T. MURAI
KODAI MATH. J.
5 (1982), 313—317

THE VALUE-DISTRIBUTION OF RANDOM ENTIRE FUNCTIONS

BY TAKAFUMI MURAI

1. It is well-known that, for a given entire function $f(z)$, $\delta(a, f)=0$ $(a\in \boldsymbol{C})$ holds except possibly for a countable set, where " δ " denotes the deficiency and $\boldsymbol{C}$ the complex plane. We cannot generally remove the above exceptional set. The purpose of this paper is to show that the totality of entire functions $f(z)$ with $\delta^*(f)=\sup_{a\in C}\delta(a, f)>0$ is thin in a sense.

An open interval $\Omega=(-1/2, 1/2)$ is naturally a probability space. A Rademacher series $\varepsilon=(\varepsilon_k)_{k=1}^{\infty}$ in Ω is defined by $\varepsilon_k(\omega)=\text{sign}(\sin 2^k\pi\omega)$ $(\omega\in\Omega)$. For a sequence $(a_k)_{k=1}^{\infty}$ $(\not\equiv 0)\subset \boldsymbol{C}$ with $\limsup_{k\to\infty}|a_k|^{1/k}=0$, a random entire function is defined by

$$(1)\qquad f_\varepsilon(z)=\sum_{k=1}^{\infty}\varepsilon_k a_k z^k=\left\{f_\omega(z)=\sum_{k=1}^{\infty}\varepsilon_k(\omega)a_k z^k\,;\ \omega\in\Omega\right\}.$$

A random entire function $f_\varepsilon(z)$ is a probability space of entire functions. We write simply $\delta(a, \omega)=\delta(a, f_\omega)$, $\delta^*(\omega)=\delta^*(f_\omega)$. In this paper, we shall show the following

THEOREM. $\delta^*(\omega)=0$ *almost surely* (*a. s.*).

2. We denote by " Pr " the probability. Put

$$(2)\qquad \begin{cases} T(r, f_\omega)=1/2\pi\displaystyle\int_0^{2\pi}\log^+|f_\omega(re^{it})|\,dt \\ T_0(r)=\log^+A_0(r)\,,\quad A_0(r)=\left(\displaystyle\sum_{k=1}^{\infty}|a_k|^2r^{2k}\right)^{1/2} \\ m(r, a, \omega)=1/2\pi\displaystyle\int_0^{2\pi}\log^+1/|f_\omega(re^{it})-a|\,dt \qquad (a\in \boldsymbol{C},\ r>0), \end{cases}$$

where $\log^+x=\max\{\log x, 0\}$ $(x>0)$. Note that $\delta(a, \omega)=\liminf_{r\to\infty} m(r, a, \omega)/T(r, f_\omega)$ $(a\in \boldsymbol{C}, \omega\in\Omega)$. If $\#\{k\,;\,a_k\neq 0\}<\infty$, then $f_\varepsilon(z)$ is a probability space of polynomials and we see easily $\delta^*(\omega)=0$ for all $\omega\in\Omega$. The proof in the case where

Received March 20, 1981

$\sharp\{k\,;\,a_k\neq 0\}=\infty$ is essential. For the sake of simplicity, we only give the proof in the case where $a_k\neq 0$ for all k. We use the following proposition, which is an improvement of Lemma 4 in [5].

PROPOSITION. *Suppose that there exist a set* $\Omega_0\subset\Omega$ *with* $Pr(\Omega_0)=1$ *and mappings* $\delta(\cdot\,;m,q,p)$ $(p=1,\cdots,q\,;q=1,2,\cdots;m=1,2,\cdots)$ *from* Ω_0 *to an interval* $[0,1]$ *such that*:

(3) *If* $\omega\in\Omega_0$ *and* ω' *satisfy* $\varepsilon_k(\omega')=\varepsilon_k(\omega)$ *except for a finite number of* k*'s, then* $\omega'\in\Omega_0$.

(4) $\delta^*(\omega)\leqq\delta(\omega\,;m,q,p)$.

(5) $\sum_{\omega'\in\Gamma(\omega,m,p)}\delta(\omega'\,;m,q,p)\leqq 1$ *for all* $\omega\in\Omega_0$, *where* $\Gamma(\omega,m,p)=\{\omega'\in\Omega_0\,;$ $\varepsilon_k(\omega')=\varepsilon_k(\omega)$ *for all* k *with* $k\neq(p-1)m+1,(p-1)m+2,\cdots,pm\}$.

(6) $\delta(\omega\,;m,q,p)=\delta(\omega'\,;m,q,p)$ *if* $\varepsilon_k(\omega)=\varepsilon_k(\omega')$ *for all* $k\geqq(p-1)m+1$.

Then $\delta^*(\omega)=0$ *a.s.*.

Proof. For a sequence $(\varepsilon_n,\varepsilon_{n+1},\cdots)$, $\varepsilon_k=\pm1$, we put $\Omega(\varepsilon_n,\varepsilon_{n+1},\cdots)=\{\omega\in\Omega_0\,;\,\varepsilon_k(\omega)=\varepsilon_k\ (k\geqq n)\}$. Let $m,q\geqq 1$ and $(\alpha_{qm+1},\alpha_{qm+2},\cdots)$, $\alpha_k=\pm1$, be a sequence such that $\Omega(\alpha_{qm+1},\alpha_{qm+2},\cdots)\neq\Phi$. Then, by (3), $\sharp\Omega(\alpha_{qm+1},\cdots)=2^{qm}$. Now we show that there exists an m-tuple $(\beta_{(q-1)m+1},\cdots,\beta_{qm})$, $\beta_k=\pm1$, such that $\delta^*(\omega)\leqq 2^{-m}$ $(\omega\in\Omega(\beta_{(q-1)m+1},\cdots,\beta_{qm},\alpha_{qm+1},\alpha_{qm+2},\cdots))$.

To see this, choose arbitrarily $\omega_0\in\Omega(\alpha_{qm+1},\cdots)$. By (5), we have

$$\sum_{\omega\in\Gamma(\omega_0,m,q)}\delta(\omega\,;m,q,q)\leqq 1\,.$$

Since $\sharp\Gamma(\omega_0,m,q)=2^m$, there exists ω_0' such that $\delta(\omega_0'\,;m,q,q)\leqq 2^{-m}$. Put $\beta_k=\varepsilon_k(\omega_0')$ $(k=(q-1)m+1,\cdots,qm)$. Then $\delta^*(\omega)\leqq\delta(\omega\,;m,q,q)=\delta(\omega_0'\,;m,q,q)\leqq 2^{-m}$ $(\omega\in\Omega(\beta_{(q-1)m+1},\cdots,\beta_{qm},\alpha_{qm+1},\cdots))$. Thus the required m-tuple $(\beta_{(q-1)m+1},\cdots,\beta_{qm})$ is obtained.

The above fact signifies that $\delta^*(\omega)\leqq 2^{-m}$ $(\omega\in\Omega(\varepsilon_{(q-2)m+1},\cdots,\varepsilon_{qm},\alpha_{qm+1},\cdots))$ holds if $(\varepsilon_{(q-1)m+1},\cdots,\varepsilon_{qm})=(\beta_{(q-1)m+1},\cdots,\beta_{qm})$, and hence it holds for the at least 2^m number of $2m$-tuples in the 2^{2m} number of $2m$-tuples $(\varepsilon_{(q-2)m+1},\cdots,\varepsilon_{qm})$, $\varepsilon_k=\pm1$.

For every $(\gamma_{(q-1)m+1},\cdots,\gamma_{qm})\neq(\beta_{(q-1)m+1},\cdots,\beta_{qm})$, $\gamma_k=\pm1$, we can choose an m-tuple $(\sigma_{(q-2)m+1},\cdots,\sigma_{(q-1)m})$, $\sigma_k=\pm1$, such that $\delta^*(\omega)\leqq 2^{-m}$ $(\omega\in\Omega(\sigma_{(q-2)m+1},\cdots,\sigma_{(q-1)m},\gamma_{(q-1)m+1},\cdots,\gamma_{qm},\alpha_{qm+1},\cdots))$. These facts signify that $\delta^*(\omega)\leqq 2^{-m}$ $(\omega\in\Omega(\varepsilon_{(q-2)m+1},\cdots,\varepsilon_{qm},\alpha_{qm+1},\cdots))$ holds for the at least $2^m+(2^m-1)=2^{2m}-(2^m-1)^2$ number of $2m$-tuples in the 2^{2m} number of $2m$-tuples $(\varepsilon_{(q-2)m+1},\cdots,\varepsilon_{qm})$, $\varepsilon_k=\pm1$.

Repeating this discussion, we see that $\delta^*(\omega)\leqq 2^{-m}$ $(\omega\in\Omega(\varepsilon_1,\cdots,\varepsilon_{qm},\alpha_{qm+1},\cdots))$ holds for the at least $2^{qm}-(2^m-1)^q$ number of qm-tuples in the 2^{qm} number of qm-tuples $(\varepsilon_1,\cdots,\varepsilon_{qm})$, $\varepsilon_k=\pm1$.

Since $Pr(\Omega_0)=1$ and $(\alpha_{qm+1}, \alpha_{qm+2}, \cdots)$ is arbitrary as long as $\Omega(\alpha_{qm+1}, \cdots) \neq \Phi$, we have $Pr(\delta^*(\omega)\leqq 2^{-m})\geqq 1-\{(2^m-1)/2^m\}^q$. Since $q\geqq 1$ is arbitrary, we have $\delta^*(\omega)\leqq 2^{-m}$ a. s.. Since $Pr(\bigcap_{m=1}^{\infty}\{\delta^*(\omega)\leqq 2^{-m}\})=1$, we have $\delta^*(\omega)=0$ a. s..

Q. E. D.

3. By the above proposition, it is sufficient to show the existence of such a set and such mappings. To define these, we need the following two lemmas, which are analogous to Lemma 3, 6 in [5]. Since these proofs are analogous, we omit the proofs.

LEMMA 1. *There exists a constant* C_0 *such that, for any sequence* $(\rho_n)_{n=1}^{\infty}$, $\rho_n>0$, $\rho_n\to\infty$ $(n\to\infty)$,

$$\limsup_{n\to\infty} T(\rho_n, f_\omega)/T_0(\rho_n)>C_0 \quad a.\,s.. \tag{7}$$

LEMMA 2. *Put*

$$\begin{cases} A_l(r)=\left(\sum_{k=l}^{\infty}\{k\,!/(k-l)\,!\}^2|a_k|^2r^{2(k-l)}\right)^{1/2} \\ T_l(r)=\log^+A_l(r) \quad (a_0=0,\ l=0,\ 1,\ \cdots). \end{cases} \tag{8}$$

Then, for a given $K\geqq 1$, *there exists a sequence* $(r_n)_{n=1}^{\infty}=(r_n(K))_{n=1}^{\infty}$; $r_n>0$, $r_n\to\infty$ $(n\to\infty)$, *such that*:

$$A_0\left(r_n+\frac{1}{A_0(r_n)}\right)\leqq 2A_0(r_n)\,. \tag{9}$$

$$T_l\left(r_n+\frac{1}{T_l(r_n)}\right)\leqq 2T_l(r_n) \quad (l=0,\ \cdots,\ K)\,. \tag{10}$$

Now we put:

$$\Omega_0=\bigcap_{m=1}^{\infty}\bigcap_{q=1}^{\infty}\{\omega\in\Omega\,;\ \limsup_{n\to\infty} T(r_n(mq),\ f_\omega)/T_0(r_n(mq))>C_0\}\,. \tag{11}$$

$$\delta(\omega\,;\ m,\ q,\ p)=\liminf_{n\to\infty,\, n\in\Delta_{\omega mq}} T(r_n(mq),\ 1/f_\omega^{((p-1)m+1)})/T(r_n(mq),\ f_\omega)\,, \tag{12}$$

where $(r_n(mq))_{n=1}^{\infty}$ is the sequence in Lemma 2 with $K=mq$ and

$$\Delta_{\omega mq}=\{n\,;\ T(r_n(mq),\ f_\omega)/T_0(r_n(mq))>C_0\} \quad (\omega\in\Omega_0).$$

Thus Ω_0, $\delta(\cdot\,;\ m,\ q,\ p)$ $(p=1,\ \cdots,\ q\,;\ q\geqq 1\,;\ m\geqq 1)$ are defined.

4. We show that the above Ω_0, $\delta(\cdot\,;\ m,\ q,\ p)$ satisfy the conditions in Lemma 1. We see easily $Pr(\Omega_0)=1$ and (3). For given $m,\ q\geqq 1$, we must prove that $\delta_p(\cdot)=\delta(\cdot\,;\ m,\ q,\ p)$ $(p=1,\ \cdots,\ q)$ satisfy (4), (5) and (6). So we write simply $r_n=r_n(mq)$ $(n\geqq 1)$. We see easily (6). To prove (4) and (5), we need the following two lemmas. Lemma 3 is proved analogously as in THEOREM 2.1 in [1] and Lemma 4 is analogous to Lemma 8 in [5], and hence we omit the proofs.

LEMMA 3. *Let $g(z)$ be an entire function ($\neq a$ polynomial) and $\{P_j(z)\}_{j=1}^{n}$ multually distinct polynomials of degree ν. Then*

$$\sum_{j=1}^{n} T(r, 1/g_j) \leqq T(r, g^{(\nu+1)}) + \sum_{j=1}^{n} \sum_{\mu=1}^{\nu+1} T(r, g_j^{(\mu)}/g_j^{(\mu-1)}) + O(\log r), \tag{13}$$

where $g_j(z)=g(z)+P_j(z)$ $(j=1, \cdots, n)$.

LEMMA 4. *Let $\omega \in \Omega_0$. Then, for any $a \in \boldsymbol{C}$ and any l, $1 \leqq l \leqq qm+1$, we have $T(r_n, f_\omega^{(l)}/(f_\omega^{(l-1)}-a)) = o(T_0(r_n))$ $(n \to \infty)$.*

First we prove (4). Let $\omega \in \Omega_0$. For every $a \in \boldsymbol{C}$, we have

$$\begin{aligned} m(r_n, a, \omega) &= T(r_n, 1/(f_\omega - a)) \\ &= T\left(r_n, \frac{f'_\omega}{f_\omega - a} \cdot \frac{f''_\omega}{f'_\omega} \cdots \frac{f_\omega^{((p-1)m+1)}}{f_\omega^{((p-1)m)}} \cdot \frac{1}{f_\omega^{((p-1)m+1)}}\right) \\ &\leqq T(r_n, 1/f_\omega^{((p-1)m+1)}) + \{T(r_n, f'_\omega/(f_\omega - a)) + \cdots + T(r_n, f_\omega^{((p-1)m+1)}/f_\omega^{((p-1)m)})\} \\ &= T(r_n, 1/f_\omega^{((p-1)m+1)}) + o(T_0(r_n)), \end{aligned} \tag{14}$$

according to Lemma 4. Hence $\delta(a, \omega) \leqq \delta_p(\omega)$. Since this inequality holds for all $a \in \boldsymbol{C}$, we have (4).

Next we prove (5). Let $\omega \in \Omega_0$. In the same manner as in (14), we have

$$T(r_n, f_\omega^{(pm+1)}) = T(r_n, f_\omega) + o(T_0(r_n)) \quad (n \to \infty). \tag{15}$$

By Lemma 3, 4 and (15), we have

$$\begin{aligned} &\sum_{\omega' \in \Gamma(\omega, m, p)} T(r_n, 1/f_{\omega'}^{((p-1)m+1)}) \\ &\leqq T(r_n, f_\omega^{(pm+1)}) + o(T_0(r_n)) = T(r_n, f_\omega) + o(T_0(r_n)) \quad (n \to \infty). \end{aligned} \tag{16}$$

Note that $\lim_{n \to \infty} T(r_n, f_\omega)/T(r_n, f_{\omega'}) = 1$ $(\omega' \in \Gamma(\omega, m, p))$ and that there exists $n \geqq 1$ such that $\Delta_{\omega mq} \cap [n, +\infty) = \Delta_{\omega' mq} \cap [n, +\infty)$ for all $\omega' \in \Gamma(\omega, m, p)$. Divide every term in (16) by $T(r_n, f_\omega)$. Letting $n \to \infty$ $(n \in \Delta_{\omega mq})$, we have (5). This completes the proof.

REFERENCES

[1] HAYMAN, W.K., Meromorphic functions, Oxford (Clarendon press), 1964.
[2] KAHANE, J.P., Some random series of functions, Heath M.M., 1968.
[3] LITTLEWOOD, J.E. AND OFFORD, A.C., On the distribution of zeros and a-values of a random integral function II, Annals of M. **49** (1948), 885-952.
[4] MURAI, T., Une remarque sur la distribution des valeurs des séries de Taylor

aléatoires, C.R.A.S. Paris, t. **287** (20 nobembre 1978), 931-934.
[5] MURAI, T., The value-distribution of random Taylor series in the unit disk, J. London M.S. (3), **24** (1981), 480-494.
[6] NEVANLINNA, R., Analytic functions, Springer, 1970.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA, 464
JAPAN