A NOTE ON THE CONCENTRATION FUNCTIONS

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(Communicated by T.Kawata)

Let $F(x)$, $G(x)$ be distribution functions, and let $Q_F(\ell)$, $Q_G(\ell)$ be its concentration functions respectively. It is well known that, if $F(x)$ and $G(x)$ are near to each other, then $Q_F(\ell)$ and $Q_G(\ell)$ are also near to each other. (P.Lévy, Théorie de L'addition des Variables Aléatoires, 1937). In this note, I will give to this fact more exact expression. Denote by $\rho(F, G)$ the Lévy's distance between two distribution functions $F(x)$ and $G(x)$. The concentration function $Q(\ell)$ of any distribution function is properly defined only for non-negative ℓ. If let $Q(\ell)=0$ for negative ℓ, $Q(\ell)$ may be considered as a distribution function.

Lemma. In order that $\rho(F, G) \leqq \delta$, it is necessary and sufficient that

$$F\left(x-\frac{\delta}{\sqrt{2}}\right)-\frac{\delta}{\sqrt{2}} \leqq G(x) \leqq F\left(x+\frac{\delta}{\sqrt{2}}\right)+\frac{\delta}{\sqrt{2}},$$

for every $x \ (-\infty<x<\infty)$.

Proof. The condition that $\rho(F, G) \leqq \delta$, means that the curve $y=G(x)$ lies between two curves obtained by translations of the curve $y=F(x)$ in the direction of straight line $x+y=0$ by δ.

Theorem. $\rho(Q_F, Q_G) \leqq 2\rho(F, G)$.

Proof. Write $\rho(F, G)=\delta$, then from the lemma,

$$F\left(x+\frac{\delta}{\sqrt{2}}+0\right) \geqq G(x+0)-\frac{\delta}{\sqrt{2}}, \qquad (-\infty<x<\infty),$$

and

$$F\left(x-\frac{\delta}{\sqrt{2}}-0\right) \leqq G(x-0)+\frac{\delta}{\sqrt{2}}, \qquad (-\infty<x<\infty).$$

Using these inequalities, we have

$$\begin{aligned} & Q_F(\ell+\sqrt{2}\delta) \\ &= \underset{-\infty<x<\infty}{\mathrm{l.u.b}}\left[F\left(x+\ell+\frac{\delta}{\sqrt{2}}+0\right)-F\left(x-\frac{\delta}{\sqrt{2}}-0\right)\right] \\ &\geqq \underset{-\infty<x<\infty}{\mathrm{l.u.b}}\left[G(x+\ell+0)-G(x-0)-\sqrt{2}\delta\right] \\ &= Q_G(\ell)-\sqrt{2}\delta, \qquad \ell \geqq 0. \end{aligned}$$

Hence

$$Q_G(\ell) \leqq Q_F(\ell+\sqrt{2}\delta)+\sqrt{2}\delta, \qquad (-\infty<\ell<+\infty)$$

Since this relation permits the exchange of F and G, we have

$$\begin{aligned} Q_F(\ell-\sqrt{2}\delta)-\sqrt{2}\delta &\leqq Q_G(\ell) \\ &\leqq Q_F(\ell+\sqrt{2}\delta)+\sqrt{2}\delta, \\ & (-\infty<\ell<\infty). \end{aligned}$$

By the lemma, this contains that

$$\rho(Q_F, Q_G) \leqq 2\delta.$$

This completes the proof.

The above result is the best one: there exist distribution functions $F(x)$, $G(x)$ such that

$$\rho(Q_F, Q_G)=2\rho(F, G).$$

For example, write $F_a(x)$ the distribution function of a random variable which is distributed uniformely in the interval $(-a, +a)$, and let $F(x)=F_a(x)$, $G(x)=F_b(x)$, $0<a<b$ then we have the above equation.

(*) Received February 28, 1950.

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