h(U, fX) = f\nabla_U X + \omega\nabla_U X + Bh(U, X) + Ch(U, X), \tag{4.17}$$

for any $U \in \Gamma(TM)$ and $X \in \Gamma(D)$, since $\overline{\nabla} F = 0$. Comparing the tangential with normal parts with respect to D of both sides of (4.17), we have

$$\nabla_U fX = f\nabla_U X, \quad \text{i.e.,} \quad (\nabla_U f)X = 0$$

and

$$Bh(U, X) = 0.$$

In the same way,

(4.18) $$-A_{FY}U + \nabla^{\perp}_{U}FY = f\nabla_U Y + \omega\nabla_U Y + Bh(U, Y) + Ch(U, Y),$$

for any $U \in \Gamma(TM)$ and $Y \in \Gamma(D')$. Comparing the tangential with transversal parts with respect to TM of both sides of (4.18), we have

$$-A_{FY}U = f\nabla_U Y + Bh(U, Y).$$

For any $X \in \Gamma(D)$ we have

$$\begin{aligned} g(f\nabla_U Y, X) &= -g(A_{FY}U, X) = \bar{g}(\overline{\nabla}_U FY, X) \\ &= g(\nabla_U fY, X), \end{aligned}$$

which implies that $(\nabla_U f)Y = 0$.

Conversely, we assume that $\nabla f = 0$. Then we have

$$\nabla_Z fX = f\nabla_Z X$$

for any $Z \in \Gamma(TM)$ and $X \in \Gamma(D)$. Thus $\nabla_Z fX \in \Gamma(D)$. Similarly, we get

$$\nabla_Z fY = f\nabla_Z Y$$

for any $Z \in \Gamma(TM)$ and $Y \in \Gamma(D')$. Thus $\nabla_Z fY \in \Gamma(D')$, that is, the distributions D and D' are parallel and the leaves of their are totally geodesic in M. This completes the proof of the Theorem. □

Theorem 4.5. *Let M be a proper semi-invariant r-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is a locally lightlike Riemannian product if and only if*

(4.19) $$Bh(Z, X) = 0, \quad \forall Z \in \Gamma(TM) \text{ and } X \in \Gamma(D).$$

Proof. We suppose that M is a locally r-lightlike Riemannian product. Then we have

(4.20) $$\begin{aligned} &\overline{\nabla}_Z FX = F\overline{\nabla}_Z X \\ &\nabla_Z fX + h(Z, FX) = f\nabla_Z X + \omega\nabla_Z X + Bh(Z, X) + Ch(Z, X), \end{aligned}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(TM)$. Taking account of Theorem 4.4 and considering equation (4.20), we conclude $Bh(X, Z) = 0$.

Conversely, suppose (4.19) is satisfied. Then from equation (4.17), we have $(\nabla_Z f)X = 0$, for any $Z \in \Gamma(TM)$ and $X \in \Gamma(D)$. It follows that D is totally geodesic in M. Furthermore, we have

(4.21) $$\begin{aligned} &\overline{\nabla}_Z FW = F\overline{\nabla}_Z W \\ &-A_{FW}Z + \nabla^{\perp}_{Z}FW = f\nabla_Z W + \omega\nabla_Z W + Bh(Y, W) + Ch(Z, W), \end{aligned}$$

for any $W, Z \in \Gamma(D')$. Thus we get

$$\begin{aligned} g(f\nabla_Z W, X) &= -g(A_{FW}Z, X) = \bar{g}(\overline{\nabla}_Z FW, X) \\ &= -\bar{g}(\overline{\nabla}_Z X, FW) = -\bar{g}(F\overline{\nabla}_Z X, W) \\ &= -\bar{g}(f\nabla_Z X + Bh(Z, X), W) = 0, \end{aligned}$$

for all $X \in \Gamma(D)$. Thus we have $f\nabla_Z W = 0$, i.e., $\nabla_Z W \in \Gamma(D')$, which implies that D' is totally geodesic in M. □

As a consequence of Theorem 4.4 and Theorem 4.5 we have the following Theorem.

Theorem 4.6. *Let M be a proper semi-invariant r-lightlike totally umbilical submanifold of a semi-Riemannian prodcut manifold $\overline{M}$. Then M is a locally lightlike Riemannian product if M is totally geodesic lightlike submanifold in $\overline{M}$.*

Definition 4.2. A semi-invariant submanifold M of a semi-Riemannian product manifold is said to be D-totally geodesic (resp. D'-totally geodesic) if its the second fundamental form h satisfies $h(X, Y) = 0$ (resp. $h(Z, W) = 0$), for any $X, Y \in \Gamma(D)(Z, W \in \Gamma(D'))$.

Theorem 4.7. *Let M be a proper semi invariant r-lightlike submanifold of a semi-Riemannian product $(\overline{M}, \bar{g})$. M is D-totally geodesic submanifold if and only if*

1) *$A^*_\xi X$ has no component in $\Gamma(FL_2 \perp D_o)$*

2) *$A_W X$ has no component in $\Gamma(D' \oplus D_o)$,*

for any $X \in \Gamma(D_o)$, $\xi \in \Gamma(\mathrm{Rad}\, TM)$ and $W \in \Gamma(S(TM^\perp))$.

Proof.

$$\begin{aligned} \bar{g}(h(X, FY), \xi) &= \bar{g}(\overline{\nabla}_X FY, \xi) = -\bar{g}(\overline{\nabla}_X \xi, FY) \\ &= \bar{g}(A^*_\xi X, FY) \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \bar{g}(h(X, FY), W) &= \bar{g}(\overline{\nabla}_X FY, W) = -\bar{g}(\overline{\nabla}_X W, FY) \\ &= \bar{g}(A_W X, FY), \end{aligned} \tag{4.23}$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(\mathrm{Rad}\, TM)$ and $W \in \Gamma(S(TM^\perp))$. Thus from the equation (4.22) and (4.23), we conclude that $h(X, FY) = 0$ if and only if the conditions (1) and (2) are satisfied. □

Theorem 4.8. *Let M be a proper semi invariant r-lightlike submanifold of a semi-Riemannian product $(\overline{M}, \bar{g})$. M is D'-totally geodesic submanifold if and only if*

1) *$A^*_\xi Z$ has no component in* $\Gamma(FL_2 \oplus F\,\mathrm{Rad}\,TM)$
2) *$A_W Y$ has no component in* $\Gamma(FL_2 \oplus \mathrm{Rad}\,TM)$,
for any $Y \in \Gamma(D')$, $\xi \in \Gamma(\mathrm{Rad}\,TM)$ *and* $W \in \Gamma(S(TM^\perp))$.

Proof.

$$\bar{g}(h(Z,Y),\xi) = \bar{g}(\overline{\nabla}_Z Y,\xi) = -\bar{g}(\overline{\nabla}_Y \xi, Z) \\ = \bar{g}(A^*_\xi Y, Z), \tag{4.24}$$

and

$$\bar{g}(h(Z,Y),W) = \bar{g}(\overline{\nabla}_Y Z, W) = -\bar{g}(\overline{\nabla}_Y W, Z) \\ = \bar{g}(A_W Y, Z) \tag{4.25}$$

for any $Y, Z \in \Gamma(D')$, $\xi \in \Gamma(\mathrm{Rad}\,TM)$ and $W \in \Gamma(S(TM^\perp))$. Thus from the equations (4.24) and (4.25) we derive $h(Y,Z) = 0$ if and only if the conditions (1) and (2) are satisfied. □

Now, we characterize a totally geodesic submanifolds in terms of killing distributions.

THEOREM 4.9. *Let M be a proper semi invariant r-lightlike submanifold of a semi-Riemannian product $(\overline{M},\bar{g})$. Then M is totally geodesic submanifold if and only if* Rad TM *and* $S(TM^\perp)$ *are killing distributions on* $\overline{M}$.

Proof.

$$\begin{aligned} \bar{g}(h(X,Y),\xi) &= \bar{g}(\overline{\nabla}_X Y,\xi) = X\bar{g}(Y,\xi) - \bar{g}(\overline{\nabla}_X \xi, Y) \\ &= \bar{g}([\xi,X],Y) - \bar{g}(\overline{\nabla}_\xi X, Y) \\ &= \bar{g}([\xi,X],Y) - \xi\bar{g}(X,Y) + \bar{g}(\overline{\nabla}_\xi Y, X) \\ &= -\xi\bar{g}(X,Y) + \bar{g}([\xi,X],Y) + \bar{g}([\xi,Y],X) - \bar{g}(\overline{\nabla}_Y X,\xi) \\ &= -(L_\xi \bar{g})(X,Y) - \bar{g}(h(X,Y),\xi), \end{aligned}$$

that is,

$$2\bar{g}(h(X,Y),\xi) = -(L_\xi \bar{g})(X,Y), \tag{4.26}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\,TM)$. In the same way,

$$\begin{aligned} \bar{g}(h(X,Y),W) &= \bar{g}(\overline{\nabla}_X Y, W) = X\bar{g}(Y,W) - \bar{g}(\overline{\nabla}_X W, Y) \\ &= \bar{g}([W,X],Y) - W\bar{g}(X,Y) + \bar{g}(\overline{\nabla}_W Y, X) \\ &= -W\bar{g}(X,Y) + \bar{g}([W,X],Y) + \bar{g}([W,Y],X) + \bar{g}(\overline{\nabla}_Y W, X) \\ &= -(L_W \bar{g})(X,Y) - \bar{g}(\overline{\nabla}_Y X, W), \end{aligned}$$

that is,

$$2\bar{g}(h(X, Y), W) = -(L_W\bar{g})(X, Y), \tag{4.27}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. Thus from the equations (4.26) and (4.27), we conclude that $h(X, Y) = 0$ if and only if $(L_{\xi}\bar{g})(X, Y) = (L_W\bar{g})(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(\mathrm{Rad}\ TM)$ and $W \in \Gamma(S(TM^{\perp}))$. Thus the proof is complete. □

Definition 4.3. Let M be a proper semi-invariant r-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. M is said to be mixed-geodesic submanifold if the second fundamental form of $\overline{M}$ satisfies $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Theorem 4.10. *Let M be a proper semi-invariant r-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is mixed-geodesic submanifold if and only if the shape operator of M satisfies*

1) $A_V X$ has only component in $\Gamma(D)$

2) $A_U X$ has no component in $\Gamma(FL_2)$,

for any $X \in \Gamma(D)$ and $V \in \Gamma(L_1 \perp L_2)$ and $U \in \Gamma(S(TM^{\perp}))$.

Proof. Let $X \in \Gamma(D)$. Choosing $Y \in \Gamma(D')$, there is a vector field $V \in \Gamma(L_1 \perp L_2)$ such that $Y = FV$. Thus we have

$$\begin{aligned} \bar{g}(h(X, Y), \xi) &= \bar{g}(\overline{\nabla}_X Y, \xi) = \bar{g}(F\overline{\nabla}_X V, \xi) \\ &= -\bar{g}(FA_V X, \xi) \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} \bar{g}(h(X, Y), U) &= \bar{g}(\overline{\nabla}_X Y, U) = -\bar{g}(\overline{\nabla}_X U, Y) = \bar{g}(A_U X, Y) \\ &= \bar{g}(A_U X, FV), \end{aligned} \tag{4.29}$$

for any $U \in \Gamma(S(TM^{\perp}))$. From the equations (4.28) and (4.29), we conclude that

$$h(X, Y) = 0$$

if and only if the conditions (1) and (2) are satisfied. □

A Lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \bar{g})$ is said to be totally umbilical if there exists a smooth transversal vector field $H \in \Gamma(\mathrm{tr}(TM))$ on $\overline{M}$, called the transversal curvature vector field of M, such that

$$h(X, Y) = \bar{g}(X, Y)H \tag{4.30}$$

for any $X, Y \in \Gamma(TM)$. Thus we can give the following Theorem.

THEOREM 4.11. *Let M be a proper semi-invariant r-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then there exist no totally umbilical proper semi-invariant lightlike submanifolds in any real product space forms $\overline{M} = M_1(c_1) \times M_2(c_2)$ with $c_1 + c_2 \neq 0$.*

Proof. We suppose that M is a totally umbilical proper semi-invariant r-lightlike submanifold of $M_1(c_1) \times M_2(c_2)$. Then from equation (2.21) we have

$$\bar{g}(\overline{R}(X, Y)FZ, FW) = \bar{g}((\overline{\nabla}_X h)(Y, FZ), FW) - \bar{g}((\overline{\nabla}_Y h)(X, FZ), FW)$$

for any $X, Y, Z \in \Gamma(TM)$ and $FW \in \Gamma(L_2)$. Moreover, from the equation (4.30), we obtain

$$(\overline{\nabla}_X h)(Y, Z) = \bar{g}(Y, Z)\nabla_X^{\perp} H.$$

Thus we infer

$$\bar{g}(\overline{R}(X, Y)FZ, FW) = \bar{g}(Y, FZ)\bar{g}(\nabla_X^{\perp} H, FW) - \bar{g}(X, FZ)\bar{g}(\nabla_Y^{\perp} H, FW). \tag{4.31}$$

Taking $Z \in \Gamma(D_o)$ and W instead of X and Y in the equation (4.31), respectively, we conclude

$$\overline{K}(Z, W, FZ, FW) = g(W, FZ)\bar{g}(\nabla_Z^{\perp} H, FW) - g(Z, FZ)\bar{g}(\nabla_W^{\perp} H, FW) = 0.$$

Moreover, we can easily see that

$$\overline{K}(Z, W, FZ, FW) = \overline{K}(Z, W, Z, W) = 0.$$

Furthermore, from the equation (3.2) we have

$$\overline{K}(Z, W, FZ, FW) = -\frac{1}{16}(c_1 + c_2),$$

which proves our assertion. □

THEOREM 4.12. *Let M be a proper semi-invariant r-lightlike totally umbilical submanifold of a semi-Riemannian prodcut manifold $\overline{M}$. Then following statements are equivalent.*

1) *The distribution D is parallel in M*

2) *$g(Z, FY)H = g(Z, Y)CH$, for any $Y \in \Gamma(D)$ and $Z \in \Gamma(TM)$, that is, $BH = 0$.*

3) *The transversal vector field H is invariant with respect to F.*

4) *f is parallel in M.*

Proof. Since D is an invariant distribution, we have $FX = fX$ for any $X \in \Gamma(D)$. If M is totally umbilical, from equations (4.9) and (4.30), we can write

$$\nabla_X fY + g(X, FY)H = f\nabla_X Y + \omega\nabla_X Y + g(X, Y)BH + g(X, Y)CH$$

for any $X, Y \in \Gamma(D)$. Considering the tangential with transversal components of both sides of the last equation, we obtain

$$\nabla_X fY = f\nabla_X Y + g(X, Y)BH$$

and

$$\omega\nabla_X Y = g(X, FY)H - g(X, Y)CH.$$

Thus $\nabla_X Y \in \Gamma(D)$ if and only if $g(X, FY)H = g(X, Y)CH$. Thus the proof is complete. □

Corollary 4.2. *Let M be a proper semi-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. The distribution D is always integrable if M is a totally umbilical proper semi-invariant lightlike submanifold.*

Now, by using the equation (3.2), we get

$$\begin{aligned}\bar{R}(X, Y)Z = {} & \frac{1}{16}(c_1 + c_2)\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(FY, Z)fX \\ & + \bar{g}(FY, Z)\omega X - \bar{g}(FX, Z)fY - \bar{g}(FX, Z)\omega Y\} \\ = {} & \frac{1}{16}(c_1 - c_2)\{\bar{g}(FY, Z)X - \bar{g}(FX, Z)Y + \bar{g}(Y, Z)fX \\ & + \bar{g}(Y, Z)\omega X - \bar{g}(X, Z)fY - \bar{g}(X, Z)\omega Y\},\end{aligned} \tag{4.32}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. Now, considering the fact that the curvature tensor field of $\overline{M} = M_1(c_1) \times M_2(c_2)$ is given by (3.2), we have special forms for the structure equations of Gauss and Codazzi for the submanifold M in $\overline{M}$. Thus the gauss equation becomes

$$\begin{aligned}R(X, Y)Z = {} & \frac{1}{16}(c_1 + c_2)\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y \\ & + \bar{g}(FY, Z)fX - \bar{g}(FX, Z)fY\} \\ = {} & \frac{1}{16}(c_1 - c_2)\{\bar{g}(FY, Z)X - \bar{g}(FX, Z)Y \\ & + \bar{g}(Y, Z)fX - \bar{g}(X, Z)fY\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y,\end{aligned} \tag{4.33}$$

for any $X, Y, Z \in \Gamma(TM)$. Finally, the Ricci equation becomes

$$\begin{aligned}(\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z) = {} & \frac{1}{16}(c_1 + c_2)\{\bar{g}(FY, Z)\omega X - \bar{g}(FX, Z)\omega Y\} \\ & + \frac{1}{16}(c_1 - c_2)\{\bar{g}(Y, Z)\omega X - \bar{g}(X, Z)\omega Y\},\end{aligned} \tag{4.34}$$

for any $X, Y, Z \in \Gamma(TM)$.

Definition 4.4. Let M be a r-lightlike submanifold of any semi-Riemannian manifold $\overline{M}$. M is said to be curvature-invariant lightlike submanifold if the covariant derivative of the second fundamental form h of M satisfies

$$(\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$.

Theorem 4.13. *Let M be a proper semi-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then there exist no curvature-invariant proper semi-invariant lightlike submanifolds in any semi-Riemannian product real space form $\overline{M} = \overline{M}_1(c_1) \times \overline{M}_2(c_2)$ with $c_1, c_2 \neq 0$.*

Proof. Let us suppose that M be a semi curvature-invariant lightlike submanifold of a semi-Riemannian product real space form $\overline{M} = M_1(c_1) \times M_2(c_2)$ with $c_1, c_2 \neq 0$. Then from the (4.34) we have

$$(c_1 + c_2)\{\bar{g}(FY,Z)\omega X - \bar{g}(FX,Z)\omega Y\} + (c_1 - c_2)\{\bar{g}(Y,Z)\omega X - \bar{g}(X,Z)\omega Y\} = 0. \tag{4.35}$$

Let $X \in \Gamma(FL_1)$ and $Y \in \Gamma(FL_2)$ in (4.35). Then we have

$$(c_1 + c_2)\bar{g}(FY,Z)\omega X + (c_1 - c_2)\bar{g}(Y,Z)\omega X = 0 \tag{4.36}$$

and

$$(c_1 + c_2)\bar{g}(FX,Z)\omega Y + (c_1 - c_2)\bar{g}(X,Z)\omega Y = 0. \tag{4.37}$$

From the solutions of the equations (4.36) and (4.37), we get

$$(c_1 + c_2)FY + (c_1 - c_2)Y = 0$$

and

$$(c_1 + c_2)FX + (c_1 - c_2)X = 0.$$

This is imposible for $L_1 = \ell\ \mathrm{tr}(TM) \neq 0$ and $L_2 \neq 0$. This is a complete proof of the Theorem.

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