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MEROMORPHIC FUNCTIONS WHOSE DERIVATIVES SHARE SMALL FUNCTIONS

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Abstract

In this paper, we prove that if the derivatives of two nonconstant meromorphic functions f and g share three small functions CM*, or share two small functions CM* and another two small functions IM*, then $f' = g'$ or f' is a quasi-Möbius transformation of g' mostly.

1. Introduction

Let $f(z)$ be a nonconstant meromorphic function in the complex plane $\boldsymbol{C}$. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f), N(r,f)$ and $m(r,f)$ (see, e.g., [1]). In this paper, we use $N_{k)}(r, 1/(f-a))$ to denote the counting function of a-points of f with multiplicities less than or equal to k, and $N_{(k}(r, 1/(f-a))$ to denote the counting function of a-points of f with multiplicities great than or equal to k. We also use $\overline{N}_{k)}(r, 1/(f-a))$ and $\overline{N}_{(k}(r, 1/(f-a))$ to denote the correspondent reduced counting function, respectively. The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure. A meromorphic function $c(z)$ is called a small function with respect to $f(z)$ provided that $T(r,c) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and $c(z)$ a small function with respect to both $f(z)$ and $g(z)$. If $f(z) - c(z)$ and $g(z) - c(z)$ have the same zeros ignoring (counting) multiplicities, then we say that $f(z)$ and $g(z)$ share $c(z)$ IM (CM). We say $f(z)$ and $g(z)$ share ∞ IM (CM) if $1/f$ and $1/g$ share 0 IM (CM).

Let $S(f = c = g)$ be the set of all common zeros of $f(z) - c(z)$ and $g(z) - c(z)$ ignoring multiplicities, $S_E(f = c = g)$ the set of all common zeros of $f(z) - c(z)$ and $g(z) - c(z)$ with the same multiplicities. Denote $\overline{N}(r, f = c = g)$, $\overline{N}_E(r, f = c = g)$ the reduced counting functions of f and g correspondent to the sets $S(f = c = g)$, and $S_E(f = c = g)$ respectively. If

$$\overline{N}\left(r, \frac{1}{f-c}\right) + \overline{N}\left(r, \frac{1}{g-c}\right) - 2\overline{N}(r, f = c = g) = S(r,f)$$

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then we say that f and g share c IM*. If

$$\overline{N}\left(r, \frac{1}{f-c}\right) + \overline{N}\left(r, \frac{1}{g-c}\right) - 2\overline{N}_E(r, f = c = g) = S(r, f)$$

then we say that f and g share c CM*. Obviously, any IM (CM) shared small function must be an IM* (CM*) shared small function.

In 1926, R. Nevanlinna [3] proved that if two meromorphic functions f and g share four values a_1, a_2, a_3, a_4 CM, then f is a Möbius transformation of g. Since then there have many papers been published on uniqueness theory and sharing values. It is easy to prove (see, e.g., [4] p. 184) that if the derivatives f' and g' of two meromorphic functions share four distinct finite values IM, then $f' = g'$. It is natural to ask what happens when f' and g' share three or four small functions. In this paper, we prove the following theorems:

THEOREM 1. *Let f and g be two nonconstant meromorphic functions sharing three small functions a_1, a_2, a_3 CM*. Let c $(\not\equiv a_1, a_2, a_3)$ be a small function with respect to f and g such that $(c, a_3, a_2, a_1) \not\equiv (1 \pm \sqrt{2})/2, -2 \pm 2\sqrt{2}, 3 \pm 2\sqrt{2}$. If*

$$N_{1)}\left(r, \frac{1}{f-c}\right) + N_{1)}\left(r, \frac{1}{g-c}\right) = S(r, f),$$

then $f = g$ or

$$f = \frac{(a_i + a_j)g - 2a_i a_j}{2g - a_i - a_j}.$$

The latter occurs only if the cross ratio (c, a_k, a_j, a_i) is equal to -1 for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

THEOREM 2. *Let f and g be two nonconstant meromorphic functions, a_1, a_2, a_3 $(\not\equiv \infty)$ be small functions with respect to f' and g', and $(a_3 - a_1)/(a_3 - a_2) \not\equiv (1 \pm \sqrt{2})/2, -2 \pm 2\sqrt{2}, 3 \pm 2\sqrt{2}$. If f' and g' share a_1, a_2, a_3 CM* then $f' = g'$ or*

$$f' = \frac{(a_i + a_j)g' - 2a_i a_j}{2g' - a_i - a_j}.$$

The latter occurs only if $2a_k - a_i - a_j = 0$ for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

Remark. There exist two meromorphic functions f and g such that f' and g' share three small functions but $f' \neq g'$. For example, the derivatives of the functions $f = e^z$ and $g = -e^{-z}$ share $0, 1, -1$ CM, but $f' \neq g'$.

THEOREM 3. *Let f and g be two nonconstant meromorphic functions. If f' and g' share two small functions a_1, a_2 $(\not\equiv \infty)$ CM*, and share another two small functions a_3, a_4 $(\not\equiv \infty)$ IM*, then $f' = g'$.*

2. Lemmas

LEMMA 1 ([2]). *Let f be a nonconstant meromorphic function and b_i, $i = 0, 1, \dots, n$ be small functions of f. If*

$$b_n f^n + b_{n-1} f^{n-1} + \cdots + b_0 \equiv 0,$$

then $b_i \equiv 0$, $i = 0, 1, \dots n$.

LEMMA 2 ([5]). *Let f and g be two nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4 be four distinct small functions with respect to f and g. If f and g share a_1, a_2 CM* and share a_3, a_4 IM*, then f is a quasi-Möbius transformation of g, i.e., there exist four small functions α_i $(i = 1, 2, 3, 4)$ such that*

$$f = \frac{\alpha_1 g + \alpha_2}{\alpha_3 g + \alpha_4}.$$

LEMMA 3 ([6]). *Let f and g be nonconstant meromorphic functions and a_0, a_i, b_i, $i = 1, 2$ be small functions of f and g such that $a_i \not\equiv a_j$, $b_i \not\equiv b_j$ $(i \neq j)$, $f - a_i$ share 0 CM* with $g - b_i$ $(i = 1, 2)$ and $f - a_i$ share ∞ CM* with $g - b_i$, $(i = 1, 2)$. If*

$$T(r, f) \neq N(r, 1/(f - a_0)) + S(r, f),$$

then f is a quasi-Möbius transformation of g.

LEMMA 4 ([7]). *Let $f_1, f_2, \dots, f_n$ be nonconstant meromorphic functions such that $f_1 + f_2 + \cdots + f_n = 1$. If $f_1, f_2, \dots, f_n$ are linearly independent, then the following inequality holds*

$$T(r, f_1) < \sum_{i=1}^{n} N_{n-1}\left(r, \frac{1}{f_i}\right) + (n - 1) \sum_{i=1}^{n} \overline{N}_{n-1}(r, f_i) + o(T(r)), \quad r \notin E.$$

Here and in the sequel, $N_{n-1}(r, f)$ is the counting function of f which counts a pole of f according to its multiplicity if that multiplicity is less than or equal to $n - 1$ and counts a pole $n - 1$ times if the multiplicity is greater than $n - 1$. Here $T(r) = \sum_{i=1}^{n} T(r, f_i)$.

LEMMA 5 ([8]). *Let f_1 and f_2 be two nonconstant meromorphic functions satisfying*

$$\overline{N}(r, f_i) + \overline{N}\left(r, \frac{1}{f_i}\right) = S(r), \quad i = 1, 2.$$

If $f_1^s f_2^t - 1$ is not identically zero for all integers s and t $(|s| + |t| > 0)$, then for any positive number ε, we have

$$N_0(r, 1; f_1, f_2) \leq \varepsilon T(r) + S(r)$$

where $N_0(r, 1; f_1, f_2)$ *denotes the reduced counting function of* f_1 *and* f_2 *related to the common* 1*-points and* $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))$ *as* $r \to \infty$ *possibly outside a set of* r *of finite linear measure.*

Lemma 6. *Let* f *and* g *be nonconstant meromorphic functions, and* a_1, a_2, a_3, a_4 *be small functions with respect to* f *and* g. *If* f *and* g *share* a_1, a_2 *CM**, *and share* a_3, a_4 *IM**, *and if there exists a small function* c $(\not\equiv a_1, a_2, a_3, a_4)$ *with respect to* f *and* g *such that*

$$N_{1)}\left(r, \frac{1}{f-c}\right) = S(r, f),$$

then $f = g$.

Proof. If two of a_1, a_2, a_3, a_4, say a_i, a_j, $i \neq j$, satisfy

$$\overline{N}\left(r, \frac{1}{f-a_i}\right) + \overline{N}\left(r, \frac{1}{f-a_j}\right) = S(r),$$

here and in the sequel, $S(r) := S(r, f) = S(r, g)$, where the equality follows from the assumption that f and g share three small functions IM*. Then we have

$$\begin{aligned} T(r,f) &\le \overline{N}\left(r, \frac{1}{f-a_i}\right) + \overline{N}\left(r, \frac{1}{f-a_j}\right) + \overline{N}\left(r, \frac{1}{f-c}\right) + S(r) \\ &\le \overline{N}_{(2}\left(r, \frac{1}{f-c}\right) + S(r) \\ &\le \frac{1}{2} N\left(r, \frac{1}{f-c}\right) + S(r) \\ &\le \frac{1}{2} T(r,f) + S(r). \end{aligned}$$

It is a contradiction. So, without loss of generality, we can assume

$$\overline{N}\left(r, \frac{1}{f-a_j}\right) \neq S(r), \quad j = 1, 2, 3.$$

By Lemma 1, f is a quasi-Möbius transformation of g, i.e., $f = (\alpha_1 g + \alpha_2)/(\alpha_3 g + \alpha_4)$ where α_i $(i = 1, 2, 3, 4)$ are small functions with respect to f and g. Since f and g share a_1, a_2, a_3 IM*, the quasi-Möbius transformation $M(x) = (\alpha_1 x + \alpha_2)/(\alpha_3 x + \alpha_4)$ will have three fixed small functions, i.e., $M(a_i) = a_i$, $i = 1, 2, 3$, which implies that $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3 \equiv 0$. Hence $f = g$. □

Lemma 7. *Let* f *be a nonconstant meromorphic functions,* g *a quasi-Möbius transformation of* f. *Let* a_1, a_2, a_3, *and* c $(\not\equiv a_1, a_2, a_3)$ *be small functions with respective to* f. *If* f *and* g *share* a_1, a_2, a_3 *CM**, *and if*

$$N_{1)}\left(r, \frac{1}{f-c}\right) + N_{1)}\left(r, \frac{1}{g-c}\right) = S(r),$$

then $f = g$ *or*

$$f = \frac{(a_i + a_j)g - 2a_i a_j}{2g - a_i - a_j}.$$

The latter occurs only when the cross ratio (c, a_k, a_j, a_i) *is equal to* -1 *for some permutation* $\{i, j, k\}$ *of* $\{1, 2, 3\}$.

Proof. By the argument similar to that in the proof of Lemma 5, without loss of generality, we can assume that

$$\overline{N}\left(r, \frac{1}{f - a_1}\right) \neq S(r), \quad \overline{N}\left(r, \frac{1}{f - a_2}\right) \neq S(r). \tag{1}$$

Let

$$F = \frac{f - a_2}{f - a_1} \frac{a_3 - a_1}{a_3 - a_2}, \quad G = \frac{g - a_2}{g - a_1} \frac{a_3 - a_1}{a_3 - a_2}.$$

Then F and G share $\infty, 0, 1$ CM*, and

$$N_{1)}\left(r, \frac{1}{F - c_0}\right) + N_{1)}\left(r, \frac{1}{G - c_0}\right) = S(r), \tag{2}$$

where $c_0 = ((c - a_2)/(c - a_1))(a_3 - a_1)/(a_3 - a_2)$, and

$$\overline{N}(r, F) \neq S(r), \quad \overline{N}\left(r, \frac{1}{F}\right) \neq S(r) \tag{3}$$

by (1). Since g is a quasi-Möbius transformations of f, G is also a quasi-Möbius transformations of F, accordingly. From these, we get $F = \alpha G$, where α is a small function. If $\alpha \equiv 1$, then $F = G$ which implies that $f = g$. Assume that $\alpha \not\equiv 1$. Then

$$N\left(r, \frac{1}{F - 1}\right) \leq N\left(r, \frac{1}{F/G - 1}\right) + S(r) = N\left(r, \frac{1}{\alpha - 1}\right) + S(r) = S(r).$$

Hence we have

$$\overline{N}\left(r, \frac{1}{F - 1}\right) + \overline{N}\left(r, \frac{1}{G - 1}\right) = S(r). \tag{4}$$

If $\alpha \not\equiv c_0$, then we have, by (4)

$$\begin{aligned} T(r, F) &\leq \overline{N}\left(r, \frac{1}{F - 1}\right) + \overline{N}\left(r, \frac{1}{F - \alpha}\right) + \overline{N}\left(r, \frac{1}{F - c_0}\right) + S(r) \\ &= \overline{N}\left(r, \frac{1}{G - 1}\right) + \overline{N}\left(r, \frac{1}{F - c_0}\right) + S(r) \\ &= \overline{N}\left(r, \frac{1}{F - c_0}\right) + S(r) \\ &\leq \frac{1}{2} T(r, F) + S(r). \end{aligned}$$

It is a contradiction. If $\alpha \equiv c_0$, then we get $F = c_0 G$. By (4), we get

$$\overline{N}\left(r, \frac{1}{F - c_0}\right) + \overline{N}\left(r, \frac{1}{G - 1/c_0}\right) = S(r). \tag{5}$$

If $c_0 \not\equiv -1$, then $1, c_0, 1/c_0$ are different from each other. By the second fundamental theorem, we have by (2), (4) and (5)

$$\begin{aligned} T(r, G) &\le \overline{N}\left(r, \frac{1}{G - 1}\right) + \overline{N}\left(r, \frac{1}{G - c_0}\right) + \overline{N}\left(r, \frac{1}{G - 1/c_0}\right) + S(r) \\ &= \overline{N}\left(r, \frac{1}{G - c_0}\right) + S(r) \\ &\le \frac{1}{2} T(r, G) + S(r), \end{aligned}$$

which is a contradiction. If $c_0 \equiv -1$, i.e., $(c, a_3, a_2, a_1) = -1$, then $F = -G$, which implies that

$$f = \frac{(a_1 + a_2)g - 2a_1 a_2}{2g - a_1 - a_2}. \qquad \square$$

Lemma 8. *Let f and g be two nonconstant meromorphic functions, and a_1, a_2, a_3 be three distinct small functions with respect to f and g. If f and g share a_1, a_2, a_3 CM*, and if f is not a quasi-Möbius transformation of g, then for any small function c ($\not\equiv a_1, a_2, a_3$) with respect to f and g, we have*

$$N_{(3}\left(r, \frac{1}{f - c}\right) = S(r).$$

Proof. Let

$$H_1 = \frac{g - a_1}{f - a_1} \frac{f - a_3}{g - a_3}, \quad H_2 = \frac{g - a_2}{f - a_2} \frac{f - a_3}{g - a_3}. \tag{6}$$

Since f and g share a_1, a_2, a_3 CM* and f is not a quasi-Möbius transformation of g, we have

$$N(r, H_j) + N\left(r, \frac{1}{H_j}\right) = S(r), \quad j = 1, 2,$$

and

$$T(r, H_1) \ne S(r), \quad T(r, H_2) \ne S(r), \quad T\left(r, \frac{H_1}{H_2}\right) \ne S(r)$$

by eliminating g from the two equations in (6), we get

$$f - c = \frac{(a_1 - c)(a_2 - a_3)(H_1 - 1) - (a_2 - c)(a_1 - a_3)(H_2 - 1)}{(a_2 - a_3)(H_1 - 1) - (a_1 - a_3)(H_2 - 1)}.$$

Set $H = (a_1 - c)(a_2 - a_3)(H_1 - 1) - (a_2 - c)(a_1 - a_3)(H_2 - 1)$, then

$$f_1 + f_2 + f_3 = 1, \tag{7}$$

where

$$f_1 = \frac{H}{(a_1 - a_2)(a_3 - c)}, \quad f_2 = -\frac{(a_2 - a_3)(a_1 - c)}{(a_1 - a_2)(a_3 - c)} H_1, \quad f_3 = \frac{(a_1 - a_3)(a_2 - c)}{(a_1 - a_2)(a_3 - c)} H_2.$$

If f_1, f_2, f_3 are linearly dependent over $\boldsymbol{C}$, then there exist three constants c_1, c_2 and c_3 such that one of them is not zero, and

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{8}$$

Obviously, $c_1 \neq 0$, otherwise $T(r, H_1/H_2) = S(r)$. It follows from (7) and (8) that

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1,$$

and hence $c_1 \neq c_2$. Thus by the second fundamental theorem, we have

$$\begin{aligned} T(r, f_2) &\leq \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}\left(r, \frac{1}{f_2 - c_1/(c_1 - c_2)}\right) + \overline{N}(r, f_2) + S(r) \\ &\leq \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}\left(r, \frac{1}{f_3}\right) + \overline{N}(r, f_2) + S(r). \end{aligned}$$

Note that

$$T(r, f_2) = T(r, H_1) + S(r), \quad \overline{N}\left(r, \frac{1}{f_j}\right) + \overline{N}(r, f_j) = S(r), \quad j = 2, 3.$$

We get $T(r, H_1) = S(r)$. This is impossible. We are led to the case: f_1, f_2, f_3 are linearly independent over $\boldsymbol{C}$. From Lemma 4, we have

$$T(r, f_1) \leq N_2\left(r, \frac{1}{f_1}\right) + S(r).$$

Therefore,

$$N\left(r, \frac{1}{H}\right) \leq T(r, f_1) + S(r) \leq N_2\left(r, \frac{1}{H}\right) + S(r).$$

It follows that

$$\sum_{n=3}^{\infty} (n - 2) \overline{N}_{(n)}\left(r, \frac{1}{H}\right) = S(r),$$

where $\overline{N}_{(n)}(r, 1/H)$ denote the counting function of zeros of H with multiplicity of n. Hence

$$N_{(3}\left(r, \frac{1}{f - c}\right) \leq N_{(3}\left(r, \frac{1}{H}\right) \leq 3 \sum_{n=3}^{\infty} (n - 2) \overline{N}_{(n)}\left(r, \frac{1}{H}\right) = S(r),$$

which completes the proof of Lemma 8. □

3. Proof of the Theorems

Proof of Theorem 1. By Lemma 7, we only need to prove that f is a quasi-Möbius transformation of g. Moreover, we only need to consider the case that $a_1 = \infty$, $a_2 = 0$ and $a_3 = 1$, otherwise, a quasi-Möbius transformation will do.

Suppose that f is not any quasi-Möbius transformation of g. By Lemma 3 and the assumption, we have

$$T(r,f) = N\left(r, \frac{1}{f-c}\right) + S(r,f) = N_{(2}\left(r, \frac{1}{f-c}\right) + S(r,f). \tag{9}$$

Let

$$h_1 := \frac{f-1}{g-1} \quad \text{and} \quad h_2 := \frac{g(f-1)}{f(g-1)}. \tag{10}$$

Then we have

$$N(r,h_j) + N\left(r, \frac{1}{h_j}\right) = S(r), \quad j = 1, 2, \tag{11}$$

and

$$T(r,h_1) \neq S(r), \quad T(r,h_2) \neq S(r), \quad T\left(r, \frac{h_1}{h_2}\right) \neq S(r).$$

It follows from (10) that

$$f = \frac{h_1 - 1}{h_2 - 1}, \quad g = \frac{1/h_1 - 1}{1/h_2 - 1}, \tag{12}$$

which leads to

$$f - c = \frac{h_1 - ch_2 + c - 1}{h_2 - 1}. \tag{13}$$

Let $H = h_1 - ch_2 + c - 1$. Then we get

$$N_{(2}\left(r, \frac{1}{H}\right) \geq N_{(2}\left(r, \frac{1}{f-c}\right) = T(r,f) + S(r).$$

Suppose that z_0 is a multiple zero of $f - c$. We have

$$\begin{aligned} H(z_0) &= h_1(z_0) - c(z_0)h_2(z_0) + c(z_0) - 1 = 0, \\ H'(z_0) &= \frac{h_1'(z_0)}{h_1(z_0)} h_1(z_0) - \left(c'(z_0) + c(z_0)\frac{h_2'(z_0)}{h_2(z_0)}\right) h_2(z_0) + c'(z_0) = 0, \end{aligned}$$

which lead to

$$\begin{aligned} h_1(z_0) &= \frac{c(z_0)(c(z_0)-1)h_2'(z_0)/h_2(z_0) - c'(z_0)}{c(z_0)(h_1'(z_0)/h_1(z_0) - h_2'(z_0)/h_2(z_0)) - c'(z_0)}, \\ h_2(z_0) &= \frac{(c(z_0)-1)h_1'(z_0)/h_1(z_0) - c'(z_0)}{c(z_0)(h_1'(z_0)/h_1(z_0) - h_2'(z_0)/h_2(z_0)) - c'(z_0)}. \end{aligned}$$

Let

$$F_1 = \alpha_1 h_1, \quad F_2 = \alpha_2 h_2, \tag{14}$$

where

$$\alpha_1 = \frac{c(h_1'/h_1 - h_2'/h_2) - c'}{c(c-1)h_2'/h_2 - c'}, \quad \alpha_2 = \frac{c(h_1'/h_1 - h_2'/h_2) - c'}{(c-1)h_1'/h_1 - c'}. \tag{15}$$

Then we have $T(r, \alpha_j) = S(r)$ $(j = 1, 2)$ by (11), (15) and the Lemma of the Logarithmic Derivative, and hence

$$T(r, F_1) = T(r, h_1) + S(r), \quad T(r, F_2) = T(r, h_2) + S(r), \tag{16}$$

and thus $S(r, F_1) = S(r, F_2) = S(r)$. Since $F_1(z_0) = 1$, $F_2(z_0) = 1$, we get

$$\overline{N}_{(2}\left(r, \frac{1}{f - c}\right) \le N_0(r, 1; F_1, F_2) + S(r). \tag{17}$$

Since f and g share $0, 1, \infty$ CM*, we have $T(r, g) \le 3T(r, f) + S(r)$. From (9), (10), the assumption and Lemma 8, we get

$$\begin{aligned} T(r, F_1) + T(r, F_2) &= T(r, h_1) + T(r, h_2) + S(r) \\ &\le 8T(r, f) + S(r) \\ &= 16\overline{N}_{(2}\left(r, \frac{1}{f - c}\right) + S(r) \\ &\le 16N_0(r, 1; F_1, F_2) + S(r). \end{aligned}$$

It is obvious that

$$\overline{N}(r, F_i) + \overline{N}\left(r, \frac{1}{F_i}\right) = S(r), \quad i = 1, 2.$$

Hence by Lemma 5, we see that there exist two non-zero and mutually prime integers s, t $(t > 0)$ such that $F_1^s F_2^{-t} = 1$, i.e.,

$$(\alpha_1 h_1)^s = (\alpha_2 h_2)^t \quad \text{and} \quad us + vt = 1 \tag{18}$$

for some integers u, v. Set $h = (\alpha_1 h_1)^v (\alpha_2 h_2)^u$. Then

$$h_1 = \frac{h^t}{\alpha_1}, \quad h_2 = \frac{h^s}{\alpha_2},$$

thus

$$H = \frac{h^t}{\alpha_1} - c\frac{h^s}{\alpha_2} + c - 1. \tag{19}$$

It follows from (15) that

$$\frac{1}{\alpha_1} - \frac{c}{\alpha_2} + c - 1 = 0.$$

Therefore,

$$H = h^m(h-1)P(h) = h^m(h-1)(p_0h^{n+1} + p_1h^n + \cdots + p_{n+1}), \tag{20}$$

where p_i, $i = 0, \ldots, n+1$ are small functions and m, n $(n \geq 0)$ are integers such that $p_{n+1} \neq 0$. We will show $P(1) = 0$ that is to say $h-1$ is a factor of $P(h)$. By (11), (12) and the definition of h, we can see that h is not a constant and $\overline{N}(r,h) + \overline{N}(r,1/h) = S(r)$, which implies $T(r,h'/h) = S(r)$ by the Lemma of the Logarithmic derivative, and thus

$$N_{(2}\left(r,\frac{1}{h-1}\right) \leq 2N\left(r,\frac{1}{h'/h}\right) = S(r). \tag{21}$$

At the multiple zero z_0 of $f-c$, we have $h(z_0) = 1$, hence by (13) such points are also multiple zeros of H, and by (9) and Lemma 8, we have

$$T(r,f) \leq 2T(r,h) + S(r). \tag{22}$$

From (20) and (21), we see that "almost all" such points are zeros of $p_0 + p_1 + \cdots + p_{n+1}$. Therefore, considering (9) we get

$$p_0 + p_1 + \cdots + p_{n+1} \equiv 0, \tag{23}$$

which implies that $P(h) = (h-1)P_1(h)$, where $P_1(h) = b_0h^n + b_1h^{n-1} + \cdots + b_n$ is a polynomial in h whose coefficients are small functions of f. From (19) and (20), we get $h^m(h-1)^2P_1(h) = h^t/\alpha_1 - ch^s/\alpha_2 + c - 1$. From this and by Lemma 1, we can see that $P_1(h)$ must be a monomial in h. Therefore, H can be expressed as $Ah^q(h-1)^2$, i.e.,

$$\frac{h^t}{\alpha_1} - c\frac{h^s}{\alpha_2} + c - 1 = Ah^q(h-1)^2, \tag{24}$$

where A is a small function of f and q an integer. Thus $\{t-q, s-q, -q\}$ is a permutation of $\{0,1,2\}$. There have three cases only: (i) $q=0$, $t=1$, $s=2$. (ii) $q=0$, $t=2$, $s=1$. (iii) $q=-1$, $t=1$, $s=-1$. Hence by considering (22) $\alpha_1 = 1/2(1-c)$, $\alpha_2 = c/(1-c)$, or $\alpha_1 = 1/(c-1)$, $\alpha_2 = c/2(c-1)$, or $\alpha_1 = 2/(1-c)$, $\alpha_2 = 2c/(c-1)$. By (12) and (18), we get

$$f(f-1) = 4c(1-c)g(g-1), \quad \text{or}$$

$$\frac{f^2}{f-1} = \frac{c^2}{4(c-1)}\frac{g^2}{g-1}, \quad \text{or}$$

$$\frac{f}{(f-1)^2} = \frac{-4c}{(c-1)^2}\frac{g}{(g-1)^2}.$$

Note that g assume the same condition with f. The above three equations remain valid if we interchange f and g. Therefore, $4c(1-c) = \pm 1$ or $c^2/4(c-1) = \pm 1$ or $-4c/(c-1)^2 = \pm 1$. From the condition about c, we get $4c(1-c) = 1$ or $c^2/4(c-1) = 1$ or $-4c/(c-1)^2 = 1$. Hence f is a quasi-

Möbius transformation of g in any case. This is a contradiction, and completes the proof of Theorem 1.

Proof of Theorem 2. Let $c = \infty$ in Theorem 1, Theorem 2 is an obvious corollary since all poles of f' and g' are multiple.

Proof of Theorem 3. By Lemma 2 and Lemma 6, we can easily get this result.

We propose the following conjecture for further study.

CONJECTURE. *Let f and g be two nonconstant meromorphic functions. If f' and g' share four small functions a_1, a_2, a_3, a_4 $(\not\equiv \infty)$ IM^*, then $f' = g'$.*

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