M. SHIROSAKI
KODAI MATH. J.
23 (2000), 224–233

HYPERBOLIC HYPERSURFACES IN THE COMPLEX PROJECTIVE SPACES OF LOW DIMENSIONS

MANABU SHIROSAKI

§1. Introduction

There have been a number of results for hyperbolic hypersurfaces in the complex projective spaces (cf. [AS], [BG], [D], [K], [MN], [N], [S] and [Z]). In particular, J. P. Demailly [D] constructed a remarkable example of hyperbolic hypersurfaces of degree 11 in $\boldsymbol{P}^3(\boldsymbol{C})$. On the other hand, the author [S] gave hyperbolic hypersurfaces of degree 13^n in $\boldsymbol{P}^n(\boldsymbol{C})$ whose complements are complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^n(\boldsymbol{C})$. In this paper, we give hyperbolic hypersurfaces in the complex projective spaces of dimension 2, 3 and 4. For example, we construct hyperbolic hypersurfaces in $\boldsymbol{P}^3(\boldsymbol{C})$ of degree 31 whose complements are complete hyprebolic and hyperbolically imbedded in $\boldsymbol{P}^3(\boldsymbol{C})$, and hyperbolic hypersurface of degree 36 in $\boldsymbol{P}^4(\boldsymbol{C})$.

Acknowledgment. The author would like to thank the referee for many helpful comments.

§2. A holomorphic mapping into a hypersurface in $\boldsymbol{P}^n(\boldsymbol{C})$

Let n, q and d be positive integers such that $q \geq n+1$ and $d \geq (q-1)^2$. Let V be a set of q column vectors in $\boldsymbol{C}^{n+1}$. We make the following assumptions.

(A1) The vectors in V are in general position.

(A2) Take any k with $0 \leq k \leq \min\{n, q-n-2\}$. Then, for any distinct vectors $\boldsymbol{v}_0, \dots, \boldsymbol{v}_n, \boldsymbol{u}_0, \dots, \boldsymbol{u}_k$ in V and any d-th roots of $\omega_0, \dots, \omega_k$ of -1, the $n+1$ vectors $\boldsymbol{v}_j - \omega_j \boldsymbol{u}_j$ $(0 \leq j \leq k)$ and $\boldsymbol{v}_j$ $(k+1 \leq j \leq n)$ are linearly independent.

(A3) Take any k with $1 \leq k \leq \min\{n, q-n-1\}$. Then, for any distinct vectors $\boldsymbol{v}_0, \dots, \boldsymbol{v}_n, \boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ in V

$$\sum_{j=1}^{k} \left\{ \frac{\det(\boldsymbol{u}_j, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n)}{\det(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n)} \right\}^d + 1 \neq 0.$$

Received April 23, 1999; revised January 28, 2000.

We define a hypersurface $S(V)$ assoicated with V in $\boldsymbol{P}^n(\boldsymbol{C})$ by

$$\sum_{\boldsymbol{v}\in V}(\boldsymbol{z}\cdot\boldsymbol{v})^d=0,$$

where $\boldsymbol{z}\cdot\boldsymbol{v}=v_0z_0+\cdots+v_nz_n$ for $\boldsymbol{v}={}^t(v_0,\ldots,v_n)$.

Now, let f be a holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ with a reduced representation $\boldsymbol{f}=(f_0,\ldots,f_n)$ such that $f(\boldsymbol{C})\subset S(V)$, *i.e.*,

$$\sum_{\boldsymbol{v}\in V}(\boldsymbol{f}\cdot\boldsymbol{v})^d\equiv 0,$$

where $\boldsymbol{f}\cdot\boldsymbol{v}=v_0f_0+\cdots+v_nf_n$ for $\boldsymbol{v}={}^t(v_0,\ldots,v_n)$.

The following generalized Borel's lemma is due to H. Fujimoto and M. Green (cf. [F, Corollary 6.4] and [G, p. 70]):

Lemma 2.1. *Let $f:\boldsymbol{C}\to S(V)$ be a holomorphic mapping. If $d\geq(q-1)^2$, then there exists a decomposition $V=V_0\cup V_1\cup\cdots\cup V_\ell$ of V with $\sharp V\geq 2$ $(1\leq j\leq\ell)$ such that*

(i) *$\boldsymbol{v}\in V_0$ if and only if $\boldsymbol{f}\cdot\boldsymbol{v}\equiv 0$,*

(ii) *for $\boldsymbol{u},\boldsymbol{v}\in V\backslash V_0$, $(\boldsymbol{f}\cdot\boldsymbol{u})/(\boldsymbol{f}\cdot\boldsymbol{v})$ is constant if and only if there exists j with $1\leq j\leq\ell$ such that $\boldsymbol{u},\boldsymbol{v}\in V_j$,*

(iii) *$\sum_{\boldsymbol{v}\in V_j}(\boldsymbol{f}\cdot\boldsymbol{v})^d\equiv 0$ for each j with $1\leq j\leq\ell$.*

Put $k_j:=\sharp V_j-1$ for $1\leq j\leq\ell$ and $k_0=\sharp V_0$. Then, we may assume $k_1\geq\cdots\geq k_\ell$ by changing indices. In this situation, we call $V=V_0\cup V_1\cup\cdots\cup V_\ell$ the first kind of decomposition of V by f, and $(k_0;k_1+1,\ldots,k_\ell+1)$ its type. Set

$$V_0=\{\boldsymbol{v}_1^{(0)},\ldots,\boldsymbol{v}_{k_0}^{(0)}\}$$

and

$$V_j=\{\boldsymbol{v}_0^{(j)},\ldots,\boldsymbol{v}_{k_j}^{(j)}\}\quad(1\leq j\leq\ell).$$

Then, by Lemma 2.1, there exist nonzero constants $\omega_\mu^{(j)}$ such that

$$\boldsymbol{f}\cdot(\boldsymbol{v}_\mu^{(j)}-\omega_\mu^{(j)}\boldsymbol{v}_0^{(j)})\equiv 0\quad(1\leq\mu\leq k_j,1\leq j\leq\ell)\tag{2.1}$$

and

$$1+(\omega_1^{(j)})^d+\cdots+(\omega_{k_j}^{(j)})^d=0\quad(1\leq j\leq\ell).\tag{2.2}$$

For brevity, we write $\boldsymbol{w}_\mu^{(j)}$ for $\boldsymbol{v}_\mu^{(j)}-\omega_\mu^{(j)}\boldsymbol{v}_\mu^{(j)}$. The equations of (2.1) and $\boldsymbol{f}\cdot\boldsymbol{v}_\mu^{(0)}\equiv 0$ $(1\leq\mu\leq k_0)$ can be represented as

$$\boldsymbol{f}(\boldsymbol{v}_1^{(0)},\ldots,\boldsymbol{v}_{k_0}^{(0)},\boldsymbol{w}_1^{(1)},\ldots,\boldsymbol{w}_{k_1}^{(1)},\ldots,\boldsymbol{w}_1^{(\ell)},\ldots,\boldsymbol{w}_{k_\ell}^{(\ell)})\equiv(0,\ldots,0).$$

Lemma 2.2. *If there exists a nonconstant holomorphic mapping $f:\boldsymbol{C}\to S(V)$, then*

(i) $0 \leq k_0 \leq n-1$,
(ii) $1 \leq k_j \leq n-1$ *for* $1 \leq j \leq \ell$,
(iii) *the rank of the matrix*

$$(\boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{w}_1^{(1)}, \dots, \boldsymbol{w}_{k_1}^{(1)}, \dots, \boldsymbol{w}_1^{(\ell)}, \dots, \boldsymbol{w}_{k_\ell}^{(\ell)})$$

is not greater than $n-1$.

Proof. The assertions (i) and (ii) follow from the assertion (iii) and the assumptions (A1) and (A2). Hence we prove only (iii).

We assume that the rank of the matrix $(\boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{w}_1^{(1)}, \dots, \boldsymbol{w}_{k_1}^{(1)}, \dots, \boldsymbol{w}_1^{(\ell)}, \dots, \boldsymbol{w}_{k_\ell}^{(\ell)})$ is greater than $n-1$. Then there exist a non-singular $(n+1) \times (n+1)$-matrix P and an entire function h such that $\boldsymbol{f}P = (0, \dots, 0, h)$. Hence, $f = (c_0 : \dots : c_n)$, where $(c_0, \dots, c_n)$ is the $(n+1)$-th row of P^{-1}. Q.E.D.

In the rest of this section, we assume that there exists a nonconstant holomorphic mapping $f : \boldsymbol{C} \to S(V)$.

LEMMA 2.3. $k_0 + k_1 \leq n-2$.

Proof. Assume the contrary. Then $k_0 + k_1 \geq n-1$.

(i) The case of $k_0 + k_1 \geq n$. In this case, by Lemma 2.2, there exist not all zero constants $a_1, \dots, a_{k_1}, b_1 \dots, b_{n-k_1}$ such that

$$\begin{aligned} &a_1 \boldsymbol{w}_1^{(1)} + \dots + a_{k_1} \boldsymbol{w}_{k_1}^{(1)} + b_1 \boldsymbol{v}_1^{(0)} + \dots + b_{n-k_1} \boldsymbol{v}_{n-k_1}^{(1)} \\ &\quad = a_1 (\boldsymbol{v}_1^{(1)} - \omega_1^{(1)} \boldsymbol{v}_0^{(1)}) + \dots + a_{k_1} (\boldsymbol{v}_{k_1}^{(1)} - \omega_{k_1}^{(1)} \boldsymbol{v}_0^{(1)}) + b_1 \boldsymbol{v}_1^{(0)} + \dots + b_{n-k_1} \boldsymbol{v}_{n-k_1}^{(1)} = \boldsymbol{0}. \end{aligned}$$

This contradicts (A1).

(ii) The case of $k_0 + k_1 = n-1$. In this case, $\ell \geq 2$. It follows from Lemma 2.2 that there exist constants $a_{j1}, \dots, a_{jk_1}, b_{j1}, \dots, b_{jk_0}$ such that

$$\begin{aligned} &\boldsymbol{v}_j^{(2)} - \omega_j^{(2)} \boldsymbol{v}_0^{(2)} \\ &\quad = a_{j1} \boldsymbol{w}_1^{(1)} + \dots + a_{jk_1} \boldsymbol{w}_{k_1}^{(1)} + b_{j1} \boldsymbol{v}_1^{(0)} + \dots + b_{jk_0} \boldsymbol{v}_{k_0}^{(1)} \\ &\quad = a_{j1} (\boldsymbol{v}_1^{(1)} - \omega_1^{(1)} \boldsymbol{v}_0^{(1)}) + \dots + a_{jk_1} (\boldsymbol{v}_{k_1}^{(1)} - \omega_{k_1}^{(1)} \boldsymbol{v}_0^{(1)}) + b_{j1} \boldsymbol{v}_1^{(0)} + \dots + b_{jk_0} \boldsymbol{v}_{k_0}^{(1)} \end{aligned}$$

for $1 \leq j \leq k_2$. By Cramer's formula we have

$$\omega_j^{(2)} = \frac{\det(\boldsymbol{v}_j^{(2)}, \boldsymbol{v}_0^{(1)}, \dots, \boldsymbol{v}_{k_1}^{(1)}, \boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)})}{\det(\boldsymbol{v}_0^{(2)}, \boldsymbol{v}_0^{(1)}, \dots, \boldsymbol{v}_{k_1}^{(1)}, \boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)})},$$

which contradicts (2.2) and (A3). Q.E.D.

LEMMA 2.4. $k_0 + \sharp\{j; 2 \leq j \leq \ell \text{ and } k_j = 1\} \leq n-2$.

Proof. Assume $k_0 + \sharp\{j; 2 \le j \le \ell$ and $k_j = 1\} \ge n-1$. Take m such that $2 \le m \le \ell$, $\ell - m + 1 + k_0 = n - 1$ and $k_m = \cdots = k_\ell = 1$. Then, by Lemma 2.2, $\boldsymbol{v}_1^{(1)} - \omega_1^{(1)}\boldsymbol{v}_0^{(1)}, \boldsymbol{v}_1^{(0)}, \ldots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{v}_1^{(m)} - \omega_1^{(m)}\boldsymbol{v}_0^{(m)}, \ldots, \boldsymbol{v}_1^{(\ell)} - \omega_1^{(\ell)}\boldsymbol{v}_0^{(\ell)}$ are linearly dependent, which contradicts (A2). Q.E.D.

Lemma 2.5. $k_0 + k_1 + \sharp\{j; 2 \le j \le \ell$ and $k_j = 1\} \le n-1$.

Proof. By Lemma 2.3, it is enough to consider the case where $\ell \ge 2$. Assume $k_0 + k_1 + \sharp\{j; 2 \le j \le \ell$ and $k_j = 1\} \ge n$. Take m such that $2 \le m \le \ell$, $\ell - m + 1 + k_0 + k_1 = n$ and $k_m = \cdots = k_\ell = 1$. Then, by Lemma 2.2, $\boldsymbol{v}_1^{(0)}, \ldots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{v}_1^{(1)} - \omega_1^{(1)}\boldsymbol{v}_0^{(1)}, \ldots, \boldsymbol{v}_{k_1}^{(1)} - \omega_{k_1}^{(1)}\boldsymbol{v}_0^{(1)}, \boldsymbol{v}_1^{(m)} - \omega_1^{(m)}\boldsymbol{v}_0^{(m)}, \ldots, \boldsymbol{v}_1^{(\ell)} - \omega_1^{(\ell)}\boldsymbol{v}_0^{(\ell)}$ are linearly dependent. This contradicts (A2). Q.E.D.

§3. Hyperbolicity of hypersurfaces in $P^n(C)$ ($n = 2, 3$ and 4)

In this section, we prove the hyperbolicity of $S(V)$ in the case where $n = 2, 3$ and 4. By Brody's criterion for hyperbolicity ([B]), it surffices to show that there exists no nonconstant holomorphic mapping $f : \boldsymbol{C} \to S(V)$.

Theorem 3.1. *If $n = 2$, then $S(V)$ is hyperbolic.*

Proof. Suppose that $S(V)$ is not hyperbolic. Then there exists a nonconstant holomorphic mapping $f : \boldsymbol{C} \to S(V)$. By Lemma 2.3, we see $k_0 + k_1 \le 0$. This contradicts $k_1 \ge 1$. Q.E.D.

The least degree of the hyperbolic hypersurfaces in Theorem 3.1 is 4.

Theorem 3.2. *If $n = 3$ and $q \ge 5$, then $S(V)$ is hyperbolic.*

Proof. Suppose that $S(V)$ is not hyperbolic. By Lemma 2.3, we see $k_0 + k_1 \le 1$. This implies that $k_0 = 0$ and $k_1 = 1$. Since $k_1 \ge \cdots \ge k_\ell \ge 1$, we have that $k_1 = \cdots = k_\ell = 1$ and hence $q = 2\ell$. If $q \ge 5$ is odd, then no such decomposition occurs. If $q \ge 5$ is even, we get

$$k_0 + \sharp\{j; 1 \le j \le \ell \text{ and } k_j = 1\} \ge 3,$$

which contradicts Lemma 2.4. Q.E.D.

The least degree of the hyperbolic hypersurfaces in Theorem 3.2 is 16.

Theorem 3.3. *If $n = 4$ and $q = 7$, then $S(V)$ is hyperbolic.*

Proof. Suppose that $S(V)$ is not hyperbolic. By Lemma 2.3, we see $k_0 + k_1 \le 2$. Since $q = 7$, it is easy to see that the only possible types of decomposition of the first type are of types $(0; 3, 2, 2)$ and $(1; 2, 2, 2)$. For these two types, we have

$$k_0 + k_1 + \sharp\{j; 2 \leq j \leq 3 \text{ and } k_j = 1\} \geq 4.$$

This contradicts Lemma 2.5. Q.E.D.

K. Masuda and J. Noguchi [MN] gave an example of hyperbolic hypersurface of degree 196 in $\boldsymbol{P}^4(\boldsymbol{C})$. The least degree of hyperbolic hypersurfaces in Theorem 3.3 is 36.

§4. A holomorphic mapping omitting a hypersurface

Let n, q and d be positive integers such that $q \geq n+1$ and $d \geq (q-1)q+1$. Let V be as in §2. Assume that (A1), (A2) and (A3). Let $f : \boldsymbol{C} \to \boldsymbol{P}^n(\boldsymbol{C})$ with a reduced representation $\boldsymbol{f} = (f_0, \dots, f_n)$ such that $f(\boldsymbol{C}) \cap S(V) = \emptyset$, *i.e.*,

$$\sum_{\boldsymbol{v} \in V} (\boldsymbol{f} \cdot \boldsymbol{v})^d \equiv \alpha^d$$

for an entire function α without zeros. The following Lemma due to M. Green plays an essential role (cf. [G, p. 73]):

LEMMA 4.1. *Let $f : \boldsymbol{C} \to \boldsymbol{P}^n(\boldsymbol{C}) \backslash S(V)$ be a holomorphic mapping. If $d \geq (q-1)q+1$, then there exists a decomposition $V = V_0 \cup V_1 \cup \cdots \cup V_\ell \cup V_{\ell+1}$ of V with $V_{\ell+1} \neq \emptyset$ such that*

(i) $\sharp V_j \geq 2$ $(1 \leq j \leq \ell)$,

(ii) *$\boldsymbol{v} \in V_0$ if and only if $\boldsymbol{f} \cdot \boldsymbol{v} \equiv 0$,*

(iii) *for $\boldsymbol{u}, \boldsymbol{v} \in V \backslash V_0$, $(\boldsymbol{f} \cdot \boldsymbol{u})/(\boldsymbol{f} \cdot \boldsymbol{v})$ is constant if and only if there exists j with $1 \leq j \leq \ell + 1$ such that $\boldsymbol{u}, \boldsymbol{v} \in V_j$,*

(iv) *$\sum_{\boldsymbol{v} \in V_j} (\boldsymbol{f} \cdot \boldsymbol{v})^d \equiv 0$ for each $1 \leq j \leq \ell$,*

(v) *$\boldsymbol{v} \in V_{\ell+1}$ if and only if $(\boldsymbol{f} \cdot \boldsymbol{v})/\alpha$ is constant,*

(vi) *$\sum_{\boldsymbol{v} \in V_{\ell+1}} (\boldsymbol{f} \cdot \boldsymbol{v})^d = \alpha^d$.*

Put $k_j := \sharp V_j - 1$ for $1 \leq j \leq \ell + 1$ and $k_0 := \sharp V_0$. We may assume that $k_1 \geq \cdots \geq k_\ell$. We call $V = V_0 \cup V_1 \cup \cdots \cup V_\ell \cup V_{\ell+1}$ the second kind of decomposition of V by f, and $(k_0; k_1 + 1, \dots, k_\ell + 1; k_{\ell+1} + 1)$ its type.

Set

$$V_0 = \{\boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)}\}$$

and

$$V_j = \{\boldsymbol{v}_0^{(j)}, \dots, \boldsymbol{v}_{k_j}^{(j)}\} \quad (1 \leq j \leq \ell + 1).$$

Then, by Lemma 4.1, there exist nonzero constants $\omega_\mu^{(j)}$ such that

$$\boldsymbol{f} \cdot (\boldsymbol{v}_\mu^{(j)} - \omega_\mu^{(j)} \boldsymbol{v}_0^{(j)}) \equiv 0 \quad (1 \leq \mu \leq k_j, 1 \leq j \leq \ell) \tag{4.1}$$

and

$$1 + (\omega_1^{(j)})^d + \cdots + (\omega_{k_j}^{(j)})^d = 0 \quad (1 \le j \le \ell). \tag{4.2}$$

If $k_{\ell+1} \ge 1$, then there exist nonzero constants $\omega_\mu^{(\ell+1)}$ such that

$$\boldsymbol{f} \cdot (\boldsymbol{v}_\mu^{(\ell+1)} - \omega_\mu^{(\ell+1)} \boldsymbol{v}_0^{(\ell+1)}) \equiv 0 \quad (1 \le \mu \le k_{\ell+1}) \tag{4.3}$$

However, there is no relation corresponding to (4.2) for $\omega_\mu^{(\ell+1)}$. By (4.1), (4.3) and $\boldsymbol{f} \cdot \boldsymbol{v}_\mu^{(0)} \equiv 0$, we have the following: If $k_{\ell+1} = 0$, then

$$\boldsymbol{f}(\boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{w}_1^{(1)}, \dots, \boldsymbol{w}_{k_1}^{(1)}, \dots, \boldsymbol{w}_1^{(\ell)}, \dots, \boldsymbol{w}_{k_\ell}^{(\ell)}) \equiv (0, \dots, 0),$$

and if $k_{\ell+1} \ge 1$, then

$$\boldsymbol{f}(\boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{w}_1^{(1)}, \dots, \boldsymbol{w}_{k_1}^{(1)}, \dots, \boldsymbol{w}_1^{(\ell+1)}, \dots, \boldsymbol{w}_{k_{\ell+1}}^{(\ell+1)}) \equiv (0, \dots, 0).$$

LEMMA 4.2. *If there exists a nonconstant holomorphic mapping $f : \boldsymbol{C} \to \boldsymbol{P}^n(\boldsymbol{C}) \backslash S(V)$, then*

(i) $0 \le k_0 \le n - 1$,
(ii) $1 \le k_j \le n - 1$ $(1 \le j \le \ell)$,
(iii) *if $k_{\ell+1} = 0$, the rank of the matrix*

$$(\boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{w}_1^{(1)}, \dots, \boldsymbol{w}_{k_1}^{(1)}, \dots, \boldsymbol{w}_1^{(\ell)}, \dots, \boldsymbol{w}_{k_\ell}^{(\ell)})$$

is not greater than $n - 1$, and if $k_{\ell+1} \ge 1$, the rank of the matrix

$$(\boldsymbol{v}_1^{(0)}, \dots, \boldsymbol{v}_{k_0}^{(0)}, \boldsymbol{w}_1^{(1)}, \dots, \boldsymbol{w}_{k_1}^{(1)}, \dots, \boldsymbol{w}_1^{(\ell+1)}, \dots, \boldsymbol{w}_{k_{\ell+1}}^{(\ell+1)})$$

is not greater than $n - 1$,
(iv) $0 \le k_{\ell+1} \le n - 1$.

In the rest of this section, we assume that there exists a nonconstant holomorphic mapping $f : \boldsymbol{C} \to \boldsymbol{P}^n(\boldsymbol{C}) \backslash S(V)$. By the same way as in the proof of Lemma 2.3, we have the following lemma:

LEMMA 4.3. (i) $k_0 + k_1 \le n - 1$.
(ii) *If $\ell \ge 2$, then* $k_0 + k_1 \le n - 2$.
(iii) $k_0 + k_{\ell+1} \le n - 1$.
(iv) *If $\ell \ge 1$, then* $k_0 + k_{\ell+1} \le n - 2$.

By the same argument as in the proof of Lemma 2.4, we also get the following lemma:

LEMMA 4.4. $k_0 + \sharp\{j; 2 \le j \le \ell$ *and* $k_j = 1\} \le n - 2$.

Furthermore, we can show the following lemma by the same method as in the proof of Lemma 2.5:

LEMMA 4.5. (i) $k_0 + k_1 + \sharp\{j; 2 \le j \le \ell \text{ and } k_j = 1\} \le n - 1$.
(ii) $k_0 + k_{\ell+1} + \sharp\{j; 1 \le j \le \ell \text{ and } k_j = 1\} \le n - 1$.

§5. Hyperbolicity of complements of hypersurfaces

In this section, we prove that $\boldsymbol{P}^n(\boldsymbol{C})\backslash S(V)$ is complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^n(\boldsymbol{C})$ in the case where $n = 2, 3$ and 4. We first recall the following criterion for hyperbolicity of complements of hypersurfaces (cf. [L, Theorem 3.3]):

LEMMA 5.1. *Let S be a hyperbolic hypersurface in $\boldsymbol{P}^n(\boldsymbol{C})$. Then $\boldsymbol{P}^n(\boldsymbol{C})\backslash S$ is complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^n(\boldsymbol{C})$ if and only if there exists no nonconstant holomorphic mapping $f : \boldsymbol{C} \to \boldsymbol{P}^n(\boldsymbol{C})\backslash S$.*

THEOREM 5.2. *If $n = 2$ and $q \ge 4$, then $\boldsymbol{P}^2(\boldsymbol{C})\backslash S(V)$ is complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^2(\boldsymbol{C})$.*

Proof. By Theorem 3.1, $S(V)$ is hyperbolic. Suppose that there exists a nonconstant holomorphic mapping $f : \boldsymbol{C} \to \boldsymbol{P}^2(\boldsymbol{C})\backslash S(V)$. If $\ell = 0$, then $k_0 + k_{\ell+1} \le 1$ by Lemma 4.3. Since $q \ge 4$, this is absurd. Hence $\ell \ge 1$. By Lemma 4.3, we see $k_0 + k_1 \le 1$. We also have $k_{\ell+1} = 0$ by Lemma 4.3. Thus we conclude that the only possible type of decomposition of the second kind is of type $(0; 2, 2, \dots, 2; 1)$. This contradicts Lemma 4.5. Q.E.D.

THEOREM 5.3. *If $n = 3$ and $q \ge 6$, then $\boldsymbol{P}^3(\boldsymbol{C})\backslash S(V)$ is complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^3(\boldsymbol{C})$.*

Proof. By Theorem 3.2, $S(V)$ is hyperbolic. Suppose that there exists a nonconstant holomorphic mapping $f : \boldsymbol{C} \to \boldsymbol{P}^3(\boldsymbol{C})\backslash S(V)$. As in the proof of Theorem 5.2, we have $\ell \ge 1$. If $\ell = 1$, then $k_0 + k_1 \le 2$ by Lemma 4.3. In this case, it is clear that

$$k_0 + (k_1 + 1) + (k_2 + 1) = q \ge 6.$$

Thus we see $k_2 \ge 2$, which contradicts Lemma 4.3. If $\ell \ge 2$, then $k_0 + k_1 \le 1$ by Lemma 4.3, (ii). This gives that $k_0 = 0$ and $k_1 = 1$. Thus the only possible type of decomposition of the second kind is of $(0; 2, 2, \dots, 2; k_{\ell+1} + 1)$. Hence $2\ell + k_{\ell+1} + 1 = q \ge 6$. On the other hand, by Lemma 4.5, we get $k_{\ell+1} + \ell \le 2$, and hence $k_{\ell+1} \le -1$. This is absurd. Q.E.D.

The least degree of the hypersurfaces in Theorem 5.3 is 31.

We next consider the case where $n = 4$ and $q = 9$. We first notice the following: Suppose that there exists a decomposition of the first kind is of $(0; 3, 3, 3)$ of V by f. Then there exist nonzero polynomials $F_1, \dots, F_s$ of the determinants of column vectors such that F_j are independent of f and

$F_j(v_1,\dots,v_9)=0$ for some j, where $V=\{v_1,\dots,v_9\}$. Indeed, by Lemma 2.2, we have

$$\boldsymbol{v}_2^{(j)}-\omega_2^{(j)}\boldsymbol{v}_0^{(j)}=a_{j0}(\boldsymbol{v}_1^{(j)}-\omega_1^{(j)}\boldsymbol{v}_0^{(j)})+a_{j1}(\boldsymbol{v}_1^{(1)}-\omega_1^{(1)}\boldsymbol{v}_0^{(1)})+a_{j2}(\boldsymbol{v}_2^{(1)}-\omega_2^{(1)}\boldsymbol{v}_0^{(1)})$$

for $j=2,3$. By applying Cramer's formula to

$$\boldsymbol{v}_2^{(j)}=-(a_{j1}\omega_1^{(1)}+a_{j2}\omega_2^{(1)})\boldsymbol{v}_0^{(1)}+a_{j1}\boldsymbol{v}_1^{(1)}+a_{j2}\boldsymbol{v}_2^{(1)}+(\omega_2^{(j)}-a_{j0}\omega_1^{(j)})\boldsymbol{v}_0^{(j)}+a_{j0}\boldsymbol{v}_1^{(j)},$$

we have

$$a_{j1}=\frac{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_2^{(j)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(j)},\boldsymbol{v}_1^{(j)})}{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(j)},\boldsymbol{v}_1^{(j)})},\quad a_{j2}=\frac{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(j)},\boldsymbol{v}_0^{(j)},\boldsymbol{v}_1^{(j)})}{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(j)},\boldsymbol{v}_1^{(j)})}$$

and

$$-(a_{j1}\omega_1^{(1)}+a_{j2}\omega_2^{(1)})=\frac{\det(\boldsymbol{v}_2^{(j)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(j)},\boldsymbol{v}_1^{(j)})}{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(j)},\boldsymbol{v}_1^{(j)})}.$$

Hence, $\omega_1^{(1)}$ and $\omega_2^{(1)}$ can be written as rational functions of the above determinants. Then, we have by (2.2)

$$\begin{aligned}
&\{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(2)},\boldsymbol{v}_0^{(2)},\boldsymbol{v}_1^{(2)})\det(\boldsymbol{v}_2^{(3)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(3)},\boldsymbol{v}_1^{(1)})\\
&\quad-\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(3)},\boldsymbol{v}_0^{(3)},\boldsymbol{v}_1^{(3)})\det(\boldsymbol{v}_2^{(2)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(2)},\boldsymbol{v}_1^{(2)})\}^d\\
&+\{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_2^{(3)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(3)},\boldsymbol{v}_1^{(3)})\det(\boldsymbol{v}_2^{(2)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(2)},\boldsymbol{v}_1^{(2)})\\
&\quad-\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_2^{(2)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(2)},\boldsymbol{v}_1^{(2)})\det(\boldsymbol{v}_2^{(3)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(3)},\boldsymbol{v}_1^{(3)})\}^d\\
&+\{\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_2^{(2)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(2)},\boldsymbol{v}_1^{(2)})\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(3)},\boldsymbol{v}_0^{(3)},\boldsymbol{v}_1^{(3)})\\
&\quad-\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(2)},\boldsymbol{v}_0^{(2)},\boldsymbol{v}_1^{(2)})\det(\boldsymbol{v}_0^{(1)},\boldsymbol{v}_2^{(3)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_0^{(3)},\boldsymbol{v}_1^{(3)})\}^d\\
&=0.
\end{aligned}$$

Hence, by permutations, we get polynomials F_j with the above property, and the number of polynomials s is 9!. Also, if there exists a decomposition of the second kind of $(0;3,3;3)$ of V by f, then $F_j(v_1,\dots,v_9)=0$ for some j. By the same way, if there exists a decomposition of the second kind of $(0;3,3,2;1)$ of V by f, then we have nonzero polynomials $G_1,\dots,G_t$ such that G_k are independent of f and $G_k(v_1,\dots,v_9)=0$ for some k. Indeed, by Lemma 4.2, we have

$$\boldsymbol{v}_j^{(2)}-\omega_j^{(2)}\boldsymbol{v}_0^{(2)}=a_{j1}(\boldsymbol{v}_1^{(1)}-\omega_1^{(1)}\boldsymbol{v}_0^{(1)})+a_{j2}(\boldsymbol{v}_2^{(1)}-\omega_2^{(1)}\boldsymbol{v}_0^{(1)})+a_{j3}(\boldsymbol{v}_1^{(3)}-\omega_1^{(3)}\boldsymbol{v}_0^{(3)})$$

for $j=1,2$. By Cramer's formula, we get

$$\omega_j^{(2)}=\frac{\det(\boldsymbol{v}_j^{(2)},\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_1^{(3)}-\omega_1^{(3)}\boldsymbol{v}_0^{(3)})}{\det(\boldsymbol{v}_0^{(2)},\boldsymbol{v}_0^{(1)},\boldsymbol{v}_1^{(1)},\boldsymbol{v}_2^{(1)},\boldsymbol{v}_1^{(3)}-\omega_1^{(3)}\boldsymbol{v}_0^{(3)})}.$$

Note that $\omega_1^{(3)}$ is a d-th root of -1. By (2.2) we get polynomials G_k, and the number of polynomials t is $9! \times d$. Let $\mathscr{V}$ be a proper algebraic subset of $\boldsymbol{C}^{45}$ defind by

$$\mathscr{V} = \left(\bigcup_{j=1}^{s} \{F_j(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_9) = 0\} \right) \cup \left(\bigcup_{j=1}^{t} \{G_j(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_9) = 0\} \right).$$

THEOREM 5.4. *If* $q = 9$ *and* $V \notin \mathscr{V}$, *then* $\boldsymbol{P}^4(\boldsymbol{C}) \backslash S(V)$ *is complete hyperbolic and hyperbolically imbedded in* $\boldsymbol{P}^4(\boldsymbol{C})$.

Proof. Suppose that there exists a nonconstant holomorphic mapping $f : \boldsymbol{C} \to S(V)$. Then by Lemmas 2.3 and 2.5, it is easy that the only possible type of decomposition of the first kind by f is of type $(0; 3, 3, 3)$. If $\boldsymbol{P}^4(\boldsymbol{C}) \backslash S(V)$ is not Brody hyperbolic, then we also see that the only possible types of decomposition of the second kind are of types $(0; 3, 3; 3)$ and $(0; 3, 3, 2; 1)$ by Lemmas 4.3 and 4.4. Hence $V \notin \mathscr{V}$ yields our assertion. Q.E.D.

By Theorem 5.4, we obtain hyperbolic hypersufaces of degree $d \geq 73$ in $\boldsymbol{P}^4(\boldsymbol{C})$ whose complements are complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^4(\boldsymbol{C})$.

Example 5.5. Let $\boldsymbol{v}_1 = {}^t(1, 0, 0)$, $\boldsymbol{v}_2 = {}^t(0, 1, 0)$, $\boldsymbol{v}_3 = {}^t(0, 0, 1)$, $\boldsymbol{v}_4 = {}^t(a, b, c)$ and $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. The condition (A1) is equivalent to $abc \neq 0$, (A2) to $a^d \neq -1$, $b^d \neq -1$, $c^d \neq -1$, and (A3) to $a^d + (-b)^d \neq 0$, $b^d + (-c)^d \neq 0$, $c^d + (-a)^d \neq 0$. Hence, if a, b, c satisfy the above conditions and $d \geq 13$, then the hypersurface in $\boldsymbol{P}^2(\boldsymbol{C})$ defined by

$$z_0^d + z_1^d + z_2^d + (az_0 + bz_1 + cz_2)^d = 0$$

is hyperbolic and its complement is complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^2(\boldsymbol{C})$. Suppose that this hypersurface has a singular point $(p_0 : p_1 : p_2)$. Then $p_0^{d-1} + a(ap_0 + bp_1 + cp_2)^{d-1} = 0$, $p_1^{d-1} + b(ap_0 + bp_1 + cp_2)^{d-1} = 0$ and $p_2^{d-1} + c(ap_0 + bp_1 + cp_2)^{d-1} = 0$. It is trivial that $P := ap_0 + bp_1 + cp_2 \neq 0$. Hence we have $p_0 = \omega P$, $p_1 = \eta P$ and $p_2 = \xi P$ for some $(d-1)$-th roots ω, η, ξ of $-a, -b, -c$, respectively. Hence we get $\omega a + \eta b + \xi c = 1$. Therefore, this hypersurface is smooth if $\omega a + \eta b + \xi c \neq 1$ for any $(d-1)$-th roots ω, η, ξ of $-a, -b, -c$, respectively. For example, if $a = 1/2$, $b = 1/4, c = 1/8$, these four conditions are satisfied.

M. B. Zaindenberg showed in [Z] that the existence of a smooth hyperbolic hypersurface of degree 5 in $\boldsymbol{P}^2(\boldsymbol{C})$ whose complement is complete hyperbolic and hyperbolically imbedded in $\boldsymbol{P}^2(\boldsymbol{C})$. On the other hand, K. Azukawa and M. Suzuki [AS] gave an explicit equation defining such a hypersurface of degree 14. The degree 13 of our example is lower than it.

References

[AS] K. Azukawa and M. Suzuki, Some examples of algebraic degeneracy and hyperbolic manifolds, Rocky Mountain J. Math., **10** (1980), 655–659.

[B] R. Brody, Compact manifolds and hyperbolicity, Trans. Amer. Math. Soc., **235** (1978), 213–219.

[BG] R. Brody and M. Green, A family of smooth hyperbolic hypersurfaces in $\boldsymbol{P}_3$, Duke Math. J., **44** (1975), 873–874.

[D] J.-P. Demailly, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, Algebraic Geometry held at Santa Cruz, July 1995 (J. Kollár et al. eds.), Proc. Sympos. Pure Math., **62-2**, Amer. Math. Soc., Providence, 1997, 285–360.

[F] H. Fujimoto, On meromorphic maps into the complex projective space, J. Math. Soc. Japan, **26** (1974), 272–288.

[G] M. L. Green, Some Picard theorems for holomorphic maps to algebraic varieties, Amer. J. Math., **97** (1975), 43–75.

[K] H. H. Khoai, Hyperbolic surfaces in $\boldsymbol{P}^3(\boldsymbol{C})$, Proc. Amer. Math. Soc., **125** (1997), 3527–3532.

[L] S. Lang, Introduction to Complex Hyperbolic Spaces, Springer-Verlag, New York, 1987

[MN] K. Masuda and J. Noguchi, A construction of hyperbolic hypersurface of $\boldsymbol{P}^n(\boldsymbol{C})$, Math. Ann., **304** (1996), 339–362.

[N] A. Nadel, Hyperbolic surfaces in $\boldsymbol{P}^3$, Duke Math. J., **58** (1989), 749–771.

[S] M. Shirosaki, On some hypersurfaces and holomorphic mappings, Kodai Math. J., **21** (1998), 29–34.

[Z] M. G. Zaidenberg, Stability of hyperbolic imbeddedness and constraction of examples, Math. USSR-Sb., **63** (1989), 351–361.

Department of Mathematical Sciences
College of Engineering
Osaka Prefecture University
Sakai, 599-8531, Japan