H. UEDA
KODAI MATH. J.
7 (1984), 153—184

ON THE GROWTH OF MEROMORPHIC FUNCTIONS OF ORDER LESS THAN 1/2, III

Dedicated to Professor Mitsuru Ozawa on his 60 th birthday

BY HIDEHARU UEDA

Introduction.

This paper is concerned with one aspect of the Nevanlinna theory of meromorphic functions in the plane $\boldsymbol{C}$. We shall assume acquaintance with the standard terminology of the Nevanlinna theory

$$T(r, f), \quad m(r, a, f), \quad n(r, a, f), \quad N(r, a, f), \cdots .$$

If $f(z)$ is meromorphic, we define

$$M(r, f)=\sup_{|z|=r} |f(z)|, \qquad m^*(r, f)=\inf_{|z|=r} |f(z)| .$$

A nonconstant function $f(z)$ of finite order ρ is further classified as having *maximal, mean,* or *minimal type* according as

$$\limsup_{r\to\infty} T(r, f)/r^{\rho}$$

is infinite, positive, or zero, respectively.

Now, let ρ and δ be numbers with $0\leqq\rho<1/2$, $1-\cos\pi\rho<\delta\leqq 1$, and let $\mathfrak{M}_{\rho,\delta}$ be the set consisting of all meromorphic functions $f(z)$ of order ρ with the property that there is an $a\in\boldsymbol{C}$ satisfying $f(0)\neq a$ and

$$(1) \qquad N(r, \infty, f)<(1-\delta)N(r, a, f)+O(1) \qquad (r\to\infty).$$

The following result is well known.

THEOREM A. *Let $f(z)\in\mathfrak{M}_{\rho,\delta}$. Then given $\varepsilon>0$, there is a sequence of $r\to\infty$ such that*

$$(2) \qquad \log m^*(r, f)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-\varepsilon)T(r, f).$$

This result was conjectured by Teichmüller [7], and Gol'dberg [4] obtained (2) in the weaker form: $\log m^*(r, f)>K\, T(r, f)$, where K is a positive constant. The determination of the exact value of K is due to Ostrowskii [6].

Received April 15, 1983

At this stage it is convenient to introduce some notations. Let S be the set consisting of all functions $h(r)$ $(r\geqq 0)$ which are positive, decreasing, continuous and tend to zero as $r\to\infty$. We further classify a function $h(r)\in S$ as $h(r)\in S_1$ or $h(r)\in S_2$ according as the integral $\int_1^\infty h(t)t^{-1}dt$ is finite or not.

As is easily seen, we may restate Theorem A as in the following manner.

THEOREM A′. *Let $f(z)\in \mathfrak{m}_{\rho,\delta}$. Then there is an $h(r)\in S$ such that*

$$\log m^*(r,f)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r))T(r,f) \tag{3}$$

for certain arbitrarily large values of r.

In our previous papers [8], [9] we considered the following problem: Are there any functions $h(r)\in S$ with which the estimate (3) holds for all $f(z)\in \mathfrak{m}_{\rho,\delta}$? The answer was no at least for the cases $\rho\in(0,1/2)$.

THEOREM B. *Let $\rho\in(0,1/2)$, $\delta\in(1\text{-}\cos\pi\rho,1]$ and $h(r)\in S$ be given. Then there is a function $f(z)\in \mathfrak{m}_{\rho,\delta}$ such that for all sufficiently large values of r*

$$\log m^*(r,f)\leqq \frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r))T(r,f)\,. \tag{4}$$

This implies that there are functions $f(z)\in \mathfrak{m}_{\rho,\delta}$ with the property that $\log m^*(r,f)/T(r,f)$ tends to $(\pi\rho/\sin\pi\rho)(\cos\pi\rho-1+\delta)$ from below arbitrarily slowly through a sequence of $r\to\infty$. For the proof of Theorem B, an important role was played by slowly varying functions. A real-valued function $L(r)$ defined for all $r\geqq 0$ belongs to the class of *slowly varying functions* (at ∞) if

(i) $L(r)$ is positive and continuous in $0\leqq r<\infty$,

and

(ii) $\lim_{r\to\infty} L(\lambda r)/L(r)=1$ for every fixed $\lambda>0$.

In [8], we showed the following results.

THEOREM C. *Let $h(r)\in S_2$ be a slowly varying function, and let ρ, δ be given as in Theorem B. Then there is a function $f(z)\in \mathfrak{m}_{\rho,\delta}$ satisfying*

$$T(r,f)=o\left(r^\rho\exp\left\{\frac{1}{(1-\varepsilon)C(\rho,\delta)}\int_1^r\frac{h(t)}{t}dt\right\}\right)\qquad(r\to\infty)$$

for any $\varepsilon>0$, and the estimate (4) for all sufficiently large values of r, where

$$C(\rho,\delta)=\frac{\pi(1-\delta)\tan\pi\rho}{\cos\pi\rho-1+\delta}+\frac{2\pi\rho-\sin 2\pi\rho}{\rho\sin 2\pi\rho}\,. \tag{5}$$

THEOREM D. *Let $h(r)\in S_2$, ρ, and δ be given as in Theorem C. Then there is a function $f(z)\in \mathfrak{m}_{\rho,\delta}$ with the property that*

$$T(r,\ f)=o\left(r^{\rho}\exp\left\{-\frac{1}{(1+\varepsilon)C(\rho,\ \delta)}\int_{1}^{r}\frac{h(t)}{t}dt\right\}\right)\qquad(r\rightarrow\infty)$$

for any $\varepsilon>0$, *and that for all sufficiently large values of* r

$$\log m^{*}(r,\ f)\leqq\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1+h(r))T(r,\ f)\,.$$

The situation discussed here complements the above Theorems B, C and D. In § 1, we state Theorem 1 (and Corollary 1) which complements Theorems B and C. §§ 2-4 are devoted to the proof of Theorem 1. Our Theorems 2 and 3 are stated in § 5, the former complements Theorem D and the latter corresponds to Corollary 1. The proof of Theorem 2 is given in §§ 6-7. In § 8 we give two counterexamples to Theorem 1 and Corollary 1. Finally in § 9 we consider the case $\rho=0$.

In what follows, we use the restrictions such as $r\geqq r_{0}$, $n\geqq n_{0}$, $\cdots$, immediately after certain relations. It is understood that the quantities r_{0}, n_{0}, $\cdots$ which appear in this way are not necessarily the same ones each time they occur. Whenever we wish to stress the importance of certain parameters, say α, D, ε, $\cdots$ on which r_{0}, n_{0}, $\cdots$ may depend, we write, for instance, $r_{0}=r_{0}(\alpha,\ D)$, $n_{0}=n_{0}(\varepsilon)$, $\cdots$.

1. Statement of Theorem 1 and Corollary 1.

Our first result is

THEOREM 1. *Let* $h(r)\in S_{2}$, *and let* ρ *and* δ *be numbers with* $0<\rho<1/2$, $1-\cos\pi\rho<\delta\leqq 1$. *If* $f(z)\in\mathfrak{m}_{\rho,\delta}$ *satisfies the growth restriction*

$$T(r,\ f)=O\left(r^{\rho}\exp\left\{\frac{1}{(1+\varepsilon)C(\rho,\ \delta)}\int_{1}^{r}\frac{h(t)}{t}dt\right\}\right)\qquad(r\rightarrow\infty)\tag{1.1}$$

with some $\varepsilon>0$, *where* $C(\rho,\ \delta)$ *is defined by* (5), *then the estimate* (3) *holds for a sequence of* $r\rightarrow\infty$.

This result complements Theorems B and C. From Theorem 1 we immediately deduce the following fact.

COROLLARY 1. *Let* $h(r)$, ρ *and* δ *be given as in Theorem* 1. *If* $f(z)\in\mathfrak{m}_{\rho,\delta}$ *is of mean type, then the estimate* (3) *holds on an unbounded sequence of* r.

Remark. Our argument in the proof of Theorem 1 yields the following result.

Let $\rho\in(0,\ 1/2)$ *and* $h(r)\in S_{2}$ *be given, and let* $f(z)$ *be an entire function which satisfies the growth condition*

$$\log M(r, f)=O\left(r^{\rho}\exp\left\{\frac{1}{(1+\varepsilon)\pi\tan\pi\rho}\int_1^r\frac{h(t)}{t}dt\right\}\right) \qquad (r\to\infty)$$

with some $\varepsilon>0$. *Then on an unbounded sequence of* r

$$\log m^*(r, f)>\cos\pi\rho(1-h(r))\log M(r, f).$$

2. Auxiliary functions.

In this section we develop the necessary material to prove Theorem 1. Let $g(z)$ be a nonconstant entire function of order less than 1/2, all of whose zeros are real and negative and such that $g(0)=1$. Assume that, corresponding to $g(z)$, there is a function $H(z)$ in the whole plane satisfying the following conditions.

(2.1) $H(z)$ is a one-valued positive continuous function in the whole plane, and is harmonic in $|\arg z|<\pi$.

(2.2) $$\max_{|\theta|\leqq\pi} H(re^{i\theta}) \text{ is of order less than } 1/2.$$

(2.3) $$\log g(r)=o(H(-r)) \qquad (r\to\infty).$$

LEMMA 1. *Let* $g(z)$ *and* $H(z)$ *be functions as we stated above. Then there are two sequences* $\{r_n\}_1^\infty\to\infty$, $\{a_n\}_1^\infty\to\infty$ *such that for* $|\theta|<\pi$

(2.4) $$\log|g(-r_n)|-\frac{H(-r_n)}{H(r_ne^{i\theta})}\log|g(r_ne^{i\theta})|\geqq a_n\left\{1-\frac{H(-r_n)}{H(r_ne^{i\theta})}\right\}.$$

The proof is quite similar to the one of Lemma 5 on [1]. This lemma will play an important role in estimating $N(r, a, f)$ from above and $\log m^*(r, f)$ from below for a sequence of $r\to\infty$. To realize this, we first prepare the following lemma.

LEMMA 2. *Let* $A>1$ *and* $h(r)\in S_2$ *be given. Then there exists a function* $h_1(r)\in S_2$ *satisfying the following* (2.5)-(2.7).

(2.5) $$h_1(r)\leqq h(r) \qquad (r\geqq 0).$$

(2.6) $h_1(r)$ *is differentiable off a discrete set* S' (*where* S' *has no finite accumulation points*), *and* $rh_1'(r)\to 0$ *as* $r(\notin S')\to\infty$.

(2.7) $\int_1^r h(t)t^{-1}dt<A\int_1^r h_1(t)t^{-1}dt+B$ $(r>1)$, *where* $B=B(A, h)$ *is a positive constant.*

Proof. Put $r_0=1$ and $M=h(1)$. Let r_n $(n=1, 2, 3, \cdots)$ be the least positive number with the property that $h(r_n)=MA^{-n}$. Since $h(r)\in S_2$, $\int_1^\infty h(t)t^{-1}dt=\infty$, from which we deduce that

$$\sum_{k=1}^{\infty} A^{-k} \log (r_k/r_{k-1})=\infty . \tag{2.8}$$

Let I be the set consisting of all positive integers k satisfying $r_k/r_{k-1}\geqq 2$, and denote all the elements of I by k_l $(l\geqq 1)$ in order of increasing magnitude. Then clearly $\sum_{k\notin I} A^{-k} \log (r_k/r_{k-1})<\log 2\sum_{k=1}^{\infty} A^{-k}=(\log 2)/(A-1)\equiv C$, and so by (2.8) $\sum_{k\in I} A^{-k} \log (r_k/r_{k-1})=\infty$. This implies that $\sharp I=\infty$.

Now, define $h_1(r)$ by $h_1(0)=MA^{-k_1}$, $h_1(r_{k_l-1})=MA^{-k_l}$, $h_1(r_{k_l})=MA^{-k_{l+1}}$, and by linear interpolation otherwise. Then $h_1(r)$ belongs to S and satisfies (2.5). Further, $h_1(r)$ is differentiable off a discrete set $S'\equiv\{r_{k_l-1}, r_{k_l}\}_{l=1}^{\infty}$. In order to verify $rh_1'(r)\to 0$ as $r(\notin S')\to\infty$, note that for $r_{k_l-1}<r<r_{k_l}$

$$0>rh_1'(r)>-r_{k_l}MA^{-k_l}(1-A^{k_l-k_{l+1}})/(r_{k_l}-r_{k_l-1})>-MA^{-k_l}/(1-r_{k_l-1}/r_{k_l})$$

and use the fact that $r_{k_l}/r_{k_l-1}\geqq 2$. It remains to prove (2.7). From the definition of $h_1(r)$, it follows that for $r_{k_j-1}\leqq r\leqq r_{k_j}$ $(j=1, 2, 3, \cdots)$

$$\begin{aligned}\int_{r_{k_j-1}}^{r} h_1(t)t^{-1}dt &\geqq \int_{r_{k_j-1}}^{r} \frac{M}{A^{k_j}}\frac{r_{k_j}-t}{r_{k_j}-r_{k_j-1}}\frac{dt}{t} \\ &\geqq \frac{M}{A^{k_j}}\left\{\frac{r_{k_j}}{r_{k_j}-r_{k_j-1}}\log\left(\frac{r}{r_{k_j-1}}\right)-1\right\}>\frac{M}{A^{k_j}}\left\{\log\left(\frac{r}{r_{k_j-1}}\right)-1\right\}.\end{aligned} \tag{2.9}$$

Suppose now that $r_n\leqq r<r_{n+1}$. There are two cases to be considered.

Case 1. Assume that $n=k_l-1$ with some l. Then

$$\begin{aligned}\int_1^r h(t)t^{-1}dt &\leqq \sum_{k=1}^{n} MA^{-k+1}\log (r_k/r_{k-1})+MA^{-n}\log (r/r_n) \\ &< A\sum_{j=1}^{l-1} MA^{-k_j}\log (r_{k_j}/r_{k_j-1})+ACM+MA^{-k_j+1}\log (r/r_n).\end{aligned} \tag{2.10}$$

Incorporating (2.9) into (2.10), we have

$$\begin{aligned}\int_1^r h(t)t^{-1}dt &\leqq A\sum_{j=1}^{l-1}\left\{\int_{r_{k_j-1}}^{r_{k_j}} h_1(t)t^{-1}dt+MA^{-k_j}\right\}+ACM \\ &\quad +A\int_{r_{k_l-1}}^{r} h_1(t)t^{-1}dt<A\int_1^r h_1(t)t^{-1}dt+2ACM.\end{aligned} \tag{2.11}$$

Case 2. Assume that $n\neq k_l-1$ for all l $(=1, 2, 3, \cdots)$. Then

$$\begin{aligned}\int_1^r h(t)t^{-1}dt &\leqq A\sum_{j=1}^{l-1} MA^{-k_j}\log (r_{k_j}/r_{k_j-1})+ACM \\ &< A\int_1^r h_1(t)t^{-1}dt+ACM.\end{aligned} \tag{2.12}$$

Thus (2.7) with $B=2ACM$ follows from (2.11) and (2.12). This completes the proof of Lemma 2.

3. Estimates on $H(re^{i\theta})$.

In this and the next section, the letter $h_1(r)$ denotes the function which is constructed from $A>1$ and $h(r)\in S_2$ according to the procedure in the proof of Lemma 2. Define

$$L(r)=\exp\left\{\tilde{\delta}\int_1^r h_1(t)t^{-1}dt\right\} \tag{3.1}$$

with a positive constant $\tilde{\delta}$. Since $h_1(t)\to 0$ as $t\to\infty$, $L(r)$ is slowly varying.

Our aim of this section is to give two estimates (See (3.6) and (3.7).) on the function

$$H(re^{i\theta})=\frac{1}{\pi}\int_0^\infty \frac{r^{1/2}(r+s)s^\rho L(s)\cos(\theta/2)}{s^{1/2}(s^2+r^2+2rs\cos\theta)}ds \qquad (r>0,\ |\theta|<\pi), \tag{3.2}$$

where $\rho\in(0, 1/2)$ is a constant and $L(s)$ is defined by (3.1). For this purpose, we need two properties of slowly varying functions.

LEMMA 3. ([5]) *Let $L(r)$ be a slowly varying function. Then $L(\lambda r)/L(r)\to 1$ uniformly, as $r\to\infty$, in any interval $A^{-1}\leqq\lambda\leqq A$, $A>1$.*

The following Lemma 4 is an easy consequence of Lemma 3.

LEMMA 4. *Let $L(r)$ be a slowly varying function. Then given $\alpha>0$ and $C>1$, there is a number $R_0=R_0(\alpha, C)>0$ such that $y>x\geqq R_0$ implies*

$$L(y)/L(x)<C(y/x)^\alpha. \tag{3.3}$$

Proof. From Lemma 3 it follows that for any $A>1$ and $\varepsilon>0$ there is a number $r_0=r_0(A, \varepsilon)>0$ such that

$$L(\lambda r)/L(r)<1+\varepsilon, \tag{3.4}$$

whenever $\lambda\in(1, A]$ and $r\geqq r_0$. Now, if $y>x\geqq r_0$, choose $a\in[0, 1)$ to satisfy $y/x=A^{m+a}$, where m is a nonnegative integer. Then iteration of (3.4) gives

$$\begin{aligned}L(y)/L(x)<(1+\varepsilon)^{m+1}\leqq(1+\varepsilon)^{m+a+1}&=(1+\varepsilon)(1+\varepsilon)^{\log(y/x)\cdot(\log A)^{-1}}\\ &=(1+\varepsilon)(y/x)^{\log(1+\varepsilon)\cdot(\log A)^{-1}}.\end{aligned}$$

Hence, if we take $A>1$ and $\varepsilon>0$ such that $1+\varepsilon\leqq C$, $\log(1+\varepsilon)\cdot(\log A)^{-1}\leqq\alpha$, we obtain (3.3) with $R_0(\alpha, C)=r_0(A, \varepsilon)$.

Now, we return to (3.2). From Lemma 4, it follows that for any fixed $\alpha\in(0, 1/2-\rho)$, $L(r)=o(r^\alpha)$ $(r\to\infty)$. Hence $H(re^{i\theta})$ provides a solution of the Dirichlet problem with boundary values

$$H(-r)=r^{\rho}L(r) \qquad (r\geqq 0) \tag{3.5}$$

in the plane slit along the real axis from 0 to $-\infty$. It is clear that $H(re^{i\theta})$ is an even function of θ. Further, we have the following

LEMMA 5. *Let $\rho\in(0, 1/2)$, $A>1$ and $h(r)\in S_2$ be given, and let $h_1(r)\in S_2$, $L(r)$ and $H(re^{i\theta})$ be defined as above. Then we have the following two estimates on $H(re^{i\theta})$.*

(i) *$H(re^{i\theta})$ is a monotonic decreasing function of $|\theta|$ for $0\leqq|\theta|\leqq\pi$, in particular,*

$$H(re^{i\theta})\geqq H(-r) \qquad (r>0, |\theta|<\pi). \tag{3.6}$$

(ii) *For $\varepsilon>0$, there is an $R_0=R_0(\varepsilon)$ such that $r\geqq R_0$ implies*

$$\int_0^{\pi}\frac{H(re^{i\theta})}{H(-r)}d\theta<\frac{\tan\pi\rho}{\rho}+\frac{\tilde{\delta}}{\pi}h_1(r)(1+\varepsilon)\left\{\frac{\pi^2}{\rho\cos^2\pi\rho}-\frac{\pi\tan\pi\rho}{\rho^2}\right\}. \tag{3.7}$$

Proof. (i) It is convenient to introduce the notation

$$\phi_1(r)=\frac{d\phi(r)}{d\log r}, \qquad \phi_2(r)=\frac{d^2\phi(r)}{d\log^2 r} \qquad (r>0)$$

when $\phi(r)$ is defined for $r>0$ and these derivatives exist. Now, put $\phi(r)=r^{\rho}L(r)$. Clearly

$$\phi_1(r)=r^{\rho}L(r)\{\rho+\tilde{\delta}h_1(r)\},$$

$$\phi_2(r)=r^{\rho}L(r)[\{\rho+\tilde{\delta}h_1(r)\}^2+\tilde{\delta}rh_1'(r)]$$

$$\geqq r^{\rho}L(r)\{\rho^2+\tilde{\delta}rh_1'(r)\} \qquad (r\notin S').$$

By redefining $h_1(r)$ if necessary for small r, we may assume that $\phi_2(r)\geqq 0$ for $r\notin S'$ $(r>0)$. (In this case, we may assume that also this "modified" $h_1(r)$ satisfies the conditions (2.5)-(2.7).) Hence $\phi_1(r)$ is monotonic increasing, so the argument in [1, pp 461-462] shows that $H(re^{i\theta})$ is a monotonic decreasing function of $|\theta|$ for $0\leqq|\theta|\leqq\pi$.

(ii) Take $\alpha\in(0, 1/2-\rho)$ and $C>1$ arbitrarily. Choose $\varepsilon'>0$ and $D>1$ with the property that

$$\varepsilon'\left\{\int_0^1(\log t^{-1})t^{-1}(t^{\rho}+t^{-\rho})\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt\right\}<(\varepsilon/2)C_1(\rho) \tag{3.8}$$

and

$$\int_0^{D^{-1}}t^{-\rho-1}(C\alpha^{-1}t^{-\alpha}+\log t^{-1})\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt<(\varepsilon/2)C_1(\rho), \tag{3.9}$$

where

$$C_1(\rho)=\frac{\pi^2}{\rho\cos^2\pi\rho}-\frac{\pi\tan\pi\rho}{\rho^2}(>0).$$

Since $\log(1+\sqrt{t})(1-\sqrt{t})^{-1}\sim 2\sqrt{t}$ as $t\to 0$, $\sim\log(1-t)^{-1}$ as $t\to 1-$, (3.8) and (3.9) are possible.

Now, we write $H(re^{i\theta})=I_1(r,\theta)+I_2(r,\theta)+I_3(r,\theta)$, where

$$I_1(r,\theta)=\frac{1}{\pi}\int_0^\infty \frac{r^{1/2}(r+s)s^\rho L(r)\cos(\theta/2)}{s^{1/2}(s^2+r^2+2rs\cos\theta)}\,ds\,,$$

$$I_2(r,\theta)=\frac{1}{\pi}\int_0^r \frac{r^{1/2}(r+s)s^\rho[L(s)-L(r)]\cos(\theta/2)}{s^{1/2}(s^2+r^2+2rs\cos\theta)}\,ds\,,$$

$$I_3(r,\theta)=\frac{1}{\pi}\int_r^\infty \frac{r^{1/2}(r+s)s^\rho[L(s)-L(r)]\cos(\theta/2)}{s^{1/2}(s^2+r^2+2rs\cos\theta)}\,ds\,.$$

Consider first $I_1(r,\theta)$. Residue calculation gives

$$\frac{1}{\pi}\int_0^\infty \frac{t^\beta\sin\theta}{t^2+2t\cos\theta+1}\,dt=\frac{\sin\theta\beta}{\sin\pi\beta}\qquad(-1<\beta<1)\,. \tag{3.10}$$

Putting $s=rt$, we have

$$I_1(r,\theta)=\psi(r)\Big(\cos\frac{\theta}{2}\Big)\frac{1}{\pi}\int_0^\infty \frac{t^{\rho+1/2}+t^{\rho-1/2}}{t^2+2t\cos\theta+1}\,dt\,. \tag{3.11}$$

Incorporating (3.10) into (3.11), it follows that

$$\begin{aligned} I_1(r,\theta)&=\psi(r)\Big(\cos\frac{\theta}{2}\Big)\frac{1}{\sin\theta}\,\frac{1}{\cos\pi\rho}\,\{\sin\theta(\rho+1/2)-\sin\theta(\rho-1/2)\}\\ &=\psi(r)\frac{\cos\theta\rho}{\cos\pi\rho}\,. \end{aligned} \tag{3.12}$$

In view of (3.5) and (3.12)

$$\int_0^\pi \frac{I_1(r,\theta)}{H(-r)}\,d\theta=\frac{\tan\pi\rho}{\rho}\,. \tag{3.13}$$

Next, we estimate $I_2(r,\theta)$. It is convenient to introduce the function

$$G_1(t,\theta)=\int_0^t \frac{(1+u)u^{\rho-1/2}}{u^2+2u\cos\theta+1}\,du\qquad(0\leqq t\leqq 1,\ 0\leqq\theta<\pi)\,. \tag{3.14}$$

It is clear that $G_1(t,\theta)$ is positive and increasing for $t>0$, and satisfies

$$G_1(t,\theta)\sim\frac{t^{\rho+1/2}}{\rho+1/2}\qquad(t\to 0)\,. \tag{3.15}$$

Putting $s=rt$, we have

$$I_2(r,\theta)=\frac{r^\rho}{\pi}\Big(\cos\frac{\theta}{2}\Big)\int_0^1[L(rt)-L(r)]\frac{(1+t)t^{\rho-1/2}}{t^2+2t\cos\theta+1}\,dt\,.$$

After (3.14) and (3.15) are taken into account, this becomes

$$I_2(r, \theta) = -\frac{\tilde{\delta}}{\pi} r^\rho \left(\cos\frac{\theta}{2}\right) \int_0^1 \frac{h_1(rt)L(rt)}{t} G_1(t, \theta)\,dt \tag{3.16}$$
$$< -\frac{\tilde{\delta}}{\pi} r^\rho \left(\cos\frac{\theta}{2}\right) h_1(r) \int_0^1 \frac{L(rt)}{t} G_1(t, \theta)\,dt$$
$$= -\frac{\tilde{\delta}}{\pi} \phi(r) h_1(r) \left(\cos\frac{\theta}{2}\right) \int_0^1 \frac{G_1(t, \theta)}{t}\,dt$$
$$+ \frac{\tilde{\delta}}{\pi} r^\rho h_1(r) \left(\cos\frac{\theta}{2}\right) \int_0^1 [L(r) - L(rt)] \frac{G_1(t, \theta)}{t}\,dt.$$

This last integral requires further attention. From Lemma 3 it follows that

$$|L(rt)/L(r) - 1| < \varepsilon' \qquad (D^{-1} \leqq t \leqq D,\ r \geqq r_0 = r_0(D, \varepsilon')). \tag{3.17}$$

Hence

$$\int_0^1 [L(r) - L(rt)] \frac{G_1(t, \theta)}{t}\,dt < L(r)\left\{\varepsilon' \int_0^1 \frac{G_1(t, \theta)}{t}\,dt + \int_0^{D^{-1}} \frac{G_1(t, \theta)}{t}\,dt\right\}. \tag{3.18}$$

Finally, using (3.15) again, we deduce that

$$\int_0^1 \frac{G_1(t, \theta)}{t}\,dt = \int_0^1 (\log t^{-1}) \frac{(1+t)t^{\rho-1/2}}{t^2 + 2t\cos\theta + 1}\,dt \tag{3.19}$$

and

$$\int_0^{D^{-1}} \frac{G_1(t, \theta)}{t}\,dt < \int_0^{D^{-1}} (\log t^{-1}) \frac{(1+t)t^{\rho-1/2}}{t^2 + 2t\cos\theta + 1}\,dt. \tag{3.20}$$

On combining (3.18)-(3.20) with (3.16), it follows that

$$I_2(r, \theta) < -\frac{\tilde{\delta}}{\pi} h_1(r)\phi(r)\left(\cos\frac{\theta}{2}\right)\left\{(1-\varepsilon')\int_0^1 (\log t^{-1}) \frac{(1+t)t^{\rho-1/2}}{t^2 + 2t\cos\theta + 1}\,dt \right. \tag{3.21}$$
$$\left. - \int_0^{D^{-1}} (\log t^{-1}) \frac{(1+t)t^{\rho-1/2}}{t^2 + 2t\cos\theta + 1}\,dt\right\} \qquad (r \geqq r_0).$$

In order to estimate $\int_0^\pi I_2(re^{i\theta})/H(-r)\,d\theta$, we use the Fubini's theorem. Then

$$\int_0^\pi \frac{I_2(re^{i\theta})}{H(-r)}\,d\theta < -\frac{\tilde{\delta}}{\pi} h_1(r)\left\{(1-\varepsilon')\int_0^1 (\log t^{-1})(1+t)t^{\rho-1/2} \right. \tag{3.22}$$
$$\times \left(\int_0^\pi \frac{\cos(\theta/2)}{t^2 + 2t\cos\theta + 1}\,d\theta\right) dt - \int_0^{D^{-1}} (\log t^{-1})(1+t)t^{\rho-1/2}$$
$$\left. \times \left(\int_0^\pi \frac{\cos(\theta/2)}{t^2 + 2t\cos\theta + 1}\,d\theta\right) dt\right\} \qquad (r \geqq r_0).$$

Further,

$$(3.23)\qquad \int_0^{\pi}\frac{\cos(\theta/2)}{t^2+2t\cos\theta+1}\,d\theta=\int_0^{\pi}\frac{\cos(\theta/2)}{(t+1)^2-4t\sin^2(\theta/2)}\,d\theta=\int_0^1\frac{2}{(t+1)^2-4tu^2}\,du$$
$$=\frac{1}{t+1}\int_0^1\Big(\frac{1}{t+1-2\sqrt{t}\,u}+\frac{1}{t+1+2\sqrt{t}\,u}\Big)du$$
$$=\frac{1}{\sqrt{t}\,(t+1)}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big).$$

Substituting this into (3.22), we obtain

$$(3.24)\qquad \int_0^{\pi}\frac{I_2(r,\theta)}{H(-r)}\,d\theta<-\frac{\tilde{\delta}}{\pi}h_1(r)\Big\{(1-\varepsilon')\int_0^1(\log t^{-1})t^{\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt$$
$$-\int_0^{D^{-1}}(\log t^{-1})t^{\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt\Big\}\qquad (r\geqq r_0).$$

We turn to $I_3(r,\theta)$. In this case we introduce the function

$$(3.25)\qquad G_2(t,\theta)=\int_0^t\frac{(1+u)u^{-\rho-1/2}}{u^2+2u\cos\theta+1}\,du\qquad (0\leqq t\leqq 1,\ 0\leqq\theta<\pi).$$

Clearly, $G_2(t,\theta)$ is positive and increasing for $t>0$, and satisfies

$$(3.26)\qquad G_2(t,\theta)\sim\frac{t^{1/2-\rho}}{1/2-\rho}\qquad (t\to 0).$$

In $I_3(r,\theta)$ we put $s=rt^{-1}$ and integrate by parts to get

$$(3.27)\qquad I_3(r,\theta)=\frac{r^{\rho}}{\pi}\Big(\cos\frac{\theta}{2}\Big)\int_0^1[L(rt^{-1})-L(r)]\frac{(1+t)t^{-\rho-1/2}}{t^2+2t\cos\theta+1}\,dt$$
$$=\frac{\tilde{\delta}}{\pi}r^{\rho}\Big(\cos\frac{\theta}{2}\Big)\int_0^1\frac{h_1(rt^{-1})L(rt^{-1})}{t}G_2(t,\theta)dt$$
$$<\frac{\tilde{\delta}}{\pi}r^{\rho}h_1(r)\Big(\cos\frac{\theta}{2}\Big)\int_0^1\frac{L(rt^{-1})}{t}G_2(t,\theta)dt$$
$$=\frac{\tilde{\delta}}{\pi}\phi(r)h_1(r)\Big(\cos\frac{\theta}{2}\Big)\int_0^1\frac{G_2(t,\theta)}{t}\,dt$$
$$+\frac{\tilde{\delta}}{\pi}r^{\rho}h_1(r)\Big(\cos\frac{\theta}{2}\Big)\int_0^1[L(rt^{-1})-L(r)]\frac{G_2(t,\theta)}{t}\,dt.$$

Using (3.26), we deduce that

$$(3.28)\qquad \int_0^1\frac{G_2(t,\theta)}{t}\,dt=\int_0^1(\log t^{-1})\frac{(1+t)t^{-\rho-1/2}}{t^2+2t\cos\theta+1}\,dt$$

and

$$(3.29)\qquad \int_0^{D^{-1}}\frac{G_2(t,\theta)}{t^{1+\alpha}}\,dt<\frac{1}{\alpha}\int_0^{D^{-1}}\frac{(1+t)t^{-\rho-1/2-\alpha}}{t^2+2t\cos\theta+1}\,dt.$$

In view of (3.3), $L(rt^{-1})<CL(r)t^{-\alpha}$ $(0<t\leqq 1,\ r\geqq R_0(\alpha, C))$. This and (3.17) give for $r\geqq R_0$

$$(3.30)\qquad \int_0^1 [L(rt^{-1})-L(r)]\frac{G_2(t,\ \theta)}{t}dt<L(r)\left\{C\int_0^{D^{-1}}\frac{G_2(t,\ \theta)}{t^{1+\alpha}}dt+\varepsilon'\int_0^1\frac{G_2(t,\ \theta)}{t}dt\right\}.$$

Combining (3.27)-(3.30), it follows that

$$(3.31)\qquad I_3(r,\ \theta)<\frac{\tilde{\delta}}{\pi}h_1(r)\psi(r)\left(\cos\frac{\theta}{2}\right)\left\{(1+\varepsilon')\int_0^1(\log t^{-1})\frac{(1+t)t^{-\rho-1/2}}{t^2+2t\cos\theta+1}dt\right.$$
$$\left.+\frac{C}{\alpha}\int_0^{D^{-1}}\frac{(1+t)t^{-\rho-1/2-\alpha}}{t^2+2t\cos\theta+1}dt\right\}\qquad (r\geqq R_0).$$

Using the Fubini's theorem and (3.23) again, we have

$$(3.32)\qquad \int_0^\pi\frac{I_3(r,\ \theta)}{H(-r)}d\theta<\frac{\tilde{\delta}}{\pi}h_1(r)\left\{(1+\varepsilon')\int_0^1(\log t^{-1})t^{-\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt\right.$$
$$\left.+\frac{C}{\alpha}\int_0^{D^{-1}}t^{-\rho-1-\alpha}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt\right\}\qquad (r\geqq R_0).$$

Hence from (3.13), (3.24) and (3.32), we obtain

$$(3.33)\qquad \int_0^\pi\frac{H(re^{i\theta})}{H(-r)}d\theta<\frac{\tan\pi\rho}{\rho}+\frac{\tilde{\delta}}{\pi}h_1(r)\left\{(1+\varepsilon')\int_0^1(\log t^{-1})t^{-\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt\right.$$
$$-(1-\varepsilon')\int_0^1(\log t^{-1})t^{\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt+\int_0^{D^{-1}}(\log t^{-1})t^{\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt$$
$$\left.+\frac{C}{\alpha}\int_0^{D^{-1}}t^{-\rho-1-\alpha}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt\right\}.$$

Since

$$\log\frac{1+x}{1-x}=2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}\qquad (0\leqq x<1),$$

we easily see that

$$(3.34)\qquad \int_0^1(\log t^{-1})t^{-1}(t^{-\rho}-t^{\rho})\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt$$
$$=2\sum_{n=1}^{\infty}\frac{1}{2n-1}\int_0^1(\log t^{-1})t^{n-3/2}(t^{-\rho}-t^{\rho})dt$$
$$=2\sum_{n=1}^{\infty}\frac{1}{2n-1}\int_0^1\left\{\frac{t^{n-3/2-\rho}}{n-1/2-\rho}-\frac{t^{n-3/2+\rho}}{n-1/2+\rho}\right\}dt$$
$$=2\sum_{n=1}^{\infty}\frac{1}{2n-1}\left\{\frac{1}{(n-\rho-1/2)^2}-\frac{1}{(n+\rho-1/2)^2}\right\}$$
$$=\frac{1}{\rho}\sum_{n=0}^{\infty}\left\{\frac{1}{(n-\rho+1/2)^2}+\frac{1}{(n+\rho+1/2)^2}\right\}$$

$$-\frac{1}{\rho^2}\sum_{n=0}^{\infty}\left\{\frac{1}{n-\rho+1/2}-\frac{1}{n+\rho+1/2}\right\}=C_1(\rho).$$

Thus (3.7) follows from (3.33), (3.34), (3.8) and (3.9). This completes the proof of Lemma 5.

Combining Lemma 5 with Lemma 1, we obtain the following result, which will be used in the next section.

Assume that $H(z)$ defined by (3.2) satisfies (2.3). Then by Lemma 1, (2.4) holds. Using (3.6), we conclude that the right hand side of (2.4) is nonnegative for $|\theta|<\pi$. Hence

$$\frac{\log|g(r_n e^{i\theta})|}{\log|g(-r_n)|}\leqq\frac{H(r_n e^{i\theta})}{H(-r_n)}\qquad(|\theta|<\pi).$$

It follows from this and (3.7) that

$$\frac{N(r_n,\,0,\,g)}{\log|g(-r_n)|}<\frac{\tan\pi\rho}{\rho}+\frac{\tilde{\delta}}{\pi}h_1(r_n)(1+\varepsilon)C_1(\rho)\qquad(n\geqq n_0(\varepsilon))$$

for a suitable sequence $\{r_n\}\to\infty$.

4. Proof of Theorem 1.

We are now in position to prove Theorem 1. We set

$$F(z)=f(z)-a=cz^{-p}\frac{\Pi(1-z/a_n)}{\Pi(1-z/b_n)}=cz^{-p}\frac{P(z)}{Q(z)},\tag{4.1}$$

where c is a nonzero constant and p is a nonnegative integer. It is convenient to introduce the notation

$$\hat{P}(z)=\Pi(1+z/|a_n|),\qquad \hat{Q}(z)=\Pi(1-z/|b_n|).\tag{4.2}$$

Choose $\varepsilon'>0$ and $A>1$ with the property that

$$(1+\varepsilon')A<1+\varepsilon,\tag{4.3}$$

and then determine $\tilde{\delta}>0$ by

$$\tilde{\delta}^{-1}=(1+\varepsilon')C(\rho,\,\delta).\tag{4.4}$$

Let $h_1(r)\in S_2$ be constructed in Lemma 2 corresponding to this A and $h(r)\in S_2$. Then from (4.1), (1.1), (4.3), (4.4), (2.7), (3.1) and (3.5) it follows that

$$\begin{aligned}T(r,\,F)&=T(r,\,f)+O(1)\\&=o\left(r^{\rho}\exp\left\{\frac{\tilde{\delta}}{A}\int_1^r h(t)t^{-1}dt\right\}\right)\\&=o(r^{\rho}L(r))=o(H(-r))\qquad(r\to\infty).\end{aligned}\tag{4.5}$$

Since

$$\log \hat{P}(r)=r\int_0^\infty \frac{N(t,\ 0,\ \hat{P})}{(t+r)^2}dt \leqq N(r,\ 0,\ \hat{P})+r\int_r^\infty \frac{N(t,\ 0,\ \hat{P})}{t^2}dt$$

$$\leqq T(r,\ F)+r\int_r^\infty \frac{T(t,\ F)}{t^2}dt\,,$$

we deduce from (4.5) that

$$\log \hat{P}(r)=o(H(-r)) \qquad (r\to\infty)\,.$$

Further, it is easy to see that $H(z)$ satisfies (2.1) and (2.2). Hence we may apply Lemma 1 to the pair of $\hat{P}(z)$ and $H(z)$. Incorporating (3.6) and (3.7) into (2.4) with $g=\hat{P}$, we deduce that there are two sequences $\{r_n\}_1^\infty\to\infty$, $\{a_n\}_1^\infty\to\infty$ such that

$$\text{(4.6)}\qquad \frac{N(r_n,\ 0,\ \hat{P})}{\log|\hat{P}(-r_n)|}<\frac{\tan\pi\rho}{\pi\rho}\Big\{1+\tilde{\delta}(1+\varepsilon')\frac{2\pi\rho-\sin 2\pi\rho}{\rho\sin 2\pi\rho}h_1(r)\Big\}\,,$$

$$\text{(4.7)}\qquad -\log \hat{P}(r_n)\geqq -\frac{H(r_n)}{H(-r_n)}\log|\hat{P}(-r_n)|+\Big\{\frac{H(r_n)}{H(-r_n)}-1\Big\}a_n\,.$$

Now, we estimate $H(r)/H(-r)$. First, using (3.12), (3.21) and (3.31), we easily obtain

$$\text{(4.8)}\qquad \frac{H(r)}{H(-r)}<\frac{1}{\cos\pi\rho}+\frac{\tilde{\delta}}{\pi}(1+\varepsilon')h_1(r)\Big\{\int_0^1(\log t^{-1})\frac{t^{-\rho-1/2}}{t+1}dt-\int_0^1(\log t^{-1})\frac{t^{\rho-1/2}}{t+1}dt\Big\}$$

$$=\frac{1}{\cos\pi\rho}+\frac{\tilde{\delta}}{\pi}(1+\varepsilon')h_1(r)\sum_{n=0}^\infty(-1)^n\Big\{\frac{1}{(n+1/2-\rho)^2}-\frac{1}{(n+1/2+\rho)^2}\Big\}$$

$$=\frac{1}{\cos\pi\rho}+\tilde{\delta}(1+\varepsilon')h_1(r)\frac{\pi\sin\pi\rho}{\cos^2\pi\rho} \qquad (r\geqq R_0(\varepsilon'))\,.$$

Next,

$$\text{(4.9)}\qquad H(r)=\frac{r^\rho}{\pi}\int_0^\infty\frac{t^\rho L(rt)}{t^{1/2}(1+t)}dt$$

$$=\frac{\psi(r)}{\pi}\int_0^\infty\frac{t^\rho}{t^{1/2}(1+t)}dt+\frac{r^\rho}{\pi}\int_0^\infty\frac{t^\rho[L(rt)-L(r)]}{t^{1/2}(1+t)}dt$$

$$>\psi(r)\frac{1}{\cos\pi\rho}-\frac{r^\rho}{\pi}\int_0^1\frac{t^{\rho-1/2}}{1+t}[L(r)-L(rt)]dt\,.$$

From (4.9) and (3.17) it follows that

$$\text{(4.10)}\qquad \frac{H(r)}{H(-r)}>\frac{1-\varepsilon'}{\cos\pi\rho} \qquad (r\geqq R_1(\varepsilon'))\,.$$

Substituting (4.8) and (4.10) into (4.7), we have

$$\text{(4.11)}\qquad -\log \hat{P}(r_n)\geqq -\Big\{\frac{1}{\cos\pi\rho}+\tilde{\delta}(1+\varepsilon')h_1(r_n)\frac{\pi\sin\pi\rho}{\cos^2\pi\rho}\Big\}\log|\hat{P}(-r_n)|$$

$$+\frac{1-\cos\pi\rho}{1+\cos\pi\rho}a_n \qquad (n\geqq n_0(\varepsilon')).$$

We proceed to estimate $\log m^*(r_n, F)$. By (4.2) and (1)

$$\log \hat{Q}(-r)=r\int_0^\infty \frac{N(t, 0, \hat{Q})}{(t+r)^2}dt$$

$$\leqq r\int_0^\infty \frac{(1-\delta)N(t, 0, \hat{P})-p\log t+O(1)}{(t+r)^2}dt$$

$$=(1-\delta)\log \hat{P}(r)-p\log r+O(1) \qquad (r\to\infty),$$

and so we deduce from (4.11), (4.1) and (4.2) that for $r=r_n$ $(n\geqq n_0)$

$$\log m^*(r, F)\geqq \log|\hat{P}(-r)|-\log \hat{Q}(-r)-p\log r-O(1) \tag{4.12}$$

$$\geqq \log|\hat{P}(-r)|\left\{1-(1-\delta)\frac{\log \hat{P}(r)}{\log|\hat{P}(-r)|}-\frac{O(1)}{\log|\hat{P}(-r)|}\right\}$$

$$\geqq \log|\hat{P}(-r)|\left[1-(1-\delta)\left\{\frac{1}{\cos\pi\rho}+\tilde{\delta}(1+\varepsilon')\frac{\pi\sin\pi\rho}{\cos^2\pi\rho}h_1(r)\right\}\right.$$

$$\left.+\frac{O(a_n)}{\log|\hat{P}(-r)|}\right].$$

Since $\log m^*(r_n, F)>0$ for $n\geqq n_0$, $m(r_n, 0, F)=0$ $(n\geqq n_0)$. Hence by the first fundamental theorem $T(r_n, F)=N(r_n, 0, F)+O(1)$ $(n\to\infty)$. It follows from this and (4.6) that for $r=r_n$ $(n\geqq n_1)$

$$T(r, f)\leqq T(r, F)+O(1)\leqq N(r, 0, F)+O(1) \tag{4.13}$$

$$<\frac{\tan\pi\rho}{\pi\rho}\log|\hat{P}(-r)|\left\{1+\tilde{\delta}(1+\varepsilon')\frac{2\pi\rho-\sin 2\pi\rho}{\rho\sin 2\pi\rho}h_1(r)\right.$$

$$\left.+\frac{O(1)}{\log|\hat{P}(-r)|}\right\}.$$

Recall that $a_n\to\infty$ as $n\to\infty$ and $\tilde{\delta}$ is defined by (4.4). Then we obtain from (4.12), (4.13) and (2.5) that for $r=r_n$ $(n\geqq n_0)$

$$\frac{\log m^*(r, f)}{T(r, f)}\geqq\frac{\log m^*(r, F)-O(1)}{T(r, f)}>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h_1(r))$$

$$>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r)).$$

This completes the proof of Theorem 1.

5. Statement of Theorems 2 and 3.

Our second result complements Theorem D.

THEOREM 2. *Let* $h(r)\in S_2$ *and let* ρ, δ *be numbers with* $0<\rho<1/2$, $1-\cos\pi\rho<\delta\leqq 1$. *If* $f(z)\in \mathfrak{M}_{\rho,\delta}$ *satisfies*

$$(5.1)\qquad T(r,\ f)=O\left(r^{\rho}\exp\left\{-\frac{1}{(1-\varepsilon)C(\rho,\ \delta)}\int_{1}^{r}\frac{h(t)}{t}dt\right\}\right)\qquad (r\rightarrow\infty)$$

with some $\varepsilon>0$, *then on a sequence of* $r\rightarrow\infty$,

$$(5.2)\qquad \log m^{*}(r,\ f)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1+h(r))T(r,\ f)\,.$$

For $h(r)\in S_1$ we have the following result, which should be compared with Corollary 1.

THEOREM 3. *Let* $h(r)\in S_1$ *and let* ρ, δ *be given as in Theorem* 2. *Then if* $f(z)\in \mathfrak{M}_{\rho,\delta}$ *is of minimal type, the estimate* (5.2) *holds on an unbounded sequence of* r.

We remark that Theorem 3 is an improvement of Theorem 1 in [8]. The proof of Theorem 3 is similar to the one of Theorem 2, so we will prove only Theorem 2.

6. Two lemmas for the proof of Theorem 2.

This and the next section are devoted to the proof of Theorem 2. The following lemma parallels Lemma 2 in the proof of Theorem 1, and will be used also in §8.

LEMMA 6. *For each* $\alpha>0$ *and* $h(r)\in S$, *there is a function* $h_1(r)\in S$ *such that*

$$(6.1)\qquad h_1(r)\geqq h(r)\qquad (r\geqq 0)\,,$$

$$(6.2)\qquad h_1(\lambda r)/h_1(r)\geqq(2\lambda)^{-\alpha}\qquad (r\geqq 0,\ \lambda>1)\,,$$

while

$$(6.3)\qquad \int_{1}^{r}h_1(t)t^{-1}dt\leqq 2^{\alpha}\int_{0}^{r}h(t)t^{-1}dt+B\quad (r>1)\,,\quad \textit{while } B=B(\alpha,\ h)$$

is a positive constant.

Proof. We first put $h_1(r)=h(0)$ for $0\leqq r\leqq 1$, and we define $h_1(r)$ for $I_n=\{r\,;2^n\leqq r\leqq 2^{n+1}\}$ $(n=0,1,2,\cdots)$ by induction. Assume that $h_1(r)$ is determined for $r\leqq 2^n$. Then if $h_1(2^n)<2^{\alpha}h(2^n)$, we set $h_1(r)=h_1(2^n)$ $(r\in I_n)$, and otherwise $h_1(r)=h_1(2^n)\{1-2^{-n}(1-2^{-\alpha})(r-2^n)\}$ $(r\in I_n)$. Clearly $h_1(r)\in S$ and (6.1) holds. To see (6.2), we note that $h_1(2r)/h_1(r)\geqq 2^{-\alpha}$ $(r\geqq 0)$, and appeal to the reasoning as in the proof of Lemma 4. It remains to show (6.3). There are three cases to be considered.

Case 1. Assume that n satisfies $h_1(2^n)<2^\alpha h(2^{n+1})$. In this case $h_1(t)=h_1(2^n)<2^\alpha h(t)$ for $t\in I_n$, so we have

$$\int_{2^n}^r h_1(t)t^{-1}dt<2^\alpha\int_{2^n}^r h(t)t^{-1}dt \qquad (r\in I_n)\,. \tag{6.4}$$

Case 2. Define $J_1=\{n\,;\,2^\alpha h(2^{n+1})\leqq h_1(2^n)<2^\alpha h(2^n)\}$. If $n\in J_1$, then $h_1(t)=h_1(2^n)$ for $t\in I_n$, $h_1(t)=h_1(2^n)\{1-2^{-n-1}(1-2^{-\alpha})(r-2^{n+1})\}$ for $t\in I_{n+1}$. Hence

$$\sum_{n\in J_1}\int_{I_n} h_1(t)t^{-1}dt=\sum_{n\in J_1} h_1(2^n)\log 2\leqq \sum_{n=0}^{\infty} 2^{-n\alpha}h(0)\log 2\equiv B/2\,. \tag{6.5}$$

Case 3. Define $J_2=\{n\,;\,h_1(2^n)>2^\alpha h(2^n)\}$. In this case $h_1(t)=h_1(2^n)\{1-2^{-n}(1-2^{-\alpha})(r-2^n)\}$ for $t\in I_n$. Hence

$$\sum_{n\in J_2}\int_{I_n} h_1(t)t^{-1}dt<\sum_{n\in J_2} h_1(2^n)\log 2\leqq B/2\,. \tag{6.6}$$

On combining (6.4)-(6.6), we deduce (6.3). This completes the proof of Lemma 6.

Let ρ and M' be numbers with $0<\rho<1/2$, $0<M'<1/2+\rho$. For $\varepsilon>0$, we choose $\varepsilon'>0$, $\alpha\in(0,\,1/2+\rho-M')$ and $D>1$ with the property that

$$2^\alpha(1+\varepsilon')\int_0^1\frac{t^{-\alpha}-1}{\alpha}t^{\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt <\int_0^1(\log t^{-1})t^{\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt+(\varepsilon/4)C_1(\rho)\,, \tag{6.7}$$

$$2^{-\alpha}(1-\varepsilon')\int_0^1\frac{1-t^\alpha}{\alpha}t^{-\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt >\int_0^1(\log t^{-1})t^{-\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt-(\varepsilon/4)C_1(\rho)\,, \tag{6.8}$$

$$\int_0^{D^{-1}}\frac{1-t^\alpha}{\alpha}t^{-\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt<(\varepsilon/4)C_1(\rho)\,, \tag{6.9}$$

and

$$\frac{2^\alpha}{\alpha+M'}\int_0^{D^{-1}}t^{\rho-1-\alpha-M'}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)dt<(\varepsilon/4)C_1(\rho)\,, \tag{6.10}$$

where $C_1(\rho)$ is defined in §3. Inequality (6.10) is immediate since $\log\{(1+\sqrt{t})(1-\sqrt{t})^{-1}\}\sim 2\sqrt{t}$ $(t\to 0)$ and $\rho-1-\alpha-M'>-3/2$. To see that a pair of ε' and α may be chosen to satisfy (6.7), we observe the following facts (i)-(iii).

(i) For any fixed $\alpha\in[-1/2,\,1/2]$, the function

$$g(t,\,\alpha)\equiv\frac{t^{-\alpha}-1}{\alpha}t^{\rho-1}\log\Big(\frac{1+\sqrt{t}}{1-\sqrt{t}}\Big)$$

is Lebesgue integrable in (0, 1), here we interpret $(t^{-\alpha}-1)/\alpha$ for $\alpha=0$ as $\log t^{-1}$.

(ii) For any fixed $t\in(0, 1)$, $g(t, \alpha)$ is a continuous function of α.

(iii) For any $\alpha\in[-1/2, 1/2]$

$$|g(t, \alpha)|\leqq g(t, 1/2).$$

It is well known that under the above conditions (i)-(iii), the function $G(\alpha)\equiv\int_0^1 g(t, \alpha)dt$ is continuous in $[-1/2, 1/2]$, in particular $G(\alpha)\rightarrow G(0)$ $(\alpha\rightarrow 0)$, from which (6.7) follows at once. Also, the existence of a pair of ε' and α (α and D) satisfying the inequality (6.8) ((6.9)) is shown analogously.

Now, we give a lemma, which corresponds to Lemma 5 in the proof of Theorem 1.

LEMMA 7. *Let $\rho\in(0, 1/2)$, $\varepsilon>0$ and $h(r)\in S_2$ be given, and let $\tilde{\delta}>0$ be a number such that $M'\equiv\tilde{\delta}h(0)<1/2+\rho$. Choose $\varepsilon'>0$, $\alpha\in(0, 1/2+\rho-M')$ and $D>1$ so that the above inequalities* (6.7)-(6.10) *hold. Further, let $h_1(r)\in S_2$ be constructed in Lemma* 6, *and define $H(re^{i\theta})$ by* (3.2) *with*

$$L(r)=\exp\left\{-\tilde{\delta}\int_1^r h_1(t)t^{-1}dt\right\}. \tag{6.11}$$

Then $H(re^{i\theta})$ satisfies

$$H(re^{i\theta})\geqq H(-r) \qquad (r>0,\ -\pi\leqq\theta\leqq\pi), \tag{6.12}$$

and

$$\int_0^\pi \frac{H(re^{i\theta})}{H(-r)}d\theta<\frac{\tan\pi\rho}{\rho}-(1-\varepsilon)\frac{\tilde{\delta}}{\pi}C_1(\rho)h_1(r) \qquad (r\geqq R_0(\varepsilon)). \tag{6.13}$$

Proof. The proof of (6.12) is quite similar to the one of (3.6), so only the proof of (6.13) need to be given. We define $G_k(r, \theta)$ $(k=1, 2)$ and $I_j(r, \theta)$ $(j=1, 2, 3)$ as in the proof of Lemma 5. (Note that $L(r)$ is defined by (6.11) in place of (3.1).) For $I_1(r, \theta)$ we have (3.12). Consider now $I_2(r, \theta)$. It is easily seen that

$$I_2(r, \theta)=\frac{\tilde{\delta}}{\pi}r^\rho\left(\cos\frac{\theta}{2}\right)\int_0^1\frac{h_1(rt)L(rt)}{t}G_1(t, \theta)dt.$$

By (6.2), $h_1(rt)\leqq 2^\alpha h_1(r)t^{-\alpha}$ for $0<t<1$, so

$$\begin{aligned}I_2(r, \theta)\leqq&\frac{\tilde{\delta}}{\pi}2^\alpha h_1(r)r^\rho L(r)\left(\cos\frac{\theta}{2}\right)\int_0^1\frac{G_1(t, \theta)}{t^{1+\alpha}}dt\\&+\frac{\tilde{\delta}}{\pi}2^\alpha h_1(r)r^\rho\left(\cos\frac{\theta}{2}\right)\int_0^1\frac{L(rt)-L(r)}{t^{1+\alpha}}G_1(t, \theta)dt,\end{aligned} \tag{6.14}$$

and the last integral invites further attention. In view of (3.17) and the fact that

$$L(rt)/L(r)=\exp\left\{\tilde{\delta}\int_{rt}^r h_1(t)t^{-1}dt\right\}\leqq t^{-\tilde{\delta}h_1(0)}=t^{-M'} \qquad (0<t<1),$$

$$(6.15)\quad \int_0^1 \frac{L(rt)-L(r)}{t^{1+\alpha}} G_1(t,\ \theta)dt$$

$$< L(r)\left\{\varepsilon'\int_0^1 \frac{G_1(t,\ \theta)}{t^{1+\alpha}}dt + \int_0^{D^{-1}} \frac{G_1(t,\ \theta)}{t^{1+\alpha+M'}}dt\right\} \qquad (r \geqq r_0(\varepsilon',\ D)).$$

Substituting (6.15) into (6.14), we have

$$(6.16)\quad I_2(r,\ \theta) < \frac{\tilde{\delta}}{\pi} 2^{\alpha} h_1(r) r^{\rho} L(r)\left(\cos\frac{\theta}{2}\right)\left\{(1+\varepsilon')\int_0^1 \frac{G_1(t,\ \theta)}{t^{1+\alpha}}dt\right.$$

$$\left. + \int_0^{D^{-1}} \frac{G_1(t,\ \theta)}{t^{1+\alpha+M'}}dt\right\}$$

$$< \frac{\tilde{\delta}}{\pi} 2^{\alpha} h_1(r) r^{\rho} L(r)\left\{(1+\varepsilon')\int_0^1 \frac{t^{-\alpha}-1}{\alpha}(1+t)t^{\rho-1/2}\frac{\cos(\theta/2)}{t^2+2t\cos\theta+1}dt\right.$$

$$\left. + \frac{1}{\alpha+M'}\int_0^{D^{-1}}(1+t)t^{\rho-1/2-\alpha-M'}\frac{\cos(\theta/2)}{t^2+2t\cos\theta+1}dt\right\}.$$

Using the Fubini's theorem, we deduce from (3.23), (6.7) and (6.10) that

$$(6.17)\quad \int_0^{\pi} \frac{I_2(r,\ \theta)}{H(-r)}d\theta < \frac{\tilde{\delta}}{\pi} 2^{\alpha} h_1(r)\left\{(1+\varepsilon')\int_0^1 \frac{t^{-\alpha}-1}{\alpha} t^{\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt\right.$$

$$\left. + \frac{1}{\alpha+M'}\int_0^{D^{-1}} t^{\rho-1-\alpha-M'}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt\right\}$$

$$< \frac{\tilde{\delta}}{\pi} h_1(r)\left\{\int_0^1 (\log t^{-1}) t^{\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt + (\varepsilon/2)C_1(\rho)\right\}$$

$$(r \geqq r_0).$$

We turn to $I_3(r,\ \theta)$. In view of (6.2) and (3.17)

$$(6.18)\quad I_3(r,\ \theta) = -\frac{\tilde{\delta}}{\pi} r^{\rho}\left(\cos\frac{\theta}{2}\right)\int_0^1 \frac{h_1(rt^{-1})L(rt^{-1})}{t} G_2(t,\ \theta)dt$$

$$< -\frac{\tilde{\delta}}{\pi} 2^{-\alpha} h_1(r) r^{\rho} L(r)\left(\cos\frac{\theta}{2}\right)\int_0^1 \frac{G_2(t,\ \theta)}{t^{1-\alpha}}dt$$

$$+ \frac{\tilde{\delta}}{\pi} 2^{-\alpha} h_1(r) r^{\rho}\left(\cos\frac{\theta}{2}\right)\int_0^1 \frac{L(r)-L(rt^{-1})}{t^{1-\alpha}} G_2(t,\ \theta)dt$$

$$< -\frac{\tilde{\delta}}{\pi} 2^{-\alpha} h_1(r) r^{\rho} L(r)\left(\cos\frac{\theta}{2}\right)(1-\varepsilon')\int_0^1 \frac{G_2(t,\ \theta)}{t^{1-\alpha}}dt$$

$$+ \frac{\tilde{\delta}}{\pi} 2^{-\alpha} h_1(r) r^{\rho} L(r)\left(\cos\frac{\theta}{2}\right)\int_0^{D^{-1}} \frac{G_2(t,\ \theta)}{t^{1-\alpha}}dt$$

$$< -\frac{\tilde{\delta}}{\pi} 2^{-\alpha} h_1(r) r^{\rho} L(r)(1-\varepsilon')\int_0^1 \frac{1-t^{\alpha}}{\alpha}(1+t)t^{-\rho-1/2}\frac{\cos(\theta/2)}{t^2+2t\cos\theta+1}dt$$

$$+\frac{\tilde{\delta}}{\pi}2^{-\alpha}h_1(r)r^{\rho}L(r)\int_0^{D^{-1}}\frac{1-t^{\alpha}}{\alpha}(1+t)t^{-\rho-1/2}\frac{\cos(\theta/2)}{t^2+2t\cos\theta+1}dt$$
$$(r\geqq r_0).$$

Again we use the Fubini's theorem to get

$$\int_0^{\pi}\frac{I_3(r,\theta)}{H(-r)}d\theta<-\frac{\tilde{\delta}}{\pi}2^{-\alpha}h_1(r)(1-\varepsilon')\int_0^1\frac{1-t^{\alpha}}{\alpha}t^{-\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt$$
$$+\frac{\tilde{\delta}}{\pi}2^{-\alpha}h_1(r)\int_0^{D^{-1}}\frac{1-t^{\alpha}}{\alpha}t^{-\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt,$$

so we deduce from (6.8) and (6.9) that

$$\int_0^{\pi}\frac{I_3(r,\theta)}{H(-r)}d\theta<-\frac{\tilde{\delta}}{\pi}h_1(r)\left\{\int_0^1(\log t^{-1})t^{-\rho-1}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)dt-(\varepsilon/2)C_1(\rho)\right\} \tag{6.19}$$
$$(r\geqq r_0).$$

Thus (6.13) follows from (3.13), (6.17), (6.19) and (3.34). This completes the proof of Lemma 7.

7. Proof of Theorem 2.

Define $F(z)$, $P(z)$, $Q(z)$, $\hat{P}(z)$ and $\hat{Q}(z)$ as in § 4, and put

$$\tilde{\delta}^{-1}=(1-\varepsilon/2)C(\rho,\delta). \tag{7.1}$$

Let $\alpha\in(0, 1/2+\rho)$, to be determined later. Since we are interested in results for large values of r, we may assume that $h(0)<(1/2+\rho-\alpha)\tilde{\delta}^{-1}$ by modifying $h(r)$ if necessary for small values of r. Now, choose $\alpha>0$, $\varepsilon'\in(0, \varepsilon/2)$ and $D>1$ such that

$$2^{\alpha}<(1-\varepsilon/2)(1-\varepsilon)^{-1}, \tag{7.2}$$

$$2^{\alpha}(1+\varepsilon')\int_0^1\frac{t^{-\alpha}-1}{\alpha}\frac{t^{\rho-1/2}}{t+1}dt<\int_0^1(\log t^{-1})\frac{t^{\rho-1/2}}{t+1}dt+(\varepsilon/8)C_2(\rho), \tag{7.3}$$

$$\frac{2^{\alpha}}{\alpha+M'}\int_0^{D^{-1}}\frac{t^{\rho-1/2-\alpha-M'}}{t+1}dt<(\varepsilon/8)C_2(\rho), \tag{7.4}$$

$$2^{-\alpha}(1-\varepsilon')\int_0^1\frac{1-t^{\alpha}}{\alpha}\frac{t^{-\rho-1/2}}{t+1}dt>\int_0^1(\log t^{-1})\frac{t^{-\rho-1/2}}{t+1}dt-(\varepsilon/8)C_2(\rho), \tag{7.5}$$

and

$$\int_0^{D^{-1}}\frac{1-t^{\alpha}}{\alpha}\frac{t^{-\rho-1/2}}{t+1}dt<(\varepsilon/8)C_2(\rho), \tag{7.6}$$

where

$$C_2(\rho)=\frac{\pi^2\sin\pi\rho}{\cos^2\pi\rho}.$$

Next, let $h_1(r)\in S_2$ be constructed in Lemma 6 corresponding to this α and $h(r)$. Then from (5.1), (7.2), (6.3) and (6.11) it follows that

$$T(r,\ F)=o(r^{\rho}L(r)) \qquad (r\to\infty). \tag{7.7}$$

As we saw in §4

$$\log\hat{P}(r)\leqq T(r,\ F)+r\int_r^{\infty}\frac{T(t,\ F)}{t^2}dt,$$

so we deduce from (7.7) that $\log\hat{P}(r)=o(r^{\rho}L(r))=o(H(-r))$ $(r\to\infty)$. This shows that we may apply Lemma 2 to $\hat{P}(r)$. Upon incorporating (6.12) and (6.13) into (2.4), it follows that there are two sequences $\{r_n\}_1^{\infty}\to\infty$ and $\{a_n\}_1^{\infty}\to\infty$ such that

$$\frac{N(r_n,\ 0,\ \hat{P})}{\log|\hat{P}(-r_n)|}<\frac{\sin\pi\rho}{\pi\rho}\left\{1-\tilde{\delta}(1-\varepsilon')\frac{2\pi\rho-\sin 2\pi\rho}{\rho\sin 2\pi\rho}h_1(r_n)\right\} \qquad (n\geqq n_0(\varepsilon')) \tag{7.8}$$

and

$$-\log\hat{P}(r_n)\geqq-\frac{H(r_n)}{H(-r_n)}\log|\hat{P}(-r_n)|+a_n\left\{\frac{H(r_n)}{H(-r_n)}-1\right\}. \tag{7.9}$$

Here we need to estimate $H(r)/H(-r)$. In view of (3.12), (6.16) and (6.18)

$$\begin{aligned}\frac{H(r)}{H(-r)}<\frac{1}{\cos\pi\rho}-\frac{\tilde{\delta}}{\pi}h_1(r)\Big\{&2^{-\alpha}(1-\varepsilon')\int_0^1\frac{1-t^{\alpha}}{\alpha}\frac{t^{-\rho-1/2}}{t+1}dt\\ &-2^{\alpha}(1+\varepsilon')\int_0^1\frac{t^{-\alpha}-1}{\alpha}\frac{t^{\rho-1/2}}{t+1}dt-2^{-\alpha}\int_0^{D^{-1}}\frac{1-t^{\alpha}}{\alpha}\frac{t^{-\rho-1/2}}{t+1}dt\\ &-2^{\alpha}\frac{1}{\alpha+M'}\int_0^1\frac{t^{\rho-1/2-\alpha-M'}}{t+1}dt\Big\} \qquad (r\geqq r_0(\alpha,\ \varepsilon',\ D)).\end{aligned}$$

After (7.3)-(7.6) are taken into account, this becomes

$$\frac{H(r)}{H(-r)}<\frac{1}{\cos\pi\rho}-\tilde{\delta}(1-\varepsilon/2)\frac{\pi\sin\pi\rho}{\cos^2\pi\rho}h_1(r) \qquad (r\geqq r_0). \tag{7.10}$$

On the other hand,

$$\begin{aligned}H(r)&=\frac{r^{\rho}L(r)}{\cos\pi\rho}-\frac{r^{\rho}}{\pi}\int_0^{\infty}\frac{L(rt)-L(r)}{t^{1/2-\rho}(t+1)}dt\\ &\geqq\frac{r^{\rho}L(r)}{\cos\pi\rho}-\frac{r^{\rho}}{\pi}\int_0^1\frac{L(rt)-L(r)}{t^{1/2-\rho}(t+1)}dt,\end{aligned}$$

so from (3.17) we have

$$\frac{H(r)}{H(-r)}>(1-\varepsilon')\frac{1}{\cos\pi\rho}-\int_0^{D^{-1}}\frac{1}{t^{1/2-\rho}(t+1)}dt \qquad (r\geqq R_0(D,\ \varepsilon')). \tag{7.11}$$

We remark that (7.8)-(7.11) correspond to (4.6)-(4.8) and (4.10) in §4, respectively. Hence similar calculations as in the final part of §4 give

$$\log m^*(r_n, F) \geqq \frac{\log|\hat{P}(-r_n)|}{\cos \pi\rho}(\cos \pi\rho - 1+\delta)\Big\{1+\frac{\tilde{\delta}(1-\delta)\pi \sin \pi\rho}{(\cos \pi\rho-1+\delta)\cos \pi\rho} \times(1-\varepsilon/2)h_1(r_n)+\frac{O(a_n)}{\log|\hat{P}(-r_n)|}\Big\}>0 \qquad (n \geqq n_1), \tag{7.12}$$

and

$$T(r_n, f) < \frac{\tan \pi\rho}{\pi\rho}\log|\hat{P}(-r_n)|\Big\{1-\tilde{\delta}(1-\varepsilon')\frac{2\pi\rho-\sin 2\pi\rho}{\rho \sin 2\pi\rho}h_1(r_n) +\frac{O(1)}{\log|\hat{P}(-r_n)|}\Big\}. \tag{7.13}$$

Thus (5.2) follows from (7.12), (7.13), (7.1), (6.1) and the fact that $\varepsilon' < \varepsilon/2$. This completes the proof of Theorem 2.

8. Two counterexamples to Theorem 1 and Corollary 1.

EXAMPLE 1. *Let* $\varepsilon \in (0, 1)$ *and* $h(r) \in S_2$ *be given, and let* ρ, δ *be numbers with* $0<\rho<1/2$, $1-\cos \pi\rho < \delta \leqq 1$. *Then there is a function* $f(z) \in \mathfrak{m}_{\rho,\delta}$ *with the property that*

$$T(r, f) = o\Big(r^\rho \exp\Big\{\frac{1}{(1-\varepsilon)C(\rho, \delta)}\int_1^r \frac{h(t)}{t}dt\Big\}\Big) \qquad (r \to \infty),$$

and that for all sufficiently large values of r *the estimate* (4) *holds.*

EXAMPLE 2. *For given* $\rho \in (0, 1/2)$, $\delta \in (1-\cos \pi\rho, 1]$ *and* $h(r) \in S_1$, *there is a function* $f(z) \in \mathfrak{m}_{\rho,\delta}$ *which is of mean type and such that for all sufficiently large values of* r *the estimate* (4) *holds.*

Since the proofs of the above two examples are essentially the same, we prove only Example 1.

Let $\varepsilon>0$, $\rho \in (0, 1/2)$ and $h(r) \in S_2$ be given, and let $\tilde{\delta}$ be a positive constant such that $M' \equiv \tilde{\delta}h(0) < 1-\rho$. Choose $\alpha>0$, $\varepsilon'>0$ and $D>1$ with the property that

$$2^{-\alpha}\int_0^1 \frac{1-t^\alpha}{\alpha}\frac{t^{-\rho}}{1+t}dt - 2^\alpha \int_0^1 \frac{t^{-\alpha}-1}{\alpha}\frac{t^{\rho-1}}{1+t}dt > -(1+2\varepsilon/3)\frac{\pi^2 \cos \pi\rho}{\sin^2 \pi\rho}, \tag{8.1}$$

$$(1+\varepsilon')\int_0^1 (\log t^{-1})\frac{t^{-\rho}}{t+1}dt-(1-\varepsilon')\int_0^1(\log t^{-1})\frac{t^{\rho-1}}{t+1}dt < -(1-\varepsilon/4)\frac{\pi^2\cos \pi\rho}{\sin^2 \pi\rho}, \tag{8.2}$$

$$\frac{1}{M'}\int_0^{D^{-1}} \frac{1}{(1+t)t^{\rho+M'}}dt < (\varepsilon/4)\frac{\pi^2 \cos \pi\rho}{\sin^2 \pi\rho}, \tag{8.3}$$

$$\int_0^{D^{-1}}(\log t^{-1})\frac{t^{\rho-1}}{t+1}dt < (\varepsilon/4)\frac{\pi^2 \cos \pi\rho}{\sin^2 \pi\rho}, \tag{8.4}$$

$$\frac{2^{\alpha}}{\rho-\alpha}<\frac{1+\varepsilon/2}{\rho}, \tag{8.5}$$

$$1-\varepsilon'-D^{-\rho}>1-\varepsilon/2, \tag{8.6}$$

$$(1-\varepsilon')\sum_{n=0}^{\infty}\frac{1}{(n+\rho)^2}+2^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{(n+1-\rho)(n+1-\rho+\alpha)}-\sum_{n=0}^{\infty}\frac{D^{-n-\rho}}{(n+\rho)^2}$$
$$>(1-\varepsilon/4)\frac{\pi^2}{\sin^2\pi\rho}, \tag{8.7}$$

$$\sum_{n=0}^{\infty}\frac{D^{\rho+M'-n-1}}{(n+1-\rho)(n+1-\rho-M')}<(\varepsilon/16)\frac{\pi^2}{\sin^2\pi\rho}, \tag{8.8}$$

$$2^{\alpha}\sum_{n=0}^{\infty}\frac{1}{(\rho+n)(\rho+n-\alpha)}+\sum_{n=0}^{\infty}\frac{1+\varepsilon'}{(n+1-\rho)^2}+\sum_{n=0}^{\infty}\frac{D^{\rho+M'-n-1}}{(n+1-\rho)(n+1-\rho-M')}$$
$$<(1+\varepsilon/4)\frac{\pi^2}{\sin^2\pi\rho}. \tag{8.9}$$

To verify that an $\alpha>0$ may be chosen to satisfy (8.1), we may note that

$$\frac{1-t^{\alpha}}{\alpha}\longrightarrow\log t^{-1}\quad(\alpha\to 0),\qquad \frac{t^{-\alpha}-1}{\alpha}\longrightarrow\log t^{-1}\quad(\alpha\to 0)$$

and

$$\int_0^1(\log t^{-1})\frac{t^{-\rho}-t^{\rho-1}}{1+t}dt=-\sum_{n=0}^{\infty}(-1)^n\left\{\frac{1}{(n+\rho)^2}-\frac{1}{(n+1-\rho)^2}\right\}$$
$$=-\frac{\pi^2\cos\pi\rho}{\sin^2\pi\rho}.$$

In the same way, (8.7) is immediate from the facts that

$$\sum_{n=0}^{\infty}\frac{1}{(n+1-\rho)(n+1-\rho+\alpha)}\longrightarrow\sum_{n=0}^{\infty}\frac{1}{(n+1-\rho)^2}\qquad(\alpha\to 0)$$

and

$$\sum_{n=0}^{\infty}\left\{\frac{1}{(n+\rho)^2}+\frac{1}{(n+1-\rho)^2}\right\}=\frac{\pi^2}{\sin^2\pi\rho}.$$

Now, let $h_1(r)\in S_2$ be constructed in Lemma 6, and put

$$L(r)=\exp\left\{\tilde{\delta}\int_1^r h_1(t)t^{-1}dt\right\}.$$

Further, we choose r_0 so large that $r\geqq r_0$ implies

$$2\log r+2/\rho+2\log 4+1<(\varepsilon/3)\tilde{\delta}\frac{\pi^2\cos\pi\rho}{\sin^2\pi\rho}h_1(r)r^{\rho}L(r), \tag{8.10}$$

$$\rho L(r)+\rho^2\log r<(\varepsilon/2)\tilde{\delta}h_1(r)r^{\rho}L(r), \tag{8.11}$$

$$(8.12)\quad 2^{\alpha}\left\{\frac{1}{r}\sum_{n=0}^{[rK-1]}\frac{1}{(n+\rho)(n+\rho-\alpha)}+2\sum_{n=1+[rK-1]}^{\infty}\frac{1}{(n+\rho)(n+\rho-\alpha)}\right\}<(\varepsilon/4)\frac{\pi^2}{\sin^2\pi\rho},$$

$$(8.13)\quad (1+\varepsilon')\left\{\frac{1}{r}\sum_{n=0}^{[rK-1]}\frac{1}{(n+1-\rho)^2}+2\sum_{n=1+[rK-1]}^{\infty}\frac{1}{(n+1-\rho)^2}\right\}<(\varepsilon/8)\frac{\pi^2}{\sin^2\pi\rho},$$

$$(8.14)\quad 2(1+K+\varepsilon')\log r+2/\rho<(\varepsilon/4)\tilde{\delta}\frac{\pi^2}{\sin^2\pi\rho}h_1(r)r^{\rho}L(r),$$

where K (>1) is a positive constant.

(8.10), (8.11) and (8.14) are possible because

$$(8.15)\quad \frac{\log r}{h_1(r)r^{\rho}}\longrightarrow 0 \qquad (r\to\infty).$$

To see this, we use (6.2) with $\alpha=\rho/2$, $r=1$. Then we have $h_1(r)\geqq 2^{-\rho/2}h_1(0)r^{-\rho/2}$, from which $r^{\rho}h_1(r)\geqq O(r^{\rho/2})$ $(r\to\infty)$. This yields (8.15).

Under the above preparations, we prove the following

LEMMA 8. *Let $\varepsilon>0$, $\rho\in(0, 1/2)$ and $h(r)\in S_2$ be given, and let $\tilde{\delta}$ be a positive constant such that $M'\equiv\tilde{\delta}h(0)<1-\rho$. Further let $\alpha>0$, $\varepsilon'>0$, and $D>1$ be chosen to satisfy* (8.1)-(8.9), *and let $h_1(r)\in S_2$ be constructed in Lemma* 6. *Define $P(z)$ as a canonical product with only negative zeros whose zero-counting function $n(r, 0, P)=[r^{\rho}L(r)]$. Then we have for $r\geqq r_0(\varepsilon)$,*

$$(8.16)\quad \left|N(r, 0, P)-\left(1-\frac{\tilde{\delta}}{\rho}h_1(r)\right)\frac{r^{\rho}L(r)}{\rho}\right|<\varepsilon\frac{\tilde{\delta}}{\rho^2}h_1(r)r^{\rho}L(r),$$

$$(8.17)\quad \left|\log P(r)-\left\{\frac{\pi}{\sin\pi\rho}-\tilde{\delta}\frac{\pi^2\cos\pi\rho}{\sin^2\pi\rho}h_1(r)\right\}r^{\rho}L(r)\right|<\varepsilon\tilde{\delta}\frac{\pi^2\cos\pi\rho}{\sin^2\pi\rho}h_1(r)r^{\rho}L(r),$$

and

$$(8.18)\quad \left|\log|P(re^{i\theta(r)})|-\left\{\frac{\pi\cos\pi\rho}{\sin\pi\rho}-\tilde{\delta}\frac{\pi^2}{\sin^2\pi\rho}h_1(r)\right\}r^{\rho}L(r)\right|$$
$$<\varepsilon\tilde{\delta}\frac{\pi^2}{\sin^2\pi\rho}h_1(r)r^{\rho}L(r),$$

where $\theta(r)=\pi-r^{-K}$ with a positive constant $K>1$.

Proof. We remark that if $h_1(r)$ is slowly varying, the estimates (8.16)-(8.18) have already been proved by Barry [2, pp 55-58]. In what follows, only one-sided inequality of (8.18) will be proved, since the other inequalities are more easily seen. The branch of $\log P(z)$ in $|\arg z|<\pi$ for which $\log P(0)=0$ may be represented by Valiron's formula:

$$\log P(z)=\int_0^{\infty}\log(1+z/t)d[t^{\rho}L(t)]=z\int_0^{\infty}\frac{[t^{\rho}L(t)]}{t(t+z)}dt.$$

Then

$$\log P(z)=z\int_1^\infty \frac{[t^\rho L(t)]}{t(t+z)}dt=z\int_0^\infty \frac{t^\rho L(t)}{t(t+z)}dt+z\int_1^\infty \frac{[t^\rho L(t)]-t^\rho L(t)}{t(t+z)}dt \tag{8.19}$$

$$=zL(r)\int_0^\infty \frac{t^\rho}{t(t+z)}dt+z\int_0^r \frac{t^\rho\{L(t)-L(r)\}}{t(t+z)}dt$$

$$+z\int_r^\infty \frac{t^\rho\{L(t)-L(r)\}}{t(t+z)}dt-z\int_0^1 \frac{t^\rho L(t)}{t(t+z)}dt$$

$$+z\int_1^\infty \frac{[t^\rho L(t)]-t^\rho L(t)}{t(t+z)}dt\equiv J_1(r,\ \theta)+\cdots+J_5(r,\ \theta),\quad \text{say.}$$

Here we take $\theta=\theta(r)\equiv\pi-r^{-K}$. Elementary calculations give

$$\operatorname{Re} J_1(r,\ \theta)=\pi r^\rho L(r)\frac{\cos\rho\theta}{\sin\pi\rho}\leqq\pi r^\rho L(r)\frac{\cos\pi\rho}{\sin\pi\rho}+o(1)\qquad (r\to\infty), \tag{8.20}$$

$$|J_4(r,\ \theta)|<2/\rho\qquad (r\geqq 2), \tag{8.21}$$

$$|J_5(r,\ \theta)|\leqq 2(1+K+\varepsilon')\log r\qquad (r\geqq r_0(\varepsilon')). \tag{8.22}$$

Next, we proceed to estimate $J_2(r,\ \theta)$. Clearly

$$\operatorname{Re}(z/(t+z))=\operatorname{Re}\sum_{n=0}^\infty(-1)^n(t/z)^n=\sum_{n=0}^\infty(-1)^n(t/r)^n\cos n\theta\qquad (t<r),$$

so we have

$$\operatorname{Re} J_2(r,\ \theta)=\int_0^r t^{\rho-1}\{L(t)-L(r)\}\sum_{n=0}^\infty(-1)^n(t/r)^n\cos n\theta\, dt \tag{8.23}$$

$$=r^\rho\int_0^1 s^{\rho-1}\{L(rs)-L(r)\}\sum_{n=0}^\infty(-1)^n s^n\cos n\theta\, ds$$

$$=r^\rho\sum_{n=0}^\infty(-1)^n\cos n\theta\int_0^1 s^{\rho-1+n}\{L(rs)-L(r)\}\,ds$$

$$=-\tilde{\delta}r^\rho\sum_{n=0}^\infty\frac{(-1)^n\cos n\theta}{\rho+n}\int_0^1 s^{\rho+n-1}h_1(rs)L(rs)ds$$

$$=-\tilde{\delta}r^\rho\sum_{n=0}^\infty\frac{1}{\rho+n}\int_0^1 s^{\rho+n-1}h_1(rs)L(rs)ds$$

$$+\tilde{\delta}r^\rho\sum_{n=0}^\infty\frac{1-(-1)^n\cos n\theta}{\rho+n}\int_0^1 s^{\rho+n-1}h_1(rs)L(rs)ds$$

$$\equiv-\tilde{\delta}r^\rho I_1(r)+\tilde{\delta}r^\rho I_2(r,\ \theta),\qquad \text{say.}$$

The estimates of $I_1(r)$ from below and $I_2(r,\ \theta)$ from above are derived by the same way as we used in §6:

$$I_1(r)\geqq\sum_{n=0}^\infty\frac{h_1(r)}{\rho+n}\left\{L(r)\int_0^1 s^{\rho+n-1}ds+\int_0^1 s^{\rho+n-1}\{L(rs)-L(r)\}\,ds\right\} \tag{8.24}$$

$$>h_1(r)L(r)\sum_{n=0}^{\infty}\left\{(1-\varepsilon')\frac{1}{(n+\rho)^2}-\frac{D^{-n-\rho}}{(n+\rho)^2}\right\} \qquad (r\geqq r_0(D,\ \varepsilon')),$$

$$I_2(r,\ \theta)\leqq\sum_{n=0}^{\infty}\frac{1-(-1)^n\cos n\theta}{\rho+n}\,L(r)\int_0^1 h_1(rs)s^{\rho+n-1}ds \tag{8.25}$$

$$\leqq 2^{\alpha}h_1(r)L(r)\sum_{n=0}^{\infty}\frac{1-(-1)^n\cos n\theta}{(\rho+n)(\rho+n-\alpha)}.$$

This last term requires further attention. Since $\theta=\pi-r^{-K}$, we deduce that $|1-(-1)^n\cos n\theta|=|1-\cos n\pi\cos(n-nr^{-K})|=|1-\cos nr^{-K}|\leqq nr^{-K}\leqq r^{-1}$ for $n\leqq [r^{K-1}]$. Hence

$$\sum_{n=0}^{\infty}\frac{1-(-1)^n\cos n\theta}{(n+\rho)(n+\rho-\alpha)}\leqq\frac{1}{r}\sum_{n=0}^{[r^{K-1}]}\frac{1}{(n+\rho)(n+\rho-\alpha)} \tag{8.26}$$

$$+2\sum_{n=[r^{K-1}]+1}^{\infty}\frac{1}{(n+\rho)(n+\rho-\alpha)}.$$

Upon incorporating (8.24)-(8.26) into (8.23), it follows that

$$\operatorname{Re}J_2(r,\ \theta)\leqq-\tilde{\delta}r^{\rho}L(r)h_1(r)\left\{(1-\varepsilon')\sum_{n=0}^{\infty}\frac{1}{(n+\rho)^2}-\sum_{n=0}^{\infty}\frac{D^{-n-\rho}}{(n+\rho)^2}\right\} \tag{8.27}$$

$$+\tilde{\delta}r^{\rho}L(r)h_1(r)2^{\alpha}\left\{\frac{1}{r}\sum_{n=0}^{[r^{K-1}]}\frac{1}{(n+\rho)(n+\rho-\alpha)}\right.$$

$$\left.+2\sum_{n=[r^{K-1}]+1}^{\infty}\frac{1}{(n+\rho)(n+\rho-\alpha)}\right\}.$$

The estimate of $\operatorname{Re}J_3(r,\ \theta)$ is similar to the one of $\operatorname{Re}J_2(r,\ \theta)$. The corresponding inequality to (8.27) is

$$\operatorname{Re}J_3(r,\ \theta)\leqq-\tilde{\delta}r^{\rho}L(r)h_1(r)2^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{(n+1-\rho)(n+1-\rho+\alpha)} \tag{8.28}$$

$$+\tilde{\delta}r^{\rho}L(r)h_1(r)\left[(1+\varepsilon')\left\{\frac{1}{r}\sum_{n=0}^{[r^{K-1}]-1}\frac{1}{(n+1-\rho)^2}\right.\right.$$

$$\left.\left.+2\sum_{n=[r^{K-1}]}^{\infty}\frac{1}{(n+1-\rho)^2}\right\}+\sum_{n=0}^{\infty}\frac{D^{\rho+M'-n-1}}{(n+1-\rho)(n+1-\rho-M')}\right].$$

After combining (8.20), (8.21), (8.22), (8.27) and (8.28), we deduce the one-sided inequality of (8.18) from (8.7), (8.8), (8.12), (8.13) and (8.14).

Further we need the following lemma due to Edrei and Fuchs [3].

LEMMA 9. *Let $f(z)$ be meromorphic in the plane. For a measurable set $I\subset[0,\ 2\pi)$, define*

$$m(r,\ f,\ I)=\frac{1}{2\pi}\int_I\log^{+}|f(re^{i\theta})|\,d\theta \qquad (r>0).$$

Then

$$m(r, f, I) \leqq 22T(2r, f)|I|\left\{1+\log^+ \frac{1}{|I|}\right\},$$

where $|I|$ *is the Lebesgue measure of* I.

We are now able to construct a function $f(z)$ which satisfies the conditions as stated in Example 1.

We first choose $\alpha>0$, $\tilde{\delta}>0$, and $\varepsilon'>0$ in turn as in the following manner:

$$2^{\alpha}(1-\varepsilon)<1, \quad 1<C(\rho, \delta)\tilde{\delta}<2^{-\alpha}(1-\varepsilon)^{-1}, \quad 2\tilde{\delta}C_1(\rho, \varepsilon)\varepsilon'<C(\rho, \delta)\tilde{\delta}-1,$$

where $C_1(\rho, \delta)=\pi(1+(1-\delta)\cos\pi\rho)/\{\sin\pi\rho(\cos\pi\rho-1+\delta)\}+1/\rho$. Since we are interested in results for large values of r, we may assume that $h(0)<(1-\rho)\tilde{\delta}^{-1}$. We next choose $\alpha'\in(0, \alpha]$, $\varepsilon''>0$ and $D>1$ with the property that (8.1)-(8.9) hold with α, ε' and ε replaced by α', ε'' and ε', respectively. Let $h_1(r)\in S_2$ be constructed in Lemma 6 corresponding to $\alpha'>0$ and $h(r)\in S_2$, and put $L(r)=\exp\left\{\tilde{\delta}\int_1^r h_1(t)t^{-1}dt\right\}$. Now, define

$$P(z)=\prod(1+z/a_n), \quad Q(z)=\prod(1-z/b_n) \quad (a_n, b_n>0),$$

where $n(r, 0, P)=[r^{\rho}L(r)]$ and $n(r, 0, Q)=[(1-\delta)|r^{\rho}L(r)-1|]$. Then we will show that $f(z)\equiv P(z)/Q(z)$ is one of the desired functions.

Using (8.17) and (8.18), we have

$$\log|f(re^{i\theta(r)})| \geqq \log|P(re^{i\theta(r)})| - \log Q(-r)>0 \qquad (r>R_1).$$

Hence by Lemma 9

$$(8.29) \qquad m(r, 0, f)=\frac{1}{\pi}\int_{\theta(r)}^{\pi} \log^+\frac{1}{|f(re^{i\theta})|}d\theta$$
$$\leqq 44T(2r, 1/f)(\pi-\theta(r))\left\{1+\log^+\frac{1}{\pi-\theta(r)}\right\}$$
$$\leqq 44T(2r, f)r^{-K}\{1+K\log r\} \qquad (r>R_1).$$

Since $T(r, f)\leqq m(r, P)+m(r, Q)\leqq \log M(r, P)+\log M(r, Q)$, we deduce from (8.17) that

$$(8.30) \qquad T(r, f)=o(r^{\rho'}) \qquad (r\to\infty),$$

for any fixed $\rho'>\rho$. In view of (8.29) and (8.30) we have $m(r, 0, f)=o(1)$ $(r\to\infty)$. From this and (8.16) it follows that

$$T(r, f)=T(r, 1/f)=N(r, 0, f)+m(r, 0, f)$$
$$<\frac{r^{\rho}L(r)}{\rho}\left\{1-\frac{\tilde{\delta}(1-\varepsilon')}{\rho}h_1(r)\right\}=O(r^{\rho}L(r))=o\left(r^{\rho}\exp\left\{\frac{\int_1^r h(t)t^{-1}dt}{(1-\varepsilon)C(\rho, \delta)}\right\}\right).$$

Further,

$$N(r,\ \infty,\ f)=N(r,\ 0,\ Q)=\int_1^r \frac{[(1-\delta)|t^\rho L(t)-1|]}{t}dt$$

$$\leqq\int_1^r \frac{(1-\delta)(t^\rho L(t)-1)}{t}dt\leqq(1-\delta)\int_1^r \frac{[t^\rho L(t)]}{t}dt$$

$$=(1-\delta)N(r,\ 0,\ P)=(1-\delta)N(r,\ 0,\ f)\,.$$

It remains to show (4). Using (8.17) and (8.18), we have

$$\log m^*(r,\ f)<\log|P(re^{i\theta(r)})|-\log Q(-r)$$

$$<\left\{\frac{\pi(\cos\pi\rho-1+\delta)}{\sin\pi\rho}-\frac{\tilde{\delta}\pi^2}{\sin^2\pi\rho}[(1-\varepsilon')-(1+\varepsilon')(1-\delta)\cos\pi\rho]h_1(r)\right\}$$

$$\times r^\rho L(r)+O(1)$$

$$<\frac{\pi}{\sin\pi\rho}(\cos\pi\rho-1+\delta)\left\{1-\frac{\pi\tilde{\delta}}{\sin\pi\rho}[1-(1-\delta)\cos\pi\rho\right.$$

$$\left.-2\varepsilon'(1+(1-\delta)\cos\pi\rho)](\cos\pi\rho-1+\delta)^{-1}h_1(r)\right\}r^\rho L(r)\qquad(r>R_1)\,.$$

On the other hand, by (8.16)

$$N(r,\ 0,\ f)>\frac{r^\rho L(r)}{\rho}\left\{1-(1+\varepsilon')\frac{\tilde{\delta}}{\rho}h_1(r)\right\}\qquad(r>R_1)\,.$$

Thus

$$\frac{\log m^*(r,\ f)}{T(r,\ f)}<\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)\left\{1-\frac{h_1(r)\pi\tilde{\delta}}{\sin\pi\rho}[1-(1-\delta)\cos\pi\rho\right.$$

$$\left.-2\varepsilon'(1+(1-\delta)\cos\pi\rho)](\cos\pi\rho-1+\delta)^{-1}\right\}\left\{1+(1+2\varepsilon')\frac{\tilde{\delta}}{\rho}h_1(r)\right\}$$

$$<\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)\{1-(C(\rho,\delta)-2\varepsilon' C_1(\rho,\delta))\tilde{\delta}h_1(r)-O(h_1^2(r))\}$$

$$<\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h_1(r))$$

$$<\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r))\qquad(r\geqq r_0)\,.$$

9. The case $\rho=0$.

In this section we simply make mention of the case $\rho=0$. The following result corresponds to Theorem B in the cases $\rho\in(0,\ 1/2)$.

THEOREM 4. *Let* $\delta\in(0, 1]$ *and* $h(r)\in S$ *be given. Then there is a function* $f(z)\in\mathfrak{m}_{0,\delta}$ *such that for all sufficiently large values of* r

$$\log m^*(r, f)<\delta(1-h(r))T(r, f).$$

First, we prove the following

LEMMA 10. *Given* $h(r)\in S$, *there is a function* $h_1(r)\in S$ *satisfying the following* (9.1)-(9.6).

$$h_1(r)\geqq h(r) \qquad (r\geqq 0). \tag{9.1}$$

(9.2) $h_1(r)$ *is a slowly varying function which is differentiable off a discrete set* S' (*where* S' *has no finite accumulation points*).

$$\sqrt{h_1(r)}\log r \longrightarrow \infty \qquad as \quad r\to\infty. \tag{9.3}$$

$$\sqrt{h_1(r)}\in S_2. \tag{9.4}$$

(9.5) $h_1'(r)$ *is continuous off* S', *and for each* $r\in S'$, $h_1'(r-0)$ *and* $h_1'(r+0)$ *exist.*

(9.6) *If we put* $\tilde{h}_1'(r)=h_1'(r+0)$, *then* $r\tilde{h}_1'(r)/\{h_1(r)\}^{3/2}\to 0$ *as* $r\to\infty$.

Proof. First, define $h_2(r)=h(r)$ $(r<e)$, $h_2(r)=\max\{h(r), h(e)(\log r)^{-1}\}$ $(r\geqq e)$. Then $h_2(r)\in S$ satisfies

$$h_2(r)\geqq h(r) \qquad (r\geqq 0), \tag{9.7}$$

$$\sqrt{h_2(r)}\log r \longrightarrow \infty \qquad (r\to\infty), \tag{9.8}$$

and

$$\sqrt{h_2(r)}\in S_2. \tag{9.9}$$

Next, choose a positive sequence $\{r_n\}_1^\infty$ such that

$$r_{n+1}/r_n\geqq e^{2^n} \qquad (n=1, 2, 3, \cdots) \tag{9.10}$$

and

$$h_2(r)\leqq h(0)/2^n \qquad (r\geqq r_n). \tag{9.11}$$

Now, define $h_1(r)\in S$ as follows:

$$h_1(r)=\begin{cases} h(0) & (0\leqq r\leqq r_1) \\ \dfrac{h(0)(\log r_{n+1}-\log r_n)}{2^{n-1}(\log r+\log r_{n+1}-2\log r_n)} & (r_n\leqq r\leqq r_{n+1}). \end{cases} \tag{9.12}$$

In view of (9.12), $h_1(r)\geqq h(0)/2^n$ for $r\leqq r_{n+1}$, so by (9.11)

$$h_1(r)\geqq h_2(r) \qquad (r>0). \tag{9.13}$$

(9.1), (9.3)-(9.5) are immediate consequences of (9.7)-(9.9), (9.12) and (9.13). Assume that $r_n \leqq r < r_{n+1}$ $(n=1, 2, \cdots)$. Then we have

$$0 < -r\tilde{h}_1'(r) = \frac{h(0)(\log r_{n+1} - \log r_n)}{2^{n-1}(\log r + \log r_{n+1} - 2\log r_n)^2} \leqq \frac{h(0)}{2^{n-1}(\log r_{n+1} - \log r_n)}$$

and $h_1(r) > h(0)/2^n$. Hence by (9.10)

$$0 < \frac{-r\tilde{h}_1'(r)}{\{h_1(r)\}^{3/2}} < \frac{2(\sqrt{2})^n}{\sqrt{h(0)}\log(r_{n+1}/r_n)} \leqq \frac{2}{h(0)(\sqrt{2})^n} \qquad (r_n \leqq r < r_{n+1}),$$

from which (9.6) follows. It remains to prove that $h_1(r)$ is slowly varying. Using (9.12), we easily see that for every fixed $\lambda > 1$

$$1 > \frac{h_1(\lambda r)}{h_1(r)} \geqq \frac{\log(r_{n+1}/r_n)}{\log(r_{n+1}/r_n) + \log\lambda} \qquad (r_n \leqq r < r_{n+1}/\lambda)$$

and

$$1 > \frac{h_1(\lambda r)}{h_1(r)} \geqq \frac{\log(r_{n+2}/r_{n+1})}{\log(r_{n+2}/r_{n+1}) + \log\lambda} \; \frac{\log(r_{n+1}/r_n) - (\log\lambda)/2}{\log(r_{n+1}/r_n)}$$
$$(r_{n+1}/\lambda \leqq r < r_{n+1}).$$

These and (9.10) imply that $h_1(r)$ is slowly varying. This completes the proof of Lemma 10.

Theorem 4 is an easy consequence of Lemma 10 and the following

LEMMA 11. *Suppose that $h_1(r) \in S$ satisfies* (9.2)-(9.6). *Put*

$$L(r) = \exp\left\{\tilde{\delta}\int_1^r \sqrt{h_1(t)}\, t^{-1}dt\right\} \tag{9.14}$$

with any fixed $\tilde{\delta} > 0$, and define

$$\phi(r) = (\log r)L(r) \qquad (r > 1). \tag{9.15}$$

Then, given $\varepsilon \in (0, 1)$ and $\delta \in (0, 1]$, there is a function $f(z) \in \mathfrak{M}_{0,\delta}$ such that

$$T(r, f) = O(\phi(r)) \qquad (r \to \infty) \tag{9.16}$$

and

$$\log m^*(r, f) < \delta - (1-\varepsilon)(1-\delta/3)(\pi^2/2)\tilde{\delta}^2 h_1(r) \qquad (r \geqq r_0(\varepsilon)). \tag{9.17}$$

Proof. For given $\varepsilon \in (0, 1)$, choose $\varepsilon' > 0$ with the property that

$$(1-\varepsilon')\left[\frac{\pi^2}{2} - \frac{\delta\pi^2}{6} - \left\{\pi^2 + \frac{\delta\pi^2}{2(1-\varepsilon')} + \frac{1+\varepsilon'}{1-\varepsilon'}(2-\delta)\right\}\varepsilon'\right] > (1-\varepsilon)(\pi^2/2)(1-\delta/3). \tag{9.18}$$

By (9.14) and (9.15)

$$\phi_1(r) \equiv r\phi'(r) = L(r)\{1 + \tilde{\delta}\sqrt{h_1(r)}\log r\}, \tag{9.19}$$

so that

$$(9.20)\qquad \psi_2(r)\equiv r\psi_1'(r+0)=\tilde{\delta}^2 h_1(r)\log r\,L(r)\Big[1+\frac{2}{\tilde{\delta}}\frac{1}{\sqrt{h_1(r)}\log r}+\frac{r\tilde{h}_1'(r)}{2\tilde{\delta}\{h_1(r)\}^{3/2}}\Big].$$

From (9.2), (9.3), (9.6), (9.14) and (9.20) it follows that for each fixed $\lambda>1$

$$(9.21)\qquad \lim_{r\to\infty}\psi_2(\lambda r)/\psi_2(r)=1\,.$$

In view of (9.3), we have $\sqrt{h_1(r)}\geqq 2\tilde{\delta}^{-1}(\log r)^{-1}$ $(r\geqq r_0>1)$. From this and (9.14) we deduce that

$$(9.22)\qquad L(r)>\exp\left\{\int_{r_0}^{r}\frac{2}{t\log t}\,dt\right\}=\left(\frac{\log r}{\log r_0}\right)^2\qquad (r\geqq r_0)\,.$$

Hence by (9.3), (9.20) and (9.22)

$$(9.23)\qquad \psi_2(r)\longrightarrow\infty\qquad (r\to\infty)\,.$$

Define $P(z)$ and $Q(z)$ by

$$(9.24)\qquad \log P(z)=\int_{r_0}^{\infty}\log(1+z/t)\,d[\psi_1(t)]\,,$$

$$\log Q(z)=\int_{r_0}^{\infty}\log(1-z/t)\,d[(1-\delta)|\psi_1(t)-1|]\,.$$

Then, since (9.21) and (9.23) hold, the arguments in [1, pp 466-469] and [9, Proof of Theorem 2] show that

$$(9.25)\qquad \log m^*(r,\,P)<\left\{1-(1-2\varepsilon')\frac{\pi^2}{2}\frac{\psi_2(r)}{\psi(r)}\right\}\log M(r,\,P)\qquad (r\geqq r_0(\varepsilon'))\,,$$

and

$$(9.26)\qquad \log M(r,\,P)\leqq N(r,\,0,\,P)+(\pi^2/6)(1+\varepsilon')\psi_2(r)+\log 2\qquad (r\geqq r_0(\varepsilon'))\,.$$

From (9.24) we have

$$(9.27)\qquad \psi(r)-\log r<N(r,\,0,\,P)<\psi(r)$$

and

$$(9.28)\qquad N(r,\,0,\,Q)<(1-\delta)N(r,\,0,\,P)\,.$$

Now, put $f(z)=P(z)/Q(z)$. Since $T(r,\,f)\leqq m(r,\,P)+m(r,\,Q)\leqq\log M(r,\,P)+\log M(r,\,Q)$, we obtain (9.16) from (9.26), (9.27), (9.15) and (9.20). Using (9.16) and (9.28), we have $f(z)\in\mathfrak{M}_{0,\delta}$. We proceed to estimate $\log m^*(r,\,f)$ from above. By (9.23), (9.14) and (9.15)

$$(9.29)\qquad \frac{(\pi^2/6)(1+\varepsilon')\psi_2(r)+\log 2}{\psi(r)-\log r}<\frac{(\pi^2/6)(1+2\varepsilon')\psi_2(r)}{(1-\varepsilon')\psi(r)}\qquad (r\geqq r_0(\varepsilon'))\,.$$

We easily see from (9.20), (9.3) and (9.22) that

(9.30) $$(\log r)/\psi_2(r)<\varepsilon' \qquad (r\geqq r_0(\varepsilon')).$$

In view of (9.15) and (9.20)

(9.31) $$\psi_2(r)/\psi(r)>(1-\varepsilon')\tilde{\delta}^2 h_1(r) \qquad (r\geqq r_0(\varepsilon')).$$

Therefore from (9.25)-(9.27), (9.29)-(9.31) and (9.18) it follows that

$$\begin{aligned}
\log m^*(r, f) &= \log m^*(r, P)-\log M(r, Q)\\
&<\left\{1-(1-2\varepsilon')\frac{\pi^2}{2}\frac{\psi_2(r)}{\psi(r)}\right\}\log M(r, P)-(1-\delta)\log M(r, P)\\
&\quad+(2-\delta)\log(r+1)\\
&<\left\{\delta-(1-2\varepsilon')\frac{\pi^2}{2}\frac{\psi_2(r)}{\psi(r)}\right\}\left\{N(r, 0, P)+\frac{\pi^2}{6}(1+\varepsilon')\psi_2(r)+\log 2\right\}\\
&\quad+(2-\delta)\log(r+1)\\
&<\left\{\delta-(1-2\varepsilon')\frac{\pi^2}{2}\frac{\psi_2(r)}{\psi(r)}\right\}\left\{1+\frac{(\pi^2/6)(1+\varepsilon')\psi_2(r)+\log 2}{\psi(r)-\log r}\right\}N(r, 0, P)\\
&\quad+(2-\delta)\log(r+1)\\
&<\left[\delta-\left\{(1-2\varepsilon')\frac{\pi^2}{2}-\left(\frac{1+2\varepsilon'}{1-\varepsilon'}\right)\delta\frac{\pi^2}{6}-\frac{\varepsilon'(1+\varepsilon')}{1-\varepsilon'}(2-\delta)\right\}\frac{\psi_2(r)}{\psi(r)}\right]\\
&\quad\times N(r, 0, P)\\
&<\delta-(1-\varepsilon)(\pi^2/2)(1-\delta/3)\tilde{\delta}^2 h_1(r) \qquad (r\geqq r_0(\varepsilon')),
\end{aligned}$$

which implies (9.17). This completes the proof of Lemma 11.

Completion of the proof of Theorem 4. Let $\delta\in(0, 1]$ and $h(r)\in S$ be given, and let $h_1(r)\in S$ be constructed in Lemma 10 corresponding to $h(r)$. Further, let $f(z)\in \boldsymbol{m}_{0,\delta}$ be constructed in Lemma 11. Then we have from (9.17) that for any $\varepsilon\in(0, 1)$

$$\log m^*(r, f)<\delta\left\{1-\frac{(1-\varepsilon)(1-\delta/3)(\pi^2/2)}{\delta}\tilde{\delta}^2 h_1(r)\right\} \qquad (r\geqq r_0(\varepsilon)),$$

so if we choose $\tilde{\delta}(>0)$ small enough, we deduce from (9.1) that

$$\log m^*(r, f)<\delta(1-h_1(r))\leqq\delta(1-h(r)) \qquad (r\geqq r_0).$$

This completes the proof of Theorem 4.

Finally, without proof we state the following result, which should be compared with Lemma 11.

THEOREM 5. *Let* $\delta\in(0, 1]$ *be given, and suppose that* $h_1(r)\in S$ *satisfies* (9.2)-

(9.6). *If* $f(z)\in\mathfrak{m}_{0,\delta}$ *satisfies the growth condition*

$$T(r,\ f)=O\left((\log r)\exp\left\{\frac{\sqrt{2\delta}}{\pi\sqrt{(1+\varepsilon)(1-\delta/3)}}\int_1^r\frac{\sqrt{h_1(t)}}{t}dt\right\}\right)\qquad (r\to\infty)$$

with some $\varepsilon>0$, *then for a suitable sequence of* $r\to\infty$

$$\log m^*(r,\ f)>\delta(1-h_1(r))T(r,\ f)\,.$$

Although the proof is more complicated than the one of Theorem 1, they are essentially the same.

References

[1] Barry, P. D., The minimum modulus of small integral and subharmonic functions, Proc. London Math. Soc., (3) **12** (1962), 445-495.

[2] Barry, P. D., On the growth of entire functions, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press (1970), 43-60.

[3] Edrei, A. and Fuchs, W. H. J., Bounds for the number of deficient values of certain classes of functions, Proc. London Math. Soc., **12** (1962), 315-344.

[4] Gol'dberg, A. A., On defects of meromorphic functions, Dokl. Akad. Nauk SSSR, **98** (1954), 893-895.

[5] Karamata, J., Sur un mode de croissance régulière des fonctions, Mathematica (Cluj), **4** (1930), 38-53.

[6] Ostrowskii, I. V., Deficiencies of meromorphic functions of order less than one, Dokl. Akad. Nauk SSSR, **150** (1963), 32-35.

[7] Teichmüller, O., Vermutungen und Sätze über die Wertverteilung gebrochener Funktionen endlicher Ordnung, Duetsche Mathematik, **4** (1939), 163-190.

[8] Ueda, H., On the growth of meromorphic functions of order less than 1/2, Kodai Math. J. **6** no. 2 (1983), 135-146.

[9] Ueda, H., On the growth of meromorphic functions of order less than 1/2, II, Kodai Math. J. **6** no. 3 (1983), 397-407.

Department of Mathematics
Daido Institute of Technology
Daido-cho, Minami-ku, Nagoya, Japan