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FIXED POINTS OF REVERSIBLE SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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1. Introduction.

Takahashi [7, p. 384] proved that if K is a compact convex subset of a Banach space, and S is a left amenable semigroup of nonexpansive self-maps of K, then K contains a common fixed point of S. This theorem generalizes a result of DeMarr [2, p. 1139], who obtained the above implication for the case where S is commutative. In this note, we observe that Takahashi's theorem can be further extended, and the proof slightly simplified, by considering a purely algebraic property that every left amenable semigroup must possess, that of left reversibility. The proof employs suitable modifications of the methods of [2] and [7].

2. Fixed point theorem.

A semigroup S is called *left reversible* if for every pair of elements $a, b \in S$, there exists a pair $c, d \in S$ such that $ac=bd$. (This terminology is due to Dubreil, see [1, p. 34]).

THEOREM. *Let K be a nonempty compact convex subset of a Banach space. If S is a left reversible semigroup of nonexpansive self-maps of K, then K contains a common fixed point of S.*

Proof. A Zorn's lemma argument establishes that there exists a minimal S-invariant nonempty compact convex set $X \subseteq K$. A second application of Zorn's lemma yields that there exists a minimal S-invariant nonempty compact set $M \subseteq X$.

Since S is left reversible, a straightforward induction argument shows that if $\{s_1, s_2, \cdots, s_n\}$ is any finite subset of S, there exists a finite subset $\{t_1, t_2, \cdots, t_n\}$ of S such that $s_1t_1 = s_2t_2 = \cdots = s_nt_n$. Hence

$$\bigcap_{i=1}^{n} s_iM \supseteq \bigcap_{i=1}^{n} s_i(t_iM) = s_1t_1M \neq \phi.$$

Thus the family $\{sM; s \in S\}$ has the finite intersection property, so defining $F = \cap \{sM; s \in S\}$, we get that F is nonempty by compactness of M.

Let $x \in F$. For each pair $a, b \in S$, there exists $c, d \in S$ such that $ac=bd$. Since $F \subseteq cM$, then $x=cy$ for some $y \in M$. Hence

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$$ax = a(cy) = b(dy) \in bM,$$

So $aF \subseteq F$ for all $a \in S$. Since F is also a compact nonempty subset of M, then $F = M$ by minimality of M. Thus $M \cap \{sM;\ s \in S\}$, therefore $M = sM$ for all $s \in S$.

Continuing now as in [7], suppose M contains more than one point. Then by [2, Lemma 1], there exists an element u in the closed convex hull of M such that $\rho = \sup\{||u - x||;\ x \in M\} < \delta$ where δ is the diameter of M. For $y \in M$, let $Y_y = \{x \in X;\ ||y - x|| \leqq \rho\}$. Define $Y = \cap\{Y_y;\ y \in M\}$, then Y is a proper nonempty compact convex subset of X. Use of the result that $M = sM$ for each of the contraction maps $s \in S$ yields that Y is S-invariant, which contradicts the minimality of X. Hence M is a singleton, which proves the theorem.

COROLLARY 1 (Takahashi [7]). *Let K be a nonempty compact convex subset of a Banach space B and let S be a left amenable semigroup of nonexpansive mappings of K into K. Then there exists an element z in K such that $sz = z$ for each s in S.*

Proof. Every left amenable semigroup is left reversible (Granirer [4, p. 47] and [5, p. 371]), so the Theorem applies.

COROLLARY 2. *Let K be a nonempty convex compact subset of a Banach space. If S is a group of nonexpansive self-maps of K, then K contains a common fixed point of S.*

Proof. Trivially, all groups are left reversible.

We remark that not every group is left amenable. For examples, see [3].

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