APPROXIMATING ZARISKI DECOMPOSITION
OF BIG LINE BUNDLES

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In this note we prove the following

THEOREM. *Let L be a line bundle on a variety V with $n=\dim V$. Suppose that $h^0(V, tL) \geq dt^n/n! + \phi(t)$ for infinitely many positive integers t, where $\phi(t)$ is a function with $\lim_{t \to \infty} \phi(t)/t^n = 0$ and d is a positive real number. Then, for any $\varepsilon > 0$, there is a birational morphism $\pi: M \to V$ together with an effective Q-divisor E on M such that $H = \pi^* L - E$ is a semiample Q-bundle with $H^n > d - \varepsilon$.*

If $H^0(t\pi^* L) = H^0(tH)$ for every $t > 0$ such that tE is a Z-divisor, then $\pi^* L = E + H$ is a Zariski decomposition of L. In such a case we have $h^0(V, tL) = dt^n/n! + $ (lower order terms) for $d = H^n$. However, unfortunately, such a decomposition does not always exist. Our result can be viewed as an approximation of Zariski decomposition.

Combined with other techniques, our theorem can be used to derive various inequalities concerning intersection numbers of line bundles. In many cases it is almost as powerful as the theory of singular Hermitian metrics (cf. [D1], [D2]) or the theory of analytic Zariski decomposition (cf. [T]). Details of such applications will be discussed elsewhere.

For the proof of the theorem, we use the following

LEMMA 1. *Let L be a line bundle on V. Suppose that $H = L - E + G$ is nef and big Q-bundle for some effective Q-divisor E, G such that $H^{n-1}E = 0$. Then any component of E is a fixed component of $|L|$.*

Proof. The following argument is just a variant of [F; (2.5)]. Replacing V by some birational model and pulling thing back if necessary, we can assume that $|L| = F + M$ with Bs $M = \emptyset$, where F is the fixed part of $|L|$. It suffices to derive a contradiction assuming that $G + F - E$ is not effective as a Q-divisor.

Take an effective Q-divisor D such that $H - \varepsilon D$ is ample for any small $\varepsilon > 0$. Take small enough ε such that $\varepsilon D + G + F - E$ is not effective. Since $L - F$ is nef, $A = H - \varepsilon D + L - F$ is an ample Q-bundle. Now, as in [F], take a

birational morphism $\pi : V' \to V$ such that $\pi^*E = E_0 + E_1$, $\pi^*(\varepsilon D + G + F) = E_0 + F_1$ and $E_1 \cap F_1 = \emptyset$. Note that $\pi_* E_1 \neq 0$.

As in [F], $B = F_1 + \pi^*A = E_1 + 2\pi^*H$ is nef as a $\boldsymbol{Q}$-bundle. Moreover, $H^{n-1}\pi^*E = 0$ implies $H^{n-1}E_1 = 0$, so $(2H)^n = (2H)^{n-1}B$, which yields $2HB^{n-1} \geq B^n$ by the Index Theorem as in [F]. Hence $0 = B^{n-1}E_1 = (\pi^*A)^{n-1}E_1 = A^{n-1}\pi_* E_1$, contradiction.

Next we prove the following

LEMMA 2. *Let B be a nef and big line bundle on a normal variety V with* $\dim V = n$ *and let* $E = \sum \mu_i E_i$ *be an effective divisor such that $A = B + E$ is nef.* *Then* $h^0(V, A) - h^0(V, B) \leq n(A^n - B^n)$.

Proof. By virtue of desingularization theory we may assume that V is smooth. Let I^+ be the set of i's such that $(A+B)^{n-1}E_i > 0$, and set $m_i = \mathrm{Max}_{a+b=n-1} A^a B^b E_i$ for each i. Since both A and B are nef, $m_i > 0$ if and only if $i \in I^+$. Now we choose a sequence $S_0 = E, S_1, \cdots$ of divisors in the following way: Each S_j is a $\boldsymbol{Z}$-linear combination of E_i's and $S_j = S_{j-1} - E_{i(j)}$ for some $i(j)$. If $S_{j-1} = \sum \nu_i E_i$, $i(j)$ is chosen from the i's where ν_i/μ_i attains the maximum value r_j. Moreover, if $r_j = \nu_i/\mu_i$ for some $i \in I^+$, $i(j)$ should be chosen from such indices in I^+. Thus, finally, we have $S_\mu = 0$ for $\mu = \sum \mu_i$. We claim $h^0(B + S_{j-1}) - h^0(B + S_j) \leq nm_{i(j)}$ for any j.

First we consider the case $i(j) \notin I^+$. Then $q_j = \mathrm{Max}_{i \in I^+} \nu_i/\mu_i < r_j$ by the choice of $i(j)$. We set $S_{j-1} - q_j E = F - G$ for some effective $\boldsymbol{Q}$-divisors F, G. Then $i \notin I^+$ for any component E_i of F, so $H_j^{n-1}E_i = 0$ for $H_j = q_j A + (1 - q_j)B = B + q_j E$. H_j is nef and big, and $H_j = B + S_{j-1} - F + G$, hence any component of F is a fixed component of $|B + S_{j-1}|$ by Lemma 1. In particular $h^0(B + S_{j-1}) = h^0(B + S_j)$, as claimed.

Next we consider the case $i(j) \in I^+$. Take a birational morphism $\pi : M \to E_{i(j)}$ such that M is smooth and $\pi^*|(B + S_{j-1})_{E_{i(j)}}| = F + |H|$ with $\mathrm{Bs}|H| = \emptyset$, where F is the fixed part. Let $\rho : M \to \boldsymbol{P}^N$ be the morphism induced by $|H|$ and let $W = \rho(M)$, $k = \dim W$ and $w = \deg W$. Let X be a general fiber of $M \to W$. Since $A + B$ is nef and big on M, it is so on X too, hence $A^a B^b X > 0$ for some a, b with $a + b = \dim X = n - 1 - k$. Then $A^a B^b X \geq 1$, so $w \leq wA^a B^b X = A^a B^b H^k\{M\} \leq A^a B^b P_j^k\{M\}$ for $P_j = r_j A + (1 - r_j)B = B + r_j E$, since P_j is nef and $P_j - H = (r_j E - S_{j-1}) + F$ is an effective $\boldsymbol{Q}$-divisor on M. By computation we have

$$A^a B^b P_j^k\{M\} = \sum_{\alpha=0}^k \binom{k}{\alpha} r_j^\alpha (1 - r_j)^{k-\alpha} A^{a+\alpha} B^{b+k-\alpha}\{E_{i(j)}\} \leq \sum_{\alpha=0}^k \binom{k}{\alpha} r_j^\alpha (1 - r_j)^{k-\alpha} m_{i(j)}$$

$= m_{i(j)}$, so $w \leq m_{i(j)}$. Since $\Delta(W, \mathcal{O}(1)) = k + w - N - 1 \geq 0$, we infer $h^0(E_{i(j)}, B + S_{j-1}) \leq k + w \leq n - 1 + m_{i(j)} \leq nm_{i(j)}$. This proves the claim.

Thus the claim is established in any case. Now we infer $h^0(A) - h^0(B) \leq \sum_j (h^0(B + S_{j-1}) - h^0(B + S_j)) \leq \sum_j nm_{i(j)} = n \sum_i \mu_i m_i \leq n \sum_i \mu_i (\sum_{a+b=n-1} A^a B^b E_i) = n \sum_{a+b=n-1} A^a B^b E = n \sum_{a+b=n-1} (A^{a+1}B^b - A^a B^{b+1}) = n(A^n - B^n)$. Thus the lemma is proved.

Proof of the Theorem. Since $\kappa(L)=n$ by assumption, there is a number c such that $h^0(tL)>0$ for all $t \geq c$. We will derive a contradiction assuming that $H^n < d-\varepsilon$ for all π and E.

Replacing ε if necessary, we may assume that $H_0^n \geq d-\varepsilon-\varepsilon/(2n)!$ for some $H_0 = \pi_0^* L - E_0$, $\pi_0 : V_0 \to V$. Let $m>0$ be an integer such that mE_0 is a $\boldsymbol{Z}$-divisor and $\mathrm{Bs}|mH_0|=\emptyset$. Now, for any integer $s>0$, we claim $h^0(smL) \leq h^0(smH_0) + ns^n m^n \varepsilon/2n!$. To see this, take a birational morphism $\pi : V' \to V$ such that $\pi^*|smL| = F+|A|$ with $\mathrm{Bs}|A|=\emptyset$, where F is the fixed part. We may assume that π factors through $\pi_0 : V_0 \to V$ and $|A-(smH_0)_{V'}| \neq \emptyset$. Hence $h^0(smL) = h^0(A) \leq h^0(smH_0) + n(A^n - s^n m^n H_0^n)$ by Lemma 2. By assumption $(A/sm)^n \leq d-\varepsilon$, hence $A^n - s^n m^n H_0^n \leq s^n m^n \varepsilon/2n!$, and the claim follows.

By the hypothesis of the theorem we have a sequence $\{t_j\}$ such that $\lim_{j \to \infty} t_j = \infty$ and $h^0(V, t_j L) \geq dt_j^n/n! + \phi(t_j)$. For each t_j, let s_j be the least integer such that $s_j m \geq t_j + c$. Then $h^0(t_j L) \leq h^0(s_j mL)$, so $dt_j^n/n! + \phi(t_j) \leq h^0(s_j mH_0) + ns_j^n m^n \varepsilon/2n! \leq s_j^n m^n (d-\varepsilon)/n! + ns_j^n m^n \varepsilon/2n! + \phi(s_j m)$ for some polynomial ϕ of degree $\leq n-1$. By the choice of s_j, we have $t_j > (s_j-1)m-c$, so $dt_j^n/n! + \phi(t_j) = ds_j^n m^n/n! + \phi'(s_j)$ with $\lim \phi'(s_j)/s_j^n = 0$. Comparing the coefficients of s_j^n in both sides, we get $d < d-\varepsilon+n\varepsilon/n$, a contradiction as desired.

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