ON COMPLEMENTED MODULAR LATTICES MEET-HOMOMORPHIC TO A MODULAR LATTICE

By Yuzo UTUMI

Let L be a modular lattice. An element e of L is called a μ-element when it satisfies the condition that, $x \wedge y \leqq e$ implies $(e \vee x) \wedge y \leqq e$. For instance, every neutral element is a μ-element.

The purpose of this note is to prove the following

Theorem I. Let L be a modular lattice with the maximum condition and θ be a meet-homomorphism of L onto a complemented modular lattice such that

(1) $a^\theta = o$ implies $(a \vee b)^\theta = b^\theta$ for every b.

Then the inverse image of 0 forms an ideal generated by some μ-element e and a condition for $a^\theta = b^\theta$ is that

(2) $a \wedge x \leqq e$ if and only if $b \wedge x \leqq e$.

Conversely, for any μ-element e the condition (2) defines an equivalence in L. The set of all the corresponding classes in L forms a complemented modular lattice meet-homomorphic to L satisfying (I) where x^θ means the class containing x.

First, let e be any element of L. By $a \geqq_e b$, is meant that $a \wedge x \leqq e$ implies $b \wedge x \leqq e$. This relation is a quasi-order. Then the following properties hold obviously.

(3) If $a \geqq b$ then $a \geqq_e b$

(4) If $a \geqq_e b$ and $c \geqq_e d$ then $a \wedge c \geqq_e b \wedge d$

(5) $a \leqq_e e$ if and only if $a \leqq e$.

(6) $e \leqq_e x$ for every x.

The relation " $a \geqq_e b$ and $b \geqq_e a$ " is an equivalence. The totality of the corresponding classes in L forms a partially ordered set M by the order induced from $\leqq_e$. From (4), any two elements of M have the meet such that $a^\theta \wedge b^\theta = (a \wedge b)^\theta$ where x^θ is the class containing x. (5) and (6) imply that the ideal generated by e forms a class e^θ which is 0-element of M. If e is a μ-element, then $a^\theta = o$ implies $(a \vee b)^\theta = b^\theta$, because of the definition of μ-element.

Lemma I. An element e of L is a μ-element if and only if $b \leqq a \vee e$ implies $b \leqq_e a$.

Proof. Let e be a μ-element. Assume $b \leqq a \vee e$ and $a \wedge x \leqq e$. Then $e \geqq (e \vee a) \wedge x \geqq b \wedge x$. Thus $b \leqq_e a$. Conversely, assume that the condition of the lemma is fulfilled. Let $a \wedge b \leqq e$. Since $a \vee e \leqq_e a$ we get $(a \vee e) \wedge b \leqq e$ which implies that e is a μ-element.

Lemma 2. If e is a μ-element, then $(a \vee b) \wedge c \leqq e$ implies $a \wedge (b \vee c) \leqq_e b$.

Proof. $a \wedge (b \vee c) \leqq (a \wedge (b \vee c)) \vee b = (b \vee c) \wedge (a \vee b) = b \vee ((a \vee b) \wedge c)$. From the Lemma I, we get $a \wedge (b \vee c) \leqq_e b$ if $(a \vee b) \wedge c \leqq e$.

Corollary. If e is a μ-element, then $a \wedge b \leqq e$ and $(a \vee b) \wedge c \leqq e$ imply $a \wedge (b \vee c) \leqq e$.

Let e be a μ-element, and a' be one of the maximal elements of L among x such that $a \wedge x \leqq e$. Now, to prove that $a^\theta \vee b^\theta = (a \vee (b \wedge (a \wedge b)'))^\theta$ we assume $a \leqq_e x$, $b \leqq_e x$ and $x \wedge y \leqq e$. Let $c = (a \wedge b)'$ and $n = (a \vee (b \wedge c)) \wedge y$. Since $a \wedge (x \wedge b \wedge c) \leqq (a \wedge b) \wedge c \leqq e$ and e is a μ-element we obtain $a \wedge ((x \wedge b \wedge c) \vee e) \leqq e$. But, $n \wedge x \leqq y \wedge x \leqq e$. Thus $a \wedge ((x \wedge b \wedge c) \vee n) \wedge x = a \wedge ((x \wedge b \wedge c) \vee (n \wedge x)) \leqq e$. Since $a \leqq_e x$ we get $a \wedge ((x \wedge b \wedge c) \vee n) \leqq e$. Hence $x \wedge b \wedge c \wedge (a \vee n) \leqq e$ from the corollary of the Lemma 2. Since $b \leqq_e x$ we have $b \wedge c \wedge (a \vee n) \leqq e$. But, $a \wedge n \leqq a \wedge y \leqq e$ since $a \leqq_e x$ and $x \wedge y \leqq e$. Thus, again from the corollary of the Lemma 2, it follows that $n = n \wedge (a \vee (b \wedge c)) \leqq e$ which proves $a \vee (b \wedge (a \wedge b)') \leqq_e x$. Next, we assume $(a \vee (b \wedge c)) \wedge z \leqq e$. Then $a \wedge ((b \wedge c) \vee z) \leqq e$ from the corollary of the Lemma 2. We have $a\wedge b\wedge((b\wedge z)\vee c) = a\wedge((b\wedge c)\vee(b\wedge z)) \leq a\wedge((b\wedge c)\vee z) \leq e$. From the maximality of $c=(a\wedge b)'$ we get $(b\wedge z)\vee c = c$ and $b\wedge z \leq c$. Thus $b\wedge z \leq (a\vee(b\wedge c))\wedge z \leq e$ which proves $b \leq_e a\vee(b\wedge c)$. But, from (3) we get $a \leq_e a\vee(b\wedge c)$. Whence $a^\theta\vee b^\theta = (a\vee(b\wedge c))^\theta$ and M forms a lattice.

Now, $a^\theta\vee a'^\theta = (a\vee(a'\wedge(a\wedge a')'))^\theta = (a\vee(a'\wedge x)) = (a\vee a')^\theta$ since $x\leq e$ implies $x' = I$. If $(a\vee a')\wedge u \not\leq e$ then $a\wedge(a'\vee u) \not\leq e$ by the corollary of the Lemma 2. From the maximality of a', we get $u \leq a'$. Thus, $u = u\wedge(a\vee a') \leq e$ which implies $a\vee a' \geq_e I$ or $(a\vee a')^\theta = I^\theta$. Hence $a^\theta\vee a'^\theta = I^\theta$. Evidently $a^\theta\wedge a'^\theta = (a\wedge a')^\theta = 0^\theta$. Whence a'^θ is a complement of a^θ.

To prove the modularity of M, we assume $a\leq_e d$ and $v\wedge(a\vee(b\wedge d\wedge(a\wedge b\wedge d)')) \leq e$ where as $(a\wedge b\wedge d)'$ we take one containing $c=(a\wedge b)'$. Let $c\wedge b = p$ and $v\wedge(a\vee p)\wedge d = w$. Since $a\vee(b\wedge d\wedge(a\wedge b\wedge d)') \geq a\vee(b\wedge d\wedge c) = a\vee(d\wedge p)$ we get $v\wedge(a\vee(d\wedge p))\leq e$. Then, $e \geq w\wedge(a\vee(d\wedge p)) \geq w\wedge((a\wedge d)\vee(d\wedge p)) = w\wedge d\wedge((a\wedge d)\vee p) = w\wedge((a\wedge d)\vee p)$. Since $(a\wedge d)\wedge p \leq a\wedge p = a\wedge b\wedge c \leq e$ we have $a\wedge d\wedge(p\vee w) \leq e$ from the corollary of the Lemma 2. By the assumption $a\leq_e d$ we get $a\wedge(p\vee w) \leq e$. From the Lemma 2 $(a\vee p)\wedge w \leq_e p$. Since $a\vee p \geq w$ we have $w\leq_e p$. Thus $w\leq_e d\wedge p$ and $w\leq_e v\wedge(a\vee(d\wedge p))\leq e$ which implies $(a\vee((a\wedge b)'\wedge b))\wedge d \leq_e a\vee(b\wedge d\wedge(a\wedge b\wedge d)')$. Therefore $(a^\theta\vee b^\theta)\wedge d^\theta \leq a^\theta\vee(b^\theta\wedge d^\theta)$ and M is modular.

Conversely, let M be a complemented modular lattice meet-homomorphic to L by the mapping θ. Assume the condition (I) is true. If e is a maximal element of the inverse image of 0, then e is a μ-element. For, if $a\wedge b \not\leq e$ then $((e\vee a)\wedge b)^\theta = (e\vee a)^\theta\wedge b^\theta = a^\theta\wedge b^\theta = (a\wedge b)^\theta \neq 0$. Next, let $a^\theta = b^\theta$. If $a\wedge x \leq e$ then $(b\wedge x)^\theta = b^\theta\wedge x^\theta = a^\theta\wedge x^\theta = (a\wedge x)^\theta = 0$ or $b\wedge x \leq e$, and conversely. In the case $a^\theta \neq b^\theta$, we may assume $a^\theta \not\geq b^\theta$. Then there exists a relative complement x^θ of $a^\theta\wedge b^\theta$ in the interval $[0, b^\theta]$. If $x^\theta = 0$ then $b^\theta = a^\theta\wedge b^\theta$ which contradicts our assumption. Hence $x^\theta \neq 0$. We have $(a\wedge x)^\theta = a^\theta\wedge x^\theta = a^\theta\wedge b^\theta\wedge x^\theta = 0$ and $(b\wedge x)^\theta = b^\theta\wedge x^\theta = x^\theta \neq 0$, which imply $a \not\equiv_e b$. Thus the proof of the Theorem I is complete.

By a c-element, it is meant an element a' for some a and fixed e. Then,

Theorem 2. In the classification, of L, of the Theorem I, every class contains at least one c-element. And an element of L is a c-element if and only if it is a maximal element in the class containing it.

Proof. We can select $(a')'$ such that $(a')' \geq a$. Then $((a')')^\theta \geq a^\theta$. But, since both $((a')')^\theta$ and a^θ are complements of $(a')^\theta$, the modularity implies $((a')')^\theta = a^\theta$ and a^θ contains $(a')'$. Now, let $(a')^\theta = x^\theta$ and $a' \leq x$. If $a' < x$ then $x\wedge a \not\leq e$. But $a'\wedge a \leq e$, hence $a' \not\equiv_e x$ which contradicts our assumption and we get $a' = x$. Conversely, let b be a maximal element of the class containing it. Then for $(b')'$ such that $(b')' \geq b$ we have $(b')' = b$ and b is a c-element.

Remark. Above we assumed the maximum condition for L, but used it essentially only for the existence of a'. If we have the condition $\bigcup_\alpha(a_\alpha\wedge b) = \bigcup_\alpha a_\alpha\wedge b$, where $\{a_\alpha\}$ is a simply ordered subset of L, then all the results above hold in almost same form, for instance, after defining a μ-ideal instead of our μ-element.

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Osaka Women's College.