A simple proof of the existence of tangent bicharacteristics for noneffectively hyperbolic operators

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Abstract The behavior of orbits of the Hamilton vector field H_p of the principal symbol p of a second-order hyperbolic differential operator is discussed. In our previous paper, assuming that p is noneffectively hyperbolic on the doubly characteristic manifold Σ of p, we have proved that if $H_S^3 p = 0$ on Σ with the Hamilton vector field H_S of some specified S, then there exists a bicharacteristic landing on Σ tangentially. The aim of this paper is to provide a much more simple proof of this result since the previous proof was fairly long and rather complicated.

1. Introduction

Let

$$P(x,D) = -D_0^2 + \sum_{|\alpha|\le 2, \alpha_0<2} a_\alpha(x) D^\alpha$$

be a second-order differential operator defined in a neighborhood of the origin of $\mathbb{R}^{n+1}$ with principal symbol $p(x,\xi)$, where $x = (x_0, x') = (x_0, x_1, \dots, x_n)$ is a system of local coordinates near the origin of $\mathbb{R}^{n+1}$ with $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_n)$ and $D^\alpha = (-i\partial/\partial x_0)^{\alpha_0} \cdots (-i\partial/\partial x_n)^{\alpha_n}$, $a_\alpha(x) \in C^\infty$. We assume that $p(x,\xi)$ is hyperbolic with respect to the x_0-direction so that $p(x, \xi_0, \xi') = 0$ has only real zeros for any real (x, ξ') with x near the origin. On integral curves of the Hamilton vector field $H_p = \sum_{j=0}^n (\partial p/\partial \xi_j\, \partial/\partial x_j - \partial p/\partial x_j\, \partial/\partial \xi_j)$ of p, that is, the solutions of the Hamilton equations

$$\frac{dx_j}{ds} = \frac{\partial p}{\partial \xi_j}(x,\xi), \qquad \frac{d\xi_j}{ds} = -\frac{\partial p}{\partial x_j}(x,\xi) \tag{1.1}$$

in local coordinates, p is constant on such a curve. If p vanishes on the curve, then the curve is called a bicharacteristic of p (e.g., see [3, p. 154]). Let $\rho = (\bar{x}, \bar{\xi})$ be a double characteristic of p, that is, p vanishes at ρ of order 2. Since ρ is a singular (stationary) point of the Hamilton equations (1.1), to take a closer look at the behavior of bicharacteristics near ρ we linearize the Hamilton equations

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at ρ, which yields $d(x,\xi)/ds = F_p(\rho)(x,\xi)$, where $F_p(\rho)$ is given by

$$F_p(\rho) = \begin{pmatrix} \frac{\partial^2 p}{\partial x\,\partial \xi}(\rho) & \frac{\partial^2 p}{\partial \xi\,\partial \xi}(\rho) \\ -\frac{\partial^2 p}{\partial x\,\partial x}(\rho) & -\frac{\partial^2 p}{\partial \xi\,\partial x}(\rho) \end{pmatrix}$$

in local coordinates and is called the Hamilton map (or the fundamental matrix) of p at ρ (see [7], [2]). About the spectral properties of $F_p(\rho)$ the following result is well known.

LEMMA 1.1 ([7, P. 15], [2, LEMMA 1.4.4])
All eigenvalues of $F_p(\rho)$ are pure imaginary except possibly for a pair of nonzero real eigenvalues $\pm\lambda$, $\lambda > 0$.

When $F_p(\rho)$ has nonzero real eigenvalues, p is called effectively hyperbolic at ρ. Otherwise p is called noneffectively hyperbolic at ρ.

We now assume that p is effectively hyperbolic at ρ and we consider bicharacteristics of p tending to ρ as $s \uparrow +\infty$ or $s \downarrow -\infty$. Then we have the following.

THEOREM 1.1 ([8, THEOREM 2.1], [11, LEMMA 3.2], [9, THEOREM 1])
*There are exactly four such bicharacteristics and there is a hypersurface containing the doubly characteristic set of p near ρ to which these four bicharacteristics are transversal. Two of them are incoming toward ρ with respect to the parameter s, and the other two are outgoing. Each one of the incoming (*resp.*, outgoing) bicharacteristics is naturally continued to the other one, and the resulting two curves are regular, C^∞, or analytic corresponding to the assumption on the principal symbol.*

Here we note that the tangents of the resulting two smooth curves at ρ are parallel to the eigenvectors corresponding to the nonzero real eigenvalues $\pm\lambda$ of $F_p(\rho)$, respectively.

We turn to consider the case in which p is noneffectively hyperbolic. We assume that the doubly characteristic set Σ of p is a C^∞-manifold and

$$\begin{cases} p(x,\xi) \text{ vanishes exactly of order 2 on } \Sigma, \\ F_p \text{ has no nonzero real eigenvalues on } \Sigma, \\ \operatorname{rank}(\sum_{j=0}^{n} d\xi_j \wedge dx_j|_\Sigma) = \text{constant on } \Sigma. \end{cases} \tag{1.2}$$

According to the spectral structure of F_p, two different possible cases may arise:

$$\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 = \{0\} \quad \text{on } \Sigma, \tag{1.3}$$

$$\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 \neq \{0\} \quad \text{on } \Sigma. \tag{1.4}$$

In the case (1.3) there is no bicharacteristic landing to Σ as $s \uparrow +\infty$ or $s \downarrow -\infty$. Indeed, we have the following result.

PROPOSITION 1.1 ([6, PROPOSITION 0.3])

Assume (1.2) and (1.3). Then there are no bicharacteristics emanating from a simple characteristic which has a limit point in Σ.

We give a sketch of the proof. Without restrictions we can assume that $p(x,\xi)$ can be written in the form $p(x,\xi) = -\xi_0^2 + q(x,\xi')$, where $q(x,\xi') \geq 0$. Under the assumptions (1.2) and (1.3) one can find a real-valued symbol $\lambda(x,\xi')$ defined near the origin and homogeneous of degree 1 such that in writing

$$p(x,\xi) = -(\xi_0+\lambda)(\xi_0-\lambda) + \big(q(x,\xi') - \lambda^2(x,\xi')\big)$$

we have

$$\begin{cases} Q(x,\xi') = q(x,\xi') - \lambda^2(x,\xi') \geq 0, \\ |\{\xi_0-\lambda, Q\}| \leq CQ, \\ |\{\xi_0+\lambda, \xi_0-\lambda\}| \leq C(\sqrt{Q}+|\lambda|) \end{cases} \tag{1.5}$$

with some $C>0$ near the origin (see [5] and also [13, Proposition 3.2.1]), where $\{f,g\}$ denotes the Poisson bracket

$$\{f,g\} = \sum_{j=0}^{n} (\partial f/\partial \xi_j \, \partial g/\partial x_j - \partial f/\partial x_j \, \partial g/\partial \xi_j).$$

Let $\gamma(s)$ be a bicharacteristic of p which lies outside Σ. Since $dx_0(s)/ds = -2\xi_0(\gamma(s))$ and $\xi_0^2(\gamma(s)) = q(\gamma(s)) \neq 0$ outside Σ one can take x_0 as a new parameter. Thanks to (1.5) we have with $\Lambda = \xi_0 - \lambda$

$$\Big|\frac{d}{dx_0}\Lambda\big(\gamma(x_0)\big)\Big| = \Big|\{p,\Lambda\}\big(\gamma(s)\big)\frac{ds}{dx_0}\Big| \leq C\big|\Lambda\big(\gamma(x_0)\big)\big|. \tag{1.6}$$

From this we conclude that $\Lambda(\gamma(x_0)) = 0$ for all x_0 if $\gamma(x_0)$ touches Σ at some point. This shows that $Q(\gamma(x_0)) = 0$ for all x_0. Since Q is nonnegative it follows that $\{Q,M\}(\gamma(x_0)) = 0$ with $M = \xi_0+\lambda$. This proves the same inequality (1.6) for M and hence $\gamma(x_0) \in \Sigma$ for any x_0 if $\gamma(x_0) \in \Sigma$ at some x_0, which is a contradiction.

In the case (1.4) the situation is completely different. Indeed, there could exist bicharacteristics of p having a limit point in Σ. We give examples. Let $1 \leq k \leq n-1$ and q_i, r_i, $i = 1,2,\ldots,k$, be positive constants. Consider

$$\begin{aligned} p(x,\xi) &= -\xi_0^2 + \sum_{i=1}^{k} q_i(x_{i-1}-x_i)^2\xi_n^2 + \sum_{i=1}^{k} r_i\xi_i^2 + \sum_{i=1}^{k} \epsilon_i\xi_i\xi_k^2 \\ &= -\xi_0^2 + q(x,\xi'), \end{aligned}$$

where $\sum_{i=1}^{k} r_i^{-1} = 1$, which is equivalent to the condition (1.4), and the double characteristic manifold of p is given by $\Sigma = \{\xi_i = 0, 0 \leq i \leq k, x_i = x_{i+1}, 0 \leq i \leq k-1\}$. In [10] it is proved that, by choosing $\{\epsilon_i\}$ suitably, there exists a bicharacteristic $(x(s),\xi(s))$ of p such that $\xi_n(s) = 1$ and $\xi_i(s) \to 0$, $x_i(s) \to 0$ for $0 \leq i \leq k$ as $s \to -\infty$. Note that $|\sum_{i=1}^{k} \epsilon_i\xi_i(s)\xi_k^2(s)| < \sum_{i=1}^{k} r_i\xi_i^2(s)$ for large $|s|$ so that $q(x(s),\xi'(s)) \geq 0$.

The spectral property (1.4) itself is not enough to determine completely the behavior of bicharacteristics and we need to look at the third-order term of the Taylor expansion of p around double characteristics to obtain a complete picture of the behavior of bicharacteristics. Let $S(x,\xi)$ be a smooth function vanishing on Σ such that

$$H_S \in \operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2, \quad F_p H_S \neq 0, \quad \text{on } \Sigma. \tag{1.7}$$

Using such a Hamilton vector field H_S of S one can characterize when there is a bicharacteristic emanating from a simple characteristic which has a limit point in Σ.

THEOREM 1.2 ([12, THEOREM 1.1])
Assume (1.2) and (1.4). Then the following assertions are equivalent.

(a) $H_S^3 p = 0$ *on* Σ *for any smooth* S *vanishing on* Σ *and satisfying (1.7).*
(b) *There is no bicharacteristic of* p *emanating from a simple characteristic and having a limit point in* Σ.

It is enough to check Theorem 1.2(a) for one S because of the following result.

LEMMA 1.2 ([1, PROPOSITION 2.3])
Let S_i, $i=1,2$, *be smooth functions vanishing on* Σ *and verifying (1.7). Then we have that*

$$H_{S_1}^3 p|_\Sigma = c H_{S_2}^3 p|_\Sigma$$

with some nonvanishing function c.

The proof of Theorem 1.2 goes as follows. If $H_S^3 p = 0$ on Σ for some such S, then p admits a (microlocal) decomposition (1.5), which has been proved in [11] under some restrictions and in [1] in full generality, removing these restrictions. Then repeating the same arguments proving Proposition 1.1 we conclude that (a) implies (b). Thus to prove Theorem 1.2 it suffices to show that there is a bicharacteristic of p with a limit point in Σ if the condition (a) fails at some point on Σ. Actually in the previous paper [12], assuming that the condition (a) fails we look for a bicharacteristic $(x(s),\xi(s))$ such that

$$\lim_{s\to\infty} s^2\big(x(s),\xi(s)\big) = v \neq 0,$$

$$v \in \operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2, \qquad 0 \neq F_p v \in \operatorname{Ker} F_p \cap \operatorname{Im} F_p^3.$$

To put the above conditions in evidence, we have proved that one can choose a system of local symplectic coordinates so that the line spanned by $z(\rho)$ verifying

$$z(\rho) \in \operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho), \qquad 0 \neq F_p(\rho) z(\rho) \in \operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p^3(\rho)$$

(note that $z(\rho)$ is unique up to a multiple factor and hence proportional to v) is given by $m_j(x,\xi) = 0$ on Σ and the representation of p, in these coordinates, contains the sum of m_j^2. Then our desired bicharacteristic $(x(s),\xi(s))$

could be expected to satisfy $m_j(x(s),\xi(s))=0$ approximately. We write down our Hamilton equations supposing that the m_j's were unknowns. We look for a solution $(x(s),\xi(s),m(s))$ of the resulting "equations" such that $\xi(s)=O(s^{-2})$, $x'(s)=O(s^{-3})$ $(x=(x_0,x'))$, and $m_j(x(s),\xi(s))=O(s^{-4})$, that is, $m_j(s)$ goes to zero faster than both $x(s)$ and $\xi'(s)$ as $s\to\infty$ or $s\to-\infty$.

This proof was fairly long and rather complicated. The aim of this paper is to provide a much simpler proof without introducing such new unknowns m_j.

2. Lemmas

From the assumption (1.2), for any $\rho\in\Sigma$, one can find $\phi_j(x,\xi')$, $j=1,\dots,r$ such that we have

$$\begin{cases} p=-\xi_0^2+\sum_{j=1}^r\phi_j^2(x,\xi'), \qquad \Sigma=\{\phi_j=0, j=0,\dots,r\},\\ \operatorname{rank}(\{\phi_i,\phi_j\})_{0\le i,j\le r}=\text{constant on }\Sigma\end{cases}$$

in a conic neighborhood of ρ with linearly independent $\{d\phi_j(\rho)\}$ where we have set $\xi_0=\phi_0$.

In this section we write $f=O(|\phi|)$ in some open set U if f is a linear combination of $\phi_1,\dots,\phi_r$ in U. It is also understood that every open set has nonempty intersection with Σ. To simplify notations we often denote by $\{\phi_j\}_{1\le j\le r}$ some other $\{\tilde\phi_j\}_{1\le j\le r}$ which is related to $\{\phi_j\}$ by a smooth orthogonal transformation if which one $\{\phi_j\}$ means is clear in context.

DEFINITION 2.1

Let I_k, $k=1,\dots,t$ be subsets of a finite-index set $\hat I$ which are mutually disjoint. We say that $\{\phi_j\}_{j\in I_k}$ $(k=1,\dots,t)$ are symplectically independent in U if $\{\phi_i,\phi_j\}=O(|\phi|)$ in U for any $i\in I_p$, $j\in I_q$, $p\ne q$.

Let $A=(a_{ij})$ be an $(m\times m)$-antisymmetric matrix of the special form

$$\begin{cases} a_{ij}\ne 0 & \text{if } |i-j|=1,\\ a_{ij}=0 & \text{if } |i-j|\ne 1.\end{cases} \tag{2.1}$$

Then the next lemma is easily examined.

LEMMA 2.1

Let A be an $(m\times m)$-antisymmetric matrix satisfying (2.1). Then $\det A\ne 0$ *if m is even while* $\operatorname{rank} A=m-1$ *if m is odd.*

Let us consider

$$Q=\sum_{j=1}^r\phi_j^2,$$

where it is assumed that $\phi_j(x,\xi)$ are defined in U and $\{d\phi_j\}$ are linearly independent there. Then we have the following.

LEMMA 2.2

Assume that there exist $i, j \in \hat{I} = \{1, \dots, r\}$ *and* $\rho \in U \cap \Sigma$ *such that* $\{\phi_i, \phi_j\}(\rho) \neq 0$. *Then there are an open set* $V \subset U$ *and* $\{\phi_i\}_{i\in I}$, $\{\phi_j\}_{j\in J}$ *which are symplectically independent,* $\hat{I} = I \cup J$, *such that we can write*

$$Q = \sum_{i\in I} \phi_i^2 + \sum_{j\in J} \phi_j^2 \tag{2.2}$$

and

$$\det\big(\{\phi_i, \phi_j\}\big)_{i,j\in I} \neq 0 \tag{2.3}$$

in V.

Proof

We first prove that one can find an open set $V \subset U$ and $\{\phi_i\}_{i\in I}$, $\{\phi_j\}_{j\in J}$ which are symplectically independent and satisfy (2.2) and

$$\begin{cases} \{\phi_i, \phi_j\} \neq 0 & \text{if } |i-j| = 1, i, j \in I, \\ \{\phi_i, \phi_j\} = O(|\phi|) & \text{if } |i-j| \neq 1, i, j \in I \end{cases} \tag{2.4}$$

in V. Without restrictions, we may assume that $\{\phi_1, \phi_j\}(\rho_1) \neq 0$ with some j and $\rho_1 \in U \cap \Sigma$. Consider a smooth orthogonal transformation of $\{\phi_2, \dots, \phi_r\}$ to $\{\tilde{\phi}_2, \dots, \tilde{\phi}_r\}$;

$$\tilde{\phi}_i = \sum_{k=2}^{r} O_{ik}\phi_k, \quad i = 2, \dots, r.$$

Noting

$$\Big\{\phi_1, \sum_{k=2}^{r} O_{ik}\phi_k\Big\} = \sum_{k=2}^{r} O_{ik}\{\phi_1, \phi_k\} + O\big(|\phi|\big)$$

we choose O_{ik} so that

$$\sum_{k=2}^{r} O_{2k}\{\phi_1, \phi_k\} \neq 0, \qquad \sum_{k=2}^{r} O_{ik}\{\phi_1, \phi_k\} = 0, \quad i = 3, \dots, r,$$

in some open set $U_1 \subset U$. Switching the notation from $\{\tilde{\phi}_j\}_{j=2}^r$ to $\{\phi_j\}_{j=2}^r$ we may assume that $Q = \sum \phi_j^2$ and

$$\{\phi_1, \phi_2\} \neq 0, \qquad \{\phi_1, \phi_j\} = O\big(|\phi|\big), \quad j = 3, \dots, r,$$

in U_1. Consider $\{\phi_2, \phi_j\}$, $j \geq 3$. If $\{\phi_2, \phi_j\} = 0$ in $U_1 \cap \Sigma$ for all $j \geq 3$, then it is enough to take

$$I = \{1, 2\}, \qquad J = \{3, \dots, r\}.$$

If not, then there exist $\rho_2 \in U_1 \cap \Sigma$ and $j_2 \geq 3$ such that $\{\phi_2, \phi_{j_2}\}(\rho_1) \neq 0$. Continuing this procedure we can conclude that there exist an open set $V \subset U$ and $\{\phi_i\}_{i\in I}$, $\{\phi_j\}_{j\in J}$ which are symplectically independent and verify (2.2) and (2.4) in V.

We turn to the next step. Take $\rho \in V \cap \Sigma$. If $|I|$ is even, then from Lemma 2.1 and (2.4) it follows that $(\{\phi_i, \phi_j\}(\rho))_{i,j\in I}$ is nonsingular and hence is near ρ. If $|I|$ is odd, then from Lemma 2.1 and (2.4) it follows that $\mathrm{rank}(\{\phi_i, \phi_j\}(\rho))_{i,j\in I} = |I| - 1$. Note that $\mathrm{rank}(\{\phi_i, \phi_j\})_{i,j\in I} \le |I| - 1$ because $(\{\phi_i, \phi_j\})_{i,j\in I}$ is an antisymmetric matrix of odd order; then we have

$$\mathrm{rank}\big(\{\phi_i, \phi_j\}\big)_{i,j\in I} = |I| - 1 \tag{2.5}$$

in some neighborhood V' of ρ. Let $I = \{i_1, i_2, \dots, i_\ell\}$. From (2.5) we have $\dim \mathrm{Ker}(\{\phi_i, \phi_j\})_{i,j\in I} = 1$, and hence we can choose smooth $c_i(x, \xi)$, $i \in I$, such that $\sum_{j\in I} c_j^2 = 1$ and

$$\sum_{j\in I} \{\phi_i, \phi_j\} c_j = 0, \quad i \in I,$$

holds in V'. Choosing a smooth orthogonal matrix $(O_{ij})_{ij\in I}$ such that $O_{i_1 j} = c_j$ and considering

$$\tilde{\phi}_i = \sum_{j\in I} O_{ij}\phi_j, \quad i \in I,$$

we may assume that $\{\tilde{\phi}_j, \tilde{\phi}_{i_1}\} = O(|\phi|)$ in V' for all $j \in I$. Since $\mathrm{rank}(\{\phi_i, \phi_j\})_{i,j\in I} = \mathrm{rank}(\{\tilde{\phi}_i, \tilde{\phi}_j\})_{i,j\in I}$ it follows from (2.5) that

$$\det\big(\{\tilde{\phi}_i, \tilde{\phi}_j\}\big)_{i,j\in I'} \neq 0,$$

where $I' = I \setminus \{i_1\}$. Thus $\{\tilde{\phi}_i\}_{i\in I'}$ and $\{\tilde{\phi}_j\}_{j\in J'}$, $J' = J \cup \{i_1\}$, verify the desired assertion. □

LEMMA 2.3

There exist an open set $V \subset U$ and $\{\phi_i\}_{i\in I}$, $\{\phi_j\}_{j\in K}$ which are symplectically independent, $\hat{I} = I \cup K$, such that we can write

$$Q = \sum_{i\in I} \phi_i^2 + \sum_{j\in K} \phi_j^2,$$

where $\det(\{\phi_i, \phi_j\})_{i,j\in I} \neq 0$ while we have

$$\{\phi_i, \phi_j\} = O\big(|\phi|\big) \quad \textit{for all } i, j \in K$$

in V.

Proof

From Lemma 2.2 there are an open set $V_1 \subset U$ and $\{\phi_j\}_{j\in I_1}$, $\{\phi_j\}_{j\in J_1}$, symplectically independent in V_1, which verify (2.2) and (2.3). If $\{\phi_i, \phi_j\} = 0$ in $V_1 \cap \Sigma$ for all $i, j \in J_1$, then it is enough to choose $I = I_1$ and $K = J_1$. Otherwise we apply Lemma 2.2 to $Q_1 = \sum_{j\in J_1} \phi_j^2$ to find an open set $V_2 \subset V_1$ and $\{\phi_j\}_{j\in I_2}$, $\{\phi_j\}_{j\in J_2}$, $J_1 = I_2 \cup J_2$, which are symplectically independent in V_2 and verify

$$Q_1 = \sum_{j\in I_2} \phi_j^2 + \sum_{j\in J_2} \phi_j^2, \qquad \det\big(\{\phi_i, \phi_j\}\big)_{i,j\in I_2} \neq 0.$$

Repeating this argument at most $[r/2]$ times we conclude that there are an open set $V \subset U$ and $\{\phi_j\}_{j \in I_k}$ $(k = 1, \dots, t)$, $\{\phi_j\}_{j \in K}$, which are symplectically independent in V and satisfy $Q = \sum_{i=1}^{t} \sum_{j \in I_i} \phi_j^2 + \sum_{j \in K} \phi_j^2$ and

$$\det\bigl(\{\phi_i, \phi_j\}\bigr)_{i,j \in I_p} \neq 0, \quad p = 1, \dots, t, \qquad \{\phi_i, \phi_j\} = 0, \quad \forall i, j \in K,$$

in V. Let us set $I = \bigcup_{i=1}^{t} I_i$; then it is obvious that $\{\phi_j\}_{j \in I}$, $\{\phi_j\}_{j \in K}$ are symplectically independent in V. Note that $(\{\phi_i, \phi_j\}(\rho))_{i,j \in I}$ is the direct sum of $(\{\phi_i, \phi_j\}(\rho))_{i,j \in I_k}$ $(k = 1, \dots, t)$ if $\rho \in V \cap \Sigma$ and hence $\det(\{\phi_i, \phi_j\})_{i,j \in I} \neq 0$ in some open set, which proves the assertion. $\square$

PROPOSITION 2.1

Assume (1.2) and (1.4). Let $\rho \in \Sigma$, and let U be any neighborhood of ρ. Then there exist an open set $V \subset U$ and $\{\phi_j\}_{j \in I_0}$, $\{\phi_j\}_{j \in I_1}$, $\{\phi_j\}_{j \in K}$ which are symplectically independent in V, where $\{0, 1, \dots, r\} = I_0 \cup I_1 \cup K, I_0 = \{0, 1, \dots, l\}$ with even l (≥ 2), such that one can write

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \sum_{k=2}^{l} \phi_j^2 + \sum_{j \in I_1} \phi_j^2 + \sum_{j \in K} \phi_j^2$$

and we have in V

$$\begin{cases} \{\xi_0 - \phi_1, \phi_j\} = O(|\phi|), & j = 0, \dots, r, \\ \{\phi_1, \phi_2\} \neq 0 & \text{if } l = 2, \\ \operatorname{rank}(\{\phi_i, \phi_j\})_{2 \leq i,j \leq l} = l - 2 & \text{if } l \geq 4, \\ \det(\{\phi_i, \phi_j\})_{i,j \in I_1} \neq 0, \\ \{\phi_i, \phi_j\} = O(|\phi|), & \forall i, j \in K. \end{cases}$$

Proof

As the first step we prove that one can write

$$p = -\xi_0^2 + \sum_{j=1}^{l} \phi_j^2 + \sum_{j \in I_1} \phi_j^2 + \sum_{j \in K} \phi_j^2, \tag{2.6}$$

where $\{\phi_j\}_{j \in I_0}$, $\{\phi_j\}_{j \in I_1}$, $\{\phi_j\}_{j \in K}$ are symplectically independent, $\{0, 1, \dots, r\} = I_0 \cup I_1 \cup K$, $I_0 = \{0, \dots, l\}$ with even l ($\geq$ or equal to 2), and

$$\begin{cases} \dim \operatorname{Ker}(\{\phi_i, \phi_j\})_{0 \leq i,j \leq l} = 1, \\ \det(\{\phi_i, \phi_j\})_{i,j \in I_1} \neq 0, \\ \{\phi_i, \phi_j\} = O(|\phi|), \quad \forall i, j \in K. \end{cases} \tag{2.7}$$

Recall that one can write

$$p = -\xi_0^2 + \sum_{j=1}^{r} \phi_j^2$$

near ρ. Let us write $\phi_0 = \xi_0$ as before. Suppose that $\{\phi_0, \phi_j\}(\rho) = 0$ for all j. Then with $q = \sum_{j=1}^{r} \phi_j^2$ we see easily that $\operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho) = \operatorname{Ker} F_q^2(\rho) \cap \operatorname{Im} F_q^2(\rho)$, which is $\{0\}$ because q is nonnegative. This contradicts (1.4). Thus we have $\{\phi_0, \phi_j\}(\rho) \neq 0$ with some $j \geq 1$. Now repeating the same arguments employed in the proof of Lemma 2.2 we conclude that there exist an open set $V \subset U$ and $\{\phi_j\}_{j\in\{0,\ldots,l\}}$, $\{\phi_j\}_{j\in\{l+1,\ldots,r\}}$, symplectically independent in V and satisfying (2.2) and (2.4) with $I = \{0,\ldots,l\}$, $l \geq 1$.

We now show that l is even by contradiction. Suppose that l is odd. Let us denote by p_ρ the second-order term of the Taylor expansion of p at ρ, which is a quadratic form in (x,ξ) of homogeneous of degree 2, called the localization of p at ρ. Then we have

$$p_\rho(X) = \sigma\big(X, F_p(\rho)X\big), \quad X = (x,\xi) \in \mathbb{R}^{2(n+1)},$$

where $\sigma = \sum_{j=0}^n d\xi_j \wedge dx_j$ is the symplectic 2-form and $\sigma((x,\xi),(y,\eta)) = \langle \xi, y\rangle - \langle x, \eta\rangle$ in local coordinates and $\langle x, y\rangle = \sum_{j=0}^n x_j y_j$ (see [4]). Let us consider $\psi = \sum_{1\leq 2j+1\leq l} c_{2j+1}\phi_{2j+1}$ with $c_{2j+1} \in \mathbb{R}$. We note that (see, e.g., [13])

$$\begin{aligned} p_\rho(H_\psi) &= -\{\phi_0, \psi\}^2(\rho) + \sum_{j=1}^r \{\phi_j, \psi\}^2(\rho) \\ &= -\{\phi_0, c_1\phi_1\}^2(\rho) + \sum_{2\leq 2i<l} \{\phi_{2i}, \psi\}^2(\rho). \end{aligned}$$

Since l is odd, thanks to (2.4) we can choose c_{2j+1} so that $\{\phi_{2i}, \psi\}(\rho) = 0$ for $2 \leq 2i < l$ and $c_1 = 1$. This implies that $p_\rho(H_\psi) = -\{\phi_0, \phi_1\}^2(\rho) < 0$ and hence $F_p(\rho)$ has nonzero real eigenvalues (see [2, Corollary 1.4.7]), which contradicts the assumption (1.2). Thus we have proved that l is even. Since l is even $\dim \mathrm{Ker}(\{\phi_i, \phi_j\})_{0\leq i,j\leq l} = 1$ follows from (2.4) easily. If $l = r$, then the proof is complete. Otherwise to end the proof it suffices to apply Lemma 2.3 to $\sum_{l+1}^r \phi_j^2$.

We turn to the second step. Let us write $\bar{p} = -\xi_0^2 + \sum_{j=1}^l \phi_j^2$, $\bar{q} = \sum_{j=l+1}^r \phi_j^2$. We remark that

$$\mathrm{Ker}\, F_{\bar{p}}^2(\rho) \cap \mathrm{Im}\, F_{\bar{p}}^2(\rho) \neq \{0\} \tag{2.8}$$

for $\rho \in \Sigma$. Indeed, since $\{\phi_j\}_{0\leq j\leq l}$ and $\{\phi_j\}_{l+1\leq j\leq r}$ are symplectically independent, $F_p(\rho) = F_{\bar{p}}(\rho) \oplus F_{\bar{q}}(\rho)$ (direct sum) in a suitable symplectic basis in $\mathbb{R}^{2(n+1)}$. Since $\mathrm{Ker}\, F_{\bar{q}}^2(\rho) \cap \mathrm{Im}\, F_{\bar{q}}^2(\rho) = \{0\}$ we obtain the assertion by (1.4).

Since $\{\phi_j\}_{0\leq j\leq l}$ satisfies (2.4), we see that $(\{\phi_i, \phi_j\})_{1\leq i,j\leq l}$ is nonsingular in some open set from Lemma 2.1, and then there are smooth c_j, $j = 1,\ldots,l$, such that

$$\sum_{j=1}^l \{\phi_k, \phi_j\} c_j = \{\phi_k, \phi_0\}, \quad k = 1,\ldots,l. \tag{2.9}$$

Write $c_j = C_j(\phi', \theta)$, where $\theta = (\theta_{r+1},\ldots,\theta_{2n+2})$ is chosen so that (ϕ_0, ϕ', θ), $\phi' = (\phi_1,\ldots,\phi_r)$ is a system of local coordinates, and define

$$\bar{c}_j = C_j(0,\theta)$$

so that $c_j = \bar{c}_j(\theta) + O(|\phi|)$. Thus $(1, -\bar{c}_1, \ldots, -\bar{c}_l)$ is in $\mathrm{Ker}(\{\phi_i, \phi_j\})_{0\leq i,j\leq l}$ modulo $O(|\phi|)$; then noting (2.7) we see that $\mathrm{Ker}(\{\phi_i, \phi_j\}(\rho))_{0\leq i,j\leq l}$ is spanned by $(1, -\bar{c}_1(\theta), \ldots, -\bar{c}_l(\theta))$ for $\rho = (0, \theta) \in \Sigma$. From (2.8) there exists $0 \neq X \in \operatorname{Ker} F_{\bar{p}}^2(\rho) \cap \operatorname{Im} F_{\bar{p}}^2(\rho)$ such that $F_{\bar{p}}(\rho) X \in \operatorname{Ker} F_{\bar{p}}(\rho)$. Since $X \in \operatorname{Im} F_{\bar{p}}(\rho)$ we can put

$$X = H_f(\rho), \quad f = \sum_{j=0}^{l} a_j \phi_j$$

with some $a = (a_0, \ldots, a_l) \in \mathbb{R}^{l+1}$ and note that $F_{\bar{p}}(\rho) X \in \operatorname{Ker} F_{\bar{p}}(\rho)$ implies that $(-\{\phi_0, f\}(\rho), \{\phi_1, f\}(\rho), \ldots, \{\phi_l, f\}(\rho))$ is proportional to $(1, -\bar{c}_1(\theta), \ldots, -\bar{c}_l(\theta))$. With $A = (\{\phi_i, \phi_j\}(\rho))_{0 \le i,j \le l}$ this shows that

$$A^t a = k^t\big(1, \bar{c}_1(\theta), \ldots, \bar{c}_l(\theta)\big)$$

with some $k \in \mathbb{R}$. Such $a \in \mathbb{R}^{l+1}$ exists if and only if

$$ {}^tAv = 0, \quad v = (v_0, \ldots, v_l) \in \mathbb{R}^{l+1} \Longrightarrow v_0 + \sum_{j=1}^{l} v_j \bar{c}_j(\theta) = 0. \tag{2.10}$$

Since ${}^tAv = -Av = 0$, v is proportional to $(1, -\bar{c}_1(\theta), \ldots, -\bar{c}_l(\theta))$ if ${}^tAv = 0$. Thus (2.10) gives

$$1 - \sum_{j=1}^{l} \bar{c}_j(\theta)^2 = 0. \tag{2.11}$$

Let us set

$$\tilde{\phi}_1(x, \xi') = \sum_{j=1}^{l} \bar{c}_j(x, \xi') \phi_j(x, \xi'), \quad \bar{c}_j(x, \xi') = \bar{c}_j\big(\theta(x, \xi')\big).$$

Noting (2.11) we take a smooth orthogonal matrix $O = (O_{ij})_{1 \le i,j \le l}$ of which the first row is $(\bar{c}_1, \ldots, \bar{c}_l)$ and put $\tilde{\phi}_k = \sum_{j=1}^{l} O_{kj} \phi_j$ so that we have

$$-\xi_0^2 + \sum_{j=1}^{l} \phi_j^2 = -(\xi_0 + \tilde{\phi}_1)(\xi_0 - \tilde{\phi}_1) + \sum_{j=2}^{l} \tilde{\phi}_j^2.$$

It is clear that

$$\{\xi_0 - \tilde{\phi}_1, \tilde{\phi}_j\} = O\big(|\phi|\big), \quad j = 0, \ldots, r, \tag{2.12}$$

because

$$\Big\{\phi_k, \phi_0 - \sum_{j=1}^{l} \bar{c}_j \phi_j\Big\} = O\big(|\phi|\big), \quad k = 0, \ldots, l,$$

which follows from (2.9), proves the assertion for $j = 0, \ldots, l$, and the assertion for $j = l+1, \ldots, r$ is obvious since $\{\phi_j\}_{0 \le j \le l}$ and $\{\phi_j\}_{l+1 \le j \le r}$ are symplectically independent. With $\tilde{\phi}_0 = \xi_0 - \tilde{\phi}_1$ it is clear that

$$\operatorname{rank}\big(\{\tilde{\phi}_i, \tilde{\phi}_j\}\big)_{0 \le i,j \le l} = \operatorname{rank}\big(\{\phi_i, \phi_j\}\big)_{0 \le i,j \le l} = l$$

and hence $\operatorname{rank}(\{\tilde{\phi}_i, \tilde{\phi}_j\})_{1 \le i,j \le l} = l$ by (2.12). When $l = 2$ this shows that $\{\tilde{\phi}_1, \tilde{\phi}_2\} \neq 0$. Let $l \ge 4$. Note that $\operatorname{rank}(\{\tilde{\phi}_i, \tilde{\phi}_j\})_{2 \le i,j \le l} \le l - 2$ since $l - 1$ is odd. Suppose that $\operatorname{rank}(\{\tilde{\phi}_i, \tilde{\phi}_j\}(\rho))_{2 \le i,j \le l} \le l - 3$ at some ρ. Then it is easy to see that $\mathrm{rank}(\{\tilde{\phi}_i, \tilde{\phi}_j\}(\rho))_{1\le i,j\le l} \le l-1$ and a contradiction. Thus switching the notation from $\tilde{\phi}_j$ to ϕ_j $(j=1,\dots,l)$ we get the desired assertion. □

3. A simple proof of Theorem 1.2

Assume that

$$H_S^3 p \neq 0$$

in some open set U with some smooth S vanishing on Σ and satisfying (1.7). We choose an open set $V \subset U$, $V \cap \Sigma \neq \emptyset$, where Proposition 2.1 holds. We fix a $\bar{\rho} \in V \cap \Sigma$ and work near $\bar{\rho}$. Since the case $l=2$ is a little bit easier than the case $l \ge 4$ we assume $l \ge 4$. Choose a system of symplectic coordinates (X, Ξ) such that $X_0 = x_0$ and $\Xi_0 = \xi_0 - \phi_1$. Switching the notation from (X, Ξ) to (x, ξ) one can write

$$p = -\xi_0^2 - 2\xi_0\phi_1 + \sum_{j=2}^{l} \phi_j^2 + \sum_{j=l+1}^{\ell} \phi_j^2 + \sum_{j=\ell+1}^{r} \phi_j^2.$$

Here we recall

$$\mathrm{rank}\big(\{\phi_i, \phi_j\}\big)_{0\le i,j\le l} = l, \qquad \phi_0 = \xi_0, \tag{3.1}$$

near $\bar{\rho}$. Since $\dim \mathrm{Ker}(\{\phi_i, \phi_j\})_{2\le i,j\le l} = 1$ from Proposition 2.1 one can choose a smooth $c = (c_2, \dots, c_l)$ with $\sum c_j^2 = 1$ so that c spans $\mathrm{Ker}(\{\phi_i, \phi_j\})_{2\le i,j\le l}$. We make a smooth orthogonal transformation from $\{\phi_j\}_{2\le j\le l}$ to $\{\tilde{\phi}_j\}_{2\le j\le l}$ such that $\tilde{\phi}_2 = \sum c_j \phi_j$ and switching the notation from $\{\tilde{\phi}_j\}$ to $\{\phi_j\}$ again we obtain the following result.

PROPOSITION 3.1

By choosing a suitable system of symplectic coordinates, p can be written in the form

$$p = -\xi_0^2 - 2\xi_0\phi_1 + \phi_2^2 + \sum_{j=3}^{\ell} \phi_j^2 + \sum_{j=\ell+1}^{r} \phi_j^2$$

in some open set V, where

$$\begin{cases} \{\xi_0, \phi_j\} = 0, & 0 \le j \le r, \\ \{\phi_2, \phi_j\} = 0, & j \neq 1, \qquad \{\phi_2, \phi_1\} \neq 0, \\ \{\phi_i, \phi_j\} = 0, & 0 \le i \le r, \ell+1 \le j \le r, \\ \det(\{\phi_i, \phi_j\})_{3\le i,j\le \ell} \neq 0 \end{cases}$$

holds in $V \cap \Sigma$.

Proof

The first assertion follows from (2.12). It is clear that $\{\phi_2, \phi_j\} = 0$ in $V \cap \Sigma$ for $j = 2, \dots, l$ by the definition of ϕ_2. The assertion $\{\phi_2, \phi_j\} = 0$ for $j = l + 1, \dots, r$ is clear because the original $\{\phi_j\}_{2\le j\le l}$ and $\{\phi_j\}_{l+1\le j\le r}$ are symplectically independent and the new ϕ_2 is a linear combination of the original $\{\phi_j\}_{2\le j\le l}$. If $\{\phi_2,\phi_1\}(\rho)=0$, then it is obvious that $\mathrm{rank}(\{\phi_i,\phi_j\}(\rho))_{0\le i,j\le l}\le l-1$, which is a contradiction, and hence the second assertion follows. The third assertion is clear. Since $\{\phi_0,\phi_j\}=0$, $0\le j\le r$, and $\{\phi_2,\phi_j\}=0$ unless $j=1$ we see easily that $\mathrm{rank}(\{\phi_i,\phi_j\})_{0\le i,j\le l}\le l-1$, which contradicts (3.1) if $\mathrm{rank}(\{\phi_i,\phi_j\})_{3\le i,j\le l}\le l-3$. This proves that $\det(\{\phi_i,\phi_j\})_{3\le i,j\le l}\neq 0$. Since $(\{\phi_i,\phi_j\}(\rho))_{3\le i,j\le \ell}$ is the direct sum of $(\{\phi_i,\phi_j\}(\rho))_{3\le i,j\le l}$ and $(\{\phi_i,\phi_j\}(\rho))_{l+1\le i,j\le \ell}$ for $\rho\in V\cap\Sigma$, we have the last assertion. □

We proceed to the proof of Theorem 1.2. Let us take

$$\xi_0,x_0,\phi_1,\dots,\phi_r,\psi_1,\dots,\psi_k\quad (r+k=2n)$$

to be a system of local coordinates around $\bar\rho$. Note that we can assume that the ψ_j's are independent of x_0, taking $\psi_j(0,x',\xi')$ as the new ψ_j. Moreover, we can assume that $\{\phi_2,\psi_j\}=0$ and $\{\phi_1,\psi_j\}=0$ on $V\cap\Sigma$, taking

$$\psi_j-\{\psi_j,\phi_2\}\phi_1/\{\phi_1,\phi_2\}-\{\psi_j,\phi_1\}\phi_2/\{\phi_2,\phi_1\}$$

as the new ψ_j. Thus it can be assumed that

$$\{\xi_0,\psi_j\}=0,\qquad \{\phi_2,\psi_j\}=0,\qquad \{\phi_1,\psi_j\}=0,\quad 1\le j\le k,$$

hold in $V\cap\Sigma$. Thanks to Jacobi's identity one can assume that

$$\big\{\phi_2,\{\phi_j,\xi_0\}\big\}=0,\quad j=\ell+1,\dots,r, \tag{3.2}$$

in $V\cap\Sigma$ since we have $\{\phi_j,\{\xi_0,\phi_2\}\}=O(|\phi|)$ and $\{\xi_0,\{\phi_2,\phi_j\}\}=O(|\phi|)$ for $\ell+1\le j\le r$ by Proposition 3.1.

Let $\gamma(s)=(x(s),\xi(s))$ be a solution to the Hamilton equations (1.1); then

$$\frac{d}{ds}f\big(\gamma(s)\big)=\{p,f\}\big(\gamma(s)\big).$$

Let us change the parameter from s to t:

$$t=s^{-1},$$

so that we have

$$d/ds=-tD,\quad D=t(d/dt)$$

and hence

$$\frac{d}{ds}(t^pF)=-t^{p+1}(DF+pF).$$

We now introduce new unknowns

$$\begin{cases}\xi_0(s)=t^4\Xi_0(t), & x_0(s)=tX_0(t),\\ \phi_1(\gamma(s))=t^2\Phi_1(t), & \phi_2(\gamma(s))=t^3\Phi_2(t),\\ \phi_j(\gamma(s))=t^4\Phi_j(t),\quad 3\le j\le \ell,\\ \phi_j(\gamma(s))=t^3\Phi_j,\quad \ell+1\le j\le r,\\ \psi_j(\gamma(s))=t^2\Psi_j(t),\quad 1\le j\le k,\end{cases} \tag{3.3}$$

and denote

$$w = (\Xi_0, X_0, \Phi_1, \dots, \Phi_r, \Psi_1, \dots, \Psi_k).$$

Let us put

$$\{\phi_j, \xi_0\} = \sum_{i=1}^{r} C_i^j \phi_i, \qquad \kappa_j = C_1^j(\bar\rho), \qquad \delta = \{\phi_1, \phi_2\}(\bar\rho);$$

then from (3.2) we get

$$\kappa_j = 0, \quad j = \ell+1, \dots, r. \tag{3.4}$$

Thanks to Proposition 3.1 and (3.4) the Hamilton equations (1.1) are reduced to

$$\begin{cases} D\Xi_0 = -4\Xi_0 - 2\kappa_2 \Phi_1 \Phi_2 + tG(t,w), \\ DX_0 = -X_0 + 2\Phi_1 + tG(t,w), \\ D\Phi_1 = -2\Phi_1 + 2\delta\Phi_2 + tG(t,w), \\ D\Phi_2 = -3\Phi_2 - 2\kappa_2 \Phi_1^2 + 2\delta\Xi_0 + tG(t,w), \\ tD\Phi_j = -4t\Phi_j - 2\kappa_j \Phi_1^2 \\ \qquad\qquad - 2\sum_{k=3}^{\ell} \{\phi_k, \phi_j\}(\bar\rho)\Phi_k + tG(t,w), \quad 3 \le j \le \ell, \\ D\Phi_j = -3\Phi_j + tG(t,w), \quad \ell+1 \le j \le r, \\ D\Psi_j = -2\Psi_j - 2\sum_{k=\ell+1}^{r} \{\phi_k, \psi_j\}(\bar\rho)\Phi_k + tG(t,w), \quad 1 \le j \le k, \end{cases} \tag{3.5}$$

where $G(t,w)$, which may change from line to line, denotes a smooth function in (t,w) defined near $(0,0)$ such that $G(t,0) = 0$.

LEMMA 3.1

We have

$$H_{\phi_2}(\bar\rho) \in \operatorname{Ker} F_p^2(\bar\rho) \cap \operatorname{Im} F_p^2(\bar\rho), \quad F_p(\bar\rho) H_{\phi_2}(\bar\rho) \neq 0.$$

Proof

From Proposition 3.1 it is easy to check that $F_p(\bar\rho)H_{\phi_2} = \delta H_{\xi_0}$ and $F_p(\bar\rho)H_{\xi_0} = 0$ so that $F_p^2(\bar\rho)H_{\phi_2}(\bar\rho) = 0$. Thanks to $\det(\{\phi_i, \phi_j\}(\bar\rho))_{3 \le i,j \le l} \neq 0$ we can choose $f = \phi_1 + \sum_{j=3}^{l} c_j \phi_j$ so that

$$H_p(\bar\rho) H_f(\bar\rho) = H_{\phi_2}(\bar\rho),$$

which proves $H_{\phi_2}(\bar\rho) \in \operatorname{Im} F_p^2(\bar\rho)$ since $\operatorname{Im} F_p(\bar\rho)$ is spanned by $\{H_{\phi_j}(\bar\rho)\}$. □

From Lemma 3.1 we can take $S = \phi_2$, and hence

$$\kappa_2 = C_1^2(\bar\rho) = \frac{\{\phi_2, \{\phi_2, \xi_0\}\}(\bar\rho)}{\{\phi_2, \phi_1\}(\bar\rho)} = \frac{-H_{\phi_2}^3 p(\bar\rho)}{2\delta} \neq 0.$$

Let us define some classes of formal power series in t and $\log t$ in which we look for our formal solutions to the reduced Hamilton equations (3.5):

$$\mathcal{E} = \Big\{ \sum_{0\le j\le i} t^i (\log t)^j w_{ij} \,\Big|\, w_{ij} \in \mathbb{C}^N \Big\},$$
$$\mathcal{E}^{\#} = \Big\{ \sum_{1\le i, 0\le j\le i} t^i (\log t)^j w_{ij} \,\Big|\, w_{ij} \in \mathbb{C}^N \Big\}.$$

LEMMA 3.2

Assume that $w \in \mathcal{E}$ satisfies (3.5) formally and $\Phi_2(0) \neq 0$. Then necessarily $\Phi_2(0) = -1/\kappa_2\delta^2$ and $w(0)$ is uniquely determined. In particular $X_0(0) \neq 0$.

Proof
By taking $\det(\{\phi_i, \phi_j\}(\bar\rho))_{3\le i,j\le \ell} \neq 0$ into account, the assertion follows from the special form (3.5). □

Let $\overline{w}$ be the uniquely determined $w(0)$ given by Lemma 3.2 and look for a formal solution to (3.5) of the form $\overline{w} + w$ with $w \in \mathcal{E}^{\#}$. To simplify notations we set

$$\begin{cases} w^{\mathrm{I}} = (X_0, \Phi_2, \Xi_0, \Phi_1), & w^{\mathrm{II}} = (\Phi_3, \dots, \Phi_\ell), \\ w^{\mathrm{III}} = (\Phi_{\ell+1}, \dots, \Phi_r), & w^{\mathrm{IV}} = (\Psi_1, \dots, \Psi_k); \end{cases}$$

then $w = {}^t(w^{\mathrm{I}}, w^{\mathrm{II}}, w^{\mathrm{III}}, w^{\mathrm{IV}})$ satisfies

$$HDw = Aw + tF + G(t,w), \quad A = \begin{bmatrix} A_{\mathrm{I}} & O & O & O \\ B_{\mathrm{II}} & A_{\mathrm{II}} & O & O \\ O & O & -3E & O \\ O & O & B_{\mathrm{IV}} & -2E \end{bmatrix} \tag{3.6}$$

with $H = E \oplus O \oplus E \oplus E$, where E is the identity matrix and O is the zero matrix. Moreover F is a constant vector and

$$G(t,w) = \sum_{2\le i, 0\le j\le i} G_{ij} t^i (\log t)^j, \quad G_{ij} = G_{ij}(w_{pq} \mid q \le p \le i-1).$$

Noting that

$$A_{\mathrm{I}} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -3 & 2\delta & -4\kappa_2\bar\Phi_1 \\ 0 & -2\kappa_2\bar\Phi_1 & -4 & -2\kappa_2\bar\Phi_2 \\ 0 & 2\delta & 0 & -2 \end{bmatrix}$$

and taking into account $\kappa_2\delta^2\bar\Phi_2 = -1$, $\kappa_2\delta\bar\Phi_1 = -1$, which follows from (3.6), we have the following.

LEMMA 3.3

The eigenvalues of A_{I} are $\{-6, -4, -1, 1\}$ while A_{II} is the antisymmetric nonsingular matrix $(\{\phi_i, \phi_j\}(\bar\rho))_{3\le i,j\le\ell}$ so that A_{II} is diagonalizable with nonzero pure imaginary eigenvalues.

Repeating the same proof of [12, Theorem 5.1], we get the following.

PROPOSITION 3.2

There exists a formal solution $w \in \mathcal{E}$ *to* (3.5) *with* $\Phi_2(0) \neq 0$, $X_0(0) \neq 0$.

Note that this formal solution is uniquely determined up to a term vt, $v \in \operatorname{Ker}(E - A)$. Since $A_{\mathrm{II}} = (\{\phi_i, \phi_j\}(\bar\rho))_{3 \le i,j \le l}$ is diagonalizable, choosing a nonsingular constant matrix S one can assume that

$$S^{-1} A_{\mathrm{II}} S = i\Lambda,$$

where Λ is a diagonal matrix with nonzero real diagonal entries. Here we set $u = S^{-1} w^{\mathrm{II}}$, $v = (w^{\mathrm{I}}, w^{\mathrm{III}}, w^{\mathrm{IV}})$; then (3.5) becomes

$$\begin{cases} tDu = tK_1 u + i\Lambda u + Q_1(v) + tG_1(t,u,v), \\ Dv = K_2 v + Q_2(v) + tG_2(t,u,v), \end{cases} \tag{3.7}$$

where the K_j's are constant matrices, $Q_j(v)$ are quadratic forms in v, and $G_j(t,u,v)$ are smooth functions such that $G_j(t,0,0) = 0$. Let

$$u = \sum_{0 \le j \le i} u_{ij} t^i (\log 1/t)^j, \qquad v = \sum_{0 \le j \le i} v_{ij} t^i (\log 1/t)^j \tag{3.8}$$

be a formal solution obtained in Proposition 3.2. Denote by u_N, v_N the sums obtained from (3.8). By dropping the terms $t^i (\log t)^j$ with $i \ge N+1$, for any given $m \in \mathbb{N}$ there is a $N = N(m)$ such that u_N, v_N satisfy (3.7) modulo $O(t^{m+1})$. We look for a solution to (3.7) in the form $(u_N, v_N) + t^m(u,v)$. Then the equations satisfied by (u,v) are (after dividing by t^m)

$$\begin{cases} (t^2 \frac{d}{dt} - i\Lambda) u = -t(mI - K_1)u + L_1(t)v + tR_1(t,u,v) + tF_1(t), \\ t\frac{d}{dt} v = -(mI - K_2)v + L_2(t)v + tR_2(t,u,v) + tF_2(t), \end{cases} \tag{3.9}$$

where $R_j(t,u,v)$ are C^1-functions defined near $(0,0,0) \in \mathbb{R} \times \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ satisfying

$$\left| R_j(t,u,v) \right| \le B_j \left(|u| + |v| \right)$$

and $L_j(t), F_j(t) \in C^1((0,T])$ with some $T > 0$ such that $L_j(t)$, $tL_j'(t)$ and $F_j(t)$, $tF_j'(t)$ are bounded in $(0,T]$. The equations in (3.9) comprise a coupled system which has $t = 0$ as a singular point of the first and the second kind, respectively. Repeating the same arguments proving [12, Theorem 6.1] we get the following.

PROPOSITION 3.3

If $m \in \mathbb{N}$ *is sufficiently large, then* (3.9) *has a solution* (u,v) *with* $u(0) = 0$, $v(0) = 0$.

This proves that there exists w satisfying (3.5). Switching to the original coordinates, we find that the Hamilton equations (1.1) have a solution $(x(s), \xi(s))$ such that

$$\lim_{s \to \infty} \big(x(s), \xi(s) \big) \in \Sigma.$$

From (3.3) we have

$$\frac{d\phi_j}{dx_0}\Big|_{x_0=0} = \Big(\frac{d\phi_j}{dt}\Big/\frac{dx_0}{dt}\Big)_{x_0=0} = 0$$

and hence the curve $(x(s), \xi(s))$ is actually tangent to Σ.

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