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ON RELATIONS BETWEEN BROWN-PETERSON COHOMOLOGY AND THE ORDINARY MOD p COHOMOLOGY THEORY

Dedicated to Professor Nobuo Shimada on his 60-th birthday

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Introduction.

Let $H^*(X; Z_p)$ be the ordinary mod p cohomology for odd prime and let $BP^*(X)$ be the Brown-Peterson cohomology theory with $BP^*=Z_{(p)}[v_1, v_2, \cdots]$. The spectrum $V(n)$ is defined by $H^*(V(n); Z_p)\cong\wedge[Q_0, \cdots, Q_n]$ [8] where Q_i is the Milnor operation and $V(n)$ is also defined by $BP^*(V(n))\cong BP^*/(p, v_1, \cdots, v_n)$ [6]. To consider the equivalence of the above two definitions was the begining of this paper.

We note a relation between the Q_i-action and v_i-torsion, which is an immediate consequence from the Sullivan's bordism theory of manifolds with singularities.

LEMMA 2.1. *Let $x_j\in BP^*(X)$ and $\Sigma v_j x_j=0 \mod I_\infty^2$ where $I_\infty=(p, v_1, \cdots)$. Then there is $y\in H^*(X; Z_p)$ such that $Q_j(y)=i(x_j)$ where i is the inclusion map (Thom map) $i: BP\to HZ_p$.*

The Brown-Peterson cohomology is studied by many authors, especially the Adams spectral sequence for $BP^*(S^N)$ is well researched. However, known examples of non free BP^*-module $BP^*(X)$ are not so many. Using above lemma, we consider the way to calculate $BP^*(X)$ when the Steenrod algebra structure of $H^*(X; Z_p)$ is known, and we give examples of the BP^*-module $BP^*(X)$.

In section 1 using Sullivan's original definition of the bordism theory with (cone type) singularities, we treat the Quillen's geometric approach to the cobordism theory. In § 2 main lemmas are shown. We recall some important facts about the Atiyah-Hirzebruch spectral sequence and we define an invariant which is convenient to use. Some examples are discussed in § 3. The spectrum $V(n)$ and Lens spaces are first treated. We next study about finite H-spaces and Eilenberg-Maclane space $K(Z, 3)$, in particular, BP^*-module structures of even dimensional indecomposed elements are discussed.

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§1. Cobordism theory with singularities.

We first recall the definition of manifolds with singularities due to Baas [1]. Let $S_n=(P_1, \cdots, P_n)$ be a sequence of closed manifolds. We say Z is an S_n-manifold if its boundary is decomposed as products of P_i, namely, there are manifolds $\partial_i Z$, $\partial_{(i_1,\dots,i_s)}Z$, $Z(i_1, \cdots, i_s)$, for $(i_1, \cdots, i_s)\subset(1, \cdots, n)$, and natural isomorphisms

$$(1)\qquad \partial Z\cong\cup\partial_i Z$$

$$\partial_{i_1}Z\cap\cdots\cap\partial_{i_s}Z=\partial_{(i_1,\dots,i_s)}Z\cong Z(i_1, \cdots, i_s)\times P_{i_1}\times\cdots\times P_{i_s}.$$

We next define an $\hat{S}_n$-manifold (S_n-manifold with cones) by the quotient space of Z collapsing the P_i-factors, i.e.,

$$(2)\qquad \hat{Z}=Z/(w, a)\sim(w, b)\quad\text{where}\quad (w, a), (w, b)\in\partial_{(i_1,\dots,i_s)}Z$$

$$\text{and}\quad w\in Z(i_1, \cdots, i_s),\ a,\ b\in P_{i_1}\times\cdots\times P_{i_s}.$$

Sullivan originally defined an $\hat{S}_n$-manifold as the manifold such that each point of its boundaries has a neighbourhood factored cone$(P_{i_1}*\cdots*P_{i_s})$. We show these two definition are equivalent.

Consider the tubular neighbourhoods of $\partial_{(1,\dots,s)}Z$,

$$(\text{Nei. in }\partial Z)\cong Z(1, \cdots, s)\times P_1\times\cdots\times P_s\times\Delta_{s-1},$$

$$(\text{Nei. in }Z)\cong Z(1, \cdots, s)\times P_1\times\cdots\times P_s\times\Delta_s$$

where we identify $\partial_{(1,\dots,s)}Z=Z(1, \cdots, s)\times P_1\times\cdots\times P_s\times$(center of Δ_{s-1}) and Δ_{s-1} includes the $s+1$-th face of Δ_s. Take the boundary of Nei. in Z (the link complex)

$$Lk\cong Z(1, \cdots, s)\times P_1\times\cdots\times P_s\times(\dot{\Delta}_s-\mathring{\Delta}_{s-1}).$$

Since $(\dot{\Delta}_s-\mathring{\Delta}_{s-1})\cong\Delta_{s-1}$, if we take the quotient (2), then

$$\widehat{Lk}\cong Z(1, \cdots, s)\times P_1\times\cdots\times P_s\times\Delta_{s-1}/(w, \cdots p_i, \cdots, \sigma)\sim(w, \cdots p_i', \cdots, \sigma),\ \sigma\in\dot{\Delta}_{s-1}$$
$$\cong Z(1, \cdots, s)\times P_1*\cdots*P_s.$$

Since boundary $\partial_{(1,\dots,s)}Z$ collapses to $Z(1, \cdots, s)$, (Nei. in Z) collapses to

$$(3)\qquad (\widehat{\text{Nei. } Z})\cong Z(1, \cdots, s)\times(\text{cone}\,P_1*\cdots*P_s).$$

Hence the definition (2) is the Sullivan one. Moreover we note $\hat{Z}$ is also defined by

$$(4)\qquad \hat{Z}=\hat{Z}\cup\Sigma J(i)\times Z(i)\cup\Sigma j(i, j)\times Z(i, j)\cup\cdots\cup J(i, \cdots, n)\times Z(1, \cdots, n)$$

where $J(1, \cdots, s)=\text{cone}(P_1*\cdots*P_s)$
$=\text{cone}(J(2, \cdots, s)\times P_1\cup\cdots\cup J(\cdots, \hat{t}, \cdots)\times P_t\cup\cdots\cup J(1, \cdots, s-1)\times P_s)$
and cone $P=P\times I/P\times\{1\}$ and $\partial_{(1,\dots,s)}Z=\{0\}\times Z(1, \cdots, s)$.

Hereafter let denote $\partial_{(i,\dots,s)}\hat{Z}=\hat{Z}(1, \cdots, s)$.

DEFINITION 1.2. Let X be an (open or closed) manifold and $\hat{Z}$ be an $\hat{S}_n$-manifold. A map $f: Z \to X$ is a complex oriented of dimension q if f is factored such that

$$f: \hat{Z} \xrightarrow{i} X \times R^N \xrightarrow{p} X$$

(1) i is an embedding with normal bundles $\nu_{i_1 \cdots i_s}$ having compatible stable complex structure on each $(\partial_{i_1 \cdots i_s})\hat{Z} - \partial(\partial_{(i_1 \cdots i_s)} Z))$,

(2) p is a projection,

(3) if $z \in (\partial_{(i_1, \cdots, i_s)} \hat{Z} - \partial(\partial_{(i_1, \cdots, i_s)} Z)$, then $(\dim Z \text{ at } z) - (\dim X \text{ at } f(z)) = q - (\dim P_{i_1} + \cdots + \dim P_{i_s} + s)$.

DEFINITION 1.3. Let $f: \hat{Z} \to X$ be a complex oriented map and $g: Y \to X$ be a map. Define the modified pull back $Y \times'_X Z \to Y$ as follows.

For ease of arguments, assume $n=1$, i.e., the S_1-case. Let $\hat{Z} = Z \cup$ cone $P_1 \times Z(1)$. Take $g' \times f'(1)$ transversal to the diagonal $\Delta \subset X \times X$. Then $(g' \times f'(1))^{-1}\Delta = Y \times_X Z(1)$ is a manifold and $(g' \times f'(1) \cdot \text{porj})^{-1}\Delta = Y \times_X Z(1) \times \text{cone}\, P_1$ where proj: $Z(1) \times \text{cone } P_1 \to Z(1)$ is the projection. Let $f'': Z \cup \partial Z(1) \times I \to X$ be a map so that $f'' = f$ on Z and f'' is the homotopy between f and f' on $Z(1) \times I$. Taking $g' \times f'''$ for $g' \times f''$ transversal to Δ, we can define

$$(g' \times (f''' \cup f'(1) \cdot \text{proj}))^{-1} = Y \times'_X Z\,.$$

When $n > 1$, we can also define the modified pull back by descending induction on sequences $(i_1, \cdots, i_s)$ in (3).

DEFINITION 1.4. Let $f_i: Z_i \to X$, $i=1, 0$ be complex oriented maps. Then they are cobordant if there is a proper complex oriented map $b: W \to X \times R$ such that $\varepsilon_i: X \to X \times R$, $\varepsilon_i(x) = (x, i)$ is transversal to b, and the pull back of ε_i is isomorphic to f_i.

THEOREM 1.5. *For a manifold X, the set of cobordism classes of proper complex oriented map of dimension $-q$ is $MU(S_n)^q(X)$, Here $MU(S_n)^*(X)$ is the cobordism theory with singularities and without cone due to Baas* [1], [9].

DEFINITION 1.6. (Gysin homomorphism) A proper complex oriented map $g: \hat{X} \to Y$ of dimension d induces a map

$$g_*: MU(S_n)^*(\hat{X}) \longrightarrow MU(S_n)^{*-d}(Y)$$

which sends $f: \hat{Z} \to \hat{X}$ into $gf: \hat{Z} \to Y$.

DEFINITION 1.7. (Contravariant map) Let $g: Y \to X$ be a map of manifolds, and let $f: \hat{Z} \to X$ be a proper complex oriented map. Then g induces a map

$$g^*: MU(S_n)^*(X) \longrightarrow MU(S_n)^*(Y)$$

which sends $f: \hat{Z} \to X$ into the modified pull back $Y \times'_X \hat{Z} \to Y$.

THEOREM 1.8. (Sullivan's exact sequence) *There is an MU^*-module exact sequence*

$$\begin{tikzcd} MU(S_n)^*(X) \arrow[rr, "P_{n+1}"] & & MU(S_n)^*(X) \arrow[dl, "i"] \\ & MU(S_n, P_{n+1})^*(X) \arrow[ul, "\delta"] & \end{tikzcd}$$

where i is the natural inclusion map and $\delta(\hat{A}, f)=(\partial_{n+1}\hat{A}, f)$.

COROLLARY 1.9. *If S_n is a regular sequence in MU^*, then $MU(S_n)^*(S^0) \cong MU^*/(S_n)$.*

In particular $MU(x_i | i\neq p^j-1)^*(X)_{(p)} \cong BP^*(X)$ and $MU(p, x_1, \cdots)^*(X)\cong H^*(X; Z_p)$. Identifyning $x_{p^i-1}=v_i$, we denote $MU(S_n x_i, \cdots | i\neq p^i-1)^*(X)_{(p)}$ by $BP(S_n)^*(X)$. Recall the notations $BP(p, v_1, \cdots, v_{n-1})=P(n)$, $BP(p, \cdots, v_{n-1}, v_{n+1}, \cdots)=k(n)$ and $v_n^{-1}k(n)=K(n)$.

Define an operation Q_{P_i} by $Q_{P_i}([\hat{A}, f])=[\partial_i\hat{A}, f | \partial_i\hat{A}]=[\hat{A}(i), f(i)]$. Then it is easily seen $Q_{P_i}Q_{P_j}=-Q_{P_j}Q_{P_i}$. Hereafter we fix the generators v_n such that the Chern number $c\Delta_{p^n-1}(v_n)=p$ mod p^2, namely, the Milnor manifolds.

THEOREM 1.10. *In $H^*(X; Z_p)\cong BP(p, v_1, \cdots)^*(X)$, the operation Q_{v_i} is the Milnor operation Q_i, (Q_0=the Bockstein operation and $Q_i=\mathcal{P}^{p^{i-1}}Q_{i-1}-Q_{i-1}\mathcal{P}^{p^{i-1}}$)[9].*

The cohomology operations in $MU^*(-)$ are MU^*-generated by the Landweber-Novikov operation s_α. The operation s_α is defined also in $MU(I_n)^*(-)$, $I_n=(p, v_1, \cdots, v_{n-1})$[9]. We here define it from geometric viewpoints, as follows.

Given $[\hat{A}, f]\in MU(I_n)^*(X)$, we will define $s_\alpha[\hat{A}, f]$. First suppose $\partial\hat{A}=\partial_i\hat{A}$, i.e., $\partial\hat{A}=A(i)\times v_i$. Let $\tau_{\partial A}: \partial A\to BU$ be the map which represents the tangent bundle of ∂A. Since

$$\tau_{\partial A}^*\cong(\tau_{A(i)}\times\tau_{v_i})^* \quad \text{and} \quad \tau_{\partial A}^*(c_\alpha)=\sum_{\alpha=\alpha'+\alpha''}\tau_{A(i)}^*c_{\alpha'}\tau_{v_i}^*c_{\alpha''},$$

the definition of s_α in $MU^*(-)$ theory follows

$$s_\alpha[\partial A, \partial f]=\sum_{\alpha=\alpha'+\alpha''}s_{\alpha'}A(i)\cdot s_{\alpha''}(v_i).$$

Here $s_{\alpha''}(v_i)\in I_i=(p, \cdots, v_{i-1})$ and we can write

$$s_\alpha[\partial A, \partial f]=\sum_{\alpha=\alpha'+\alpha''}s_{\alpha'}A(i)\cdot\sum_{j<i}b_{\alpha'', j}v_j \quad (1).$$

Let $[M, g]\in MU^*(A)$ be a manifold which represents $\tau_A^*(c_\alpha)$. Then there is a manifold W so that

$$\partial W=\partial M\vee(\text{right hand side of } (1)).$$

Therefore we can define $s_\alpha[A, f]$ by

$$gf: M\bigcup_{\partial M} W\cup \sum_{\alpha', j}(s_{\alpha'}(A(i))\times b_{\alpha', j})\times \text{conev}_j \longrightarrow A\cup A(i)\times \text{conev}_j \longrightarrow X.$$

The fact that if $b_{\alpha' j}$ is also in I_{i-1} then $b_{\alpha', j}=0$ in $MU(I_n)^*$ implies that $s_\alpha [A, f]$ is welldefined.

The case $\partial A=\cup\partial_i A$ is also proved by descending induction on sequences $(i_1, \cdots, i_s)$ in (3).

The cohomology operations in $MU(I_n)$-theory is known

$$MU(I_n)_*(MU(I_n))\cong MU^*/I_n\otimes MU^*(MU)\otimes \Lambda[Q_p, \cdots, Q_{v_{n-1}}].$$

Therefore we can minic the arguments in Quillen's paper [5]. In particular we can prove

THEOREM 1.11. *Let X be a finite complex. Then $MU(I_n)^*(X)$ (resp. $P(n)^*(X)$) is generated as a $MU(I_n)^*$-module (resp. $P(n)^*$-module) by elements of non negative degree.*

§ 2. Main lemmas.

In the previous section we noted the geometric mean of the Milnor operation Q_i.

LEMMA 2.1. *Let $x_j\in BP^*(X)$ and $\Sigma v_j x_j=0 \mod I_\infty^2$ where $I_\infty=(p, v_1, \cdots)$. Then there is $y\in H^*(X; Z_p)$ such that $Q_j y=i(x_j)$ where i is the natural inclusion map $i: BP\to HZ_p$.*

Proof. Think of x_j as an singular manifold. Since $\Sigma v_j x_j=0 \mod I_\infty^2$, there is a manifold y' whose boundary is

$$\partial y'=\vee v_j(x_j\vee w_j) \text{ where } w_j=0 \mod I_\infty.$$

Let $y=\hat{y}'$ be the I_∞-manifold constructed from y' attaching cones

$$\hat{y}'=y'\bigcup_{\partial y'}(\text{conev}_j)\times(x_j\vee w_j) \quad\text{and}\quad \partial_j\hat{y}'=(x_j\vee w_j).$$

Think of $H^*(X; Z_p)$ as $BP(I_\infty)^*(X)$ and we have

$$i'\partial_j\hat{y}'=Q_j\hat{y}'=i(x_j\vee w_j)=i(x_j),$$

where $i': BP(\cdots, \hat{v}_j, \cdots)^*(X)=k(n)^*(X)\to H^*(X; Z_p)$. q.e.d.

We next recall the Atiyah-Hirzebruch spectral sequence. Given multiplicative spectrum A, let denote by ${}_AE_r^{*,*}(X)$ the Atiyah-Hirzebruch spectral sequence which converges to $A^*(X)$,

$${}_AE_2^{*,*}\cong H^*(X; A^*)\Rightarrow A^*(X).$$

The following lemma is well known [9],

LEMMA 2.2. *The first non zero differential of the spectral sequence* ${}_{P(n)}E_r^{*,*}$ (*and* ${}_{k(n)}E_r^{*,*}$) *is* $d_{2p^n-1}=v_nQ_n$.

COROLLARY 2.3. *In the spectral sequence* ${}_{P(1)}E_r^{*,*}={}_{BP(-,Z)}E_r^{*,*}$, *if* $d_rx=0$ *for* $r<2p^n-1$ *then* $d_{2p^n-1}(x)=v_nQ_n(x) \bmod (p, \cdots, v_{n-1})$.

Proof. The natural inclusion map $i: P(1)\to P(n)$ and Lemma 2.2 follow the corollary. q. e. d.

Recall that an ideal I in BP^* is invariant if $\theta(I)\subset I$ for all operations $\theta\in BP^*(BP)$.

LEMMA 2.4. *Let* $x_1, \cdots, x_s\in {}_{P(1)}E_r^{t,0}$. *Then the mod annihilator* $AM(x_1)=\{a\in P(1)^* \mid ax_1=0 \bmod (x_2, \cdots, x_s)$ *in* $E_r^{t,*}\}$ *is invariant.*

Proof. This $AM(x_1)$ is indeed the mod$(x_2, \cdots, x_s)$ annihilator ideal in $P(1)^*(X^{t+1}/X^{t-r-2})$. Hence this is an invariant ideal. q. e. d.

We now consider relations between the Atiyah-Hirzebruch spectral sequence and the Sullivan exact sequence.

LEMMA 2.5. *Let* $wx=0$ *in* $P(1)^*(X)$ *for* $0\neq w\in P(1)^*$, *and let* $i(x)=x'\neq 0$ *in* $H^*(X; Z_p)$. *From the Sullivan exact sequence, there is* y *in* $BP(p, w)^*(X)$ *such that* $\delta y=x$. *Then* $d_ry'=\lambda wx'$ *in* ${}_{P(1)}E_r^{*,*}$ *where* $0\neq\lambda\in Z_p$, $i(y)=y'$ *and* $r=|w|+1$.

Proof. Since $P(1)^*(X)=BP^*(X; Z_p)\cong BP^*(X\wedge S^0\cup_p e^1)$, we consider this lemma in ${}_{BP}E_r(X\wedge S^0\cup_p e^1)$. Take the normal cells decomposition of $X\wedge S^0\cup_p e^1$, i.e.,

$$*=Y_0\subset \cdots \subset Y_\infty\cong X\wedge S^0\cup_p e^1,$$

$$Y_n=Y_{n-1}\cup_f \operatorname{cone}(\bigvee_k S^n\cup_p e^1), \qquad k=\dim_{Z_p}(H_n(Y_\infty; Z)).$$

Put $Y(n-x')=Y_{n-1}/Y_{n-r-2}$-cone$(S^{n-2}\cup_p e^{n-1})_{x'}$ where $n=|x'|$ and cone$(S^{n-2}\cup_p e^{n-1})_{x'}$ is the cone of the Moore space which represents x' in $H^*(Y_\infty; Z)$. Since $\delta y=x'=0$ in $BP^*(Y(n-x'))$ where $y\in BP(w)^*(Y(n-x'))$, y is also in $BP^*(Y(n-x'))$ and y' is a permanent cycle in ${}_{BP}E^{*,*}(Y(n-x'))$, i.e., $d_ry'=0$.

On the other hand put

$$Y(n-1+x')=Y_{n-2}/Y_{n-r-2}\cup \operatorname{cone}(S^{n-2}\cup_p e^{n-1})_{x'}.$$

Then $\delta y=x\neq 0$ and y' is not a permanent cycle in ${}_{BP}E^{*,*}(Y(n-1+x'))$. Hence $d_ry'=\lambda wx'$.

Therefore $d_ry'=\lambda wx'$ in ${}_{BP}E^{*,*}(Y_{n-1}/Y_{n-r-2})$. By the construction of the spectral sequence we have the lemma. q. e. d.

The following corollary is an analogous result of Lemma 2.1.

COROLLARY 2.6. *Let* $(w_1, \cdots, w_s)=J_s$, $|w_i|<|w_{i+1}|$ *be a regular sequence in* $P(1)^*$. *Let* $b_j\in P(1)^*(X)$ *and* $0\neq i(b_j)$ *in* $H^*(X; Z_p)$. *Suppose there is a relation in* $P(1)^*(X)$ *such that*

$$w_1b_1+w_2b_2+\cdots+w_sb_s=0.$$

Then there is $y\in{}_{F(1)}E_2^{*,*}$ *such that* $d_{r_t}(y)=\lambda_t w_t i(b_t)$ *in* ${}_{BP(p,J_{t-1})}E_{r_t}$, $0\neq\lambda_t\in Z$, *for* $1\leqq t\leqq s$.

Proof. Using the argument similar to the proof of Lemma 2.1, we can construct a $\hat{J}_s$-manifold $\hat{y}$ such that

$$\partial_j y=w_jb_j, \qquad \text{i.e.,} \qquad \partial_j\hat{y}=b_j.$$

Since $w_jb_j=0$ and $\delta_j y=b_j$ in $BP(p,\ J_s-\{w_j\})^*(X)$, it follows from Lemma 2.5 that

$$d_{r_j}(y)=\lambda_j w_j b_j \quad \text{in} \quad BP(p,\ J_s-\{w_j\})E_{r_j}^{*,*}.$$

That $BP(J_j)^*\cong BP(p,\ J_s-\{w_j\})^*$ for $*<|w_{j+1}|$ implies the lemma. q.e.d.

For the preceding of this paper, we define an index which is convenient to use. If $x\in H^*(X;\ Z_p)$ is in the image of $i: P(n)^*(X)\to H^*(X;\ Z_p)$, then x can be represented by a manifold with singularities of type $(p,\ v_1,\ \cdots,\ v_{n-1})$.

Define $t(x)=n$ if x is in Image $i: P(n)^*(X)\to H^*(X;\ Z_p)$ and is not in Image $i: P(n-1)^*(X)\to H^*(X;\ Z_p)$.

From the facts that $Q_j=i\delta_j$, $ir_\alpha=c(\mathcal{P}^\alpha)i$, and $P(n)^*(P(n))\cong P(n)^*\otimes_{BP^*}BP^*(BP)\otimes\Lambda[Q_0,\ \cdots,\ Q_{n-1}]$, we can easy see the following;

(2.1) $\quad t(x)=n$ implies $Q_m x=0$ for all $m\geqq n$.

(2.2) $\quad Q_n x\neq 0$ implies $t(x)\geqq n+1$.

(2.3) $\quad t(x)=n$ implies $t(Q_{n-1}x)\leqq n-1$.

(2.4) $\quad t(Q_n x)\leqq t(x)$.

(2.5) $\quad t(\mathcal{P}^\alpha x)\leqq t(x)$.

(2.6) $\quad t(xy)\leqq \max(t(x),\ t(y))$.

(2.7) $\quad$ Given $f: X\to Y$, $t(f^*x)\leqq t(x)$.

Question 2.7. Assume $t(b_1)\leqq 1(,\ t(b_i)\leqq 1)$ and there is a unique z in $H^*(X;\ Z_p)$ such that $Q_1z=b_1$(, respectively $v_kb_i=0$ in ${}_{P(1)}E^{*,*}$ for all $k<i$ and $Q_iz=b_i$). Then are there b_j' in $BP^*(X;\ Z_p)$ such that

$$Q_jz=i(b_j') \quad \text{and} \quad \Sigma v_jb_j'=0.$$

§ 3. Examples

3.1. The spectrum $V(n)$.

THEOREM 3.1.1. (Larry Smith) *Given a finite complex X, then $H^*(X; Z_p) \cong \Lambda[Q_0, \cdots, Q_n]$ if and only if $BP^*(X) \cong BP^*/(p, v_1, \cdots, v_n)$.*

Proof. Assume $H^*(X; Z_p) \cong \Delta[Q_0, \cdots, Q_n]$. Using Corollary 2.3, it is inductively proved that

$$_{P(1)}E_{2p^s}^{*,*} \cong (\Lambda[Q_0, Q_{s+1}, \cdots, Q_n] \otimes BP^*/(p, v_1, \cdots, v_s))Q_1 \cdots Q_s.$$

Hence we have $P(1)^*(X) \cong BP^*/(p, \cdots, v_n) \otimes \Lambda[Q_0]$ and $BP^*(X) \cong BP^*/(p, \cdots, v_n)$.

Conversely let $BP^*(X) \cong BP^*/(p, \cdots, v_n)$. From Lemma 2.1, there are $y_r \in H^*(X; Z_p)$ with $Q_r y_r = v_r x$ where x is the BP^*-module generator of $BP^*(X)$. From Lemma 2.5, $d_s y_r = v_r x$ in $_{P(1)}E_s^{*,*}$. The BP^*-module generated by y_n in E_{2p^n-1} is a $BP^*/(p, \cdots, v_{n-1})$-free module, indeed, if d_{2p^n-1} is not monic then the BP^*-module generator of $\ker d_{2p^n-1}$ is of the second degree$>-2p^n+1$ and this contradicts to that the generator is not a permanent cycle. Hence we can take $y_{n,n-1}$ in $H^*(X; Z_p)$ such that $Q_{n-1} y_{n,n-1} = y_n$.

Continuing this argument, there is z such that $Q_0 \cdots Q_n z = x$. Let $H^*(X; Z_p) \cong \Lambda[Q_0, \cdots, Q_n]z + B$. Each element in B is not a permanent cycle in $_{P(1)}E^{*,*}$. Let w be a highest dimensional non zero element in B. Then $dw = v_n Q_{i_1} \cdots Q_{i_s} \bar{z}$ and this follows the contradiction. Therefore $B=0$. q.e.d.

Remark. Theorem 3.11 is also proved more easily by using the Sullivan exact sequence.

We show that all regular invariant ideals containing p appear as annihilator ideals of some elements in $BP^*(X)$.

EXAMPLE 3.1.2. Let $J_n = (p, a_1, \cdots, a_n)$ be a regular invariant ideal of BP^*. Let $BP(J_n)$ be the spectrum of the bordism theory with the coefficient BP^*/J_n. The spectrum is inductively defined by the cofibering

$$BP(J_i) \xrightarrow{a_{i+1}} BP(J_i) \longrightarrow BP(J_{i+1}).$$

Using the fact $a_{i+1}^* = a_{i+1} \bmod J_i$, we can see (reference [9]) such as the case $J_n = I_{n+1} = (p, \cdots v_n)$

$$BP^*(BP(J_n)) \cong BP^*/J_n \underset{BP^*}{\otimes} BP^*(BP) \quad \text{and}$$

$$P(M)^*(BP(J_n)) \cong P(M)^* \underset{BP^*}{\otimes} BP^*(BP) \otimes \Lambda[\bar{Q}_0, \cdots, \bar{Q}_n]$$

for sufficient large M, e.g., $M > |a_n|$.

Let $BP(J_n)^N$ be an N-dimensional skeleton of $BP(J_n)$(, note that $BP(J_n)^N$ is equivalent to a finite complex, because $p \in J_n$). The highest degree of the nonzero differential of the spectral sequence $_{P(1)}E^{*,*}(BP(J_n))$ is $|a_n|+1$. Hence we have

$$BP^*(BP(J_n)^N) \cong BP^*/J_n \underset{BP^*}{\otimes} BP^*(BP^{N-|a_n|-2}) \oplus A$$

where A is the BP^*-module generated by generators$>N-|a_n|-2$.

3.2. Lens space.

Let X be a finite complex with $H^{\mathrm{odd}}(X; Z)=0$. Then the spectral sequence ${}_{BP}E^{*,*}(X)$ collapses.

THEOREM 3.2.1. *Let L be a $2m+1$-dimensional generalized Lens space $L(p, q_1, \cdots, q_m)$. Then there is a BP^*-algebra isomorphism*

$$BP^*(L)\cong BP^*[x]/(x^{m+1}, f(x))$$

where $f(x)=px+a_1x^2+\cdots$, and $a_{p^n-1}=v_n \bmod (p, \cdots, v_{n-1})$.

Proof. The cohomology ring is well known

$$H^*(L; Z_p)\cong Z_p[x]/(x^{m+1})\otimes \Lambda(\alpha), \qquad Q_0\alpha=x.$$

Since $H^{\mathrm{odd}}(L; Z)=0$, there is a BP^*-module isomorphism

$${}_{BP}E_\infty^{*,*}\cong BP^*[x]/(p, x^{m+1}).$$

From Lemma 2.1 and $Q_n\alpha=x^{p^n}$, we have

$$px+a_1x^2+\cdots=f(x)=0 \text{ in } BP^*(L) \quad\text{and}\quad a_{p^n-1}=v_n \bmod (p, \cdots, v_{n-1}).$$

q. e. d.

Remark. From the Gysin exact sequence, it is well known when $L=L(p, 1, \cdots, 1)$, the polynomial $f(x)$ is the p-th product $[p]$ of the formal group law.

3.3. Finite H-spaces

Suppose that W is a 1-connected (mod p) finite H-space. Let Q be the Z_p-module of indecomposed elements in $H^*(W; Z_p)$. The Kane's binary theorem [4] is stated as follows.

(1) $$Q^{2n}=\delta\mathcal{P}^{p^{s-1}}Q^{2n-p^s-1}=Q_sQ^{2n-p^s+1} \qquad \text{for } s\geqq 0$$

(2) $$\text{if } Q^{2n}\neq 0 \text{ then } n=(p^{k+1}-1)/(p-1)-p^l \qquad \text{for } 1\leqq l\leqq k.$$

Moreover for $a\in Q^{2n}$

(3) $$a^p=0 \quad \text{if } l=p^k \text{ in (1)},$$
$$a^{p^2}=0 \quad \text{otherwise.}$$

Let denote by $(y_k, \cdots, y_1)$ the system of generators such that

(4) $$|y_l|=(p^{k+1}-1)/(p-1)-p^l$$

(5) $$\mathcal{P}^{p^{(l-1)}}y_l=y_{l-1}.$$

Question 3.3.1. Is it true that $t(y_k)=0$ for all k and

$$v_hy_l'+{}_lvy_h'=0 \bmod p \text{ in } BP^*(W)$$

where $i(y_l')=y_l$, $i(y_h')=y_h$?

Remark 3.3.2. (1) Harper constructed [3] an H-space for each odd prime p such as

$$H^*(W\,;Z_p)=\wedge(x_3,\ x_{2p+1})\otimes Z_p[x_{2p+2}]/(x_{2p+2}^p).$$

Then by the arguments similar to [10],

$$BP^*(W)\cong BP^*\{1,\ y_3,\ y_a\}\oplus BP^*\{y_b,\ y_c\}/(py_b=v_1y_c)$$
$$\oplus BP^*/(p,\ v_1)[x_{2p+2}]/(x_{2p+2})^p,$$

where $a=2p^2+2p+2$, $b=a-3$, $c=2p^2+1$.

(2) The cohomology ring of the exceptional Lid group E_8 for $p=3$ is

$$H^*(E_8\,;Z_3)\cong Z_3[x_8,\ x_{20}]/(x_8^3,\ x_{20}^3)\otimes\Lambda$$

where Λ is the external product of odd dimensional generators. The BP^*-module structure of $BP^*(E_8)$ is known [11]. It holds that $t(x_8)=0$ and hence $v_1x_8=v_2x_{20}$.

It is unknown whether there exists an H-space such that $k\geqq 3$ in the binary theorem (2).

3.4. Eilenberg-MacLane space $K(Z,\ 3)$.

The mod p cohomology of $K(Z,\ 3)$ is known

$$H^*(K(Z,\ 3)\,;Z_p)\cong Z_p[\delta\mathcal{P}\tau,\ \delta\mathcal{P}^P\mathcal{P}\tau,\ \cdots]\otimes\Lambda[\tau,\ \mathcal{P}\tau,\ \cdots].$$

For simplicity of notations, let denote $\mathcal{P}^{P^{n-1}}\cdots\mathcal{P}\tau=c_n$, $\delta c_n=b_n$. Then $|c_n|=2(p^n-1)+3$, $|b_n|=2(p^n-1)+4$.

LEMMA 3.4.1. *In $H^*(K(Z,\ 3)\,;Z_p)$, the Milnor operations act*
(1) $Q_m\tau=b_m$ (2) $Q_mb_n=0$
(3) $Q_mc_n=Q_nc_m=(b_{n-m})^{p^m}$ *for* $n>m>0$ *and* $Q_mc_m=0$.

Proof. The cohomology ring $H^*(K;\ Z_p)$ is a Hopf algebra and, c_n, b_n are primitive elements. By the definition, we have (1) and $Q_0c_n=b_n$. We show $Q_m(\mathcal{P}^{P^{n-1}}c_{n-1})=(Q_{m-1}c_{n-1})^p$, indeed,

$$Q_m(\mathcal{P}^{P^{n-1}}c_{n-1})=\mathcal{P}^{P^{n-1}+P^{m-1}}Q_{m-1}c_{n-1}-Q_{m-1}\mathcal{P}^{P^{n-1}+P^{m-1}}c_{n-1}$$
$$=(Q_{m-1}c_{m-1})^p(,\text{ since }|Q_{m-1}c_{m-1}|=2(p^{n-1}+p^{m-1})).$$

Hence inductively we have $Q_mc_n=(b_{n-m}^{P^{m-1}})^p$.

Since Q_m is a derivation, $Q_m(b_n)$ is also primitive. Hence $Q_m(b_n)$ is an indecomposed element or its p-th power. By dimensional reason, we have (2).

q.e.d.

THEOREM 3.4.2. *There exist $b_j'\in BP^*(K(Z,\ 3))$ such that $i(b_j')=b_j$ and $v_1b_1'+v_2b_2'+\cdots=0$.*

To prove this theorem we recall the Wilson's theorem. Let $BP\langle n\rangle = BP(v_{n+1}, \cdots)$, namely, $BP\langle n\rangle_* = Z_{(p)}[v_1, \cdots, v_n]$.

THEOREM 3.4.3. (Wilson [13]) *For* $k \leqq 2(p^n + \cdots + p + 1)$,

$$i : BP^k(X) \longrightarrow BP\langle n\rangle^k(X) \quad \textit{is epic.}$$

Proof of Theorem 3.4.2. Since $b_1 = Q_0 c_1 = Q_1 \tau$,

$$b_1 \in \mathrm{Image}\,(BP\langle 1\rangle^*(K(Z, 3)) \longrightarrow H^*(K(Z, 3), Z_p)).$$

Moreover if $i(b_1'') = b_1$ then $v_1 b_1'' = 0$. By Wilson's theorem, $|b_1''| = 2(p+1)$ implies

$$b_1'' \in \mathrm{Image}\,(BP^*(K(Z, 3)) \longrightarrow BP\langle 1\rangle^*(K(Z, 3)).$$

Therefore $t(b_1) = 0$ and let $i(b_1') = b_1$.

From Sullivan's exact sequence, there is b_2'' such that

$$v_1 b_1 = -v_2 b_2'' \quad \text{in} \quad BP\langle 2\rangle^*(X).$$

Moreover from Lemma 2.1, $i(b_2'') = b_2$. By also Wilson's theorem, $|b_2''| < 2(p^2 + p + 1)$ implies

$$b_2'' \in \mathrm{Image}\,(BP^*(K(Z, 3)) \longrightarrow BP\langle 2\rangle^*(X)).$$

Take b_2' such as $i(b_2') = b_2''$. Continuing this argument. We have the theorem.

q. e. d.

THEOREM 3.4.4. *Let the filtration* $F_S = \mathrm{Ker}\,(P(1)^*(X) \to P(1)^*(X^S))$. *Then there is a* $P(1)^*$*-module isomorphism*

$$P(1)^*(K(Z, 3))/F_{2p^3+2} \cong P(1)^*[b_1, b_2, b_3]/(R, D)$$

where D is the ideal of elements of degree $\geqq 2p^3 + 2$ *in* $Z_p[b_1, b_2, b_3]$ *and R is the ideal generated by the following five relations*

(1) $$v_1 b_1 + v_2 b_2 + v_3 b_3 = 0 \mod I_3^2$$

(2) $$v_1^p b_2 + v_2 b_1^p + v_3 b_2^p = 0 \mod I_3^2 - \{v_1\}^2$$

(3) $$v_2 b_1^{p+1} + v_3 b_2^p b_1 = 0 \mod I_3^2$$

(4) $$v_2 b_1^{p-1} b_2 + v_3 (b_1^{p^2} - b_1^{p-1} b_3) = 0 \mod I_3^2$$

(5) $$v_1^{p^2+1} b_3 = 0 \mod I_3^2 - \{v_1^2\}\,.$$

Proof. We compute the Atiyah-Hirzebruch spectral sequence

$$E_2 = H^*(K(Z, 3), P(1)^*) \Rightarrow P(1)^*(K(Z, 3)).$$

The first non zero differential is $d_{2p-1} = v_1 \otimes Q_1$ and Q_1 acts such as

$$\tau \longrightarrow b_1, \quad c_1 \longrightarrow 0, \quad c_2 \longrightarrow b_1^p, \quad c_3 \longrightarrow b_2^p, \cdots$$

Hence we have

$$E_{2p}^{s,*} \cong P(1)^*[b_1, \cdots] \otimes \Lambda(c_1, c_2 - b_1^{p-1}\tau)/(v_1 b_1^p, v_1 b_1^p, \cdots).$$

Each element x of $|c_1|<|x|<|b_2|$ or $|c_1|+|c_2|<|x|<|b_2^p|$ is v_1-torsion in E_{2p}. Since $K(1)^*(K(Z, \ ))\cong 0$, $v_1^s b_2 \in \mathrm{Imd}_r$ and we have

$$d_{2(p-1)p+1} c_1 = v_1^p b_2.$$

Since $d_s(c_2 - b_1^{p-1}\tau)=0$ for $s \leqq 2(p-1)p+1$, we also have

$$d_{2(p-1)p+1} c_1(c_2 - b_1^{p-1}\tau) = v_1^p b_2 (c_2 - b_1^{p-1}).$$

Hence if $s \leqq 2p^3+2$, then

$$E_{2(p^2-1)}^{s,*} \cong P(1)[b_1, b_2, b_3] \otimes \wedge(c_1 b_1, c_2 b_1^{p-1})/(v_1 c_1 b_1, v_1 b_1^p, v_1^p b_2).$$

It is easily seen $v_1^p x=0$ for $|c_1|+|c_2|<|x|<|v_2|+|c_2|$. The next non zero differential is $d_{2p^2+1}=v_2 \otimes Q_2$. The operation Q_2 acts

(i) $c_1 b_1 \to v_2 b_1^{p+1}$; both sides are $P(1)^*/v_1$-free,

(ii) $c_2 - b_1^{p-1}\tau \to -v_2 b_1^{p-1} b_2$; the left side is $P(1)^*$-free and the other is v_1-torsion,

(iii) $c_1(c_2 - b_1^{p-1}\tau) \to v_2(b_1^p(c_2 - b_1^{p-1}\tau) + c_1 b_1^{p-1} b_2)$; both sides are $P(1)/v_1$-free.

Therefore if $s \leqq 2p^3+2$,

$$E_{2p^2}^{s,*} \cong P(1)^*[b_1, b_2, b_3] \otimes \{1, v_1 \otimes (c_2 - b_1^{p-1}\tau), b_1^p(c_2 - b_1^{p-1}\tau) + c_1 b_1^{p-1} b_2\} /$$

$$(v_1 b_1, v_1^p b_2, v_2 b_1^{p+1}, v_2 b_1^{p-1} b_2, (v_1, v_2)(b_1^p(c_2 - b_1^{p-1}\tau) + c_1 b_1^{p-1} b_2))$$

We will see odd dimensional elements are not permanent. Since $K(1)^*(K(Z, 3)) \cong 0$, $v_1^s b_3 \in \mathrm{Imd}_r$. The $P(1)^*$-free generator of dimension$<|b_3|$ is only one and

$$d_r(v_1 \otimes c_2 - b_1^{p-1}\tau) = v_1^{p^2+1} b_3.$$

Since $d_{2p^3+1} = v_3 \otimes Q_3 \bmod (v_1, v_2)$, we have

$$d_{2p^3+1}(b_1^p(c_2 - b_1^{p-1}\tau) + c_1 b_1^{p-1} b_2) = v_3(b_1^{p+p^2} - b_1^{2p-1} b_3 + b_2^{p+1} b_1^{p+1}).$$

Hence for $s \leqq 2p^3+2$

$$E_{2p^2}^{s,*} \cong P(1)^*[b_1, b_2, b_3]/(v_1 b_1, v_1^p b_2, v_2 b_1^{p+1}, v_2 b_1^{p-1} b_2, v_1^{p^2+1} b_3).$$

By Lemma 2.1 and Lemma 2.2, we have the relation, for example, the derivations $d_r c_1 = v_1^p b_2$, $Q_2 c_1 = b_1^p$, $Q_3 c_1 = b_2^p$ imply relation (2). q.e.d.

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