On mod p nonvanishing of special values of L-functions associated with cusp forms on GL_2 over imaginary quadratic fields

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Abstract Let f be a cusp form on GL_2 over an imaginary quadratic field F of class number 1, and let p be an odd prime which satisfies some mild conditions. Then we show the existence of a finite-order Hecke character φ of $F_{\mathbf{A}}^{\times}$ such that the algebraic part of the special value of L-functions of $f \otimes \varphi$ at $s = 1$ is a p-adic unit. This is an analogous result to the result of A. Ash and G. Stevens for GL_2 over the field of rationals obtained in [AS].

1. Introduction

Mod p nonvanishing of special values of automorphic L-functions is an interesting problem and is studied by various people. The purpose of this paper is to show the mod p nonvanishing of special values of automorphic L-functions associated with GL_2 over an imaginary quadratic field of class number 1 (see Theorem 1.1 below). This result is an analogue of [AS, Theorem 4.5] for GL_2 over the rational number field (see also [OP, appendix], [Va, Remark 1.12]).

Our result is stated as follows. Let F be an imaginary quadratic field of class number 1, and let $\mathfrak{N}$ be an integral ideal of $\mathcal{O}_F$, the ring of integers of F, which satisfies $[\mathbf{Z} : \mathfrak{N} \cap \mathbf{Z}] > 3$. Let $\Gamma := \Gamma_1^1(\mathfrak{N})$ be the subgroup of $\mathrm{GL}_2(\mathcal{O}_F)$ defined in Section 2.1. We denote the discriminant of F by D. We fix an odd prime number p which is prime to $\mathfrak{N}$, D, and the order of the group of roots of unity in F. Moreover, we assume that F does not contain ζ_p, the primitive pth root of unity. We fix an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$ and an isomorphism $\overline{\mathbf{Q}}_p \cong \mathbf{C}$. Let f be a cusp form of weight $(2,2)$ with respect to Γ which is defined in Section 2.1. Suppose that f is normalized and f is an eigenform with respect to Hecke operators $T(\mathfrak{q})$ for all prime ideals $\mathfrak{q}$ of $\mathcal{O}_F$. We denote by $\lambda_f(T(\mathfrak{q}))$ the eigenvalue of f with respect to $T(\mathfrak{q})$. Let $\Omega_f \in \mathbf{C}$ be a complex period of f which is introduced in Section 2.2. It is known that the ratio $L(1, f, \varphi)/\Omega_f$ is an algebraic number (see [Hi2, Theorem 8.1]), where $L(s, f, \varphi)$ denotes the automorphic L-function associated with f which is introduced in Section 2.3. Then we prove the following theorem.

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THEOREM 1.1

Suppose that there exists a prime element $\ell \in \mathcal{O}_F$ such that $\ell \equiv 1 \mod \mathfrak{N}$ and $\lambda_f(T(\ell)) - \ell\overline{\ell} - 1$ is a p-adic unit. Then there exist infinitely many Hecke characters φ of finite order of $F_{\mathbf{A}}^{\times}$ such that

$$\frac{\Gamma_{\mathbf{C}}(1)^2 L(1,f,\varphi)}{\Omega_f} \quad \text{is a } p\text{-adic unit},$$

where we define $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ for $s \in \mathbf{C}$.

Our proof is based on Stevens's one in [St, Theorem 2.1] (see also [Su, Section 3]). For the proof of algebraicity of the special values of the automorphic L-function, we use the Eichler-Shimura-Harder isomorphism (see [Hi2, Proposition 3.1]). By this isomorphism, we regard a cusp form f as a class $[f]$ in the first cohomology group of a certain quotient X_Γ of $\mathbf{C} \times \mathbf{R}_{>0}$ under the natural action of Γ (cf. Section 2.1). The special value $L(1,f,\varphi)$ is expressed as a pairing of $[f]$ and a certain class in the first homology of X_Γ. Hence by using Poincaré duality, we prove our main theorem by investigating the first homology group of X_Γ.

To be more precise, by reduction modulo p and Poincaré duality, a cusp form defines a nonzero homomorphism from the first homology group of X_Γ to $\overline{\mathbf{F}}_p$, where $\mathbf{F}_p$ denotes the finite field of order p and $\overline{\mathbf{F}}_p$ denotes its algebraic closure.

If we assume that $\frac{\Gamma_{\mathbf{C}}(1)^2 L(1,f,\varphi)}{\Omega_f} \bmod p$ are trivial for almost all Hecke characters φ of finite order, then we show that this homomorphism must be a zero map. Thus we get a contradiction.

In the appendix, we generalize Fricke's lemma [St, Lemma, p. 526] on generators of a congruence subgroup for GL_2 over the rational number field to a congruence subgroup for GL_2 over arbitrary number fields.

Notation

For $z \in \mathbf{C}$, z^c or $\overline{z}$ denotes the complex conjugate of z. Let F be an imaginary quadratic field, and let $\mathcal{O}_F$ be its ring of integers. Let $I_F := \{\mathrm{id}, c\}$ be the set of embeddings $F \hookrightarrow \mathbf{C}$. We denote by D the discriminant of F. We write h as the class number of F. Let $F_{\mathbf{A}}$ denote the ring of adeles of F. We put $\hat{\mathcal{O}}_F := \mathcal{O}_F \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$. We denote by $\mathbf{e}_F : F_{\mathbf{A}}/F \to \mathbf{C}^{\times}$ the usual additive character characterized by $\mathbf{e}_F(x_\infty) = \exp(2\pi\sqrt{-1}(x_\infty + \overline{x_\infty}))$, where we denote the infinite component of $x \in F_{\mathbf{A}}$ by x_∞.

We denote by $\mathbf{Z}[I_F]$ the free $\mathbf{Z}$-module generated by I_F. For $n = n_{\mathrm{id}}\,\mathrm{id} + n_c c \in \mathbf{Z}[I_F]$, we define $n^* \in \mathbf{Z}$ to be $n^* := n_{\mathrm{id}} + n_c + 2$. We set $t := \mathrm{id} + c \in \mathbf{Z}[I_F]$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{C})$, we set $g^\iota := \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, $g^c := \left(\begin{smallmatrix} a^c & b^c \\ c^c & d^c \end{smallmatrix}\right)$. For a nonnegative integer m and a commutative ring A, we define $L(m;A)$ to be a set of two-variable homogeneous polynomials of degree m with coefficients in A. For functions $f : X \to L(m;A)$, where X stands for a certain space, we sometimes denote $f(x)$ for $x \in X$ by $f\big(x, \left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)\big)$, to emphasize the dependence of f on the variables $\left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)$.

For a commutative ring A and an A-module M, we denote the largest torsion-free quotient of M by M'.

2. Special values of the automorphic L-function

We recall the definition of cusp forms on GL_2 over an imaginary quadratic field F in Section 2.1 and recall the definition of complex periods associated with cusp forms in Section 2.2. In Section 2.3, we recall a certain integral expression of L-functions associated with cusp forms. Because of some technical difficulties, Theorem 1.1 is proved only under the assumption that the class number of F is 1. However, all statements in Sections 2.1 and 2.2 are given without the assumption of class number for a future improvement. For most of the basic facts which are stated in this section, the reader may consult [Hi1], [Hi2], and [Ur].

2.1. Definition of cusp forms

We introduce the definition of cusp forms over $\mathrm{GL}_2(F_{\mathbf{A}})$. Let $n = n_{\mathrm{id}}\,\mathrm{id} + n_c c$ be an element of $\mathbf{Z}[I_F]$. We write $k := n + 2t$. Let $\chi : \mathbf{C}^\times \to \mathbf{C}^\times$ be a character such that $\chi(z) := z^{-n} := z^{-n_{\mathrm{id}}}\overline{z}^{-n_c}$. For an integral ideal $\mathfrak{N}$ of F, we define

$$K_1(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\hat{\mathcal{O}}_F); c, d-1 \in \mathfrak{N}\hat{\mathcal{O}}_F\}.$$

DEFINITION 2.1

We put $n^* = n_{\mathrm{id}} + n_c + 2$. A cusp form on $\mathrm{GL}_2(F_{\mathbf{A}})$ of weight k and level $\mathfrak{N}$ is a C^∞-function $f : \mathrm{GL}_2(F_{\mathbf{A}}) \to L(n^*; \mathbf{C})$ satisfying the following conditions:

(1) $D_\sigma f = \left(\frac{n_\sigma^2}{2} + n_\sigma\right) f$, for $\sigma \in I_F$, where we denote the Casimir operator by D_σ (cf. [Hi2, Section 2.3]).

(2) $f(\gamma z_\infty g, \mathbf{s}) = \chi(z_\infty) f(g, \mathbf{s})$ for $\gamma \in \mathrm{GL}_2(F), z_\infty \in \mathbf{C}^\times \subset F_{\mathbf{A}}^\times$. Here we identify $F_{\mathbf{A}}^\times$ with the center of $\mathrm{GL}_2(F_{\mathbf{A}})$, and we denote a pair of variables $\left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)$ by $\mathbf{s}$. For $g \in \mathrm{GL}_2(F_{\mathbf{A}})$, we set $f(g, \mathbf{s}) := \sum_{\alpha=0}^{n^*} f_\alpha(g) S^{n^*-\alpha} T^\alpha$.

(3) $f(gu, \mathbf{s}) = f(g, u_\infty \mathbf{s})$, for $u = u_\infty u_f \in \mathrm{SU}_2(\mathbf{C}) K_1(\mathfrak{N})$.

(4) $\int_{U(F)\backslash U(F_{\mathbf{A}})} f(vg, \mathbf{s})\, du = 0$, for $g \in \mathrm{GL}_2(F_{\mathbf{A}})$, where we define $U(F) = \{v = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right); u \in F\}$ and $U(F_{\mathbf{A}}) = \{v = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right); u \in F_{\mathbf{A}}\}$.

Let us denote by $S_k(\mathfrak{N})$ the space of cusp forms on $\mathrm{GL}_2(F_{\mathbf{A}})$.

If $f : \mathrm{GL}_2(F_{\mathbf{A}}) \to L(n^*; \mathbf{C})$ is a cusp form, then f has the Fourier expansion. To describe this, we define the modified Bessel function K_α to be the unique solution of the following equations:

$$\frac{d^2 K_\alpha}{dx^2} + \frac{1}{x}\frac{dK_\alpha}{dx} - \left(1 + \frac{\alpha^2}{x^2}\right) K_\alpha = 0 \qquad \text{and} \qquad K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \quad \text{as } x \to \infty.$$

We define the Whittaker function $W_k : \mathbf{C}^\times \to L(n^*; \mathbf{C})$ by

$$W_k(y) = \sum_{\alpha=0}^{n^*} \binom{n^*}{\alpha} \left(\frac{y}{\sqrt{-1}|y|}\right)^{n_c+1-\alpha} K_{\alpha-(n_c+1)}(4\pi|y|) S^{n^*-\alpha} T^\alpha.$$

Then the Fourier expansion of f is obtained as follows.

PROPOSITION 2.1 ([Hi2, THEOREM 6.1])
Let $\mathscr{I}$ be the group of fractional ideals of F. For $f \in S_k(\mathfrak{N})$, there exists a function $\mathbf{a} : \mathscr{I} \times S_k(\mathfrak{N}) \to \mathbf{C}$ such that

(1) *the function $\mathbf{a}$ vanishes outside the set of integral ideals of F;*

(2) *we have $f((\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix})) = |y|_{\mathbf{A}} \sum_{\xi \in F^\times} \mathbf{a}(\xi y \delta_F, f) W_k(\xi y_\infty) \mathbf{e}_F(\xi z)$, where δ_F is the different of $F/\mathbf{Q}$.*

In the next subsection, we describe the Eichler-Shimura-Harder isomorphism. For this purpose, we introduce a definition of cusp forms as a function on $\mathrm{GL}_2(\mathbf{C})$.

For Γ, an arithmetic subgroup of $\mathrm{GL}_2(F)$, we define cusp forms on $\mathrm{GL}_2(\mathbf{C})$.

DEFINITION 2.2
A cusp form on $\mathrm{GL}_2(\mathbf{C})$ of weight k with respect to Γ is a C^∞-function $f : \mathrm{GL}_2(\mathbf{C}) \to L(n^*; \mathbf{C})$ satisfying the following conditions:

(1) $D_\sigma f = (n_\sigma^2/2 + n_\sigma) f$, for $\sigma = \mathrm{id}$ or c, where we denote the Casimir operator by D_σ.

(2) $f(\gamma z g, \mathbf{s}) = \chi(z) f(g, \mathbf{s})$, for $\gamma \in \Gamma, z \in \mathbf{C}^\times$.

(3) $f(gu, \mathbf{s}) = f(g, u\mathbf{s})$, for $u \in \mathrm{SU}_2(\mathbf{C})$.

(4) $\int_{\xi^{-1}\Gamma\xi \cap U(\mathbf{C}) \backslash U(\mathbf{C})} f(\xi v g, \mathbf{s})\, du = 0$, for $\xi \in \mathrm{SL}_2(F)$, where we define $U(\mathbf{C}) = \{v = (\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix})\,; u \in \mathbf{C}\}$.

Let us denote by $S_k(\Gamma)$ the space of cusp forms on $\mathrm{GL}_2(\mathbf{C})$ of weight k with respect to Γ.

REMARK 1
If $n_{\mathrm{id}} \neq n_c$, $S_k(\Gamma)$ is trivial (see [Hi2, Corollary 2.2]).

We recall the relation between cusp forms in the sense of Definition 2.1 and cusp forms in the sense of Definition 2.2.

We fix representatives $\{\mathfrak{a}_i\}_{i=1,\ldots,h}$ of the class group of F. We may assume that $\mathfrak{a}_i$ is prime to $\mathfrak{N}$ for $i = 1, \ldots, h$. We fix finite ideles $\{a_i\}_{i=1,\ldots,h}$ such that the ideal of $\mathcal{O}_F$ associated with a_i is $\mathfrak{a}_i$. Then, by the strong approximation theorem, we get a disjoint decomposition:

$$\mathrm{GL}_2(F_{\mathbf{A}}) = \coprod_{i=1}^{h} \mathrm{GL}_2(F) t_i \,\mathrm{GL}_2(\mathbf{C}) K_1(\mathfrak{N}),$$

where $t_i = \left(\begin{smallmatrix} a_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Throughout this paper, we fix a system of such elements $\{t_i\}_{1 \le i \le h}$. Then we define

$$\Gamma_1^i(\mathfrak{N}) = \mathrm{GL}_2(F) \cap t_i \,\mathrm{GL}_2(\mathbf{C}) K_1(\mathfrak{N}) t_i^{-1}.$$

For $f \in S_k(\mathfrak{N})$, we define a function $f_i : \mathrm{GL}_2(\mathbf{C}) \to L(n^*; \mathbf{C})$ by $f_i(g) = f(t_i g)$. Then we can see that $f_i \in S_k(\Gamma_1^i(\mathfrak{N}))$.

In Section 3, we use the modular symbol method, so we need to interpret cusp forms as differential forms on a certain quotient of the hyperbolic 3-fold $\mathscr{H}$

defined as follows:

$$\mathscr{H}=\left\{\begin{pmatrix} x & -y \\ y & \overline{x}\end{pmatrix}; x\in\mathbf{C}, y\in\mathbf{R}_{>0}\right\}.$$

We set the action of $\mathrm{SL}_2(\mathbf{C})$ on $\mathscr{H}$ by

$$\gamma\cdot z=\big(\rho(a)z+\rho(b)\big)\big(\rho(c)z+\rho(d)\big)^{-1},$$

where $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbf{C}), z\in\mathscr{H}$, and $\rho(t):=\left(\begin{smallmatrix} t & 0 \\ 0 & \overline{t}\end{smallmatrix}\right)$ for $t\in\mathbf{C}$. Note that this action is transitive and the stabilizer of $\varepsilon:=\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ is $\mathrm{SU}_2(\mathbf{C})$. So we may identify $\mathrm{SL}_2(\mathbf{C})/\mathrm{SU}_2(\mathbf{C})$ to $\mathscr{H}$. We recall that we can identify $\mathrm{GL}_2(\mathbf{C})/\mathbf{C}^\times U_2(\mathbf{C})\cong\mathrm{SL}_2(\mathbf{C})/\mathrm{SU}_2(\mathbf{C})$. We define

$$Y_1^i(\mathfrak{N})=\Gamma_1^i(\mathfrak{N})\backslash\mathrm{GL}_2(\mathbf{C})/\mathbf{C}^\times U_2(\mathbf{C}).$$

We may identify $Y_1^i(\mathfrak{N})$ to $\overline{\Gamma_1^i(\mathfrak{N})}\backslash\mathrm{SL}_2(\mathbf{C})/\mathrm{SU}_2(\mathbf{C})$ or $\overline{\Gamma_1^i(\mathfrak{N})}\backslash\mathscr{H}$, where we denote $\mathrm{SL}_2(\mathbf{C})\cap\Gamma_1^i(\mathfrak{N})$ by $\overline{\Gamma_1^i(\mathfrak{N})}$. Then we have

$$Y_1(\mathfrak{N}):=\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(F_{\mathbf{A}})/\mathbf{C}^\times U_2(\mathbf{C})K_1(\mathfrak{N})=\coprod_{i=1}^h Y_1^i(\mathfrak{N}).$$

2.2. Eichler-Shimura-Harder isomorphism

We fix $n=n_{\mathrm{id}}\,\mathrm{id}+n_c c\in\mathbf{Z}[I_F]$ and $k:=n+2t$. In this subsection, we describe briefly the Eichler-Shimura-Harder isomorphism for $f\in S_k(\mathfrak{N})$ and define the complex period of cusp forms which we use.

We recall the definition of the sheaf $\mathscr{L}(n;A)$, where A is a certain $\mathcal{O}_F$-algebra. We define the action of $\mathrm{GL}_2(\mathbf{C})$ on $L(n_{\mathrm{id}};\mathbf{C})\otimes L(n_c;\mathbf{C})$ by

$$(\gamma\cdot P\otimes P_c)\left(\left(\begin{smallmatrix} X \\ Y\end{smallmatrix}\right),\left(\begin{smallmatrix} X_c \\ Y_c\end{smallmatrix}\right)\right)=P\left(\left(\begin{smallmatrix} d & -b \\ -c & a\end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y\end{smallmatrix}\right)\right)\otimes P_c\left(\left(\begin{smallmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a}\end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y\end{smallmatrix}\right)\right),$$

where $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbf{C})$, $P(\left(\begin{smallmatrix} X \\ Y\end{smallmatrix}\right))\in L(n_{\mathrm{id}};\mathbf{C})$, and $P_c\left(\left(\begin{smallmatrix} X_c \\ Y_c\end{smallmatrix}\right)\right)\in L(n_c;\mathbf{C})$. When we regard $L(n_{\mathrm{id}};\mathbf{C})\otimes L(n_c;\mathbf{C})$ as the $\mathrm{GL}_2(\mathbf{C})$-module, we denote the $\mathrm{GL}_2(\mathbf{C})$-module by $L(n;\mathbf{C})$. For an $\mathcal{O}_F$-subalgebra A of $\mathbf{C}$ or $\hat{\mathcal{O}}_F$, we define the $\mathrm{GL}_2(A)$-module $L(n;A)$ in a similar manner.

Let $L_i(n;\mathcal{O}_F)$ denote the set $L(n;F)\cap t_i\cdot L(n;\hat{\mathcal{O}}_F)$, and regard $L_i(n;\mathcal{O}_F)$ as the $\Gamma_1^i(\mathfrak{N})$-module. For $\mathcal{O}_F$-algebra A, we write $L_i(n;A)=L_i(n;\mathcal{O}_F)\otimes_{\mathcal{O}_F}A$. We give $L_i(n;A)$ the discrete topology and denote by $\mathscr{L}_i(n;A)$ the sheaf determined by continuous section of the following projection:

$$\overline{\Gamma_1^i(\mathfrak{N})}\backslash\big(\mathscr{H}\times L_i(n;A)\big)\to Y_1^i(\mathfrak{N}).$$

By using $\mathscr{L}_i(n;A)$, since $Y_1(\mathfrak{N})=\coprod_{i=1}^h Y_1^i(\mathfrak{N})$, we define the sheaf $\mathscr{L}(n;A)$ on $Y_1(\mathfrak{N})$.

PROPOSITION 2.2 ([Ur, LEMME 2.3.1])
If $[\mathbf{Z}:\mathfrak{N}\cap\mathbf{Z}]>3$, *then, for all* $i=1,\dots,h$, $\overline{\Gamma_1^i(\mathfrak{N})}$ *is torsion-free.*

Hereafter in this article, we assume that $[\mathbf{Z}:\mathfrak{N}\cap\mathbf{Z}]>3$. By Proposition 2.2, $\mathscr{L}(n;A)$ is a locally constant sheaf, and we have the following isomorphism.

COROLLARY 2.1
If $[\mathbf{Z}:\mathfrak{N}\cap\mathbf{Z}]>3$, *then*

$$H^*\big(Y_1(\mathfrak{N}),\mathscr{L}(n;A)\big)\cong\bigoplus_{i=1}^{h}H^*\big(\overline{\Gamma_1^i(\mathfrak{N})},L(n;A)\big).$$

For $f\in S_k(\Gamma_1^i(\mathfrak{N}))$, we define an element $\delta_{\Gamma_1^i(\mathfrak{N})}(f)$ of the parabolic cohomology group $H^1_{\mathrm{par}}(Y_1^i(\mathfrak{N}),\mathscr{L}_i(n;\mathbf{C}))$, where the parabolic cohomology is defined by the image of the compact support cohomology $H^1_c(Y_1^i(\mathfrak{N}),\mathscr{L}_i(n;\mathbf{C}))$ under the natural map $H^1_c(Y_1^i(\mathfrak{N}),\mathscr{L}_i(n;\mathbf{C}))\to H^1(Y_1^i(\mathfrak{N}),\mathscr{L}_i(n;\mathbf{C}))$.

To introduce $\delta_{\Gamma_1^i(\mathfrak{N})}(f)$, we introduce some notation. For a pair of variables $\mathbf{u}:=\binom{U}{V}$, we define an element $Q(\mathbf{u})\in L(n^*;\mathbf{C})^{n^*+1}$ by the following equation:

$$Q(\mathbf{u})={}^t\!\left(\binom{n^*}{i}(-1)^{n^*-i}U^iV^{n^*-i}\right)_{i=0,1,\dots,n^*}.$$

By using $Q(\mathbf{u})$, for variables X,Y,X_c,Y_c,A,B, we define an element $\Psi\in(L(n;\mathbf{C})\otimes L(2;\mathbf{C}))^{n^*+1}$ by the following equation:

$$(XV-YU)^{n_{\mathrm{id}}}(X_cU+Y_cV)^{n_c}(AV-BU)^2={}^tQ(\mathbf{u})\Psi(\mathbf{x},\mathbf{x}_c,\mathbf{a}),$$

where $\mathbf{x}=\binom{X}{Y}$, $\mathbf{x}_c=\binom{X_c}{Y_c}$, $\mathbf{a}=\binom{A}{B}$. For Ψ, we denote the ith component of Ψ by $\Psi_i\in L(n;\mathbf{C})\otimes L(2;\mathbf{C})$ for $i=0,\dots,n^*$, so by definition, we have

$$\Psi(X,Y,X_c,Y_c,A,B)={}^t\big(\Psi_0(X,Y,X_c,Y_c,A,B),\dots,\Psi_{n^*}(X,Y,X_c,Y_c,A,B)\big).$$

We note that Ψ_i is homogeneous in each pair of variables $(X,Y),(X_c,Y_c)$, and (A,B) of degree n_{id},n_c, and 2, respectively. For a pair of variables $\mathbf{s}=\binom{S}{T}$, we set

$$\mathbf{s}^{n^*}={}^t(S^{n^*},S^{n^*-1}T,\dots,T^{n^*})\in L(n^*;\mathbf{C})^{n^*+1},$$

and for $u\in\mathrm{SU}_2(\mathbf{C})$, we define an element $\rho_{n^*}(u)\in M_{n^*+1}(\mathbf{C})$ by the following equation:

$$\rho_{n^*}(u)\mathbf{s}^{n^*}=(u\mathbf{s})^{n^*}.$$

Then, by [Hi2, (2.8b)], we can check that Ψ have the following property:

$$\rho_{n^*}(u)\Psi(\mathbf{x},\mathbf{x}_c,\mathbf{a})=\Psi(u\mathbf{x},u^c\mathbf{x}_c,u\mathbf{a})\quad\text{for all } u\in\mathrm{SU}_2(\mathbf{C}).$$

Now, we define $\delta_{\Gamma_1^i(\mathfrak{N})}(f)$. To restrict $f\in S_k(\Gamma_1^i(\mathfrak{N}))$ to $\mathrm{SL}_2(\mathbf{C})$, we get a C^∞-function on $\mathrm{SL}_2(\mathbf{C})$. We denote this function again by f. Since f is a $L(n^*;\mathbf{C})$-valued function, we can describe f by the following form:

$$f(g,\mathbf{s})=\sum_{0\le\alpha\le n^*}f_\alpha(g)S^{n^*-\alpha}T^\alpha=\mathbf{f}(g)\mathbf{s}^{n^*}.$$

For $\mathbf{f}$ and Ψ, we write

$$\mathbf{f}'(g;\mathbf{x},\mathbf{x}_c,\mathbf{a})=\mathbf{f}(g)\Psi\big(g^\iota\mathbf{x},(g^c)^\iota\mathbf{x}_c,{}^tj(g,\varepsilon)\mathbf{a}\big),$$

where for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbf{C})$, $z\in\mathscr{H}$, we define $j\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),z\right)=\rho(c)z+\rho(d)$. By replacing the variables (A^2,AB,B^2) with $(dx,-dy,-d\overline{x})$ in $\mathbf{f}'(g;\mathbf{x},\mathbf{x}_c,\mathbf{a})$, we define

$\delta_{\Gamma_1^i(\mathfrak{N})}(f)$ for $f \in S_k(\Gamma_1^i(\mathfrak{N}))$. Then $\delta_{\Gamma_1^i(\mathfrak{N})}(f)$ gives an element of $H^1_{\mathrm{par}}(\overline{\Gamma_1^i(\mathfrak{N})}\backslash\mathscr{H}, \mathscr{L}_i(n, \mathbf{C}))$. Furthermore, we get the following isomorphism.

PROPOSITION 2.3 ([Hi2, COROLLARY 2.2])
We have an isomorphism:

$$\delta_{\Gamma_1^i(\mathfrak{N})} : S_k\big(\Gamma_1^i(\mathfrak{N})\big) \xrightarrow{\sim} H^1_{\mathrm{par}}\big(\overline{\Gamma_1^i(\mathfrak{N})}\backslash\mathscr{H}, \mathscr{L}_i(n;\mathbf{C})\big).$$

For $f \in S_k(\mathfrak{N})$, we define a function $f_i : \mathrm{GL}_2(\mathbf{C}) \to L(n^*;\mathbf{C})$ by $f_i(g) := f(t_i g)$. Thus, by using $\delta := \bigoplus_{i=1}^h \delta_{\Gamma_1^i(\mathfrak{N})}$, we get the Eichler-Shimura-Harder isomorphism.

THEOREM 2.1 ([Hi2, PROPOSITION 3.1], [Ur, THÉORÈME 3.2])
The map $\delta : S_k(\mathfrak{N}) \to H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathbf{C}))$ *is an isomorphism of Hecke modules.*

We take $\mathfrak{p}$, a prime ideal of F, and let $f \in S_k(\mathfrak{N})$ be a normalized Hecke eigenform; that is, f is an eigenform for all Hecke operators which satisfies $\mathbf{a}(\mathcal{O}_F, f) = 1$. We define K to be the field generated by all Hecke eigenvalues of f over F. We denote by $\mathfrak{P}|p$ the prime ideal of K which is induced by the fixed embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. We denote by $K_{\mathfrak{P}}$ (resp., $\mathcal{O}_{K,\mathfrak{P}}$) the completion of K at $\mathfrak{P}$ (resp., the ring of integers of $K_{\mathfrak{P}}$).

By [Hi2, Section 8], the dimension over $\mathbf{C}$ of $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathbf{C}))$ equals the rank over $\mathcal{O}_{K,\mathfrak{P}}$ of $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))$. Moreover, for a Hecke eigenform f, $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))'[f]$ is a free $\mathcal{O}_{K,\mathfrak{P}}$-module of rank 1, where we denote by $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))'$ the largest torsion-free quotient of $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))$ and we denote by $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))'[f]$ the Hecke eigenspace with respect to the Hecke algebra homomorphism corresponding to f.

We fix a generator η_f of $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))'[f]$, which is determined up to multiplication by a unit of $\mathcal{O}_{K,\mathfrak{P}}$. We define a complex period $\Omega_f \in \mathbf{C}$ of f by $\delta(f) = \Omega_f \eta_f$, where we regard η_f as an element of $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathbf{C}))[f]$ via the natural map

$$H^1_{\mathrm{par}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})\big)'[f] \hookrightarrow H^1_{\mathrm{par}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathbf{C})\big)[f].$$

We note that Ω_f is determined up to multiplication by a unit of $\mathcal{O}_{K,\mathfrak{P}}$.

2.3. Integral expressions of special values

In this subsection, we show an analogous result to [AS, Proposition 4.4]. For this purpose, we recall the integral expression of special values of L-functions according to [Hi2, Section 7].

To introduce the definition of the L-function of f and its twists, we define the Gaussian sum and the operator $R(\varphi)$. Let $\varphi : F_{\mathbf{A}}^\times \to \mathbf{C}^\times$ be a Hecke character of finite order. We denote the conductor of φ by $\mathfrak{c} = \prod_{i=1} \mathfrak{p}_i^{e_i}$ and take $\varpi_{\mathfrak{c}} \in \hat{\mathcal{O}}_F$ such that $\mathfrak{c}\hat{\mathcal{O}}_F = \varpi_{\mathfrak{c}}\hat{\mathcal{O}}_F$. For $\mathfrak{c}$, we denote by $(\mathfrak{c}^{-1}/\mathcal{O}_F)^\times$ the subset of $(\mathfrak{c}^{-1}/\mathcal{O}_F)$ consisting of elements whose annihilator coincides with $\mathfrak{c}$. We choose and fix a subset R of $F_{\mathfrak{c}} := \prod_{\mathfrak{p}|\mathfrak{c}} F_{\mathfrak{p}}$ which is a representative of $\mathrm{Im}((\mathfrak{c}^{-1}/\mathcal{O}_F)^\times \hookrightarrow$

$\bigoplus_{\mathfrak{p}|\mathfrak{c}} \mathfrak{c}_{\mathfrak{p}}^{-1}/\mathcal{O}_{F,\mathfrak{p}} \hookrightarrow \bigoplus_{\mathfrak{p}|\mathfrak{c}} F_{\mathfrak{p}}/\mathcal{O}_{F,\mathfrak{p}})$. We fix $d \in F_{\mathbf{A},f}^{\times}$ such that the fractional ideal of F generated by $d \in F_{\mathbf{A},f}^{\times}$ is the different of $F/\mathbf{Q}$. We denote $\varphi|_{F_{\mathfrak{c}}^{\times}}$ by $\varphi_{\mathfrak{c}}$. Then we define the Gaussian sum for φ by the following equation:

$$G(\varphi) = \varphi(d)^{-1} \sum_{u \in R} \varphi_{\mathfrak{c}}(\varpi_{\mathfrak{c}} u) \mathbf{e}_F(d^{-1}u).$$

The Gaussian sum $G(\varphi)$ does not depend on the choice of d (cf. [Hi2, Section 6]). For $u \in R$, we write $\alpha(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \in G(F_{\mathbf{A},f})$. Then, for $f \in S_k(\mathfrak{N})$, we define

$$f|_{R(\varphi)}(g) = \varphi\big(\det(g)\big) \sum_{u \in R} \varphi_{\mathfrak{c}}(\varpi_{\mathfrak{c}} u) f\big(g\alpha(u)\big) \in S_k(\mathfrak{N}\mathfrak{c}^2)$$

(see [Hi2, Section 6, (6.7)]).

For a cusp form f and a Hecke character φ, the L-function of f and that of f twisted by φ are defined respectively by

$$L(s,f) = \sum_{\mathfrak{a}} \lambda_f\big(T(\mathfrak{a})\big) N(\mathfrak{a})^{-s},$$

$$L(s,f,\varphi) = \sum_{\mathfrak{a}} \lambda_f\big(T(\mathfrak{a})\big) \varphi(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

where the right-hand sum runs over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_F$, we denote by $T(\mathfrak{a})$ the Hecke operator which is introduced in [Hi2, Section 4], and we denote by $\lambda_f(T(\mathfrak{a}))$ the Hecke eigenvalue of f with respect to $T(\mathfrak{a})$.

We put $\mathbf{C}^1 = \{z \in \mathbf{C}; |z| = 1\}$. We write $E = \mathbf{C}^1 \backslash \mathbf{C}^{\times}$ and define

$$\Delta_i : E \to Y_1^i(\mathfrak{N}) = \Gamma_1^i(\mathfrak{N}) \backslash \mathrm{GL}_2(\mathbf{C})/\mathbf{C}^{\times} U_2(\mathbf{C}); a \mapsto \begin{pmatrix} |a| & 0 \\ 0 & 1 \end{pmatrix}.$$

Let A be an $\mathcal{O}_F$-algebra. For each $\mathbf{j} = j_{\mathrm{id}}\,\mathrm{id} + j_c c \in \mathbf{Z}[I_F]$ satisfying $0 \le j_{\mathrm{id}} \le n_{\mathrm{id}}$, $0 \le j_c \le n_c$, and $x^{\mathbf{j}} = x^{j_{\mathrm{id}}} \overline{x}^{j_c} = 1$ for all $x \in \mathcal{O}_F^{\times}$, the map

$$L(n;A) \to A; \sum_{m_{\mathrm{id}}=0, m_c=0}^{n_{\mathrm{id}}, n_c} a_{m_{\mathrm{id}}, m_c} X^{n_{\mathrm{id}} - m_{\mathrm{id}}} Y^{m_{\mathrm{id}}} X_c^{n_c - m_c} Y_c^{m_c} \mapsto a_{\mathbf{j}}$$

induces the map $v_{\mathbf{j}} : \Delta_i^* \mathscr{L}(n;A) \to A$ of local systems on E.

To discuss the integrality of special values of L-functions, we introduce some notation. We regard $\delta_{\Gamma_1^i(\mathfrak{N})}(f_i)$ as an element of $H_{\mathrm{c}}^1(Y_1^i(\mathfrak{N}), \mathscr{L}(n;\mathbf{C}))$ via the section s_i of the natural map

$$H_{\mathrm{c}}^1\big(Y_1^i(\mathfrak{N}), \mathscr{L}(n;\mathbf{C})\big) \to H_{\mathrm{par}}^1\big(Y_1^i(\mathfrak{N}), \mathscr{L}(n;\mathbf{C})\big),$$

which is defined in [Hi3, Section 2.1]:

$$s_i : H_{\mathrm{par}}^1\big(Y_1^i(\mathfrak{N}), \mathscr{L}(n;\mathbf{C})\big) \to H_{\mathrm{c}}^1\big(Y_1^i(\mathfrak{N}), \mathscr{L}(n;\mathbf{C})\big).$$

Then, we define the map

$$s = \bigoplus_{i=1}^{h} s_i : H_{\mathrm{par}}^1\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathbf{C})\big) \to H_{\mathrm{c}}^1\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathbf{C})\big).$$

We define the cuspidal cohomology group $H_{\mathrm{cusp}}^1(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))$ with coefficient $\mathcal{O}_{K,\mathfrak{P}}$ to be $\mathrm{Im}\, s \cap \iota\big(H_{\mathrm{c}}^1(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))\big)$, where ι is the scalar extension map $H_{\mathrm{c}}^1(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})) \to H_{\mathrm{c}}^1(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathbf{C}))$. For a Hecke eigenform

$f \in S_k(\mathfrak{N})$, we denote by $H^1_{\mathrm{cusp}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))[f]$ the Hecke eigenspace with respect to the Hecke algebra homomorphism corresponding to f. Then $H^1_{\mathrm{cusp}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))[f]$ is a free $\mathcal{O}_{K,\mathfrak{P}}$-module of rank 1. We fix a generator $\eta_{f,\mathrm{c}}$ of $H^1_{\mathrm{cusp}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))[f]$. Then we define a complex number $\Omega_{f,\mathrm{c}} \in \mathbf{C}$ by $s(\delta(f)) = \Omega_{f,\mathrm{c}}\eta_{f,\mathrm{c}}$. At the end of this section, we prove that $\Omega_{f,\mathrm{c}}$ is equal to Ω_f up to multiplication by a unit of $O_{K,\mathfrak{P}}$ under the assumption that the class number of F is 1 and under some mild conditions. We denote the largest torsion-free quotient of $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))$ by $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))'$. We also regard $\eta_{f,\mathrm{c}}$ as an element of $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))$ via the pullback of the natural map

$$H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}})\big)' \hookrightarrow H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathbf{C})\big),$$

which is induced by the scalar extension map ι.

We denote by $\eta_{f,\mathrm{c},i}$ the image of $\eta_{f,\mathrm{c}}$ via the natural map

$$H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}})\big)' \to H^1_{\mathrm{c}}\big(Y_1^i(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}})\big)',$$

which is induced by the projection

$$H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}})\big) \to H^1_{\mathrm{c}}\big(Y_1^i(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}})\big).$$

For an $\mathcal{O}_F$-algebra A, the maps Δ_i and $v_{\mathbf{j}}$ induce the natural map

$$H^1_{\mathrm{c}}\big(Y_1^i(\mathfrak{N}), \mathscr{L}(n; A)\big)' \stackrel{\Delta_i^*}{\to} H^1_{\mathrm{c}}\big(E, \Delta_i^*\mathscr{L}(n; A)\big)' \stackrel{v_{j*}}{\to} H^1_{\mathrm{c}}(E, A)'.$$

We denote by $\Delta_i^*\delta^{\mathbf{j}}(f_i)$ (resp., $\Delta_i^*\eta_f^{\mathbf{j}}$) the image of $s_i(\delta_{\Gamma_1^i(\mathfrak{N})}(f_i))$ (resp., $\eta_{f,\mathrm{c},i}$) under the above map for $A = \mathbf{C}$ (resp., $A = \mathcal{O}_{K,\mathfrak{P}}$). Then we have the integral expression of special values of L-functions as follows.

THEOREM 2.2 ([Hi2, THEOREM 8.1])

We denote the different of $F/\mathbf{Q}$ by δ_F. Let $\mathbf{j}$ be an element of $\mathbf{Z}[I_F]$ satisfying $0 \le j_{\mathrm{id}} \le n_{\mathrm{id}}$, $0 \le j_c \le n_c$, and $x^{\mathbf{j}} = x^{j_{\mathrm{id}}}\overline{x}^{j_c} = 1$ for all $x \in \mathcal{O}_F^\times$. Then we have

$$\begin{aligned}\sum_{i=1}^{h} \omega_{\mathbf{j}}(\mathfrak{a}_i\delta_F) \int_E \Delta_i^*\delta^{\mathbf{j}}(f|_{R(\varphi),i}) \\ &= (-1)^{n_{\mathrm{id}}+1}\sqrt{-1}^{j_{\mathrm{id}}+j_c}2^{-1}(2\pi)^{-(j_{\mathrm{id}}+j_c+2)} \\ &\quad \times \Gamma(j_{\mathrm{id}}+1)\Gamma(j_c+1)\sharp(\mathcal{O}_F^\times)G(\varphi)|D|L(1, f, \varphi\omega_{\mathbf{j}}),\end{aligned}$$

where $\omega_{\mathbf{j}} : F_{\mathbf{A}}^\times \to \mathbf{C}^\times$ is an unramified Hecke character such that $\omega_{\mathbf{j},\infty}(z) = z^{\mathbf{j}}$ for $z \in \mathbf{C}^\times$.

The goal of the rest of this subsection (see Proposition 2.5) is to rewrite the left-hand side of the equality of Theorem 2.2. For this purpose, we recall some basic properties of φ and R as follows.

LEMMA 2.1

We denote the conductor of φ by $\mathfrak{c}$. Then the following statements hold.

(1) *If* $\mathfrak{c}$ *is nontrivial,* $\sum_{u\in R}\varphi_{\mathfrak{c}}(u)=0$.

(2) *If* $\mathfrak{c}$ *is a principal ideal, then we can take* $R\subset F_{\mathfrak{c}}$ *as the image of*

$$\left\{\frac{t}{m_{\mathfrak{c}}};t\in\left\{\textit{representative of }(\mathcal{O}_F/\mathfrak{c})^{\times}\right\}\right\}\subset F$$

via the embedding $F\hookrightarrow F_{\mathfrak{c}}$*, where we denote a generator of* $\mathfrak{c}$ *by* $m_{\mathfrak{c}}\in\mathcal{O}_F$*.*

Hereafter we assume that the conductor of φ is nontrivial. From here on, we define a map $\widetilde{R(\varphi)}$ between cohomology groups for our use below.

By the strong approximation theorem, for $u\in R$ and t_i which is introduced in Section 2.1, we can find $j_i\in\{1,\dots,h\},\alpha_u^{(i)}\in\mathrm{GL}_2(F)$, and $k_u^{(i)}=k_{u,\infty}^{(i)}k_{u,f}^{(i)}\in\mathrm{GL}_2(\mathbf{C})K_1(\mathfrak{N}\mathfrak{c}^2)$ such that

$$t_i\alpha(u)=\alpha_u^{(i)}t_{j_i}k_u^{(i)}.$$

By taking the determinants of both sides, we have $j_i=i$. By the definition of $\alpha_u^{(i)}$, we have

$$\alpha_u^{(i)}\overline{\Gamma_1^i(\mathfrak{N}\mathfrak{c}^2)}(\alpha_u^{(i)})^{-1}\subset\overline{\Gamma_1^i(\mathfrak{N})}.$$

Hence the following map,

$$\mathscr{H}\times L(n;\mathbf{C})\to\mathscr{H}\times L(n;\mathbf{C});(z,P)\mapsto(\alpha_u^{(i)}\cdot z,\alpha_u^{(i)}\cdot P),$$

induces a morphism of local systems

$$\widetilde{R(\varphi)_u^i}:\mathscr{L}_i(n;\mathbf{C})_{/Y_1^i(\mathfrak{N})}\to\mathscr{L}_i(n;\mathbf{C})_{/Y_1^i(\mathfrak{N}\mathfrak{c}^2)}.$$

Note that this map does not depend on the choice of $\alpha_u^{(i)}$. The map $\widetilde{R(\varphi)_u^i}$ induces the morphism of cohomology groups, which we denote also by $\widetilde{R(\varphi)_u^i}$. Then we define

$$\widetilde{R(\varphi)^i}=\sum_{u\in R}\varphi_{\mathfrak{c}}(\varpi_{\mathfrak{c}}u)\widetilde{R(\varphi)_u^i}:H^1\big(Y_1^i(\mathfrak{N}),\mathscr{L}(n;\mathbf{C})\big)\to H^1\big(Y_1^i(\mathfrak{N}\mathfrak{c}^2),\mathscr{L}(n;\mathbf{C})\big)$$

and

$$\widetilde{R(\varphi)}=\sum_{i=1}^{h}\varphi(a_i)\widetilde{R(\varphi)^i}:H^1\big(Y_1(\mathfrak{N}),\mathscr{L}(n;\mathbf{C})\big)\to H^1\big(Y_1(\mathfrak{N}\mathfrak{c}^2),\mathscr{L}(n;\mathbf{C})\big),$$

where a_i for $i=1,\dots,h$ is the finite idele of $F_{\mathbf{A}}^{\times}$ which is fixed in Section 2.1. In the same way, for any $\mathbf{Z}$-algebra A which contains all matrix elements of $\{\alpha_u\}_{u\in R}$, we define the map

$$\widetilde{R(\varphi)}:H^1\big(Y_1(\mathfrak{N}),\mathscr{L}(n;A)\big)\to H^1\big(Y_1(\mathfrak{N}\mathfrak{c}^2),\mathscr{L}(n;A)\big).$$

Especially, $\widetilde{R(\varphi)}$ is defined for $A=K_{\mathfrak{P}}(\varphi_0)$, where $\varphi_0:=\varphi|_{\hat{\mathcal{O}}_F^{\times}}$ and we denote by $K_{\mathfrak{P}}(\varphi_0)$ the subfield of $\overline{\mathbf{Q}}_p$ generated by the image of φ_0 over $K_{\mathfrak{P}}$. Similarly, if the conductor of φ is prime to $\mathfrak{p}:=\mathfrak{P}\cap\mathcal{O}_F$, then $\widetilde{R(\varphi)}$ is defined also for $A=\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]$, where $\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]$ is the subring of $\mathbf{C}$ which is generated by the image of φ_0 over $\mathcal{O}_{K,\mathfrak{P}}$. By abuse of notation, we also denote by $\widetilde{R(\varphi)}$ the map between compact support cohomology groups,

$$\widetilde{R(\varphi)}; H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n; A)\big) \to H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}\mathfrak{c}^2), \mathscr{L}(n; A)\big),$$

which is induced by the morphism of local systems $\widetilde{R(\varphi)}^i_u : \mathscr{L}_i(n; \mathbf{C})_{/Y_1^i(\mathfrak{N})} \to \mathscr{L}_i(n; \mathbf{C})_{/Y_1^i(\mathfrak{N}\mathfrak{c}^2)}$ for $i = 1, \dots, h$.

By the definition of $R(\varphi)$ and $\widetilde{R(\varphi)}$, we deduce the following proposition.

PROPOSITION 2.4

The following diagram is commutative:

$$\begin{array}{ccc} S_k(\mathfrak{N}) & \xrightarrow{R(\varphi)} & S_k(\mathfrak{N}\mathfrak{c}^2) \\ \delta\big\downarrow & & \delta\big\downarrow \\ H^1\big(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathbf{C})\big) & \xrightarrow{\widetilde{R(\varphi)}} & H^1\big(Y_1(\mathfrak{N}\mathfrak{c}^2), \mathscr{L}(n; \mathbf{C})\big), \end{array}$$

where δ is the map introduced in Theorem 2.1.

By Theorem 2.2 and Proposition 2.4, we obtain the following corollary.

COROLLARY 2.2

We have the following equations.

(1) *We have*

$$\begin{aligned} &\sum_{i=1}^{h} \omega_{\mathbf{j}}(\mathfrak{a}_i \delta_F) \Delta_i^* \delta^{\mathbf{j}}(f_i)|_{\widetilde{R(\varphi)}} \cap E \\ &\quad = (-1)^{n_{\mathrm{id}}+1} \sqrt{-1}^{j_{\mathrm{id}}+j_c} 2^{-1} (2\pi)^{-(j_{\mathrm{id}}+j_c+2)} \\ &\qquad \times \Gamma(j_{\mathrm{id}}+1)\Gamma(j_c+1)\sharp(\mathcal{O}_F^\times) G(\varphi)|D| L(1, f, \varphi\omega_{\mathbf{j}}), \end{aligned}$$

where $\cap$ denotes the cap product (cf. [Ur, Section 1]) and we identify $H_0^{\mathrm{c}}(E, \mathbf{C})$ with $\mathbf{C}$ via the canonical isomorphism.

(2) *By dividing the previous equation by $\Omega_{f,\mathrm{c}}$, we obtain*

$$\begin{aligned} &\sum_{i=1}^{h} \omega_{\mathbf{j}}(\mathfrak{a}_i \delta_F) \Delta_i^* \eta_f^{\mathbf{j}}|_{\widetilde{R(\varphi)}} \cap E \\ &\quad = (-1)^{n_{\mathrm{id}}+1} \sqrt{-1}^{j_{\mathrm{id}}+j_c} 2^{-1} (2\pi)^{-(j_{\mathrm{id}}+j_c+2)} \\ &\qquad \times \Gamma(j_{\mathrm{id}}+1)\Gamma(j_c+1)\sharp(\mathcal{O}_F^\times) G(\varphi)|D| L(1, f, \varphi\omega_{\mathbf{j}})/\Omega_{f,\mathrm{c}}. \end{aligned}$$

We have $\Delta_i^* \eta_f^{\mathbf{j}}|_{\widetilde{R(\varphi)}} \cap E \in H_0^{\mathrm{c}}(E, K_{\mathfrak{P}}(\varphi_0)) \cong K_{\mathfrak{P}}(\varphi_0)$. If the conductor of φ is prime to $\mathfrak{N}$, then we have $\Delta_i^* \eta_f^{\mathbf{j}}|_{\widetilde{R(\varphi)}} \cap E \in H_0^{\mathrm{c}}(E, \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]) \cong \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]$.

We rewrite the left-hand sides of the equalities of Corollary 2.2 in order to express special values of the L-function associated with f as a cap product of η_f and a *twisted* cycle in Proposition 2.5. For this purpose, we introduce an element $E^i_{\mathbf{j}}$ of $H_1(Y^i_1(\mathfrak{N}), \mathscr{L}^*_i(n; \mathcal{O}_{K,\mathfrak{P}}))$, where $\mathscr{L}^*_i(n; \mathcal{O}_{K,\mathfrak{P}})$ is the local system on $Y^i_1(\mathfrak{N})$ which is determined by $\mathrm{Hom}_{\mathcal{O}_{K,\mathfrak{P}}}(L(n; \mathcal{O}_{K,\mathfrak{P}}), \mathcal{O}_{K,\mathfrak{P}})$. Note that if $[\mathbf{Z} : \mathfrak{P} \cap \mathbf{Z}] > n$, we define a nondegenerate bilinear form:

$$[\,,\,]_n : L(n; \mathcal{O}_{K,\mathfrak{P}}) \times L(n; \mathcal{O}_{K,\mathfrak{P}}) \to \mathcal{O}_{K,\mathfrak{P}};$$

$$\big(P(X,Y,X_c,Y_c), Q(X,Y,X_c,Y_c)\big) \mapsto \sum_{j_{\mathrm{id}}=0, j_c=0}^{n_{\mathrm{id}}, n_c} \frac{(-1)^{j_{\mathrm{id}}+j_c} a_{j_{\mathrm{id}},j_c} b_{n_{\mathrm{id}}-j_{\mathrm{id}}, n_c-j_c}}{\binom{n_{\mathrm{id}}}{j_{\mathrm{id}}}\binom{n_c}{j_c}},$$

where we define

$$P(X,Y,X_c,Y_c) = \sum_{j_{\mathrm{id}}=0, j_c=0}^{n_{\mathrm{id}}, n_c} a_{j_{\mathrm{id}},j_c} X^{n_{\mathrm{id}}-j_{\mathrm{id}}} Y^{j_{\mathrm{id}}} X_c^{n_c-j_c} Y_c^{j_c},$$

$$Q(X,Y,X_c,Y_c) = \sum_{j_{\mathrm{id}}=0, j_c=0}^{n_{\mathrm{id}}, n_c} b_{j_{\mathrm{id}},j_c} X^{n_{\mathrm{id}}-j_{\mathrm{id}}} Y^{j_{\mathrm{id}}} X_c^{n_c-j_c} Y_c^{j_c}.$$

We denote by $L^*(n; \mathcal{O}_{K,\mathfrak{P}})$ the $\mathcal{O}_{K,\mathfrak{P}}$-module which is generated by

$$\left\{ \binom{n_{\mathrm{id}}}{j_{\mathrm{id}}} \binom{n_c}{j_c} X^{n_{\mathrm{id}}-j_{\mathrm{id}}} Y^{j_{\mathrm{id}}} X_c^{n_c-j_c} Y_c^{j_c} \right\}_{0 \le j_{\mathrm{id}}, j_c \le n}$$

over $\mathcal{O}_{K,\mathfrak{P}}$. Then we identify $\mathrm{Hom}_{\mathcal{O}_{K,\mathfrak{P}}}(L(n; \mathcal{O}_{K,\mathfrak{P}}), \mathcal{O}_{K,\mathfrak{P}})$ with $L^*(n; \mathcal{O}_{K,\mathfrak{P}})$ via $[\,,\,]_n$.

Hereafter in this article, we assume that the class number of F is 1. Since the class number is 1, we denote by $\Gamma_1(\mathfrak{N})$ the group $\Gamma^1_1(\mathfrak{N})$ which is defined in Section 2.1 for short. We adjoin boundaries to $\mathscr{H}$ in the same manner as in [Ur, Section 2.3], and we denote it by $\mathscr{H}^*$. (In [Ur, Section 2.3], $\mathscr{H}^*$ is denoted by $\mathcal{Z}^*$.) We denote the Borel-Serre compactification of $Y_1(\mathfrak{N})$ by $Y_1(\mathfrak{N})^* = \overline{\Gamma_1(\mathfrak{N})} \backslash \mathscr{H}^*$ and its boundary by $\partial Y_1(\mathfrak{N})^*$. We note that $Y_1(\mathfrak{N})^*$ and $Y_1(\mathfrak{N})$ are homotopy equivalent.

We introduce an element $E_{\mathbf{j},x}$ of a relative homology group $H_1(Y_1(\mathfrak{N})^*, \partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]))$ below. We define $E^* = E \cup \{0\} \cup \{\infty\}$ on which we have a natural topology induced by the isomorphism $E^* \cong \mathbf{R}_{>0} \cup \{0\} \cup \{\infty\}$. For an element of $x \in F$, we define the map $\Delta_x : E \to Y_1(\mathfrak{N})$ by

$$E \to Y_1(\mathfrak{N}); a \mapsto \left(\begin{smallmatrix} |a| & x \\ 0 & 1 \end{smallmatrix}\right)$$

and we naturally extend Δ_x to the map $E^* \to Y_1(\mathfrak{N})^*$, which we also denote by the same symbol Δ_x. We obtain a natural sequence:

$$\begin{aligned} &H^0\big(E^*, \Delta_x^* \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big) \\ &\quad \xrightarrow{\sim} H_1\big(E^*, \Delta_x^* \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big) \\ &\quad \to H_1\big(E^*, \partial E^*, \Delta_x^* \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big) \\ &\quad \to H_1\big(Y_1(\mathfrak{N})^*, \partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big). \end{aligned}$$

We define an element $E_{\mathbf{j},x}$ of $H_1(Y_1(\mathfrak{N})^*, \partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]))$ to be the image of an element $\alpha_{\mathbf{j},x}$ of $H^0(E^*, \Delta_x^* \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]))$ under the above map, which is defined as follows. Since $\Delta_x^* \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])$ is a constant sheaf on E^*, we define $\alpha_{\mathbf{j},x}$ by the following equation:

$$\alpha_{\mathbf{j},x} = (-1)^{j_{\mathrm{id}}+j_c} \binom{n_{\mathrm{id}}}{j_{\mathrm{id}}} \binom{n_c}{j_c} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \cdot (X^{n_{\mathrm{id}}-j_{\mathrm{id}}} Y^{j_{\mathrm{id}}} X_c^{n_c-j_c} Y_c^{j_c}).$$

Since the class number of F is 1, we fix a generator $m_{\mathfrak{c}} \in \mathcal{O}_F$ of the conductor $\mathfrak{c}$ of φ. Let $R \subset F$ be a subset of representatives of $(\mathfrak{c}^{-1}/\mathcal{O}_F)^\times$ which is introduced at the beginning of Section 2.3. Now we define a relative homology class:

$$c_{\varphi,\mathbf{j}} = \sum_{u \in R \subset F} \varphi_{\mathfrak{c}}(m_{\mathfrak{c}} u) E_{\mathbf{j},u} \in H_1\big(Y_1(\mathfrak{N})^*, \partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big).$$

We assume that $\mathfrak{c}$ is prime to $\mathfrak{N}$. Then, by using $c_{\varphi,\mathbf{j}}$, we introduce an element of $H_1(Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]))$ for φ. For this purpose, we need the following lemma.

LEMMA 2.2

Let $t/m_{\mathfrak{c}}, t'/m_{\mathfrak{c}}$ be elements of R. Then, if $\mathfrak{c}$ is prime to $\mathfrak{N}$, $t/m_{\mathfrak{c}}$ and $t'/m_{\mathfrak{c}}$ determine the same cusp in $\overline{\Gamma_1(\mathfrak{N}) \backslash \mathscr{H}}$.

Proof

Since $\mathfrak{c}$ is prime to $\mathfrak{N}$, there exists $\mathfrak{n} \in \mathfrak{N}$ such that $\mathfrak{n}\mathcal{O}_F$ is prime to $\mathfrak{c}$. Since t and t' are prime to $\mathfrak{c}$, $\mathfrak{c}$ is prime to $t\mathfrak{n}$ and $t'\mathfrak{n}$. Therefore we find a, a', c, and $c' \in \mathcal{O}_F$ which satisfy:

$$a m_{\mathfrak{c}} - c\mathfrak{n}t = 1 \qquad \text{and} \qquad a' m_{\mathfrak{c}} - c'\mathfrak{n}t' = 1.$$

We write $\gamma = \left(\begin{smallmatrix} a & t \\ c\mathfrak{n} & m_{\mathfrak{c}} \end{smallmatrix}\right)$ and $\gamma' = \left(\begin{smallmatrix} a' & t' \\ c'\mathfrak{n} & m_{\mathfrak{c}} \end{smallmatrix}\right)$. Then we see that

$$\gamma \cdot 0 = \frac{t}{m_{\mathfrak{c}}} \qquad \text{and} \qquad \gamma' \cdot 0 = \frac{t'}{m_{\mathfrak{c}}},$$

and $\gamma'\gamma^{-1} \in \Gamma_1(\mathfrak{N})$. This proves the lemma. $\square$

By Lemma 2.2, $c_{\varphi,\mathbf{j}}$ belongs to the kernel of the boundary map

$$H_1\big(Y_1(\mathfrak{N})^*, \partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big) \to H_0\big(\partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big).$$

Hence, $c_{\varphi,\mathbf{j}}$ falls in the image of the map

$$\begin{aligned} &H_1\big(Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big)/H_1\big(\partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big) \\ &\quad \hookrightarrow H_1\big(Y_1(\mathfrak{N})^*, \partial Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big), \end{aligned}$$

whose image is equal to the kernel of the boundary map. By abuse of notation, we also denote by $c_{\varphi,\mathbf{j}}$ a pullback of $c_{\varphi,\mathbf{j}}$ to $H_1(Y_1(\mathfrak{N})^*, \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]))$.

Then, since $\delta(f)$ is an element of the parabolic cohomology group, the cap product $\eta_{f,\mathrm{c}} \cap c_{\varphi,\mathbf{j}}$ does not depend on the choice of the pullback. We regard $\eta_{f,\mathrm{c}} \cap c_{\varphi,\mathbf{j}} \in H_0^{\mathrm{c}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]) \otimes \mathscr{L}^*(n, \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]))$ as an element of $\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]$ via the following composition of maps:

$$\begin{aligned}&H_0^{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]) \otimes \mathscr{L}^*(n; \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])\big)\\ &\quad \to H_0^{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big) \xrightarrow{\sim} \mathcal{O}_{K,\mathfrak{P}}[\varphi_0],\end{aligned}$$

where the first map is induced by the bilinear form $[\,,\,]_n$. Then we obtain the desired expression of special values.

PROPOSITION 2.5

We have

$$\begin{aligned}&\omega_{\mathbf{j}}(\delta_F) s\big(\delta(f)\big) \cap c_{\varphi,\mathbf{j}}\\ &\quad = (-1)^{n_{\mathrm{id}}+1} \sqrt{-1}^{j_{\mathrm{id}}+j_c} 2^{-1} (2\pi)^{-(j_{\mathrm{id}}+j_c+2)}\\ &\qquad \times \Gamma(j_{\mathrm{id}}+1)\Gamma(j_c+1) \sharp(\mathcal{O}_F^\times) G(\varphi) |D| L(1, f, \varphi\omega_{\mathbf{j}}),\\ &\omega_{\mathbf{j}}(\delta_F) \eta_{f,\mathrm{c}} \cap c_{\varphi,\mathbf{j}}\\ &\quad = (-1)^{n_{\mathrm{id}}+1} \sqrt{-1}^{j_{\mathrm{id}}+j_c} 2^{-1} (2\pi)^{-(j_{\mathrm{id}}+j_c+2)}\\ &\qquad \times \Gamma(j_{\mathrm{id}}+1)\Gamma(j_c+1) \sharp(\mathcal{O}_F^\times) G(\varphi) |D| L(1, f, \varphi\omega_{\mathbf{j}})/\Omega_{f,\mathrm{c}}.\end{aligned}$$

Proof

Note that, since the conductor $\mathfrak{c}$ of φ is principal, we have

$$\omega|_{\widetilde{R(\varphi)}} = \sum_{u \in R \subset F} \varphi_{\mathfrak{c}}(m_{\mathfrak{c}} u) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \omega,$$

for $\omega \in H^1(\overline{\Gamma_1(\mathfrak{N})} \backslash \mathscr{H}; \mathscr{L}(n; \mathbf{C}))$. By the definitions of $\widetilde{R(\varphi)}$ and c_φ, we have $\Delta_1^* \delta^{\mathbf{j}}(f)|_{\widetilde{R(\varphi)}} \cap E = s(\delta(f)) \cap c_{\varphi,\mathbf{j}}$. This proves the proposition. $\square$

We describe a relation between Ω_f and $\Omega_{f,\mathrm{c}}$ in Proposition 2.6. For this purpose, we need the following lemma.

LEMMA 2.3

Let ℓ be a prime element of $\mathcal{O}_F$ such that $\ell \equiv 1 \mod \mathfrak{N}$. Then the Hecke operator $T(\ell)$ acts on $H^0(\partial Y_1(\mathfrak{N})^*, \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))$ by the multiplication of $\ell^{n+1}+1$, where we denote $\ell^{n_{\mathrm{id}}+1}(\overline{\ell})^{n_c+1}$ by ℓ^{n+1}.

Proof

For each cusp s of $\overline{\Gamma_1(\mathfrak{N})}$, we denote by Γ_s the group $\{\gamma \in \overline{\Gamma_1(\mathfrak{N})};\ \gamma(s) = s\}$. We compute the action of the double coset $\overline{\Gamma_1(\mathfrak{N})} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right) \overline{\Gamma_1(\mathfrak{N})}$ on $H^0(\partial Y_1(\mathfrak{N})^*, \mathscr{L}(n; \mathcal{O}_F)) \cong \bigoplus_s H^0(\Gamma_s, \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))$ according to the definition of the action given in [Hi1, Section 3].

We take an element g of $\mathrm{SL}_2(\mathcal{O}_F)$ such that $g(s) = \infty$. By the assumption $\ell \equiv 1 \mod \mathfrak{N}$, we find a disjoint sum decomposition:

$$\overline{\Gamma_1(\mathfrak{N})} \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \overline{\Gamma_1(\mathfrak{N})} = \overline{\Gamma_1(\mathfrak{N})} g \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} g^{-1} \Gamma_s \cup \overline{\Gamma_1(\mathfrak{N})} g \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} g^{-1} \Gamma_s.$$

Similarly, we have disjoint sum decompositions

$$\overline{\Gamma_1(\mathfrak{N})}g\begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix}g^{-1}\Gamma_s = \bigcup_{j \bmod \ell\mathcal{O}_F}\overline{\Gamma_1(\mathfrak{N})}g\begin{pmatrix}1 & \mathfrak{n}j\\ 0 & \ell\end{pmatrix}g^{-1},$$

$$\overline{\Gamma_1(\mathfrak{N})}g\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}g^{-1}\Gamma_s = \overline{\Gamma_1(\mathfrak{N})}g\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}g^{-1},$$

where we denote a generator of $\mathfrak{N}$ by $\mathfrak{n}$. We note that $g\left(\begin{smallmatrix}\ell & 0\\ 0 & 1\end{smallmatrix}\right)g^{-1}(s) = s$ and $g\left(\begin{smallmatrix}1 & 0\\ 0 & \ell\end{smallmatrix}\right)g^{-1}(s) = s$. We put $\sigma = g\left(\begin{smallmatrix}\ell & 0\\ 0 & 1\end{smallmatrix}\right)g^{-1}$ and $\sigma_j = g\left(\begin{smallmatrix}1 & \mathfrak{n}j\\ 0 & \ell\end{smallmatrix}\right)g^{-1}$.

Hence, for $\gamma_0 \in \Gamma_s$, the action of $\left[\overline{\Gamma_1(\mathfrak{N})}\left(\begin{smallmatrix}1 & 0\\ 0 & \ell\end{smallmatrix}\right)\overline{\Gamma_1(\mathfrak{N})}\right]$ on $u = \bigoplus_s u_s \in H^0(\partial Y_1(\mathfrak{N})^*, \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))$ is calculated as follows:

$$(u|_{[\overline{\Gamma_1(\mathfrak{N})}\left(\begin{smallmatrix}1 & 0\\ 0 & \ell\end{smallmatrix}\right)\overline{\Gamma_1(\mathfrak{N})}]})_s(\gamma_0) = \sigma^\iota u_s(\gamma) + \sum_{j \bmod \ell\mathcal{O}_F}\sigma_j^\iota u_s(\gamma_j),$$

where γ (resp., γ_j for $j \bmod \ell\mathcal{O}_F$) is an element of Γ_s such that $\sigma\gamma_0 = \gamma\sigma'$ (resp., $\sigma_j\gamma_0 = \gamma_j\sigma_j'$) for some σ' (resp., σ_j') $\in \{\sigma\} \cup \{\sigma_j : j \bmod \ell\mathcal{O}_F\}$.

By using the above description, we compute the action of the double coset. We note that $H^0(\Gamma_s, \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})) = \langle g\cdot Y^{n_{\mathrm{id}}}Y_c^{n_c}\rangle_{\mathcal{O}_{K,\mathfrak{P}}}$. We have the following equalities:

$$\sigma^\iota(g\cdot Y^{n_{\mathrm{id}}}Y_c^{n_c}) = g\cdot Y^{n_{\mathrm{id}}}Y_c^{n_c},$$
$$\sigma_j^\iota(g\cdot Y^{n_{\mathrm{id}}}Y_c^{n_c}) = \ell^{n_{\mathrm{id}}}(\overline{\ell})^{n_c}g\cdot Y^{n_{\mathrm{id}}}Y_c^{n_c}.$$

This completes the lemma. □

PROPOSITION 2.6

Suppose that there exists a prime element $\ell \in \mathcal{O}_F$ *such that* $\ell \equiv 1 \bmod \mathfrak{N}$ *and* $\lambda_f(T(\ell)) - \ell^{n+1} - 1$ *is a unit of* $\mathcal{O}_{K,\mathfrak{P}}$. *Then* $\Omega_{f,\mathrm{c}}/\Omega_f$ *is a unit of* $\mathcal{O}_{K,\mathfrak{P}}$.

Proof

We denote by $H_{\mathrm{c}}^1(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))'$ the largest torsion-free quotient of $H_{\mathrm{c}}^1(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))$. Since the map

$$\iota : H_{\mathrm{c}}^1\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})\big)' \to H_{\mathrm{par}}^1\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})\big)'$$

is Hecke equivariant, there exists $\alpha \in \mathcal{O}_{K,\mathfrak{P}}$ such that $\iota(\eta_{f,\mathrm{c}}) = \alpha\eta_f$. To prove the proposition, it is enough to show that α is a unit of $\mathcal{O}_{K,\mathfrak{P}}$.

Since the map ι is surjective, there exists $\eta_f' \in H_{\mathrm{c}}^1(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}}))'$ such that $\iota(\eta_f') = \eta_f$. Since the boundary exact sequence

$$H^0\big(\partial Y_1(\mathfrak{N})^*, \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})\big) \to H_{\mathrm{c}}^1\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})\big)$$
$$\to H_{\mathrm{par}}^1\big(Y_1(\mathfrak{N}), \mathscr{L}(n;\mathcal{O}_{K,\mathfrak{P}})\big)$$

is Hecke equivariant (see [Hi1, Section 1.10]), the kernel of ι is annihilated by the operator $(T(\ell) - \ell^{n+1} - 1)$ by Lemma 2.3. Since $\eta_{f,\mathrm{c}} - \alpha\eta_f'$ belongs to the kernel of ι, we obtain $(T(\ell) - \ell^{n+1} - 1)(\eta_{f,\mathrm{c}} - \alpha\eta_f') = 0$.

We denote by $\langle \eta_{f,\mathrm{c}} \rangle_{\mathcal{O}_{K,\mathfrak{P}}}$ the $\mathcal{O}_{K,\mathfrak{P}}$-submodule of $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))'$ which is generated by $\eta_{f,\mathrm{c}}$. Since $T(\ell)(\eta_{f,\mathrm{c}}) = \lambda_f(T(\ell))\eta_{f,\mathrm{c}}$, $\alpha(T(\ell) - \ell^{n+1} - 1)\eta'_f$ is an element of $\langle \eta_{f,\mathrm{c}} \rangle_{\mathcal{O}_{K,\mathfrak{P}}}$. By freeness of $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathscr{L}(n; \mathcal{O}_{K,\mathfrak{P}}))'$, $T(\mathfrak{q})$ acts on $(T(\ell) - \ell^{n+1} - 1)\eta'_f$ by multiplication of the scalar $\lambda_f(T(\mathfrak{q}))$ for any prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$. In particular, $(T(\ell) - \ell^{n+1} - 1)\eta'_f$ belongs to $\langle \eta_{f,\mathrm{c}} \rangle_{\mathcal{O}_{K,\mathfrak{P}}}$. Hence, there exists $\beta \in \mathcal{O}_{K,\mathfrak{P}}$ such that $(T(\ell) - \ell^{n+1} - 1)\eta'_f = \beta\eta_{f,\mathrm{c}}$. By definition, we have the following equalities:

$$\begin{aligned} \iota\big((T(\ell) - \ell^{n+1} - 1)\eta'_f\big) &= \big(T(\ell) - \ell^{n+1} - 1\big)\eta_f \\ &= \big(\lambda_f(T(\ell)) - \ell^{n+1} - 1\big)\eta_f, \\ \iota(\beta\eta_{f,\mathrm{c}}) &= \alpha\beta\eta_f. \end{aligned}$$

Thus we obtain $(\lambda_f(T(\ell)) - \ell^{n+1} - 1)\eta_f = \alpha\beta\eta_f$. By assumption, $\lambda_f(T(\ell)) - \ell^{n+1} - 1$ is a unit of $\mathcal{O}_{K,\mathfrak{P}}$. This implies that α is a unit of $\mathcal{O}_{K,\mathfrak{P}}$. □

3. Proof of Theorem 1.1

In this section, we prove our main theorem, and we always assume that the class number of F is 1 and p is prime to $\sharp\mathcal{O}_F^\times D\mathfrak{N}$. Moreover, we suppose the condition of Proposition 2.6 for $n_{\mathrm{id}} = n_c = 0$.

We define the following homomorphism:

$$\mathrm{pr} : \overline{\Gamma_1(\mathfrak{N})} \to H_1\big(Y_1(\mathfrak{N})^*, \overline{\mathbf{F}}_p\big)/H_1\big(\partial Y_1(\mathfrak{N})^*, \overline{\mathbf{F}}_p\big); \gamma \mapsto \{0, \gamma \cdot 0\},$$

where $\{0, \gamma \cdot 0\}$ denotes the projection of the path from zero to $\gamma \cdot 0$ in $\mathscr{H}^*$ to the $Y_1(\mathfrak{N})^*$. Then pr is surjective.

REMARK 2

(1) The map pr does not depend on the choice of the cusp. In fact, we have the same map replacing the cusp zero by another cusp $x \in F$. This follows from the fact that $\mathscr{H}^*$ is simply connected. Hence we easily see that pr is actually a homomorphism.

(2) We have $\mathrm{pr}(\{\text{parabolic element of } \overline{\Gamma_1(\mathfrak{N})}\}) = \{0\}$. We see this from Remark 2(1) and the definition of parabolic elements.

(3) For an element γ of $\overline{\Gamma_1(\mathfrak{N})}$, an element x of $\mathcal{O}_F$, and $u = \gamma \cdot 0$, we have $u + x = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot \gamma \cdot 0$. By using Remarks 2(1) and 2(2), we have the following identities:

$$\begin{aligned} \{0, u + x\} &= \mathrm{pr}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \gamma\right) \\ &= \mathrm{pr}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) + \mathrm{pr}(\gamma) \\ &= \mathrm{pr}(\gamma) = \{0, u\}. \end{aligned}$$

Hence we have $\{0, u + x\} = \{0, u\}$ in $H_1(Y_1(\mathfrak{N})^*, \overline{\mathbf{F}}_p)/H_1(\partial Y_1(\mathfrak{N})^*, \overline{\mathbf{F}}_p)$.

At the moment, we assume the following two lemmas and we complete the proof of the main theorem.

LEMMA 3.1

There exists a natural homomorphism

$$H_{\mathrm{c}}^1\big(Y_1(\mathfrak{N}),\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big)' \to \mathrm{Hom}_{\overline{\mathbf{F}}_p}\big(H_1(Y_1(\mathfrak{N})^*,\overline{\mathbf{F}}_p),\overline{\mathbf{F}}_p\big),$$

and the image Φ_f of η_f under the homomorphism is not zero. Moreover, Φ_f satisfies the following identity:

$$\Phi_f(c_\varphi) = (-1)^{n_{\mathrm{id}}+1}2^{-3}\sharp\mathcal{O}_F^\times G(\varphi)|D|\frac{\Gamma_{\mathbf{C}}(1)^2L(1,f,\varphi)}{\Omega_{f,\mathrm{c}}} \in \overline{\mathbf{F}}_p,$$

where φ is a finite-order Hecke character of $F_{\mathbf{A}}^\times$ whose conductor is prime to $\mathfrak{p}$ and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ for $s \in \mathbf{C}$.

REMARK 3

We note that $(-1)^{n_{\mathrm{id}}+1}2^{-3}\sharp\mathcal{O}_F^\times G(\varphi)|D|$ is a p-adic unit by assumption of p and the conductor of φ. Since we suppose that there exists a prime element $\ell \in \mathcal{O}_F$ such that $\ell \equiv 1 \mod \mathfrak{N}$ and $\lambda_f(T(\ell)) - \ell\overline{\ell} - 1$ is a p-adic unit, we see that the quantity $\frac{\Gamma_{\mathbf{C}}(1)^2L(1,f,\varphi)}{\Omega_f} \in \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]$ is a p-adic unit if and only if the quantity $\Phi_f(c_\varphi) \in \overline{\mathbf{F}}_p$ is not zero by Proposition 2.6.

We denote the extension of F obtained by adding a primitive pth root of unity ζ_p by $F(\zeta_p)$. Let $\mathfrak{M}_p$ denote the conductor of the extension of $F(\zeta_p)/F$.

LEMMA 3.2

Let b and $d \in \mathcal{O}_F$ be elements satisfying that $d \equiv 1 \bmod \mathfrak{M}_p\mathfrak{N}$ and $b\mathcal{O}_F$ is prime to $d\mathfrak{M}_p$. There exist infinitely many prime elements $\pi \in \mathcal{O}_F$ which satisfy the following conditions:

(1) *the integer $N(\pi) - 1$ is prime to p;*
(2) *there exists a $\nu \in \mathfrak{N}$ such that $\pi = 1 + \nu$;*
(3) *$\{0, b/d\} = \{0, b/\pi\}$;*
(4) *$N(\pi) - 1 \neq \sharp\mathcal{O}_F^\times$.*

The proofs of Lemmas 3.1 and 3.2 are given at the end of this section.

Proof of Theorem 1.1

By the identity in Lemma 3.1 and Remark 3, it is enough to show that there exist infinitely many Hecke characters of finite order φ on $F_{\mathbf{A}}^\times$ such that $\Phi_f(c_\varphi) \neq 0$. If we suppose $\Phi_f(c_\varphi) = 0$ for almost all φ, we can prove that Φ_f is zero. This contradicts Lemma 3.1. Hence, we show below that $\Phi_f \circ \mathrm{pr}(\overline{\Gamma_1(\mathfrak{N})}) = \{0\}$ assuming that $\Phi_f(c_\varphi) = 0$ for almost all φ.

From Corollary A.2 (see the appendix) and Remark 2(2), to prove $\Phi_f \circ \mathrm{pr}(\overline{\Gamma_1(\mathfrak{N})}) = \{0\}$, it is enough to show that $\Phi_f \circ \mathrm{pr}(\overline{\Gamma_1(\mathfrak{M})}) = \{0\}$ for a suitable integral ideal $\mathfrak{M}$ of $\mathcal{O}_F$ such that $\mathfrak{N}|\mathfrak{M}$. We define $\mathfrak{M} := \mathfrak{M}_p\mathfrak{N}$ and take an arbitrary element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\overline{\Gamma_1(\mathfrak{M})}$. We show that $\Phi_f \circ \mathrm{pr}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = 0$.

Note that we may assume that $b\mathcal{O}_F$ is prime to $\mathfrak{M}_p$. This is easily checked as follows. We take $t \in \mathcal{O}_F$ such that $t\mathcal{O}_F$ is prime to $\mathfrak{M}_p$. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & t-b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+a(t-b) \\ c & d+c(t-b) \end{pmatrix},$$

$\left(\begin{smallmatrix} 1 & t-b \\ 0 & 1 \end{smallmatrix}\right)$ is a parabolic element of $\overline{\Gamma_1(\mathfrak{N})}$, and $b + a(t-b) \equiv t \mod \mathfrak{M}_p$, we may replace b (resp., d) by $b+a(t-b)$ (resp., $d+c(t-b)$). Hence, hereafter we assume that $b\mathcal{O}_F$ is prime to $\mathfrak{M}_p$. Note, moreover, that $b\mathcal{O}_F$ is prime to $d\mathfrak{M}_p$.

Since b and d satisfy the condition of Lemma 3.2, we take a prime element $\pi = 1 + \nu \in \mathcal{O}_F$ as in Lemma 3.2, where ν is an element of $\mathfrak{N}$. We may assume that $\pi\mathcal{O}_F$ is prime to $b\mathcal{O}_F$. For such a prime element $\pi \in \mathcal{O}_F$, we write

$$H_\pi = \{\varphi : F_{\mathbf{A}}^\times \to \mathbf{C}^\times; \varphi : \text{ finite-order Hecke character}, \mathfrak{c}_\varphi := \mathrm{cond}\varphi \mid \pi\mathcal{O}_F\},$$

and we denote $\sharp H_\pi$ by m. Then we have $m \mid N(\pi) - 1$; hence m is invertible in $\overline{\mathbf{F}}_p$. Since there exist infinitely many such prime elements π by Lemma 3.2, we may assume that $\Phi_f(c_\varphi) = 0$ for all $\varphi \in H_\pi \backslash \{1\}$, where 1 denotes the trivial character. Then, we have $\Phi_f(\{0, b/d\}) = \Phi_f(\{0, 1/\pi\})$. In fact, we have

$$\Phi_f\left(\left\{0, \frac{b}{d}\right\}\right) = \Phi_f\left(\left\{0, \frac{b}{\pi}\right\}\right) = \left\{\frac{1}{m}\sum_{\varphi \in H_\pi} \varphi_{\mathfrak{c}_\varphi}(b)\overline{\varphi_{\mathfrak{c}_\varphi}}(b)\right\}\Phi_f\left(\left\{0, \frac{b}{\pi}\right\}\right)$$
$$= \frac{1}{m}\sum_{\varphi \in H_\pi}\sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(\pi u)\overline{\varphi_{\mathfrak{c}_\varphi}}(b)\Phi_f(\{0, u\}).$$

The first equality follows from the fact that π satisfies condition (3) of Lemma 3.2. By Remark 2(3), we may assume that $b/\pi \in R$. The last equality follows from the following property of characters φ for $u \in R \backslash \{b/\pi\}$:

$$\frac{1}{m}\sum_{\varphi \in H_\pi} \varphi_{\mathfrak{c}_\varphi}(\pi u)\overline{\varphi_{\mathfrak{c}_\varphi}}(b) = 0.$$

We note that we have the following identities in $H_1(Y_1(\mathfrak{N})^*, \overline{\mathbf{F}}_p)$ for every nontrivial character φ:

$$\{0, u\} + \{u, \infty\} + \{\infty, 0\} = 0,$$
$$\sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(-\pi u)\{\infty, 0\} = 0,$$
$$-\sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(-\pi u)\{\infty, u\} = \sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(-\pi u)\{u, \infty\}.$$

Hence we have

$$\frac{1}{m}\sum_{\varphi \in H_\pi}\sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(\pi u)\overline{\varphi_{\mathfrak{c}_\varphi}}(b)\Phi_f(\{0, u\})$$
$$= \frac{1}{m}\sum_{\varphi \in H_\pi \backslash \{1\}}\sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(\pi u)\overline{\varphi_{\mathfrak{c}_\varphi}}(b)\Phi_f(\{0, u\}) + \frac{1}{m}\sum_{u \in R}\Phi_f(\{0, u\})$$

$$= \frac{1}{m} \sum_{\varphi \in H_\pi \setminus \{1\}} \varphi_{\mathfrak{c}_\varphi}(-1)\overline{\varphi_{\mathfrak{c}_\varphi}}(b)\Phi_f\Big(\sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(-\pi u)\{u,\infty\}\Big)$$
$$+ \frac{1}{m}\sum_{u\in R} \Phi_f(\{0,u\}).$$

By the definition of c_φ and by the assumption that $\Phi_f(c_\varphi) = 0$, we have that

$$\frac{1}{m} \sum_{\varphi \in H_\pi \setminus \{1\}} \varphi_{\mathfrak{c}_\varphi}(-1)\overline{\varphi_{\mathfrak{c}_\varphi}}(b)\Phi_f\Big(\sum_{u \in R} \varphi_{\mathfrak{c}_\varphi}(-\pi u)\{u,\infty\}\Big) + \frac{1}{m}\sum_{u\in R} \Phi_f(\{0,u\})$$
$$= \frac{1}{m} \sum_{\varphi \in H_\pi \setminus \{1\}} \varphi_{\mathfrak{c}_\varphi}(-1)\overline{\varphi_{\mathfrak{c}_\varphi}}(b)\Phi_f(c_\varphi) + \frac{1}{m}\sum_{u\in R} \Phi_f(\{0,u\})$$
$$= \frac{1}{m}\sum_{u\in R} \Phi_f(\{0,u\}).$$

Thus, we have proved the following identity:

$$\Phi_f\Big(\Big\{0, \frac{b}{d}\Big\}\Big) = \frac{1}{m}\sum_{u\in R} \Phi_f(\{0,u\}).$$

By the same manner as in the proof of the above equality, we also obtain $\Phi_f(\{0, 1/\pi\}) = (1/m)\sum_{u\in R}\Phi_f(\{0,u\})$. Therefore we conclude that $\Phi_f(\{0, b/d\}) = \Phi_f(\{0, 1/\pi\})$.

Now we easily show that $\Phi_f(\{0, b/d\}) = 0$. Since $\left(\begin{smallmatrix} 1 & 0 \\ \nu & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are parabolic elements of $\Gamma_1(\mathfrak{N})$,

$$\Phi_f\Big(\Big\{0, \frac{b}{d}\Big\}\Big) = \Phi_f\Big(\Big\{0, \frac{1}{\pi}\Big\}\Big)$$
$$= \Phi_f\Big(\Big\{0, \frac{1}{1+\nu}\Big\}\Big)$$
$$= \Phi_f \circ \mathrm{pr}\left(\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Thus, $\Phi_f \circ \mathrm{pr}(\overline{\Gamma_1(\mathfrak{M})}) = \{0\}$. This completes the proof of Theorem 1.1. □

Proof of Lemma 3.1

We take an element $\eta_{f,\mathrm{c},0}$ in the inverse image of $\eta_{f,\mathrm{c}}$ under the map $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]) \to H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])'$. We denote by $\eta_{f,0}$ the image of $\eta_{f,\mathrm{c},0}$ under the natural map $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]) \to H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])$. We note that the image of $\eta_{f,0}$ under the map $H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]) \to H^1_{\mathrm{par}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])'$ is equal to η_f up to multiplication of a unit of $\mathcal{O}_{K,\mathfrak{P}}$ by Proposition 2.6.

First we consider the following sequence:

$$0 \to H^1_{\mathrm{par}}\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big) \to H^1\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big).$$

We denote also the image of the above map by $\eta_{f,0}$. Next we consider the following map:

$$\iota : H^1\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big) \to H^1\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big) \otimes_{\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]} \overline{\mathbf{Z}}_p \otimes_{\overline{\mathbf{Z}}_p} \overline{\mathbf{F}}_p.$$

Since we assume that p is prime to $\mathfrak{N}$, $H^1(\partial Y_1(\mathfrak{N})^*, \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])$ is torsion-free over $\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]$ (see [Ur, Proposition 2.4.1]). Thus, the image of $\eta_{f,0}$ by ι is not zero.

Since the image of $\eta_{f,0}$ under the map ι is not zero, the image of $\eta_{f,\mathrm{c},0}$ under the map

$$H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big) \to H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big) \otimes_{\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]} \overline{\mathbf{Z}}_p \otimes_{\overline{\mathbf{Z}}_p} \overline{\mathbf{F}}_p$$

is not zero. By the universal coefficient theorem, we obtain the following injection:

$$0 \to H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0]\big) \otimes_{\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]} \overline{\mathbf{Z}}_p \otimes_{\overline{\mathbf{Z}}_p} \overline{\mathbf{F}}_p \to H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \overline{\mathbf{F}}_p\big).$$

Since $H^3_{\mathrm{c}}(Y_1(\mathfrak{N}), \overline{\mathbf{F}}_p)$ is isomorphic to $\overline{\mathbf{F}}_p$ and the cup product is nondegenerate, we obtain the following injection:

$$0 \to H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \overline{\mathbf{F}}_p\big) \to \mathrm{Hom}_{\overline{\mathbf{F}}_p}\big(H^2(Y_1(\mathfrak{N}), \overline{\mathbf{F}}_p), \overline{\mathbf{F}}_p\big).$$

Finally, by using the image of $\eta_{f,\mathrm{c},0}$ under the above maps, we obtain a map Φ_f by Poincaré's duality (see [Ur, Théorème 1.4,1.6]). By construction, Φ_f is not zero. We note that the map Φ_f does not depend on a choice of pullback of $\eta_{f,c} \in H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])'$ to $H^1_{\mathrm{c}}(Y_1(\mathfrak{N}), \mathcal{O}_{K,\mathfrak{P}}[\varphi_0])$. This follows from the fact that the cup product

$$H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}\big) \otimes_{\mathcal{O}} H^2\big(Y_1(\mathfrak{N}), \mathcal{O}\big) \to \mathcal{O}$$

induces the map

$$H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}\big)' \otimes_{\mathcal{O}} H^2\big(Y_1(\mathfrak{N}), \mathcal{O}\big)' \to \mathcal{O},$$

where M' denotes the largest torsion-free quotient of the $\mathcal{O}$-module M, and that we have the following commutative diagram:

$$\begin{array}{ccccc}
H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}\big) \otimes_{\mathcal{O}} H^2\big(Y_1(\mathfrak{N}), \mathcal{O}\big) & \longrightarrow & H^3_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \mathcal{O}\big) & \xrightarrow{\sim} & \mathcal{O} \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \overline{\mathbf{F}}_p\big) \otimes_{\overline{\mathbf{F}}_p} H^2\big(Y_1(\mathfrak{N}), \overline{\mathbf{F}}_p\big) & \longrightarrow & H^3_{\mathrm{c}}\big(Y_1(\mathfrak{N}), \overline{\mathbf{F}}_p\big) & \xrightarrow{\sim} & \overline{\mathbf{F}}_p
\end{array}$$

where we abbreviate $\mathcal{O}_{K,\mathfrak{P}}[\varphi_0]$ to $\mathcal{O}$.

By Proposition 2.5 and the definition of Φ_f, it is obvious to see that $\Phi_f(c_\varphi)$ satisfies the identity in the statement of the lemma. □

Proof of Lemma 3.2

We denote the ray class group for $\mathfrak{M}_p$ of F by $\mathrm{Cl}_F(\mathfrak{M}_p)$ and take their representative $\mathrm{Cl}_F(\mathfrak{M}_p) = \{\beta_1\mathcal{O}_F, \ldots, \beta_m\mathcal{O}_F\}$, where $\beta_i \in \mathcal{O}_F$ for $i = 1, \ldots, m$. Here we note that we assume the class number of F to be equal to 1.

Since $b\mathfrak{N}$ is prime to $d\mathfrak{M}_p$, we take $\alpha \in \mathfrak{N}$ such that $b\alpha \equiv 1 \mod d\mathfrak{M}_p$. We write $P_j = \{d + b\alpha(\beta_j - 1 + \mu); \mu \in \mathfrak{M}_p\}$. Then there exists an $\pi_j \in P_j$ such that

$\pi_j\mathcal{O}_F$ is a prime ideal. This follows from Chebotarev's density theorem, since $b\alpha\mathfrak{M}_p$ and $d\mathcal{O}_F$ are coprime. We take an $\mu_j \in \mathfrak{M}_p$ such that $\pi_j = d + b\alpha(\beta_j - 1 + \mu_j)$, which satisfies $N(\pi_j) - 1 \neq \sharp\mathcal{O}_F^\times$. Then we show $\pi_j\mathcal{O}_F \sim \beta_j\mathcal{O}_F \in \mathrm{Cl}_F(\mathfrak{M}_p)$. To show this, it is enough to show that $\pi_j \equiv \beta_j \mod \mathfrak{M}_p$:

$$\begin{aligned}\pi_j &= d + b\alpha(\beta_j - 1 + \mu_j)\\ &\equiv 1 + 1\cdot(\beta_j - 1 + 0) \mod \mathfrak{M}_p\\ &\equiv \beta_j.\end{aligned}$$

We denote by $F(\mathfrak{M}_p)$ the ray class field for $\mathfrak{M}_p$ of F. Then $F(\mathfrak{M}_p)$ contains $F(\zeta_p)$. By assumption, we can take $\sigma \in \mathrm{Gal}(F(\mathfrak{M}_p)/F)$ such that $\sigma|_{F(\zeta_p)} \neq \mathrm{id}_{F(\zeta_p)}$. From the class field theory, there exists an isomorphism

$$\mathrm{Art}_{F(\mathfrak{M}_p)/F} : \mathrm{Cl}(\mathfrak{M}_p) \xrightarrow{\sim} \mathrm{Gal}\big(F(\mathfrak{M}_p)/F\big),$$

where we denote the Artin map by $\mathrm{Art}_{F(\mathfrak{M}_p)/F}$. Hence there exists $j \in \{1, \dots, m\}$ such that $\beta_j\mathcal{O}_F = \mathrm{Art}^{-1}_{F(\mathfrak{M}_p)/F}(\sigma)$. Since $\pi_j\mathcal{O}_F \sim \beta_j\mathcal{O}_F \in \mathrm{Cl}_F(\mathfrak{M}_p)$, we have $\mathrm{Art}_{F(\mathfrak{M}_p)/F}(\pi_j\mathcal{O}_F) = \sigma$.

For the above j, we set $\pi := \pi_j$ and $\mathfrak{c} := \pi\mathcal{O}_F$. Then, since $\sigma|_{F(\zeta_p)} \neq \mathrm{id}_{F(\zeta_p)}$, we have the condition (1).

The conditions (2) and (4) are obvious. We verify the condition (3). This follows from the equation

$$\begin{pmatrix} 1 & 0 \\ \alpha(\beta_j - 1 + \mu_j) & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & b \\ * & d + b\alpha(\beta_j - 1 + \mu_j) \end{pmatrix}$$

and the fact that $\left(\begin{smallmatrix} 1 & 0 \\ \alpha(\beta_j - 1 + \mu_j) & 1 \end{smallmatrix}\right)$ is a parabolic element of $\Gamma_1(\mathfrak{N})$.

The existence of infinitely many such π is a consequence of Chebotarev's density theorem. □

Appendix

In the proof of the main theorem, we need the following Corollary A.2, known as Fricke's lemma for the $F = \mathbf{Q}$ case (see [St, Lemma, p. 526]). We generalize this to an arbitrary number field F. Let $\mathfrak{N}$ be an integral ideal of $\mathcal{O}_F$. We fix an integral ideal $\mathfrak{a}$ which is prime to $\mathfrak{N}$. We fix a finite idele a_0 whose associated ideal is $\mathfrak{a}$. We put $t = \left(\begin{smallmatrix} a_0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We define $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R}$. We consider F_∞ as the set of infinite adeles of $F_{\mathbf{A}}$. We define

$$\begin{aligned} K_1(\mathfrak{N}) &= \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\hat{\mathcal{O}}_F); c, d-1 \in \mathfrak{N}\hat{\mathcal{O}}_F\right\},\\ \Gamma_1^{\mathfrak{a}}(\mathfrak{N}) &= \mathrm{GL}_2(F) \cap t\,\mathrm{GL}_2(F_\infty)K_1(\mathfrak{N})t^{-1}\\ &= \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F); a, d \in \mathcal{O}_F, b \in \mathfrak{a}, c \in \mathfrak{a}^{-1}\mathfrak{N},\right.\\ &\qquad \left. d \equiv 1 \mod \mathfrak{N}, ad - bc \in \mathcal{O}_F^\times\right\},\\ \Gamma^{\mathfrak{a}}(\mathfrak{N}) &= \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^{\mathfrak{a}}(\mathfrak{N}); b \in \mathfrak{a}\mathfrak{N}, a \equiv 1 \mod \mathfrak{N}\right\}. \end{aligned}$$

We put $\overline{\Gamma_1^{\mathfrak{a}}(\mathfrak{N})} = \mathrm{SL}_2(F) \cap \Gamma_1^{\mathfrak{a}}(\mathfrak{N})$ and $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})} = \mathrm{SL}_2(F) \cap \Gamma^{\mathfrak{a}}(\mathfrak{N})$.

We note that, for the integral ideal $\mathfrak{a}_i$ and $\Gamma_1^i(\mathfrak{N})$ which are introduced in Section 2.1, we have $\Gamma_1^{\mathfrak{a}_i}(\mathfrak{N}) = \Gamma_1^i(\mathfrak{N})$.

LEMMA A.1

Let $\mathfrak{M}$ be an integral ideal of $\mathcal{O}_F$ such that $\mathfrak{N}|\mathfrak{M}$ and $\mathfrak{M}$ is prime to $\mathfrak{a}$. Then $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$ is generated by $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{M})}$ and parabolic elements of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$.

Proof

Let $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$. It suffices to show that we can get an element of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{M})}$ by multiplying some parabolic elements of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$ to γ. We show this by a four-step argument.

Step 1. First of all, we show that we can assume that $a\mathcal{O}_F$ is coprime to $\mathfrak{a}\mathfrak{M}$. Since $\gamma \in \overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$, $a\mathcal{O}_F$ is prime to $bc\mathcal{O}_F$. By Chebotarev's density theorem, we can find $\alpha \in \mathcal{O}_F$ such that $(a+\alpha bc)\mathcal{O}_F$ is prime to $\mathfrak{a}\mathfrak{M}$. Now we note that $\left(\begin{smallmatrix} 1 & \alpha b \\ 0 & 1 \end{smallmatrix}\right)$ is a parabolic element of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$. We note that

$$\begin{pmatrix} 1 & \alpha b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+\alpha bc & b+\alpha bd \\ c & d \end{pmatrix}.$$

Hence, by replacing $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} a+\alpha bc & b+\alpha bd \\ c & d \end{smallmatrix}\right)$, we may assume that $a\mathcal{O}_F$ is prime to $\mathfrak{a}\mathfrak{M}$.

Step 2. Next we show that we can assume $b \in \mathfrak{a}\mathfrak{M}$. For this purpose, we show that there exists an $\alpha \in \mathfrak{N}$ such that $b + a\alpha \in \mathfrak{a}\mathfrak{M}$. Since $a\mathcal{O}_F$ is prime to $\mathfrak{a}\mathfrak{M}$, we can find a $k \in \mathcal{O}_F$ such that $ak \equiv 1 \mod \mathfrak{a}\mathfrak{M}$. So we see $b - bak \in \mathfrak{a}\mathfrak{M}$. Since $b \in \mathfrak{a}\mathfrak{N}$, for $\alpha := -bk \in \mathfrak{a}\mathfrak{N}$, we have $b + a\alpha \in \mathfrak{a}\mathfrak{M}$. Then, since $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ is a parabolic element of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+a\alpha \\ c & d+c\alpha \end{pmatrix},$$

we may assume that $b \in \mathfrak{a}\mathfrak{M}$.

Step 3. Now we show that we can assume $a \equiv 1 \mod \mathfrak{M}$. We take an element u of $\mathfrak{a}$ such that there exists an ideal $\mathfrak{b}$ such that $u\mathcal{O}_F = \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b}$ is prime to $\mathfrak{a}$. Since $\mathfrak{a}$ is prime to $\mathfrak{M}$ by the assumption of Lemma A.1, we may assume that $\mathfrak{b}$ is prime to $\mathfrak{M}$. Since $a\mathcal{O}_F$ is prime to $\mathfrak{M}$, there exists an element t of $\mathcal{O}_F$ such that $at \equiv 1 \mod \mathfrak{M}$ and $t \equiv 1 \mod \mathfrak{b}$ by the Chinese remainder theorem. In particular, since $a \equiv 1 \mod \mathfrak{N}$, we have $t \equiv 1 \mod \mathfrak{N}$. For the above $t \in \mathcal{O}_F$, we have $u(t-1) \in \mathfrak{a}\mathfrak{N}$ and $u^{-1}(1-t) \in \mathfrak{a}^{-1}\mathfrak{N}$. Then it is easy to see that $\left(\begin{smallmatrix} t & u(t-1) \\ u^{-1}(1-t) & -t+2 \end{smallmatrix}\right)$ is a parabolic element of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$. We note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & u(t-1) \\ u^{-1}(1-t) & -t+2 \end{pmatrix} = \begin{pmatrix} at+bu^{-1}(1-t) & au(t-1)+b(-t+2) \\ ct+du^{-1}(1-t) & cu(t-1)+d(-t+2) \end{pmatrix}.$$

Since $b \in \mathfrak{a}\mathfrak{M}$, we have

$$at + bu^{-1}(1-t) \equiv 1 \mod \mathfrak{M}.$$

Hence, by replacing $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} at+bu^{-1}(1-t) & au(t-1)+b(-t+2) \\ ct+du^{-1}(1-t) & cu(t-1)+d(-t+2) \end{smallmatrix}\right)$, we may assume that $a \equiv 1 \mod \mathfrak{M}$. However, after this replacement, we might lose the condition $b \in \mathfrak{a}\mathfrak{M}$.

Step 4. By the assumption $a \equiv 1 \mod \mathfrak{M}$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-ab \\ c & d-bc \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c-ac & d-bc \end{pmatrix};$$

we conclude the lemma. □

Since any element of $\overline{\Gamma_1^{\mathfrak{a}}(\mathfrak{N})}$ is transformed to an element of $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{N})}$ by multiplying a certain unipotent element of $\overline{\Gamma_1^{\mathfrak{a}}(\mathfrak{N})}$, we have the following corollary from Lemma A.1.

COROLLARY A.2
Notation is the same as in the above lemma. Then $\overline{\Gamma_1^{\mathfrak{a}}(\mathfrak{N})}$ *is generated by* $\overline{\Gamma^{\mathfrak{a}}(\mathfrak{M})}$ *and parabolic elements of* $\overline{\Gamma_1^{\mathfrak{a}}(\mathfrak{N})}$.

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