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ON THE L^2 BOUNDEDNESS THEOREM OF NON-HOMOGENEOUS FOURIER INTEGRAL OPERATORS IN R^n

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§ 1. Introduction and Notations.

A Fourier integral operator is an integral transformation of the form

$$Af(x)=(2\pi)^{-n}\int_{R^n} e^{iS(x,\xi)}a(x, \xi)\hat{f}(\xi)d\xi\,, \tag{1}$$

where

$$\hat{f}(\xi)=\int_{R^n} e^{-iy\cdot\xi}f(y)dy$$

is the Fourier transform of f defined on $\boldsymbol{R}^n$. We call $S(x, \xi)$ its phase function and $a(x, \xi)$ its symbol function (cf. Hörmander [8]). When $S(x, \xi)=x\cdot\xi$, a Fourier integral operator becomes a pseudo-differential operator.

If a symbol function satisfies the inequalities

$$|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)|\leqq C_{\alpha,\beta}(1+|\xi|)^{\delta|\alpha|-\rho|\beta|} \tag{2}$$

$(0\leqq\delta\leqq\rho\leqq1,\ \delta<1)$, Calderón-Vaillancourt [5] proved that the pseudo-differential operator with symbol $a(x, \xi)$ is L^2 bounded, and Fujiwara [7] and Kumano-go [12] proved the L^2 boundedness theorem of the Fourier integral operator.

If we take $\lambda(\xi)=(1+|\xi|)^{(\rho+\delta)/2}$, then such a symbol function $a(x, \xi)$ satisfies the inequalities

$$|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{|\alpha|-|\beta|}. \tag{3}$$

In this paper we consider the case that a weight function $\lambda(\xi)$ is more general in ξ (See Definition 1 in Section 2), and we shall prove the L^2 boundedness theorem of the Fourier integral operator A with symbol function $a(x, \xi)$ satisfying the inequalities (3).

We use the standard notations for functions and operators. A multi-index is a sequence $\alpha=(\alpha_1, \cdots, \alpha_n)$ of non-negative integers (the number n will usually be clear from the context). If α is a multi-index and $x=(x_1, \cdots, x_n)$, $\xi=(\xi_1, \cdots, \xi_n)$ in $\boldsymbol{R}^n$, we set

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$$|\alpha|=\alpha_1+\cdots+\alpha_n\,,\quad \alpha!=\alpha_1!\cdots\alpha_n!\,,$$

$$x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n\,,\quad |x|=(x_1^2+\cdots+x_n^2)^{1/2},$$

$$\langle\xi\rangle=(1+|\xi|^2)^{1/2},\quad x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\,,\quad \partial_{x_j}=\partial/\partial_{x_j}\,,\qquad j=1,\cdots,n\,.$$

If $f=f(x)$ is a function of x, then we set

$$\nabla_x f=(\partial_{x_1}f,\cdots,\partial_{x_n}f)\,,$$

$$\|f\|=\left(\int_{\boldsymbol{R}^n}|f(x)|^2dx\right)^{1/2}.$$

For a positive number r, we denote by χ_r the characteristic function of the ball $\{x\,;\,|x|\leqq r\}$. We denote by $\mathcal{S}(\boldsymbol{R}^n)$ the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}^n$, and by $L^2(\boldsymbol{R}^n)$ the set of measurable functions on $\boldsymbol{R}^n$ such that $\|f\|$ is finite. If A is an operator, we denote the operator norm of A in $L^2(\boldsymbol{R}^n)$ by $\|A\|$.

We adopt the following convention on constants: unless otherwise stated, constants C, C' vary from statement to statement, but depend only on the constants previously chosen.

§2. Definitions and Results.

DEFINITION 1. We say that a C^∞ real valued function $\lambda_1(\xi)$ defined on $\boldsymbol{R}^n$ is a basic weight function if $\lambda_1(\xi)$ satisfies the following conditions:

(W-1) There exist a positive constant C_1 such that

$$1\leqq\lambda_1(\xi)\leqq C_1\langle\xi\rangle$$

for all ξ in $\boldsymbol{R}^n$.

(W-2) For any multi-index α there exists a positive constant C_α such that

$$|\partial_\xi^\alpha\lambda_1(\xi)|\leqq C_\alpha\lambda_1(\xi)^{1-|\alpha|}$$

for all ξ in $\boldsymbol{R}^n$.

And we set $\lambda_\varepsilon(\xi)=\lambda_1(\xi)^\varepsilon$, where $0\leqq\varepsilon\leqq1$. Then we say that $\lambda_\varepsilon(\xi)$ is a weight function of type ε induced from $\lambda_1(\xi)$.

Remark 1. The weight function of the above type is used in Boutet de Monvel [4] and Kumano-go [10]. Beals-Fefferman [2], Beals [3] and Kumano-go-Taniguchi [11] define more general weight functions also depending on the x-variables in order to develop the calculus of pseudo-differential operators. In the context of such general weight functions we would be able to consider the L^2 boundedness of Fourier integral operators. In [13] we have attempted some generalizations.

DEFINITION 2. Let $\lambda(\xi)$ be a weight function of type ε ($0\leqq\varepsilon\leqq1$) and μ a real number. We say that a C^∞ function $a(x, \xi)$ defined on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ is a symbol function of order μ if $a(x, \xi)$ satisfies the condition:

(S-μ) For any two multi-indices α and β there exists a constant $C_{\alpha,\beta}$ such that for all (x, ξ) in $\boldsymbol{R}^n\times\boldsymbol{R}^n$ the estimate

$$|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{\mu+|\alpha|-|\beta|}$$

holds.

Remark 2. The set $S^\mu(\lambda)$ of all symbol functions of order μ corresponding to a weight function $\lambda(\xi)$ is a Fréchet space with semi-norms $|\cdot|_{\mu,k}$, where

$$|a|_{\mu,k}=\sum_{|\alpha|+|\beta|\leqq k}\sup_{(x,\xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^n}\frac{|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)|}{\lambda(\xi)^{\mu+|\alpha|-|\beta|}} \tag{4}$$

for any non-negative integer k.

EXAMPLE 1. $\lambda_1(\xi)=\langle\xi\rangle$ is a basic weight function and $\lambda_\tau(\xi)=\langle\xi\rangle^\tau$ is a weight function of type τ $(0\leqq\tau<1)$. Then the symbol class $S^\mu(\lambda_\tau)$ is $S^{\tau\mu}_{\tau,\tau}$ in Hörmander [8]. And $S^\mu_{\rho,\delta}\subset S^\nu(\lambda_\tau)$ if $\delta\leqq\tau\leqq\rho$, $\tau<1$ and $\nu\geqq\tau\mu$.

EXAMPLE 2. We set

$$\lambda_1(\xi)=\Big(1+\sum_{j=1}^n\xi_j^{2m_j}\Big)^{1/2m},\qquad m=\max_{1\leqq j\leqq n}\{m_j\}.$$

Then $\lambda_1(\xi)$ is a basic weight function.

DEFINITION 3. We say that a C^∞ real valued function $S(x, \xi)$ defined on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ is a phase function if $S(x, \xi)$ satisfies the following conditions:

(P-1) For any two-multi-indices α and β such that $|\alpha|+|\beta|\geqq 2$ there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta S(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{|\alpha|-|\beta|}$$

for any (x, ξ) in $\boldsymbol{R}^n\times\boldsymbol{R}^n$.

(P-2) There exists a positive constant $\delta_0>0$ such that

$$\inf_{(x,\xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^n}|\det(\partial_{x_j}\partial_{\xi_k}S(x, \xi))|\geqq\delta_0 .$$

DEFINITION 4. Let $\lambda(\xi)$ be a weight function of type ε $(0\leqq\varepsilon<1)$. Corresponding to $a(x, \xi)$ in $S^\mu(\lambda)$ and a phase function $S(x, \xi)$ we define a Fourier integral operator A on smooth functions by the formula

$$Au(x)=\int_{\boldsymbol{R}^n}e^{iS(x,\xi)}a(x, \xi)\hat{u}(\xi)đ\xi , \tag{5}$$

and $đ\xi=(2\pi)^{-n}d\xi$.

The defining integral in the right-hand side of (5) converges absolutely at least for any function u in $\mathcal{S}(\boldsymbol{R}^n)$. For we have the estimate

$$|e^{iS(x,\xi)}a(x, \xi)\hat{u}(\xi)| \leqq C_N\langle\xi\rangle^{\varepsilon\mu-N},$$

where N is any positive integer.

Our result is:

THEOREM 1. *Let $\lambda(\xi)$ be a weight function of type ε $(0\leqq\varepsilon<1)$. Suppose that a symbol function $a(x, \xi)$ is in $S^0(\lambda)$ and a phase function $S(x, \xi)$ satisfies* (P-1) *and* (P-2). *Then the Fourier integral operator A is L^2 bounded and has the estimate that*

$$\|Au\| \leqq C_m|a|_{0,m}\|u\|, \tag{6}$$

where m is an integer such that $m>2n/(1-\varepsilon)$.

EXAMPLE 3. If $\lambda(\xi)=1$, then the Fourier integral operator turns out to be an oscillatory integral transformation in Fujiwara [6] and Asada-Fujiwara [1].

EXAMPLE 4. Fujiwara [7] and Kumano-go [12] proved the L^2 boundedness theorem of Fourier integral operators with symbol functions in $S^0_{\rho,\delta}$ $(0\leqq\delta\leqq\rho\leqq1, \delta<1)$, under the condition that $S(x, \xi)$ satisfies (P-3), not (P-1).

(P-3) For any two multi-indices α and β with $|\alpha|+|\beta|\geqq2$, there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta S(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{1-|\beta|}.$$

Applying Theorem 1 to this case we have the following

COROLLARY. *Let τ be a real number such that $0\leqq\tau<1$. We assume the following conditions:*

(i) *For any multi-indices α and β there exists a constant $C_{\alpha,\beta}$ such that*

$$|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{\tau(|\alpha|-|\beta|)}.$$

(ii) *There exists a positive constant δ_0 such that*

$$|\det(\partial_{x_j}\partial_{\xi_k}S(x, \xi))| \geqq \delta_0.$$

(iii) *For any multi-indices α and β with $|\alpha|+|\beta|\geqq2$ there exists a constant $C_{\alpha,\beta}$ such that*

$$|\partial_x^\alpha\partial_\xi^\beta S(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{\tau(|\alpha|-|\beta|)}.$$

Then the Fourier integral operator A is L^2 bounded and has the estimate that

$$\|Au\| \leqq C|a|_{0,m}\|u\|,$$

where m is an integer such that $m>2n/(1-\tau)$.

Remark 3. By Plancherel's theorem we have only to prove that the integral operator

(7) $$f(\xi)\to\int_{R^n} e^{iS(x,\xi)}a(x,\xi)f(\xi)d\xi$$

is L^2 bounded. We again denote by A this integral operator.

§3. Proof of Theorem.

LEMMA 1. *Let* $\lambda_1(\xi)$ *be a basic weight function and* $\lambda_\varepsilon(\xi)=\lambda_1(\xi)^\varepsilon$ $(0\leqq\varepsilon<1)$. *Then* $\lambda_\varepsilon(\xi)$ *satisfies the following estimates*:

(W-3) $$1\leqq\lambda_\varepsilon(\xi)\leqq C_\varepsilon\langle\xi\rangle^\varepsilon.$$

(W-4) *For any multi-index* α *there exists a constant* $C_{\varepsilon,\alpha}$ *such that*

$$|\partial_\xi^\alpha\lambda_\varepsilon(\xi)|\leqq C_{\varepsilon,\alpha}\lambda_\varepsilon(\xi)\lambda_1(\xi)^{-|\alpha|}.$$

This lemma is an immediate consequence of Definition 1. So we omit its proof.

LEMMA 2. *Let* $\lambda_1(\xi)$ *be a basic weight function. Then there exist positive constants* r_0 *and* C *such that* $C^{-1}\leqq\lambda_1(\xi)/\lambda_1(\eta)\leqq C$ *whenever* $|\xi-\eta|\leqq r_0\lambda_1(\xi)$.

Proof. We note from (W-2) that for $|\alpha|=1$

$$|\partial_\xi^\alpha\lambda_1(\xi)|\leqq C.$$

By the mean value theorem we have

$$|\lambda_1(\eta)-\lambda_1(\xi)|\leqq C|\eta-\xi|\leqq Cr_0\lambda_1(\xi).$$

Take a positive constant r_0 such that $r_0C<1/2$. Thus, if $|\xi-\eta|\leqq r_0\lambda_1(\xi)$, then $|\lambda_1(\eta)-\lambda_1(\xi)|\leqq(1/2)\lambda_1(\xi)$. Hence we have $1/2\leqq\lambda_1(\eta)/\lambda_1(\xi)\leqq3/2$.

COROLLARY. *Let* $\lambda_\varepsilon(\xi)$ *be a weight function of type* ε $(0\leqq\varepsilon<1)$. *Then we have*

(i) *If* $|\xi-\eta|\leqq r_0\lambda_\varepsilon(\xi)$, *then* $C^{-1}\leqq\lambda_\varepsilon(\eta)/\lambda_\varepsilon(\xi)\leqq C$.

(ii) *If* $|\xi-\sigma|\leqq r_0\lambda_\varepsilon(\sigma)$ *and* $|\xi-\sigma'|\leqq r_0\lambda_\varepsilon(\sigma')$, *then*

$$C^{-1}\leqq\lambda_\varepsilon(\sigma')/\lambda_\varepsilon(\sigma)\leqq C.$$

(iii) *If* $|\sigma-\sigma'|\leqq\frac{1}{2}r_0(\lambda_\varepsilon(\sigma)+\lambda_\varepsilon(\sigma'))$, *then* $C^{-1}\leqq\lambda_\varepsilon(\sigma')/\lambda_\varepsilon(\sigma)\leqq C$.

From now on we fix a weight function $\lambda_\varepsilon(\xi)$ of type ε $(0\leqq\varepsilon<1)$ and we omit the subscript ε and write $\lambda(\xi)$.

Let r be a positive real number. We set

(8) $$U_{(s,\sigma)}(r)=\{(x,\xi);\ |x-s|\leqq r\lambda(\sigma)^{-1},\ |\xi-\sigma|\leqq r\lambda(\sigma)\}$$

for (s,σ) in $\boldsymbol{R}^n\times\boldsymbol{R}^n$. This set is a neighborhood of (s,σ) in $\boldsymbol{R}^n\times\boldsymbol{R}^n$, where we endow a Riemannian metric at (s,σ) as follows

$$g_{(s,\sigma)}(x, \xi)=\lambda(\sigma)^2|x|^2+\lambda(\sigma)^{-2}|\xi|^2.$$

This Riemannian metric $g_{(s,\sigma)}$ is slowly varying, $g\leqq g^\sigma$ and σ-temperate in the sense of Hörmander [9].

Remark 4. Corollary of Lemma 2 implies that $C^{-1}\leqq\lambda(\xi)/\lambda(\sigma)\leqq C$ for all (x, ξ) in $U_{(s,\sigma)}(r_0)$.

We shall construct a partition of unity with continuous parameters subordinated to a covering $\{U_{(s,\sigma)}(r)\}_{(s,\sigma)\in \boldsymbol{R}^n\times\boldsymbol{R}^n}$ (for some $r>0$) of $\boldsymbol{R}^n\times\boldsymbol{R}^n$ which a weight function $\lambda(\xi)$ defines. This partition of unity is similar to that in Hörmander [9] which depends on discrete parameters.

LEMMA 3. *Let r_1 and r_2 be real numbers such that $0<r_2<r_1<(1/4)r_0$. Then we can choose C^∞-functions $\varphi_{(s,\sigma)}(x, \xi)$ continuously depending on (s, σ) in $\boldsymbol{R}^n\times\boldsymbol{R}^n$ and satisfying the following conditions*:

(i) *Each $\varphi_{(s,\sigma)}(x, \xi)$ is non-negative, strictly positive for all (x, ξ) in $U_{(s,\sigma)}(r_2)$ and is supported in $U_{(s,\sigma)}(r_1)$.*

(ii) $$\iint_{\boldsymbol{R}^n\times\boldsymbol{R}^n}\varphi_{(s,\sigma)}(x, \xi)dsd\sigma=1\,.$$

(iii) *For any two multi-indices α and β there exists a constant $C_{\alpha,\beta}$ such that*

$$\sup_{(x,\xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^n}|\partial_x^\alpha\partial_\xi^\beta\varphi_{(s,\sigma)}(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\sigma)^{|\alpha|-|\beta|},$$

where the constant $C_{\alpha,\beta}$ is independent of (s, σ).

Proof. Take a C^∞ function φ in $\boldsymbol{R}^1$ such that $0\leqq\varphi(t)\leqq 1$, $\varphi(t)=1$ if $t\leqq r_2$ and $\varphi(t)=0$ if $t\geqq r_1$. And set

$$\psi_{(s,\sigma)}(x, \xi)=\varphi(\lambda(\sigma)|x-s|)\cdot\varphi(\lambda(\sigma)^{-1}|\xi-\sigma|)\,, \tag{9}$$

$$\Psi(x, \xi)=\iint_{\boldsymbol{R}^n\times\boldsymbol{R}^n}\psi_{(s,\sigma)}(x, \xi)dsd\sigma\,. \tag{10}$$

First the followings are obvious:

$$0\leqq\psi_{(s,\sigma)}(x, \xi)\leqq 1\,. \tag{11}$$

$$\operatorname*{supp}_{(x,\xi)}\psi_{(s,\sigma)}\subset U_{(s,\sigma)}(r_1) \tag{12}$$

and

$$\psi_{(s,\sigma)}(x, \xi)=1 \quad \text{whenever } (x, \xi) \text{ is in } U_{(s,\sigma)}(r_2)\,. \tag{13}$$

Next we can prove (iii) for $\psi_{(s,\sigma)}(x, \xi)$. By induction we see that

$$\partial_x^\alpha\partial_\xi^\beta\psi_{(s,\sigma)}(x, \xi)=\partial_x^\alpha\varphi(\lambda(\sigma)|x-s|)\cdot\partial_\xi^\beta\varphi(\lambda(\sigma)^{-1}|\xi-\sigma|)$$

is a sum of terms of the form

$$C\varphi^{(j)}(\lambda(\sigma)|x-s|)\cdot\lambda(\sigma)^{j}\prod_{\mu=1}^{j}\partial_{x}{}^{\alpha_{\mu}}|x-s| \times\varphi^{(k)}(\lambda(\sigma)^{-1}|\xi-\sigma|)\cdot\lambda(\sigma)^{-k}\prod_{\nu=1}^{k}\partial_{\xi}{}^{\beta_{\nu}}|\xi-\sigma|, \tag{14}$$

where

$$1\leqq j\leqq|\alpha|,\quad |\alpha_1|+\cdots+|\alpha_j|=|\alpha|$$

$$1\leqq k\leqq|\beta|,\quad |\beta_1|+\cdots+|\beta_k|=|\beta|.$$

Therefore each term (14) is dominated by

$$\begin{aligned}&C_{\alpha,\beta}|\varphi^{(j)}(\lambda(\sigma)|x-s|)|\lambda(\sigma)^{j}\prod_{\mu=1}^{j}|x-s|^{1-|\alpha_{\mu}|}\\ &\qquad\times|\varphi^{(k)}(\lambda(\sigma)^{-1}|\xi-\sigma|)|\lambda(\sigma)^{-k}\prod_{\nu=1}^{k}|\xi-\sigma|^{1-|\beta_{\nu}|}\\ &\leqq C_{\alpha,\beta}\Big(\sum_{j=0}^{|\alpha|}\sup_{t}|\varphi^{(j)}(t)|t^{j-|\alpha|}\Big)\Big(\sum_{k=0}^{|\beta|}\sup_{t}|\varphi^{(k)}(t)|t^{k-|\beta|}\Big)\lambda(\sigma)^{|\alpha|-|\beta|}.\end{aligned}$$

Thus,

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}\psi_{(s,\sigma)}(x,\xi)|\leqq C_{\alpha,\beta}\lambda(\sigma)^{|\alpha|-|\beta|}. \tag{15}$$

Third we show that the inequalities

$$C\leqq\Psi(x,\xi)\leqq C' \tag{16}$$

hold for some positive constants C and C'. We note from Remark 4 that the inequalities

$$C_r\leqq\iint_{R^n\times R^n}\chi_r(\lambda(\sigma)(x-s))\chi_r(\lambda(\sigma)^{-1}(\xi-\sigma))dsd\sigma\leqq C_r' \tag{17}$$

hold for all r such that $0<r<r_0$. Here χ_r denotes the characteristics function of the ball of radius r. The properties (11), (12) and (13) imply that

$$\begin{aligned}\chi_{r_2}(\lambda(\sigma)(x-s))\chi_{r_2}(\lambda(\sigma)^{-1}(\xi-\sigma))&\leqq\psi_{(s,\sigma)}(x,\xi)\\ &\leqq\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma)).\end{aligned} \tag{18}$$

Substituting (18) into (10) and considering (17) we have the inequalities (16).

Fourth, we prove that $\Psi(x,\xi)$ is in $S^0(\lambda)$. We differentiate (10) under integral sign and use (15) and (17) in view of (12) and Remark 4. Thus,

$$\begin{aligned}|\partial_x^{\alpha}\partial_{\xi}^{\beta}\Psi(x,\xi)|&\leqq\iint_{R^n\times R^n}|\partial_x^{\alpha}\partial_{\xi}^{\beta}\psi_{(s,\sigma)}(x,\xi)|dsd\sigma\\ &\leqq C_{\alpha,\beta}\lambda(\sigma)^{|\alpha|-|\beta|}\iint_{R^n\times R^n}\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))dsd\sigma\\ &\leqq C_{\alpha,\beta}\lambda(\sigma)^{|\alpha|-|\beta|}.\end{aligned} \tag{19}$$

Now we set

$$\varphi_{(s,\sigma)}(x, \xi)=\psi_{(s,\sigma)}(x, \xi)/\Psi(x, \xi).$$

From (11), (12), (13), (15), (16) and (19) it is clear that $\varphi_{(s,\sigma)}(x, \xi)$ satisfies the required properties.

Let $t=(s, \sigma)$ be any point in $\boldsymbol{R}^n\times\boldsymbol{R}^n$ and set

$$a_t(x, \xi)=\varphi_t(x, \xi)a(x, \xi). \tag{20}$$

Each $a_t(x, \xi)$ is supported in a set $U_t(r_1)$ and for any two multi-indices α and β estimates

$$|\partial_x^\alpha\partial_\xi^\beta a_t(x, \xi)|\leqq C_{\alpha,\beta}|a|_{0;|\alpha|+|\beta|}\lambda(\sigma)^{|\alpha|-|\beta|}. \tag{21}$$

hold for some constants $C_{\alpha,\beta}$. And define

$$A_tf(x)=\int_{\boldsymbol{R}^n}e^{iS(x,\xi)}a_t(x, \xi)f(\xi)d\xi. \tag{22}$$

Then we have

$$Af(x)=\int_{\boldsymbol{R}^{2n}}A_tf(x)dt. \tag{23}$$

The adjoint operator $A_{t'}^*$ of $A_{t'}$ for $t'=(s', \sigma')$ is given by

$$A_{t'}^*g(\xi)=\int_{\boldsymbol{R}^n}e^{-iS(y,\xi)}\overline{a_{t'}(y, \xi)}g(y)dy, \tag{24}$$

where $\overline{a_{t'}(y, \xi)}$ is the complex conjugate of $a_{t'}(y, \xi)$.

Now we prepare to apply the lemma of Cotlar-Knapp-Stein formulated by Calderón-Vaillancourt [5] (See Lemma 7 below). Thus we have only to prove all of the following estimates:

1° There exists a positive constant C independent of $t=(s, \sigma)$ such that

$$\|A_t\|\leqq C.$$

2° There exist non-negative functions $h(t, t')$ and $k(t, t')$ such that

$$\|A_tA_{t'}^*\|\leqq h(t, t')^2, \quad \|A_t^*A_{t'}\|\leqq k(t, t')^2.$$

3° The above functions satisfy the following estimates

$$\sup_{t'}\int_{\boldsymbol{R}^{2n}}h(t, t')dt\leqq M, \quad \sup_{t'}\int_{\boldsymbol{R}^{2n}}k(t, t')dt\leqq M$$

for some constant M.

Proof of 1° We know from (20) and (22) that the integral kernel function $H_t(x, \xi)$ of A_t is dominated by

$$C|a|_{0,0}\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma)).$$

Thus we have

$$\int_{R^n} |H_t(x, \xi)|\, dx \leqq C_{n,r_1} |a|_{0,0} \lambda(\sigma)^{-n},$$

$$\int_{R^n} |H_t(x, \xi)|\, d\xi \leqq C_{n,r_1} |a|_{0,0} \lambda(\sigma)^{n}.$$

Hence these estimates imply the desired inequality

$$\|A_t\| \leqq C_{n,r_1} |a|_{0,0}.$$

For $t=(s, \sigma)$ and $t'=(s', \sigma')$ we denote the integral kernel functions of the operators $A_t A_{t'}^*$ and $A_t^* A_{t'}$ by $H_{t,t'}(x, y)$ and $K_{t,t'}(\xi, \eta)$ respectively. Thus, from (22) and (24) we have the following expressions.

(25) $$A_t A_{t'}^* f(x) = \int_{R^n} H_{t,t'}(x, y) f(y) dy,$$

where

(26) $$H_{t,t'}(x, y) = \int_{R^n} e^{i(S(x,\xi)-S(y,\xi))} a_t(x, \xi) \overline{a_{t'}(y, \xi)} d\xi.$$

And

(27) $$A_t^* A_{t'} g(\xi) = \int_{R^n} K_{t,t'}(\xi, \eta) g(\eta) d\eta,$$

where

(28) $$K_{t,t'}(\xi, \eta) = \int_{R^n} e^{-i(S(x,\xi)-S(x,\eta))} \overline{a_t(x, \xi)} a_{t'}(x, \eta) dx.$$

Now we shall estimate $H_{t,t'}(x, y)$ and $K_{t,t'}(\xi, \eta)$ in the following Proposition 1. Then we need two lemmas concerning the phase functions and integration by parts (See Lemmas 4 and 5 below). And we shall prove the statements 2° and 3° as Propositions 2 and 3 respectively.

LEMMA 4. 1) *There exists a positive constant* δ_1 *such that*

(29) $$|\nabla_\xi (S(x, \xi) - S(y, \xi))| \geqq \delta_1 |x-y|$$

and

(30) $$|\nabla_x (S(x, \xi) - S(x, \eta))| \geqq \delta_1 |\xi - \eta|.$$

2-i) *For any multi-index* α *such that* $|\alpha| \geqq 1$ *there exists a constant* C_α *such that the estimates*

(31) $$(\lambda(\sigma)^{-1} + \lambda(\sigma')^{-1})^{-|\alpha|} |\partial_\xi^\alpha (S(x, \xi) - S(y, \xi))| \leqq C_\alpha \rho$$

hold for all (x, ξ) *in* $U_{(s,\sigma)}(r_1)$ *and* (y, ξ) *in* $U_{(s',\sigma')}(r_1)$, *where*

(32) $$\rho = \{1 + (\lambda(\sigma)^{-1} + \lambda(\sigma')^{-1})^{-2} |\nabla_\xi (S(x, \xi) - S(y, \xi))|^2\}^{1/2}.$$

2-ii) *For any multi-index α such that $|\alpha|\geqq 1$ there exists a constant C_α such that the estimates*

$$(\lambda(\sigma)+\lambda(\sigma'))^{-|\alpha|}|\partial_x^\alpha(S(x, \xi)-S(x, \eta))|\leqq C_\alpha\tau \tag{33}$$

hold for all (x, ξ) in $U_{(s,\sigma)}(r_1)$ and (x, η) in $U_{(s',\sigma')}(r_1)$, where

$$\tau=\{1+(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\nabla_x(S(x, \xi)-S(x, \eta)|^2\}^{1/2}. \tag{34}$$

Proof. 1) Let $z=\nabla_\xi S(x, \xi)$ and $w=\nabla_\xi S(y, \xi)$. Because of (P-1) and (P-2), we can apply the global implicit function theorem to the mapping

$$T_\xi: \boldsymbol{R}^n\ni x\longrightarrow z=\nabla_\xi S(x, \xi)\in\boldsymbol{R}^n,$$

where $\xi\in\boldsymbol{R}^n$ is fixed. Thus, T_ξ is a global diffeomorphism. When we consider x as a function of (z, ξ), we write $x=x(z, \xi)$. Since the Jacobian matrix $\partial x/\partial z$ is the inverse matrix of $(\partial_{x_j}\partial_{\xi_k}S(x, \xi))$, each component of $\partial x/\partial z$ has an upper bound $\gamma=C_nC_{1,1}^{n-1}\delta_0^{-1}$. By the mean value theorem we obtain

$$|x(z, \xi)-x(w, \xi)|\leqq\gamma|z-w|.$$

Thus,

$$|x-y|\leqq\gamma|\nabla_\xi S(x, \xi)-\nabla_\xi S(y, \xi)|.$$

This is equivalent to the inequality (29) with $\delta_1=\gamma^{-1}=\delta_0/C_nC_{1,1}^{n-1}$. A similar argument shows that the inequality (30) is valid.

2) When $|\alpha|=1$, the inequality (31) is valid from the definition of ρ. When $|\alpha|\geqq 2$, we have, for (x, ξ) in $U_{(s,\sigma)}(r_1)$ and (y, ξ) in $U((_{s',\sigma'})(r_1)$,

$$|\partial_\xi^\alpha S(x, \xi)|\leqq C_\alpha\lambda(\sigma)^{-|\alpha|},\quad |\partial_\xi^\alpha S(y, \xi)|\leqq C_\alpha\lambda(\sigma')^{-|\alpha|}.$$

Then

$$\begin{aligned}&(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-|\alpha|}|\partial_\xi^\alpha S(x, \xi)-\partial_\xi^\alpha S(y, \xi)|\\&\leqq C_\alpha(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-|\alpha|}(\lambda(\sigma)^{-|\alpha|}+\lambda(\sigma')^{-|\alpha|})\\&\leqq 2C_\alpha\leqq 2C_\alpha\rho\,.\end{aligned}$$

Thus, the inequality (31) is valid. By a similar argument we know that the second inequality (33) is also valid. This completes the proof of Lemma 4.

LEMMA 5. *Let L be a partial differential operator of order* 1:

$$Lu(x)=\rho^{-2}(1-iK^{-2}\nabla_x F(x)\cdot\nabla_x)u(x)\,,$$

where K is a positive constant, $F(x)$ is a smooth real-valued function and

$$\rho=(1+K^{-2}|\nabla_x F(x)|^2)^{1/2}.$$

Then,

$$Le^{iF(x)}=e^{iF(x)}. \tag{i}$$

(ii) *We denote by ^tL the formal transposed operator of L. Then for any positive integer m, $(^tL)^m u(x)$ is a sum of terms of the form*

$$C\rho^{-k}\left\{\prod_{\nu=1}^{q} K^{-|\alpha_\nu|}\partial_x^{\alpha_\nu}F(x)\right\}K^{-|\beta|}\partial_x^{\beta}u(x), \tag{35}$$

where

$$\begin{aligned} &2m \leqq k \leqq 4m, \quad k-2m \leqq q \leqq k-m, \\ &|\alpha_\nu| \geqq 1, \quad \sum_{\nu=1}^{q}|\alpha_\nu| \leqq q+m, \quad |\beta| \leqq m. \end{aligned} \tag{36}$$

Proof. We use the same procedure as the proof of Lemma 2.5 in Asada-Fujiwara [1, p. 331].

The identity (i) follows from definition of L and ρ.

To prove (ii) we note that

$$\partial_{x_j}\rho^{-m} = -m\rho^{-m-2}K^{-2}\sum_{k=1}^{n}\partial_{x_k}F\cdot\partial_{x_j}\partial_{x_k}F.$$

Then Leibniz's rule shows that

$$\begin{aligned} {}^tLu(x) &= \rho^{-2}u(x) + iK^{-2}\sum_{j=1}^{n}\partial_{x_j}(\rho^{-2}\partial_{x_j}F(x)\cdot u(x)) \\ &= \rho^{-2}u(x) + i(-2)\rho^{-4}K^{-4}\sum_{j,k=1}^{n}\partial_{x_j}F\cdot\partial_{x_k}F\cdot\partial_{x_j}\partial_{x_k}F\cdot u(x) \\ &\quad + i\rho^{-2}K^{-2}\sum_{j=1}^{n}\partial_{x_j}^2F\cdot u(x) + i\rho^{-2}K^{-2}\sum_{j=1}^{n}\partial_{x_j}F\cdot\partial_{x_j}u. \end{aligned}$$

Thus ^tL is a linear combination of operators of the form

$$\rho^{-2}\times \tag{37}$$

$$\rho^{-4}K^{-4}\partial_{x_j}F\cdot\partial_{x_k}F\cdot\partial_{x_j}\partial_{x_k}F\times, \tag{38}$$

$$\rho^{-2}K^{-2}\partial_{x_j}^2F\times, \tag{39}$$

$$\rho^{-2}K^{-2}\partial_{x_j}F\cdot\partial_{x_j}. \tag{40}$$

Now we say that the term (35) is of the type $(k, q, \sum_{\nu=1}^{q}|\alpha_\nu|, |\beta|)$. Then ^tLu is a sum of terms of the types (2, 0, 0, 0), (4, 3, 4, 0), (2, 1, 2, 0) and (2, 1, 1, 1). When we operate (37), (38) and (39) to a term (35) of the type $(k, q, \sum|\alpha_\nu|, |\beta|)$ once, the type of the resultant term increases by (2, 0, 0, 0), (4, 3, 4, 0) and (2, 1, 2, 0), respectively. Next we examine how the types change when we operate an operator (40) to a term (35). Leibniz's rule shows that

$$\begin{aligned} &\rho^{-2}K^{-2}\partial_{x_j}F\cdot\partial_{x_j}\Big(\rho^{-k}\cdot\prod_{\nu=1}^{q}K^{-|\alpha_\nu|}\partial_x^{\alpha_\nu}F(x)\cdot K^{-|\beta|}\partial_x^{\beta}u(x)\Big) \\ &= (-k)\rho^{-(k+4)}K^{-4}\sum_{i=1}^{n}\partial_{x_i}F\cdot\partial_{x_j}F\cdot\partial_{x_i}\partial_{x_j}F\prod_{\nu=1}^{q}K^{-|\alpha_\nu|}\partial_x^{\alpha_\nu}F\times K^{-|\beta|}\partial_x^{\beta}u(x) \end{aligned}$$

$$+\rho^{-(k+2)}K^{-1}\partial_{x_j}F\times\sum_{\mu=1}^{q}\Big(\prod_{\nu\neq\mu}K^{-|\alpha_\nu|}\partial_x^{\alpha_\nu}F\Big)K^{-1}\partial_{x_j}K^{-|\alpha_\mu|}\partial_x^{\alpha_\mu}F\times K^{-|\beta|}\partial_x^\beta u(x)$$

$$+\rho^{-(k+2)}K^{-1}\partial_{x_j}F\times\prod_{\nu=1}^{q}K^{-|\alpha_\nu|}\partial_x^{\alpha_\nu}F(x)\times K^{-1}\partial_{x_j}(K^{-|\beta|}\partial_x^\beta u(x)).$$

The resultant terms under operations of (40) are a sum of terms the types of which increase by (4, 3, 4, 0), (2, 1, 2, 0) and (2, 1, 1, 1). Consequently, when we operate ${}^tL$ to a term (35), the types of the resultant terms increase by (2, 0, 0, 0), (4, 3, 4, 0), (2, 1, 2, 0) and (2, 1, 1, 1). We repeat the process; thus we have

$$({}^tL)^m u(x)=\sum C\rho^{-k}\prod_{\nu=1}^{q}K^{-|\alpha_\nu|}\partial_x^{\alpha_\nu}F(x)\times K^{-|\beta|}\partial_x^\beta u(x).$$

Here the summation is taken all over non-negative integers i_1, i_2, i_3, i_4 such that $i_1+i_2+i_3+i_4=m$. And

$$(k,\ q,\ \sum|\alpha_\nu|,\ |\beta|)=i_1(2,0,0,0)+i_2(4,3,4,0)+i_3(2,1,2,0)+i_4(2,1,1,1).$$

Then k, q, α_ν, β satisfy the condition (36). This completes the proof of Lemma 5.

Now using Lemmas 4 and 5 we obtain estimates for the integral kernel functions $H_{t,t'}(x, y)$ and $K_{t,t'}(\xi, \eta)$, where $t=(s, \sigma)$ and $t'=(s', \sigma')$ are parameters in $\boldsymbol{R}^n\times\boldsymbol{R}^n$.

PROPOSITION 1. 1) *For any non-negative integer m there exists a constant C_m such that*

$$(41)\qquad |H_{t,t'}(x,y)|\leqq C_m|a|_{0;m}^2\min\{\lambda(\sigma),\lambda(\sigma')\}^n\chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big)$$
$$\times\frac{\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma')(y-s'))}{(1+\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|x-y|^2)^{m/2}},$$

where χ_r is the characterisric function of the ball $\{x; |x|\leqq r\}$. And the above constant C_m is independent of x, y, $t=(s, \sigma)$ and $t'=(s', \sigma')$.

2) *For any non-negative integer m there exists a constant C_m such that*

$$(42)\qquad |K_{t,t'}(\xi,\eta)|\leqq C_m|a|_{0,m}^2\min\{\lambda(\sigma)^{-1},\lambda(\sigma')^{-1}\}^n\chi_{r_1}\Big(\frac{s-s'}{\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}}\Big)$$
$$\times\frac{\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))\chi_{r_1}(\lambda(\sigma')^{-1}(\eta-\sigma'))}{(1+\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\xi-\eta|^2)^{m/2}},$$

where C_m is independent of x, y, $t=(s, \sigma)$ and $t'=(s', \sigma')$.

Proof. 1) We set

$$F(\xi, x, y)=S(x,\xi)-S(y,\xi).$$

Then from (26) we have

$$(43)\qquad H_{t,t'}(x,y)=\int_{R^n}e^{iF(\xi,x,y)}a_t(x,\xi)\overline{a_{t'}(y,\xi)}d\xi.$$

Let L be a partial differential operator of order 1:

$$L=\rho^{-2}\{1-i(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}\nabla_\xi F(\xi,\ x,\ y)\cdot\nabla_\xi\},$$

where

$$\rho=\{1+(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|\nabla_\xi F|^2\}^{1/2}.$$

Then we rewrite the right-hand side of (43) using the identity (i) of Lemma 5 and integrate by parts, and repeat the process; thus we have

$$H_{t,t'}(x,\ y)=\int_{R^n}e^{iF(\xi,x,y)}({}^tL)^m[a_t(x,\ \xi)\overline{a_{t'}(y,\ \xi)}]d\xi\,, \tag{44}$$

where m is an arbitrary non-negative integer. Applying (ii) of Lemma 5 we see that $({}^tL)^m(a_t\overline{a_{t'}})$ is a sum of terms of the form

$$C\rho^{-k}(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-j}\cdot\prod_{\nu=1}^{q}\partial_\xi^{\alpha_\nu}F\cdot\partial_\xi^\beta(a_t\overline{a_{t'}})\,, \tag{45}$$

where

$$\begin{gathered}2m\leqq k\leqq 4m\,,\quad k-2m\leqq q\leqq k-m\,,\\ |\alpha_\nu|\geqq 1\,,\quad q\leqq\sum_{\nu=1}^{q}|\alpha_\nu|\leqq q+m\,,\quad |\beta|\leqq m\,,\\ j=\sum_{\nu=1}^{q}|\alpha_\nu|+|\beta|.\end{gathered} \tag{46}$$

Leibniz's rule and estimates (21) show that

$$|\partial_\xi^\beta(a_t\overline{a_{t'}})|\leqq C_\beta|a|_{0,|\beta|}^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{|\beta|}. \tag{47}$$

Estimates (31) of Lemma 4, (46) and (47) show that each term (45) is dominated by

$$\begin{aligned}&C_m\rho^{-k}\sum_{\nu=1}^{q}\frac{|\partial_\xi^{\alpha_\nu}F|}{(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{|\alpha_\nu|}}\cdot\frac{|\partial_\xi^\beta(a_t\overline{a_{t'}})|}{(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{|\beta|}}\\ &\leqq C_m\rho^{-k+q}|a|_{0,m}^2\leqq C_m\rho^{-m}|a|_{0,m}^2\,.\end{aligned}$$

Thus,

$$|({}^tL)^m(a_t\overline{a_{t'}})|\leqq C_m\rho^{-m}|a|_{0,m}^2\,. \tag{48}$$

When we apply estimates (48) and (29) of Lemma 4 to the right-hand side of (44), considering the support of the integrand in it, we have

$$\begin{aligned}|H_{t,t'}(x,\ y)|\leqq C_m|a|_{0,m}^2&\frac{\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma')(y-s))}{\{1+\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|x-y|^2\}^{m/2}}\\ &\times\int_{R^n}\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))\chi_{r_1}(\lambda(\sigma')^{-1}(\xi-\sigma'))d\xi\,.\end{aligned}$$

We note that

(49)
$$\int_{R^n}\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))\chi_{r_1}(\lambda(\sigma')^{-1}(\xi-\sigma'))d\xi$$
$$\leqq C_{n,r_1}\min\{\lambda(\sigma),\ \lambda(\sigma')\}^n\chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big).$$

Therefore we have

$$|H_{t,t'}(x,\ y)|\leqq C_{m,n,r_1}|a|^2_{0,m}\min\{\lambda(\sigma),\ \lambda(\sigma')\}^n\chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big)$$
$$\times\frac{\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma')(y-s'))}{\{1+\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|x-y|^2\}^{m/2}}.$$

This proves the estimate (41).

2) Set $G(x,\ \xi,\ \eta)=S(x,\ \xi)-S(x,\ \eta)$ and

$$L=\tau^{-2}\{1-i(\lambda(\sigma)+\lambda(\sigma'))^{-2}\nabla_x G\cdot\nabla_x\},$$

where

$$\tau=\{1+(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\nabla_x G|^2\}^{1/2}.$$

Then integrating by parts in (28), we obtain

$$K_{t,t'}(\xi,\ \eta)=\int_{R^n}e^{iG(x,\xi,\eta)}({}^tL)^m(\overline{a_t(x,\ \xi)}a_{t'}(x,\ \eta))dx.$$

By Lemma 5-2) and Leibniz's rule we have the estimate

$$|({}^tL)^m(\overline{a_t(x,\ \xi)}a_{t'}(x,\ \eta))|\leqq C_m|a|^2_{0,m}\tau^{-m}.$$

Thus, noting the support of the integrand and using the estimate (30) in Lemma 4 we have

$$|K_{t,t'}(\xi,\ \eta)|\leqq\int_{R^n}|({}^tL)^m(\overline{a_t(x,\ \xi)}a_{t'}(x,\ \eta))|dx$$
$$\leqq C_m|a|^2_{0,m}\frac{\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))\chi_{r_1}(\lambda(\sigma')^{-1}(\eta-\sigma'))}{\{1+\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\xi-\eta|^2\}^{m/2}}$$
$$\times\int_{R^n}\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma')(x-s'))dx$$
$$\leqq C_m|a|^2_{0,m}\min\{\lambda(\sigma)^{-1},\ \lambda(\sigma')^{-1}\}^n\chi_{r_1}\Big(\frac{s-s'}{\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}}\Big)$$
$$\times\frac{\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))\chi_{r_1}(\lambda(\sigma')^{-1}(\eta-\sigma'))}{\{1+\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\xi-\eta|^2\}^{m/2}}.$$

This completes the proof of Proposition 1.

Next we obtain the estimates of the L^1-norms of $H_{t,t'}(x,\ y)$ and $K_{t,t'}(\xi,\ \eta)$ with respect to the first and second variables, respectively.

PROPOSITION 2. 1) *Let m be a non-negative integer. Set*

$$h_1(t, t')=C_m|a|_{0;m}\chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big)\chi_{r_1}\Big(\frac{s-s'}{2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}}\Big), \tag{50}$$

$$h_2(t, t')=C_m|a|_{0,m}\chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big) \times\frac{1-\chi_{r_1}\Big(\dfrac{s-s'}{2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})}\Big)}{\Big\{1+\dfrac{1}{4}\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|s-s'|^2\Big\}^{m/4}} \tag{51}$$

and

$$h(t, t')=h_1(t, t')+h_2(t, t').$$

Then we have the estimates

$$\sup_y\int_{R^n}|H_{t,t'}(x, y)|dx\leqq h(t, t')^2 \tag{52}$$

and

$$\sup_x\int_{R^n}|H_{t,t'}(x, y)|dy=h(t, t')^2, \tag{53}$$

where the constant C_m is independent of $t=(s, \sigma)$ and $t'=(s', \sigma')$.

2) *For any positive integer m we set*

$$k_1(t, t')=C_m|a|_{0,m}\chi_{r_1}\Big(\frac{s-s'}{\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}}\Big)\chi_{r_1}\Big(\frac{\sigma-\sigma'}{2(\lambda(\sigma)+\lambda(\sigma'))}\Big), \tag{54}$$

$$k_2(t, t')=C_m|a|_{0,m}\chi_{r_1}\Big(\frac{s-s'}{\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}}\Big) \times\frac{1-\chi_{r_1}\Big(\dfrac{\sigma-\sigma'}{2(\lambda(\sigma)+\lambda(\sigma'))}\Big)}{\Big\{1+\dfrac{1}{4}\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\sigma-\sigma'|^2\Big\}^{m/4}} \tag{55}$$

and

$$k(t, t')=k_1(t, t')+k_2(t, t').$$

Then we have the estimates

$$\sup_\eta\int_{R^n}|K_{t,t'}(\xi, \eta)|d\xi\leqq k(t, t')^2 \tag{56}$$

and

$$\sup_\xi\int_{R^n}|K_{t,t'}(\xi, \eta)|d\eta\leqq k(t, t')^2. \tag{57}$$

Here C_m is some constant independent of $t=(s, \sigma)$ and $t'=(s', \sigma')$.

COROLLARY OF PROPOSITION 2. *We have the following estimates*

(58) $$\|A_t A_{t'}^*\|^{1/2} \leqq h(t, t'),$$

(59) $$\|A_t^* A_{t'}\|^{1/2} \leqq k(t, t').$$

Proof of Corollary. We apply Schur's lemma. Thus the estimate (58) follows from (52) and (53). And the estimate (59) follows from (56) and (57).

Proof of Proposition 2. 1) We consider separately two cases:

(60) $$|s-s'| \leqq 2r_1(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}),$$

(61) $$|s-s'| \geqq 2r_1(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}).$$

First we work out the case (60). Take $m=0$ in (41). Then

(62) $$|H_{t,t'}(x, y)| \leqq C_0 |a|_{0,0}^2 \min\{\lambda(\sigma), \lambda(\sigma')\}^n \times \chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big)\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma')(y-s')).$$

Integration of (60) in x yields

(63) $$\int_{R^n} |H_{t,t'}(x, y)| dx \leqq C_{0,n,r_1} |a|_{0,0}^2 \chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big).$$

We pass to the non-trivial case (61). We know from (41) that

$$\lambda(\sigma)|x-s| \leqq r_1 \quad \text{and} \quad \lambda(\sigma')|y-s'| \leqq r_1$$

whenever (x, y) is in the support of $H_{t,t'}$. Therefore in the case of (61) we have

$$\begin{aligned}|s-s'| &\leqq |s-x|+|x-y|+|y-s'| \\ &\leqq (\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})r_1+|x-y| \\ &\leqq \frac{1}{2}|s-s'|+|x-y|.\end{aligned}$$

Thus,

(64) $$\frac{1}{2}|s-s'| \leqq |x-y|.$$

Substitution of this inequality (64) into the right-hand side of (41) yields

(65) $$|H_{t,t'}(x, y)| \leqq C_m |a|_{0,m}^2 \min\{\lambda(\sigma), \lambda(\sigma')\}^n \times \frac{\chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big)\chi_{r_1}(\lambda(\sigma)(x-s))\chi_{r_1}(\lambda(\sigma')(y-s'))}{\Big\{1+\frac{1}{4}\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|s-s'|^2\Big\}^{m/2}}$$

Then integration of (65) in x shows that

$$\int_{R^n}|H_{t,t'}(x, y)|dx\leqq C_m|a|_{0,m}^2\chi_{r_1}\Big(\frac{\sigma-\sigma'}{\lambda(\sigma)+\lambda(\sigma')}\Big) \times\Big\{1+\frac{1}{4}\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|s-s'|^2\Big\}^{-m/2}\leqq h_2(t, t')^2. \tag{66}$$

Therefore (63) and (66) imply the desired estimate (52). By a similar argument we have the estimate (53). This completes the proof of the part 1).

2) By an argument similar to the proof of the part 1) we know that the statement 2) is valid.

PROPOSITION 3. 1) *Let m be an arbitrary integer such that $m>2n$. Then we have the estimate*

$$\int_{R^{2n}}h(t, t')dt\leqq C_m|a|_{0,m}. \tag{67}$$

2) *Let m be an arbitrary integer such that $m>2n/(1-\varepsilon)$. Then we have the estimate*

$$\int_{R^{2n}}k(t, t')dt\leqq C_m|a|_{0,m}. \tag{68}$$

Proof. 1) We note from (iii) in Corollary to Lemma 2 that $C_2^{-1}\leqq\lambda(\sigma')/\lambda(\sigma)\leqq C_2$ in the support of h_1 and h_2.

We first prove estimate (67) for h_1. Since the characteristic function $\chi_{r_1}(\sigma)$ is a monotone non-increasing function of $|\sigma|$, we dominate $h_1(t, t')$ by

$$C|a|_{0,0}\chi_{r_1}\Big(\frac{\sigma-\sigma'}{(C_2+1)\lambda(\sigma')}\Big)\chi_{r_1}\Big(\frac{s-s'}{2(C_2+2)\lambda(\sigma')^{-1}}\Big).$$

Then

$$\int_{R^{2n}}h_1(t, t')dt\leqq C_{0,n,r_1}|a|_{0,0}\lambda(\sigma')^n\lambda(\sigma')^{-n}\leqq C_{0,n,r_1}|a|_{0,0}.$$

Next we prove (67) for h_2. We bound $h_2(t, t')$ from above:

$$h_2(t, t')\leqq C_m|a|_{0,m}\chi_{r_1}\Big(\frac{\sigma-\sigma'}{(C_2+1)\lambda(\sigma')}\Big) \times\Big\{1+\frac{1}{4}\delta_1^2(1+C_2^{-1})^{-2}\lambda(\sigma')^2|s-s'|^2\Big\}^{-m/4}$$

Therefore we have

$$\int_{R^{2n}}h_2(t, t')dt\leqq C_m|a|_{0,m}\int_{R^n}\chi_{r_1}\Big(\frac{\sigma-\sigma'}{(C_2+1)\lambda(\sigma')}\Big)d\sigma \times\int_{R^n}\Big\{1+\frac{1}{4}\delta_1^2(1+C_2^{-1})^{-2}\lambda(\sigma')^2|s-s'|^2\Big\}^{-m/4}ds.$$

$$\leqq C_m|a|_{0,m}\delta_1^{-n}\int_{R^n}(1+|s|^2)^{-m/4}ds,$$

which is finite, independent of t' if $m>2n$.

Thus (67) is proved for $h=h_1+h_2$.

2) We know from (iii) in Corollary to Lemma 2 that $C_2^{-1}\leqq\lambda(\sigma')/\lambda(\sigma)\leqq C_2$ in the support of k_1. A similar argument shows that

$$k_1(t,\ t')\leqq C_0|a|_{0,0}\chi_{r_1}\Big(\frac{s-s'}{(C_2+1)\lambda(\sigma')^{-1}}\Big)\chi_{r_1}\Big(\frac{\sigma-\sigma'}{2(C_2+1)\lambda(\sigma')}\Big).$$

Then we have

$$\int_{R^{2n}}k_1(t,\ t')dt\leqq C_{0,n,r_1}|a|_{0,0}.$$

The desired estimate (68) is proved for k_1.

Next we prove (68) for k_2. We integrate (55) first with respect to s and then to σ, and we have

$$\begin{aligned}(69)\qquad &\int_{R^{2n}}k_2(t,\ t')dt\leqq C_m|a|_{0,m}\\ &\times\int_{R^n}\Big\{1+\frac{1}{4}\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\sigma-\sigma'|^2\Big\}^{-m/4}d\sigma\int_{R^n}\chi_{r_1}\Big(\frac{s-s'}{\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1}}\Big)ds\\ &\leqq C_{m,n,r_1}|a|_{0,m}\int_{R^n}\frac{(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^n d\sigma}{\Big\{1+\frac{1}{4}\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\sigma-\sigma'|^2\Big\}^{m/4}}.\end{aligned}$$

We make use of the following lemma to handle the estimate of the right-hand side of (69).

LEMMA 6. *Let $\lambda(\xi)$ be a weight function of type ε $(0\leqq\varepsilon<1)$. Then for any positive number $N\geqq\varepsilon/(1-\varepsilon)$ there exists a constant C_N such that*

$$(70)\qquad \frac{\lambda(\xi)}{\lambda(\eta)}\leqq C_N\Big(1+\frac{|\xi-\eta|^2}{\lambda(\xi)^2}\Big)^{N/2}$$

for any ξ and η in $\boldsymbol{R}^n$.

Admitting Lemma 6 for the moment, we continue the proof of Proposition 3-2). We are searching for a bound for the right-hand side of (69). We divide the right-hand side of (69) into two parts:

$$J_1=C_m|a|_{0,m}\int_{\lambda(\sigma)\leqq\lambda(\sigma')}\frac{(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^n d\sigma}{\Big\{1+\frac{1}{4}\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\sigma-\sigma'|^2\Big\}^{m/4}}$$

$$J_2=C_m|a|_{0,m}\int_{\lambda(\sigma)\geqq\lambda(\sigma')}\frac{(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^n d\sigma}{\Big\{1+\frac{1}{4}\delta_1^2(\lambda(\sigma)+\lambda(\sigma'))^{-2}|\sigma-\sigma'|^2\Big\}^{m/4}}.$$

First we work out J_1.

$$J_1 \leqq C_m |a|_{0,m} \min\left\{1, \frac{1}{16}\delta_1^2\right\}^{-m/4} \int_{R^n} \frac{\lambda(\sigma)^{-n} d\sigma}{\{1+\lambda(\sigma')^{-2}|\sigma-\sigma'|^2\}^{m/4}}.$$

We use the inequality (70) in Lemma 6. If we take $\xi=\sigma'$ and $\eta=\sigma$, then we have

$$\lambda(\sigma)^{-1} \leqq C_N \lambda(\sigma')^{-1}\left(1+\frac{|\sigma-\sigma'|^2}{\lambda(\sigma')^2}\right)^{N/2}, \qquad N=\varepsilon/(1-\varepsilon).$$

Then

$$(71) \qquad J_1 \leqq C_m \min\left\{1, \frac{1}{16}\delta_1^2\right\}^{-m/4} |a|_{0,m} \int_{R^n} \frac{\lambda(\sigma')^{-n} d\sigma}{(1+\lambda(\sigma')^{-2}|\sigma-\sigma'|^2)^{(m/4)-(nN/2)}}$$

$$\leqq C_m \min\left\{1, \frac{1}{16}\delta_1^2\right\}^{-m/4} |a|_{0,m} \int_{R^n} \frac{d\sigma}{(1+|\sigma|^2)^{(m/4)-(nN/2)}}.$$

If $m>2n/(1-\varepsilon)$, then $m>2n(1+N)$. Hence $(m/2)-nN>n$. Therefore the right-hand side of (71) is finite and independent of σ'. Thus we have the estimate

$$J_1 \leqq C_m |a|_{0,m}.$$

Next we consider J_2. Since we know from Lemma 6 that

$$\frac{1}{\lambda(\sigma')} \leqq C_N \frac{1}{\lambda(\sigma)}\left(1+\frac{|\sigma-\sigma'|^2}{\lambda(\sigma)^2}\right)^{N/2},$$

we obtain the estimate that

$$1+\frac{|\sigma-\sigma'|^2}{\lambda(\sigma')^2} \leqq C\left(1+\frac{|\sigma-\sigma'|^2}{\lambda(\sigma)^2}\right)^{N+1}.$$

Then

$$J_2 \leqq C_m |a|_{0,m} \min\left\{1, \frac{1}{16}\delta_1^2\right\}^{-m/4} \int_{R^n} \frac{\lambda(\sigma')^{-n} d\sigma}{(1+\lambda(\sigma)^{-2}|\sigma-\sigma'|^2)^{m/4}}$$

$$\leqq C_m \min\left\{1, \frac{1}{16}\delta_1^2\right\}^{-m/4} |a|_{0,m} \int_{R^n} \frac{\lambda(\sigma')^{-n} d\sigma}{(1+\lambda(\sigma')^{-2}|\sigma-\sigma'|^2)^{m/\{4(N+1)\}}}$$

$$\leqq C_m \min\left\{1, \frac{1}{16}\delta_1^2\right\}^{-m/4} |a|_{0,m} \int_{R^n} \frac{d\sigma}{(1+|\sigma|^2)^{m/\{4(N+1)\}}},$$

which is finite and independent of σ' if $m>2n(N+1)$. Thus,

$$J_2 \leqq C_m |a|_{0,m}.$$

Hence we have the estimate

$$\int_{R^{2n}} k_2(t, t')dt \leqq J_1+J_2 \leqq C_m |a|_{0,m}.$$

We have proved Proposition 3, assuming Lemma 6.

Proof of Lemma 6. We take the basic weight function $\lambda_1(\xi)$ such that $\lambda(\xi) =\lambda_1(\xi)^\varepsilon$. If $|\xi-\eta| \leqq r_0\lambda_1(\xi)$, then we have, from Lemma 2, the estimates

$$C^{-1}\leqq\lambda_1(\xi)/\lambda_1(\eta)\leqq C.$$

Thus it is clear that (70) is valid in this case.

If $|\xi-\eta|\geqq r_0\lambda_1(\xi)$, then

$$\left(1+\frac{|\xi-\eta|^2}{\lambda(\xi)^2}\right)^{\varepsilon/2(1-\varepsilon)}\geqq(1+r_0^2\lambda_1(\xi)^{2(1-\varepsilon)})^{\varepsilon/2(1-\varepsilon)}$$
$$\geqq r_0^{\varepsilon/(1-\varepsilon)}\lambda_1(\xi)^{\varepsilon}\geqq r_0^{\varepsilon/(1-\varepsilon)}\lambda_\varepsilon(\xi)\geqq r_0^{\varepsilon/(1-\varepsilon)}\lambda_\varepsilon(\xi)/\lambda_\varepsilon(\eta).$$

Therefore in this case we also have the estimate (70). This completes the proof of Lemma 6.

Now we have established Propositions 2 and 3 to apply the following lemma formulated by Calderón-Vaillancourt ([5]).

LEMMA 7. *Let $t\to A_t$ be a continuous function from $\boldsymbol{R}^n$ to bounded operators on Hilbert space, and suppose that*

$$\|A_t\|\leqq C,$$

$$\|A_tA_{t'}^*\|^{1/2}\leqq h(t, t'), \quad \|A_t^*A_{t'}\|^{1/2}\leqq k(t, t'),$$

where $h(t, t')$ and $k(t, t')$ satisfy the estimates

$$\sup_{t'}\int_{\boldsymbol{R}^n}h(t, t')dt\leqq M, \quad \sup_{t'}\int_{\boldsymbol{R}^n}k(t, t')dt\leqq M.$$

Then for any compact set K in $\boldsymbol{R}^n$ we have the estimate

$$\left\|\int_K A_tdt\right\|\leqq M,$$

where the constant M is independent of K.

Proof. If $A=\int_K A_tdt$, we have $\|A\|^2=\|A^*A\|$ and more generally, by the spectral theorem, $\|A\|^{2m}=\|(A^*A)^m\|$. We expand in an integral and use the fact that

$$\|A_1^*A_2\cdots A_{2m-1}^*A_{2m}\|$$
$$\leqq\min\{\|A_1^*A_2\|\cdots\|A_{2m-1}^*A_{2m}\|, \ \|A_1^*\|\,\|A_2A_3^*\|\cdots\|A_{2m-1}A_{2m-1}^*\|\,\|A_{2m}\|\}.$$

Taking the geometric mean of the two estimates and noting that $\|A_t\|\leqq C$ by hypothesis, we obtain

$$\|A\|^{2m}=\|(A^*A)^m\|\leqq\iint\cdots\int\|A_{t_1}^*\|^{1/2}\|A_{t_1}^*A_{t_2}\|^{1/2}\cdots\|A_{t_{2m-2}}A_{t_{2m-1}}^*\|^{1/2}$$
$$\times\|A_{t_{2m-1}}^*A_{t_{2m}}\|^{1/2}\|A_{t_{2m}}\|^{1/2}dt_1dt_2\cdots dt_{2m}$$
$$\leqq C\iint\cdots\int k(t_1, t_2)h(t_2, t_3)\cdots h(t_{2m-2}, t_{2m-1})$$
$$\times k(t_{2m-1}, t_{2m})dt_1dt_2\cdots dt_{2m}$$

$$\leqq C|K|M^{2m-1},$$

where $|K|$ is the volume of K. Hence,

$$\|A\|\leqq(C|K|/M)^{1/2m}M,$$

and letting $m\to\infty$ we obtain $\|A\|\leqq M$.

PROPOSITION 4. *Let $\lambda(\xi)$ be a weight function of type ε ($0\leqq\varepsilon<1$). Suppose that $a(x, \xi)$ is a symbol function in $S^0(\lambda)$ and $S(x, \xi)$ is a phase function. If $a(x, \xi)$ has compact support, then the Fourier integral operator A is L^2 bounded and has the following estimate*

$$\|Af\|\leqq C_m|a|_{0,m}\|f\|,$$

where m is an integer such that $m>2n/(1-\varepsilon)$ and the constant C_m is independent of the support of $a(x, \xi)$.

Proof. From the inequality 1°) and the estimates in Propositions 2 and 3 we know that A_t defined in (22) satisfies the conditions of Lemma 7 if $a(x, \xi)$ has compact support. Applying Lemma 7 we have the conclusion of Proposition 4.

Now it remains to prove Theorem 1 when a symbol function $a(x, \xi)$ has non compact support. To handle this case we make use of the following lemma.

LEMMA 8. *If $a(x, \xi)$ in $S^0(\lambda)$, then we have the estimate*

$$\|Af\|\leqq C_m|a|_{0,m}\sum_{|\alpha|\leqq m}\|\langle\xi\rangle^{m+|\alpha|\varepsilon}\partial_\xi^\alpha(e^{iS(0,\xi)}f(\xi))\| \tag{72}$$

for any function f in $\mathcal{S}(R^n)$, where m is an integer such that $m>n/(1-\varepsilon)$.

Admitting Lemma 8 for the moment, we prove Theorem 1.

Proof of Theorem 1. Let $a_j(x, \xi)$ be a bounded sequence in $S^0(\lambda)$ which converges to a symbol $a(x, \xi)$ in the topology of $S^0(\lambda)$. And suppose that each $a_j(x, \xi)$ has compact support. Then for f in $\mathcal{S}(R^n)$

$$\|Af\|\leqq\liminf_{j\to\infty}\|(A-A_j)f\|+\liminf_{j\to\infty}\|A_jf\|$$

$$\leqq\liminf_{j\to\infty}|a-a_j|_{0,m}\sum_{|\alpha|\leqq m}\|\langle\xi\rangle^{m+|\alpha|\varepsilon}\partial_\xi^\alpha(e^{iS(0,\xi)}f(\xi))\|+\liminf_{j\to\infty}C_m|a_j|_{0,m}\|f\|$$

$$\leqq C_m|a|_{0,m}\|f\|.$$

We have proved the proof of Theorem 1, assuming Lemma 8.

Proof of Lemma 8. Set

$$Bg(x)=\int_{R^n}e^{i(S(x,\xi)-S(0,\xi))}a(x, \xi)g(\xi)d\xi.$$

Then $Af(x)=B(e^{iS(0,\xi)}f(\xi))$. Hence if suffices to prove that the estimate

$$\|Bg\|\leqq C_m|a|_{0,m}\sum_{|\alpha|\leqq m}\|\langle\xi\rangle^{m+|\alpha|\varepsilon}\partial_\xi^\alpha g(\xi)\| \tag{73}$$

holds.

Let $\varphi_{(s,\sigma)}(x, \xi)$ be a partition of unity in Lemma 3. For σ' in $\boldsymbol{R}^n$ we set

$$\varphi_{\sigma'}(\xi)=\int_{\boldsymbol{R}^n}\varphi_{(s,\sigma')}(0, \xi)ds\,.$$

Then

$$|\partial_\xi^\beta\varphi_{\sigma'}(\xi)|\leqq C_\beta\lambda(\xi)^{-|\beta|}\chi_{r_1}(\lambda(\sigma')^{-1}(\xi-\sigma'))\,. \tag{74}$$

Define

$$B_{(s,\sigma,\sigma')}g(x)=\int_{\boldsymbol{R}^n}e^{i(S(x,\xi)-S(0,\xi))}a_{(s,\sigma)}(x, \xi)\varphi_{\sigma'}(\xi)g(\xi)d\xi\,. \tag{75}$$

Then we have

$$Bg(x)=\int_{\boldsymbol{R}^n}B_{(s,\sigma,\sigma')}g(x)dsd\sigma d\sigma'. \tag{76}$$

Here we note that (75) is of the form similar to $H_{(s,\sigma,0,\sigma')}(x, 0)$.

Let L_0 be a partial differential operator of order 1:

$$L_0=\rho_0^{-2}\{1-i(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}\nabla_\xi(S(x, \xi)-S(0, \xi))\cdot\nabla_\xi\},$$

where

$$\rho_0=\{1+(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|\nabla_\xi(S(x, \xi)-S(0, \xi))|^2\}^{1/2}.$$

Then integration by parts in (75) yields that

$$B_{(s,\sigma,\sigma')}g(x)=\int_{\boldsymbol{R}^n}e^{i(S(x,\xi)-S(0,\xi))}({}^tL_0)^m(a_{(s,\sigma)}(x, \xi)\varphi_{\sigma'}(\xi)g(\xi))d\xi\,.$$

By a similar argument in the proof of Proposition 1 we obtain the estimate

$$\begin{aligned}|B_{(s,\sigma,\sigma')}g(x)|\leqq\, & C_m|a|_{0,m}\chi_{r_1}(\lambda(\sigma)(x-s))\\ &\times\int_{\boldsymbol{R}^n}\frac{\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))\chi_{r_1}(\lambda(\sigma')^{-1}(\xi-\sigma'))}{\{1+\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|x|^2\}^{m/2}}\\ &\times\sum_{|\beta|\leqq m}\lambda(\sigma')^{|\beta|}\partial_\xi^\beta(\varphi_{\sigma'}(\xi)g(\xi))|d\xi\,.\end{aligned}$$

We set

$$\begin{aligned}\tilde{H}_{(s,\sigma,\sigma')}(x, \xi)=\, & C_m|a|_{0,m}\chi_{r_1}(\lambda(\sigma)(x-s))\\ &\times\frac{\chi_{r_1}(\lambda(\sigma)^{-1}(\xi-\sigma))\chi_{r_1}(\lambda(\sigma')^{-1}(\xi-\sigma'))}{\{1+\delta_1^2(\lambda(\sigma)^{-1}+\lambda(\sigma')^{-1})^{-2}|x|^2\}^{m/2}}\end{aligned}$$

and

$$G_{\sigma'}(\xi)=\sum_{|\beta|\leqq m}\lambda(\sigma')^{|\beta|}|\partial_\xi^\beta(\varphi_{\sigma'}(\xi)g(\xi))|.$$

Then

$$|B_{(s,\sigma,\sigma')}g(x)| \leqq \int_{R^n} \tilde{H}_{(s,\sigma,\sigma')}(x, \xi) G_{\sigma'}(\xi) d\xi . \tag{77}$$

By a similar argument in the proof of Propositions 2 and 3 there exists a positive function $\tilde{h}(s, \sigma, \sigma')$ such that

$$\int_{R^n} \tilde{H}_{(s,\sigma,\sigma')}(x, \xi) dx \leqq C_m |a|_{0,m} \tilde{h}(s, \sigma, \sigma') , \tag{78}$$

$$\int_{R^n} \tilde{H}_{(s,\sigma,\sigma')}(x, \xi) d\xi \leqq C_m |a|_{0,m} \tilde{h}(s, \sigma, \sigma') \tag{79}$$

and

$$\int_{R^{2n}} \tilde{h}(s, \sigma, \sigma') ds d\sigma \leqq C , \tag{80}$$

where m is an arbitrary integer such that $m > n/(1-\varepsilon)$. By Schur's lemma and (77), (78) and (79) we have

$$\|B_{(s,\sigma,\sigma')}g\| \leqq C_m |a|_{0,m} \tilde{h}(s, \sigma, \sigma') \|G_{\sigma'}(\sigma)\| .$$

Then by Minkowski's inequality and (80) we have

$$\begin{aligned} \|Bg\| &\leqq \int_{R^{3n}} \|B_{(s,\sigma,\sigma')}g\| ds d\sigma d\sigma' \\ &\leqq C_m |a|_{0,m} \int_{R^{3n}} \tilde{h}(s, \sigma, \sigma') \|G_{\sigma'}(\xi)\| ds d\sigma d\sigma' \\ &\leqq C_m |a|_{0,m} \int_{R^n} \|G_{\sigma'}(\xi)\| d\sigma'. \end{aligned}$$

If $|\xi - \sigma'| \leqq r_1 \lambda(\sigma')$, then Corollary to Lemma 2 implies

$$C_2^{-1} \leqq \lambda(\xi)/\lambda(\sigma') \leqq C_2 .$$

Hence we have the estimate

$$C^{-1} \leqq \frac{1+\lambda(\xi)^{-2}|\xi|^2}{1+\lambda(\sigma')^{-2}|\sigma'|^2} \leqq C$$

on the support of $G_{\sigma'}(\xi)$. Then

$$\int_{R^n} \|G_{\sigma'}(\xi)\| d\sigma' \leqq C_m \int_{R^n} (1+\lambda(\sigma')^{-2}|\sigma'|^2)^{-m/2} \times \|(1+\lambda(\xi)^{-2}|\xi|^2)^{m/2} G_{\sigma'}\| d\sigma'. \tag{81}$$

By Leibniz's rule and (74) we obtain the estimate

(82)
$$\begin{aligned}
&\|(1+\lambda(\xi)^{-2}|\xi|^2)^{m/2}G_{\sigma'}(\xi)\| \\
&\leqq C_m \sum_{|\beta|\leqq m} \|(1+\lambda(\xi)^{-2}|\xi|^2)^{m/2}\lambda(\xi)^{|\beta|}|\partial_\xi^\beta(\varphi_{\sigma'}(\xi)g(\xi))|\| \\
&\leqq C_m \sum_{|\beta|\leqq m}\sum_{\gamma+\delta=\beta} \|(1+\lambda(\xi)^{-2}|\xi|^2)^{m/2}\lambda(\xi)^{|\beta|}|\partial_\xi^\gamma\varphi_{\sigma'}(\xi)||\partial_\xi^\delta g(\xi)|\| \\
&\leqq C_m \sum_{|\beta|\leqq m}\sum_{|\delta|\leqq|\beta|} \|(1+\lambda(\xi)^{-2}|\xi|^2)^{m/2}\lambda(\xi)^{|\delta|}|\partial_\xi^\delta g(\xi)|\| \\
&\leqq C_m \sum_{|\alpha|\leqq m} \|\langle\xi\rangle^{m+\varepsilon|\alpha|}|\partial_\xi^\alpha g(\xi)|\|\,.
\end{aligned}$$

And from Lemma 6 we have the estimate

(83)
$$\int_{R^n}(1+\lambda(\sigma')^{-2}|\sigma'|^2)^{-m/2}d\sigma' \leqq C_{m,\varepsilon}\int_{R^n}(1+\lambda(0)^{-2}|\sigma'|^2)^{-(1-\varepsilon)m/2}d\sigma',$$

which is finite if $m>n/(1-\varepsilon)$. Hence from (81), (82) and (83) we have the estimate

$$\int_{R^n}\|G_{\sigma'}(\xi)\|d\sigma' \leqq C_m \sum_{|\beta|\leqq m}\|\langle\xi\rangle^{m+\varepsilon|\beta|}|\partial_\xi^\beta g(\xi)|\|.$$

Thus,

$$\|Bg\| \leqq C_m|a|_{0,m}\sum_{|\beta|\leqq m}\|\langle\xi\rangle^{m+\varepsilon|\beta|}|\partial_\xi^\beta g(\xi)|\|\,.$$

This completes the proof of Lemma 8 and the proof of Theorem 1.

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