Seidel's long exact sequence on Calabi-Yau manifolds

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Abstract In this paper, we generalize construction of Seidel's long exact sequence of Lagrangian Floer cohomology to that of compact Lagrangian submanifolds with vanishing Malsov class on general Calabi-Yau manifolds. We use the framework of anchored Lagrangian submanifolds and some compactness theorem of *smooth* J-holomorphic sections of Lefschetz Hamiltonian fibration for a generic choice of J. The proof of the latter compactness theorem involves a study of proper pseudoholomorphic curves in the setting of noncompact symplectic manifolds with cylindrical ends.

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1. Introduction

To put the content of this paper in perspective, we first recall a long exact sequence for symplectic Floer cohomology of Lagrangian submanifolds, which was constructed by Seidel [Se3] originally for the category of exact Lagrangian submanifolds on (noncompact) exact symplectic manifolds.

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1.1. Dehn twists and Seidel's long exact sequence

Let (M, ω, α) be an exact symplectic manifold with contact-type boundary: α is a contact 1-form on ∂M which satisfies $d\alpha = \omega|_{\partial M}$ and makes ∂M convex. Assume that $[\omega, \alpha] \in H^2(M, \partial M; \mathbb{R})$ is zero so that α can be extended to a 1-form θ on M satisfying $d\theta = \omega$.

THEOREM 1.1 (SEIDEL [Se3])

Let L be an exact Lagrangian sphere in M together with a preferred diffeomorphism $f : S^2 \to L$. Denote by $\tau_L = \tau_{(L,[f])}$ the Dehn twist associated to $(L, [f])$. For any two compact exact Lagrangian submanifolds $L_0, L_1 \subset M$, there is a long exact sequence of Floer cohomology groups

$$\longrightarrow HF\big(\tau_L(L_0), L_1\big) \longrightarrow HF(L_0, L_1) \longrightarrow HF(L, L_1) \otimes HF(L_0, L) \longrightarrow . \tag{1.1}$$

Due to the well-known difficulties in the construction of Lagrangian intersection Floer cohomology and a new nontrivial compactness issue arising from the singularities of the Lefschetz fibration used for the construction, Seidel used the exactness assumption to avoid these difficulties and work entirely with the exact Lagrangian category, and he left the extension to the more general situation, such as that for closed Calabi-Yau manifolds, as an open problem (see [Se3]). While the limitation to the exact Lagrangian category simplifies the analysis of holomorphic disc bubbles, it also forces him to work entirely with the language of *exact Lefschetz fibrations* and to make sure that he does not go out of this domain largely for the consistency of his exposition, as Seidel himself indicated. Because of this, [Se3] develops a fair amount of geometry of exact Lefschetz fibrations, some of which are not directly relevant to the construction of the long exact sequence. Partly due to this digression, it took some effort and time for the author of this paper to get to the main point of Seidel's construction in [Se3].

The cases of closed Calabi-Yau manifolds or Fano toric manifolds are the ones that are physically most relevant to mirror symmetry. According to Kontsevich [K2] and Seidel [Se3], the symplectic Dehn twists correspond to a particular class of autoequivalences, "twist functors along spherical objects," of derived categories of coherent sheaves, and this long exact sequence corresponds to an exact sequence of the same form in the mirror Calabi-Yau. Therefore it is important to establish the long exact sequence for a class of Lagrangian branes that is *closed* under the action of symplectic Dehn twists. The class of exact Lagrangian submanifolds in exact symplectic manifolds is one such class, which Seidel considered in [Se3].

One of the points Seidel tried to ensure by working with the exact Lagrangian category is to have not only single-valuedness of the action functional on the path space but *coherence* of the definition of the action functional between different exact Lagrangian submanifolds: this then allows one to have the energy estimate for the Floer trajectories and, more importantly, to have *tiny-big decomposition* of the Floer moduli spaces entering in the construction. This decomposition then enables him to apply the spectral sequence argument and derive the desired conclusion based on the contribution coming from the *tiny* part of the Floer moduli spaces which can be explicitly analyzed.

1.2. Calabi-Yau Lagrangian branes

In regard to extending Seidel's construction to closed Calabi-Yau manifolds, we highlight two points that we take in this paper.

The first point is our restriction to the class of Lagrangian submanifolds with zero Maslov class. This class is *closed* under the action by symplectic Dehn twists and enables one to consider the involved cohomology as a $\mathbb{Z}$*-graded group* which is essential in the point of view of mirror symmetry. It appears to the author that for this kind of long exact sequence to exist some condition of this sort of Calabi-Yau property is needed (see the remark at the end of Section 7.4 for the reason).

The second point is the usage of the notion of *anchors* and *anchored Lagrangian submanifolds* introduced in [FO+2]. We recall the definition of anchored Lagrangian submanifolds introduced in [FO+2].

DEFINITION 1.1

Fix a base point y of an ambient symplectic manifold (M,ω). Let L be a Lagrangian submanifold of (M,ω). We define an *anchor* of L to y as a path $\gamma:[0,1]\to M$ such that

$$\gamma(0)=y, \quad \gamma(1)\in L.$$

We call a pair (L,γ) an *anchored* Lagrangian submanifold.

This notion has its origin in the preprint [FO+3] when the authors take the *based point of view* of Lagrangian submanifolds in relation to the coherence of the definitions of various Maslov-type indices and of action functionals when one considers several Lagrangian submanifolds altogether as one studies the Fukaya category. From a technical point of view, consideration of anchored Lagrangian submanifolds enables one to maintain the consistency of the definitions both of action functionals and of the absolute gradings on the Calabi-Yau Lagrangian branes. Most importantly, this also enables us to provide a coherent filtration in the relevant Floer complexes and to have tiny-big decomposition of the relevant Floer moduli spaces of the kind Seidel considered in [Se3].

Now we introduce a class of decorated Lagrangian submanifolds on Calabi-Yau manifold (M,ω) which we call *Calabi-Yau Lagrangian branes*. It is expected that this class of Lagrangian submanifolds generates the Fukaya category of a Calabi-Yau manifold that is mirror to the derived category of coherent sheaves on the mirror Calabi-Yau. We refer readers to the main part of the paper for various undefined terms in the statement. We also omit the important datum of flat line bundles on L in this definition because it does not play much of a role in our proof but can be easily incorporated in the construction.

DEFINITION 1.2

Let $y \in M$ be a base point, and let $\Lambda_y \subset T_yM$ be a fixed Lagrangian subspace. Suppose that Θ is a quadratic complex volume form on (M, ω, J) with $\langle \Theta(y), \Lambda_y \rangle = 1$. We consider the quadruple $((L, \gamma), s, [b])$, which we call an (*anchored*) *Calabi-Yau Lagrangian brane*, that satisfies the following data:

(1) L is a Lagrangian submanifold of M such that the Maslov index of L is zero and $[\omega] \in H^2(M, L; \mathbb{Z})$; we also enhance L with a flat complex line bundle on it;

(2) γ is an anchor of L relative to y;

(3) s is a spin structure of L;

(4) $[b] \in \mathcal{M}(L)$ is a bounding cochain described in Section 8.2.

We denote by $\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$ the collection of Calabi-Yau Lagrangian branes and define $\mathrm{Fuk}(\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}})$ to be the Fukaya category generated by $\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$.

We remark that the notion of an *anchor* to L is introduced to solve the problems of grading and filtration on the Floer complex in a uniform way in [FO+2]. In particular, it provides a canonical filtration on the associated Floer complex of anchored Lagrangian submanifolds which is needed to apply some spectral sequence argument in the proof (see the end of Section 10 in particular).

1.3. Statement of the main result and compactness issue

The main purpose of this paper is to construct an exact sequence for the Calabi-Yau Lagrangian branes on Calabi-Yau manifolds which is the analogue to that of Seidel [Se3].

We first note that each Dehn twist τ_L along a given Lagrangian sphere $L \subset M$ acts on $\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$. We denote this action by

$$(\tau_L)_* : \mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}} \to \mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$$

and the image of $\mathcal{L}$ under this action by $\tau_L(\mathcal{L}) = (\tau_L)_*\mathcal{L}$. This action defines an autoequivalence on $\mathrm{Fuk}(\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}})$, whose nonanchored versions should correspond to twist functors along spherical objects of derived categories of coherent sheaves alluded to in the beginning of this introduction.

THEOREM 1.2

Let (M, ω) be a compact (symplectic) Calabi-Yau manifold, and let $y \in M$ be a base point. Let $L \subset M$ be a Lagrangian sphere together with a preferred diffeomorphism $f : S^n \to L$, and let $\mathcal{L} = ((L, \gamma), s_{st}, 0)$ be the associated Calabi-Yau Lagrangian brane. Denote by $\tau_L = \tau_{(L,[f])}$ the Dehn twist associated to $(L, [f])$.

Consider any Calabi-Yau Lagrangian branes $\mathcal{L}_0, \mathcal{L}_1$. Then there is a long exact sequence of $\mathbb{Z}$-graded Floer cohomologies

$$\longrightarrow HF\big((\tau_L)_*\mathcal{L}_0, \mathcal{L}_1\big) \longrightarrow HF(\mathcal{L}_0, \mathcal{L}_1) \longrightarrow HF(\mathcal{L}, \mathcal{L}_1) \otimes HF(\mathcal{L}_0, \mathcal{L}_1) \longrightarrow \tag{1.2}$$

as a Λ_{nov}-module, where the Floer cohomologies involved are the deformed Floer cohomologies constructed in [FO+1].

We also have the long exact sequence for the nonanchored version of the Floer cohomology (see Section 11.2).

Once we use these frameworks, construction of the long exact sequence largely follows Seidel's strategy: we use the framework of Lefschetz fibration with Lagrangian boundary conditions for the construction of various operators appearing in the Floer theory, and we use the spectral sequence for the $\mathbb{R}$-filtered groups based on the tiny-big decomposition of the Floer moduli spaces. However, unlike the exact Lagrangian case, the definition of Lagrangian Floer cohomology for Calabi-Yau Lagrangian branes meets obstruction, as described in [FO+1]. Because of this we have to use the Maurer-Cartan elements b_i and use the associated *deformed Floer cohomology* appearing in the statement of the main theorem (Theorem 1.2). (Since Lagrangian submanifolds with zero Maslov class in Calabi-Yau manifolds are *semipositive*, the related transversality issue is relatively standard, which is one of the advantages of considering this class of Lagrangian submanifolds.) For the reader's convenience and the readability of the paper, we borrow a fair amount of material from [FO+1] in our exposition. For the same reason, we also borrow much exposition from [Se3] and refer to the two for further details. In a way, most of the materials used in this paper are not new but have already been present in the literature in one way or another. We organize them in a coherent way to be able to construct the required long exact sequence. Familiarity with the scheme in [Se3] would be useful for the readers to follow the stream of the arguments used in this paper, especially those presented in Sections 9–11.

However, there is one nontrivial analytical issue that needs to be overcome. This concerns the issue of compactification of the *smooth* pseudoholomorphic section of *Lefschetz (Hamiltonian) fibration* when the fibration has nonempty critical fibers. By the definition of Lefschetz Hamiltonian fibration given in Definition 5.1, any smooth section avoids critical points of the fibration. However, a priori a sequence of smooth sections may approach critical points *if the derivatives of the sections in the sequence blow up*. When applied to a sequence of pseudoholomorphic sections, the bubble could touch the critical points. Therefore to define the relative Gromov-Witten-type invariants in the Lefschetz fibration setting, one should study the behavior of pseudoholomorphic sections approaching the critical points. This compactness result may be mathematically the most novel part of this paper and is performed in Section 7.

In this regard, we prove the following.

THEOREM 1.3 (THEOREM 7.6)

Let $\pi : E \to \Sigma$ be a Lefschetz Hamiltonian fibration with Lagrangian boundary $Q \subset E_{\partial\Sigma}$ such that E is fiberwise Calabi-Yau and Q has vanishing fiberwise Maslov class. Then there exists a dense subset of j-compatible J's for which we have a constant $C > 0$ depending only on (E, J, j) and the section class $B \in \pi_2^{\mathrm{sec}}(E, Q)$, but independent of s, such that we have

$$\operatorname{dist}(\operatorname{Im} s, E^{\mathrm{crit}}) \geq C \tag{1.3}$$

for any smooth *section* $s : \Sigma \to E$ *with* $[s, \partial s] \in B \in \pi_2^{\mathrm{sec}}(E, Q)$.

The proof of this theorem turns out to involve the compactification and the Fredholm theory in the setting of symplectic field theory (see [EGH]) in which we regard a bubble touching a critical point x_0 as a proper pseudoholomorphic curve on $\mathbb{C}$ in a punctured fiber $E_{z_0} \setminus \{x_0\}$ (see [FO+4], [OZ] for relevant studies of such compactification and Fredholm theory). This theorem may not hold without assuming that E is fiberwise Calabi-Yau and that Q has vanishing relative Maslov class (see the end of Section 7.4 for the reason).

Once this theorem is established, study of compactification of smooth pseudoholomorphic sections in the current case is essentially the same as the case of smooth Hamiltonian fibrations as studied in [En] and [MS].

The result in this paper was first announced at Eliashberg's 60th birthday conference "New Challenges and Perspectives in Symplectic Field Theory" held at Stanford University June 25–29, 2007 and then presented in various seminars and in conferences afterward. We apologize to readers for the long delay in appearance.

2. Basic facts on symplectic Dehn twists

In this section, we summarize basic facts on the Dehn twists in the symplectic point of view which Seidel [Se1], [Se3] extensively studied in a series of papers. We borrow the basic facts on the symplectic Dehn twists from them with a slight variation of the exposition that is necessary for the purpose of this paper.

Assume that $L \subset (M, \omega)$ is an embedded Lagrangian sphere together with an equivalence class $[f]$ of diffeomorphisms $f : S^n \to L$: two f_1, f_2 are equivalent if and only if $f_2^{-1} f_1$ can be deformed inside $\mathrm{Diff}(S^n)$ to an element of $O(n+1)$. To any such $(L, [f])$ Seidel associates a Dehn twist $\tau_L = \tau_{(L,[f])} \in \mathrm{Symp}(M)$ using a *model Dehn twist* on the cotangent bundle $T := T^*S^n$. Let $f : S^n \to L \subset M$ be a representative of the equivalence class $[f]$. Denote by $T(r) \subset T^*S^n$ the disc bundle of radius r in terms of the standard metric on the unit sphere $S^n = S^n(1) \subset \mathbb{R}^{n+1}$. Identifying $T = TS^n$ with respect to the standard metric, one considers the map

$$\sigma_t(u, v) = \Big(\cos(t)u - \sin(t)\|u\|v, \cos(t)v + \sin(t)\frac{u}{\|u\|}\Big) \subset \mathbb{R}^{n+1} \times S^n \tag{2.1}$$

for $0 < t < \pi$; σ_π is the antipodal involution $A(u, v) = (-u, -v)$. Next we fix a function $R \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\begin{aligned} &\operatorname{supp} R \subset T(1), \\ &R(-t) = R(t) - t \quad \text{for } |t| \le \frac{1}{2}. \end{aligned} \tag{2.2}$$

Then we consider the rescaled function

$$R_\lambda(t) = \lambda R\Big(\frac{t}{\lambda}\Big) \tag{2.3}$$

for all $0 < \lambda \leq 1$. Then R_λ is supported in $T(\lambda)$ and satisfies

$$R_\lambda(-t) = R_\lambda(t) - t \quad \text{for } |t| \leq \frac{\lambda}{2}. \tag{2.4}$$

Insertion of 1-parameter λ in our choice of R is deliberate and is later explicitly related to the parameter that enters in the Lagrangian surgery.

The following lemma is a slight variation of [Se3, Lemma 1.8], whose proof is referred thereto.

LEMMA 2.1

Let $\mu : T \setminus T(0) \to \mathbb{R}$ be the length function $\mu(u, v) = \|v\|$, and let $H_\lambda = R_\lambda \circ \mu$ on $T \setminus T(0)$. Then $\phi^{2\pi}_{H_\lambda}$ extends smoothly over $T(0)$ to a symplectic diffeomorphism ϕ_λ of T. The function $K_\lambda = 2\pi(R'_\lambda \circ \mu - R \circ \mu)$ also extends smoothly over $T(0)$ and satisfies

$$\phi^*_\lambda \theta_T - \theta_T = dK_\lambda.$$

These ϕ_λ are called model Dehn twists.

The model Dehn twists, denoted by τ_λ, have the explicit formula

$$\tau_\lambda(y) = \begin{cases} \sigma_{2\pi R'_\lambda(\mu(y))}(y), & y \in T(\lambda) \setminus T(0), \\ A(y), & y \in T(0), \end{cases} \tag{2.5}$$

where the angle of rotation goes from $2\pi R'_\lambda(0) = \pi$ to $2\pi R'_\lambda(\lambda) = 0$. Note that as $\lambda \to 0$, we have

$$2\pi R'_\lambda\big(\mu(y)\big) = 2\pi R'\Big(\frac{\mu(y)}{\lambda}\Big)$$

and it changes from $2\pi R'(0) = \pi$ to $2\pi R'(1) = 0$.

Now we take a Darboux-Weinstein chart, or a symplectic embedding $\iota : T(\lambda) \to M$ such that

$$\iota|_{o_{T^*S^n}} = f, \qquad \iota^*\omega = \omega_T (= -d\theta_T),$$

for a representative of the framed Lagrangian sphere $(L, [f])$.

Take a model Dehn twist τ supported in the interior of $T(1) \subset T^*S^n$.

We denote $U = \operatorname{im} \iota$ and fix the Darboux neighborhood once and for all, and we consider the 1-parameter family of Dehn twists τ_r, any of which we denote by τ_L:

$$\tau_L = \tau_{(L,[f];r)} = \begin{cases} \iota \circ \tau_r \circ \iota^{-1} & \text{on } \operatorname{im}(\iota) = U, \\ \mathrm{id} & \text{elsewhere}. \end{cases} \tag{2.6}$$

We quote the following basic fact on the Dehn twist $\tau_{(L,[f])}$ from [Se3] with a slight variation of the statements.

PROPOSITION 2.2 ([Se3, PROPOSITION 1.11])

Let $(L, [f])$ be a framed Lagrangian sphere in M. There is a 1-parameter family of

Lefschetz fibrations $(E^L_\lambda, \pi^L_\lambda) \to D(\lambda)$ *together with an isomorphism* $\phi^L_\lambda : E^L_\lambda \to M$ *of symplectic manifolds, such that we have the following.*

(1) *Consider the rescaling map* $R_\lambda : D(\lambda) \to D(1)$ *defined by* $z \mapsto z/\lambda$. *Then*

$$(R_\lambda)^* E^L_1 = E^L_\lambda.$$

(2) *If* ρ^L_λ *is the symplectic monodromy around* $\partial \overline{D}(\lambda)$, *then* $\phi^L_\lambda \circ \rho^L \circ (\phi^L_\lambda)^{-1}$ *is a Dehn twist along* $(L, [f])$.

(3) *There exists a decomposition*

$$E^L = E \cup \overline{D}(\lambda) \times \big(M \setminus \iota(T(\lambda)) \setminus V\big)$$

such that E *is the standard fibration* $q : \mathbb{C}^{n+1} \to \mathbb{C}$ *defined by*

$$q(z_1, \ldots, z_{n+1}) = z_1^2 + \cdots + z_{n+1}^2.$$

We denote any of these maps by τ_L.

An important point on this Dehn twist $\tau_{(L,[f];\lambda)}$ is that its support can be put into a Darboux neighborhood of the given Lagrangian sphere L which can be made as close as L by choosing $\lambda > 0$ small, whose derivative can be controlled. One can choose R such that for some $\delta > 0$ we have

$$R'(t) \geq 0 \quad \text{for all } t \geq 0, \tag{2.7}$$

$$R''(t) < 0 \quad \text{for all } t \geq 0 \text{ such that } R'(t) \geq \delta \tag{2.8}$$

and then consider R_λ for any sufficiently small $\lambda > 0$. According to Seidel's terminology, the corresponding Dehn twist is *δ-wobbly*.

3. Action, grading, and anchored Lagrangian submanifolds

In this section, we consider the general Lagrangian submanifolds treated as in [FO+1]. For the fine chain level analysis of the Floer complex, it is essential to analyze the $\mathbb{R}$-*filtration* on $CF(L_0, L_1)$ that is provided by the action functional $\mathcal{A}$ on $\Omega(L_0, L_1)$. This action functional is not single valued on $\Omega(L_0, L_1)$ itself, even for the pair (L_0, L_1) of Calabi-Yau Lagrangian branes, but single valued only on some covering space. For the purpose of studying the Fukaya category and performing various constructions in the Floer homology in a coherent manner, we need to consider a whole collection of Lagrangian submanifolds and assign these auxiliary data to each pair of the given collection in a consistent way. For this purpose, [FO+1] and [FO+2] use the notion of *anchored* Lagrangian submanifolds. This auxiliary data is important later for consistency of definition of action functionals and in turn for the analysis of *tiny-big* decomposition of the various Floer moduli spaces entering in the construction of boundary map, chain map, chain homotopy, and pants products.

3.1. Novikov covering space

We consider the space of paths

$$\Omega = \Omega(L_0, L_1) = \{\ell : [0,1] \to P \mid \ell(0) \in L_0, \ell(1) \in L_1\}.$$

On this space, we are given the *action 1-form* α defined by

$$\alpha(\ell)(\xi) = \int_0^1 \omega\big(\dot{\ell}(t), \xi(t)\big)\, dt$$

for each tangent vector $\xi \in T_\ell \Omega$. From this expression, it follows that

$$\mathrm{Zero}(\alpha) = \big\{\ell_p : [0,1] \to M \;\big|\; p \in L_0 \cap L_1, \ell_p \equiv p\big\}.$$

Using the Lagrangian property of (L_0, L_1), a straightforward calculation shows that this form is *closed*. Note that $\Omega(L_0, L_1)$ is not connected but has countably many connected components. We work on a particular fixed connected component of $\Omega(L_0, L_1)$. We pick up a based path $\ell_0 \in \Omega(L_0, L_1)$ and consider the corresponding component $\Omega(L_0, L_1; \ell_0)$; then we define a covering space

$$\pi : \widetilde{\Omega}(L_0, L_1; \ell_0) \to \Omega(L_0, L_1; \ell_0)$$

on which we have a single-valued action functional such that $d\mathcal{A} = -\pi^*\alpha$.

The base path ℓ_0 automatically picks out a connected component from each of L_0 and L_1 as its initial and final points $x_0 = \ell_0(0) \in L_0$, $x_1 = \ell_0(1) \in L_1$. Then $\Omega(L_0, L_1; \ell_0)$ is a subspace of the space of paths between the corresponding connected components of L_0 and L_1, respectively. Because of this *we always assume that L_0, L_1 are connected from now on*, unless otherwise stated.

Next we describe *the Novikov covering* of the component $\Omega(L_0, L_1; \ell_0)$ of $\Omega(L_0, L_1)$. We first start by describing the universal covering space of $\Omega(L_0, L_1; \ell_0)$. Consider the set of all pairs (ℓ, w) such that $\ell \in \Omega(L_0, L_1)$ and such that $w : [0,1]^2 \to M$ satisfies the boundary condition

$$\begin{cases} w(0,\cdot) = \ell_0, w(1,\cdot) = \ell, \\ w(s,0) \in L_0, w(s,1) \in L_1 \quad \text{for all } s \in [0,1]. \end{cases}$$

Considering w as a continuous path $s \mapsto w(s,\cdot)$ in $\Omega(L_0, L_1; \ell_0)$ from ℓ_0 and ℓ, the fiber at ℓ of the universal covering space of $\Omega(L_0, L_1; \ell_0)$ can be represented by the set of path homotopy classes of w relative to its end $s = 0, 1$.

3.2. The Γ-equivalence

We define a covering space of $\Omega(L_0, L_1; \ell_0)$ by modding out the space of paths in $\Omega(L_0, L_1; \ell_0)$ by another equivalence relation that is weaker than the homotopy. The deck transformation group of this covering space is shown to be *abelian* by construction.

Note that when we are given two pairs (ℓ, w) and (ℓ, w') from $\Omega(L_0, L_1; \ell_0)$, the concatenation

$$\overline{w} \# w' : [0,1] \times [0,1] \to M$$

defines a loop $c: S^1 \to \Omega(L_0, L_1; \ell_0)$. One may regard this loop as a map

$$C: S^1 \times [0,1] \to M \tag{3.1}$$

satisfying the boundary condition $C(s,0) \in L_0$, $C(s,1) \in L_1$. Obviously the symplectic area of C, denoted by

$$I_\omega(c) = \int_C \omega,$$

depends only on the homotopy class of C satisfying (3.1) and so defines a homomorphism on $\pi_1(\Omega(L_0, L_1; \ell_0))$, which we also denote by

$$I_\omega : \pi_1(\Omega(L_0, L_1), \ell_0) \to \mathbb{R}.$$

Next we note that for the map $C: S^1 \times [0,1] \to M$ satisfying (3.1), it associates a symplectic bundle pair $(\mathcal{V}, \lambda)$ defined by

$$\mathcal{V}_C = C^*TM, \lambda_C = c_0^*TL_0 \sqcup c_1^*TL_1,$$

where $c_i : S^1 \to L_i$ is the map given by $c_i(s) = C(s,i)$ for $i = 0,1$. This allows us to define another homomorphism

$$I_\mu : \pi_1(\Omega(L_0, L_1), \ell_0) \to \mathbb{Z}, \qquad I_\mu(c) = \mu(\mathcal{V}_C, \lambda_C),$$

where $\mu(\mathcal{V}_C, \lambda_C)$ is the Maslov index of the bundle pair $(\mathcal{V}_C, \lambda_C)$.

Using the homomorphisms I_μ and I_ω, we define an equivalence relation $\sim$ on the set of all pairs (ℓ, w) satisfying (3.1). For given such pair w, w', we denote by $\overline{w}\#w'$ the concatenation of $\overline{w}$ and w' along ℓ, which defines a loop in $\Omega(L_0, L_1; \ell_0)$ based at ℓ_0.

DEFINITION 3.1

We say that (ℓ, w) is Γ-equivalent to (ℓ, w') and write $(\ell, w) \sim (\ell, w')$ if the following conditions are satisfied: $I_\omega(\overline{w}\#w') = 0 = I_\mu(\overline{w}\#w')$. We denote the set of equivalence classes $[\ell, w]$ by $\widetilde{\Omega}(L_0, L_1; \ell_0)$ and call it the *Novikov covering space*.

There is a canonical lifting of $\ell_0 \in \Omega(L_0, L_1; \ell_0)$ to $\widetilde{\Omega}(L_0, L_1; \ell_0)$: this is just $[\ell_0, \widetilde{\ell}_0] \in \widetilde{\Omega}(L_0, L_1; \ell_0)$, where $\widetilde{\ell}_0$ is the map $\widetilde{\ell}_0 : [0,1]^2 \to M$ with $\widetilde{\ell}_0(s,t) = \ell_0(t)$. In this way, $\widetilde{\Omega}(L_0, L_1; \ell_0)$ also has a natural base point which we suppress from the notation.

We denote by $\Pi(L_0, L_1; \ell_0)$ the group of deck transformations of the covering space $\widetilde{\Omega}(L_0, L_1; \ell_0) \to \Omega(L_0, L_1; \ell_0)$. It is easy to see that the isomorphism class of $\Pi(L_0, L_1; \ell_0)$ depends only on the connected component containing ℓ_0.

The two homomorphisms I_ω and I_μ push down to homomorphisms

$$E : \Pi(L_0, L_1; \ell_0) \to \mathbb{R}, \qquad \mu : \Pi(L_0, L_1; \ell_0) \to \mathbb{Z}$$

defined by

$$E(g) = I_\omega[C], \qquad \mu(g) = I_\mu[C]$$

for any map $C: S^1 \times [0,1] \to M$ representing the class $g \in \Pi(L_0, L_1; \ell_0)$. By construction, it follows that the group $\Pi(L_0, L_1; \ell_0)$ is an abelian group. We now define the Novikov ring $\Lambda(L_0, L_1; \ell_0)$ associated to the abelian covering $\widetilde{\Omega}(L_0, L_1; \ell_0) \to \Omega(L_0, L_1; \ell_0)$ as a completion of the group ring $R[\Pi(L_0, L_1; \ell_0)]$. Here R is a commutative ring with unit.

DEFINITION 3.2

$\Lambda_k^R(L_0, L_1; \ell_0)$ denotes the set of all (infinite) sums

$$\sum_{\substack{g\in\Pi(L_0,L_1;\ell_0)\\ \mu(g)=k}} a_g[g]$$

such that $a_g \in R$ and, for each C, the set $\{g \in \Pi(L_0, L_1; \ell_0) \mid E(g) \le C, a_g \neq 0\}$ is of finite order. We put

$$\Lambda^R(L_0, L_1; \ell_0) = \bigoplus_k \Lambda_k^R(L_0, L_1; \ell_0).$$

The ring structure on $\Lambda^R(L_0, L_1; \ell_0)$ is defined by the convolution product

$$\Big(\sum_{g\in\Pi(L_0,L_1;\ell_0)} a_g[g]\Big)\cdot\Big(\sum_{g\in\Pi(L_0,L_1;\ell_0)} b_g[g]\Big) = \sum_{g_1,g_2\in\Pi(L_0,L_1;\ell_0)} a_{g_1}b_{g_2}[g_1g_2].$$

It is easy to see that the term in the right-hand side is indeed an element in $\Lambda^R(L_0, L_1; \ell_0)$. Thus $\Lambda^R(L_0, L_1; \ell_0) = \bigoplus_k \Lambda_k^R(L_0, L_1; \ell_0)$ becomes a graded ring under this multiplication. We call this graded ring the *Novikov ring* associated to the pair (L_0, L_1) and the connected component containing ℓ_0.

We also use the universal Novikov ring $\Lambda_{\rm nov}$ in this paper. We recall its definition here. An element of $\Lambda_{\rm nov}$ is a formal sum $\sum a_i T^{\lambda_i} e^{\mu_i}$ with $a_i \in \mathbb{C}$, $\lambda_i \in \mathbb{R}$, $\mu_i \in \mathbb{Z}$ such that $\lambda_i \le \lambda_{i+1}$ and $\lim_{i\to\infty} \lambda_i = \infty$. T and e are formal parameters. We define a valuation $\mathfrak{v} : \Lambda_{\rm nov} \to \mathbb{R}_{\ge 0}$ defined by

$$\mathfrak{v}\Big(\sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i}\Big) = \lambda_1.$$

We denote the corresponding valuation ring by

$$\Lambda_{0,\rm nov} = \Big\{\sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{\rm nov} \;\Big|\; \lambda_i \ge 0\Big\}.$$

It carries a unique maximal ideal consisting of $\sum a_i T^{\lambda_i} e^{\mu_i}$ with $\lambda_i > 0$ for all i which we denote by $\Lambda_{0,\rm nov}^+$. We have a natural embedding

$$\Lambda^R(L_0, L_1; \ell_0) \to \Lambda_{\rm nov}$$

given by

$$\sum_{g\in\Pi(L_0,L_1;\ell_0)} b_g[g] \mapsto \sum_{g\in\Pi(L_0,L_1;\ell_0)} b_g T^{\omega(g)} e^{\mu(g)/2}. \tag{3.2}$$

Now for a given pair (ℓ, w), we define the *action functional*

$$\mathcal{A} : \widetilde{\Omega}(L_0, L_1; \ell_0) \to \mathbb{R}$$

by the formula

$$\mathcal{A}(\ell, w) = \int w^*\omega.$$

It follows from the definition of $\Pi(L_0, L_1; \ell_0)$ that the integral depends only on the Γ-equivalence class $[\ell, w]$ and so pushes down to a well-defined functional on the covering space $\widetilde{\Omega}(L_0, L_1; \ell_0)$.

We call an intersection point *admissible to* ℓ_{01} when the associated constant path lies in the same connected component $\Omega(L_0, L_1; \ell_{01})$ as ℓ_{01} in $\Omega(L_0, L_1)$.

LEMMA 3.1

The set $\mathrm{Cr}(L_0, L_1; \ell_0)$ *of admissible critical points of* $\mathcal{A}$ *consists of the pairs of the type* $[\ell_p, w]$, *where* ℓ_p *is the constant path with* $p \in L_0 \cap L_1$ *and* w *is as in (2.2).* $\mathrm{Cr}(L_0, L_1; \ell_0)$ *is invariant under the action of* $\Pi(L_0, L_1; \ell_0)$ *and so forms a principal bundle over a subset of* $L_0 \cap L_1$ *with its fiber isomorphic to* $\Pi(L_0, L_1; \ell_0)$.

We put

$$\mathrm{Cr}(L_0, L_1) = \bigcup_{\ell_{0,i}} \mathrm{Cr}(L_0, L_1; \ell_{0,i}),$$

where $\ell_{0,i}$ runs over the set of base points of connected components of $\Omega(L_0, L_1)$.

Next we assign an *absolute* Morse index to each critical point of $\mathcal{A}$. In general, assigning such an absolute index is not a trivial matter because the obvious Morse index of $\mathcal{A}$ at any critical point is infinite. For this purpose, we use the Maslov index of a certain bundle pair naturally associated to the critical point $[\ell_p, w] \in \mathrm{Cr}(L_0, L_1; \ell_0)$.

We call this Morse index of $[\ell_p, w]$ the *Maslov-Morse index* (relative to the base path ℓ_0) of the critical point. The definition of the index somewhat resembles that of $\mathcal{A}$. However, to define this we also need to fix a section λ^0 of $\ell_0^*\Lambda(M)$ such that

$$\lambda^0(0) = T_{\ell_0(0)}L_0, \qquad \lambda^0(1) = T_{\ell_0(1)}L_1.$$

Here $\Lambda(M)$ is the bundle of Lagrangian Grassmanians of TM,

$$\Lambda(M) = \bigcup_{p \in M} \Lambda(T_pM),$$

where $\Lambda(T_pM)$ is the set of Lagrangian subspaces of the symplectic vector space (T_pM, ω_p).

Let $[\ell_p, w] \in \mathrm{Cr}(L_0, L_1; \ell_0) \subset \widetilde{\Omega}(L_0, L_1; \ell_0)$ be an element whose projection corresponds to the intersection point $p \in L_0 \cap L_1$.

As before, we associate a symplectic bundle pair $(\mathcal{V}_w, \lambda_w)$ over the square $[0,1]^2$, which is defined uniquely up to the homotopy. We first choose $\mathcal{V}_w =$

w^*TM. To define λ_w, let us choose a path $\alpha^p : [0,1] \to \Lambda(T_pM, \omega_p)$ satisfying

$$\begin{cases} \alpha^p(0) = T_pL_0, \alpha^p(1) = T_pL_1 \subset T_pM, \\ (\alpha^p)(t) \oplus T_pL_0 = T_pM, \\ \alpha^p(t) \in U_0(T_pL_0) \quad \text{for small } t, \end{cases}$$

where $U_0(T_pL_0)$ is as above.

Then we consider a continuous Lagrangian subbundle $\lambda_w \to \partial[0,1]^2$ of $\mathcal{V}|_{\partial[0,1]^2}$ by the following formula: the fiber at each point of $\partial[0,1]^2$ is given as

$$\begin{cases} \lambda_w(s,0) = T_{w(s,0)}L_0, \lambda_w(1,t) = \alpha^p(t), \\ \lambda_w(s,1) = T_{w(s,1)}L_1, \lambda_w(0,t) = \lambda^0(0,t). \end{cases}$$

It follows that the homotopy type of the bundle pair constructed as above does not depend on the choice of α^p either.

DEFINITION 3.3

We define the *Maslov-Morse index* of $[\ell_p, w]$ (relative to λ^0) by

$$\mu([\ell_p, w]; \lambda^0) = \mu(\mathcal{V}_w, \lambda_w).$$

3.3. Anchored Lagrangian submanifolds

Now we generalize this construction for a chain

$$\mathcal{L} = (L_0, \ldots, L_k)$$

of more than two Lagrangian submanifolds, that is, with $k \geq 2$.

To realize this purpose, we need the notion of *anchors* of Lagrangian submanifolds.

DEFINITION 3.4

Fix a base point y of an ambient symplectic manifold (M, ω). Let L be a Lagrangian submanifold of (M, ω). We define an *anchor* of L to y as a path $\gamma : [0,1] \to M$ such that

$$\gamma(0) = y, \quad \gamma(1) \in L.$$

We call a pair (L, γ) an *anchored* Lagrangian submanifold.

It is easy to see that any homotopy class of path in $\Omega(L, L')$ can be realized by a path that passes through the given point y. We denote the set of homotopy classes of the anchors γ to $y \in M$ by $\pi_1(y, L)$.

The following lemma is easy to check.

LEMMA 3.2

Suppose that L is connected. Then the set $\pi_1(y, L)$ of homotopy classes relative to the ends is a principal homogeneous space of $\pi_1(M, L)$; that is, it is a $\pi_1(M, L)$-torsor. We call an element of $\pi_1(y, L)$ an anchor class *of L relative to y.*

Proof
Since L is connected, the natural map $\pi_1(M) \to \pi_1(M,L)$ is surjective, and so $\pi_1(M,L) \cong \pi_1(M)/\operatorname{im}(\pi_1(L) \to \pi_1(M))$ forms a group. It is obvious to see that $\pi_1(M,L)$ acts on $\pi_1(y,L)$ by concatenation of paths on the right. By definition, this action is free. Transitivity is obvious by definition. $\square$

For a given pair $(\mathcal{L},\mathcal{L}')$ of anchored Lagrangians $\mathcal{L} = (L,\gamma), \mathcal{L}' = (L',\gamma')$, we denote

$$\Omega(\mathcal{L},\mathcal{L}') := \Omega(L,L';\overline{\gamma}\#\gamma'),$$

where the latter is the path component of $\Omega(L,L')$ containing $\overline{\gamma}\#\gamma'$. We also denote

$$\mathcal{L}\cap\mathcal{L}' = \left\{p \in L\cap L' \,\middle|\, \widehat{p} \in \Omega(\mathcal{L},\mathcal{L}')\right\}. \tag{3.3}$$

Here $\widehat{p}$ is the constant path $\widehat{p}(t) \equiv p$.

When we are given a Lagrangian chain

$$\mathfrak{L} = (L_0, L_1, \ldots, L_k),$$

we also consider a chain of anchors $\gamma_i : [0,1] \to M$ of L_i to y for $i = 0,\ldots,k$. These anchors give a systematic choice of a base path $\ell_{ij} \in \Omega(L_i,L_j)$ by concatenating γ_i and γ_j as

$$\ell_{ij} = \overline{\gamma}_i * \gamma_j,$$

where $\overline{\gamma}$ is the time reversal of γ given by $\overline{\gamma}(t) = \gamma(1-t)$. The upshot of this construction is the following overlapping property:

$$\begin{aligned} \ell_{ij}(t) &= \ell_{i\ell}(t) \quad \text{for } 0 \le t \le \frac{1}{2}, \\ \ell_{ij}(t) &= \ell_{\ell j}(t) \quad \text{for } \frac{1}{2} \le t \le 1, \end{aligned} \tag{3.4}$$

for all j,ℓ.

We write $\vec{p} = (p_{10},\ldots,p_{k(k-1)})$. Let $\chi_i = \exp(-2\pi i\sqrt{-1}/k)$. We consider the set of homotopy class of maps $v : D^2 \to M$ such that $v(\overline{\chi_{i+1}\chi_i}) \subset L_i$ and $v(\chi_i) = p_{i(i+1)}$. We denote it by $\pi_2(\mathfrak{L};\vec{p})$. If $\mathcal{E}$ is an anchored Lagrangian chain and $\mathfrak{L}$ is its underlying Lagrangian chain, we write $\pi_2(\mathcal{E};\vec{p})$ in place of $\pi_2(\mathfrak{L};\vec{p})$, sometimes by abuse of notation.

DEFINITION 3.5
Let $\mathcal{E} = \{(L_i,\gamma_i)\}_{0\le i\le k}$ be a chain of anchored Lagrangian submanifolds. A homotopy class $B \in \pi_2(\mathfrak{L};\vec{p})$ is called *admissible* to $\mathcal{E}$ if it can be obtained by a polygon that is a gluing of k bounding strips $w_{i(i+1)} : [0,1]\times[0,1] \to M$ satisfying

$$w_{i(i+1)}(0,t) = \begin{cases} \gamma_i(2t-1), & 0 \le t \le \frac{1}{2}, \\ \gamma_{i+1}(2t-1), & \frac{1}{2} \le t \le 1, \end{cases} \tag{3.5}$$

$$w_{i(i+1)}(s,0) \in L_i, \qquad w_{i(i+1)}(s,1) \in L_{i+1}, \tag{3.6}$$

$$w_{i(i+1)}(1,t) = p_{i(i+1)}. \tag{3.7}$$

When this is the case, we denote the homotopy class B as

$$B = [w_{01}]\#[w_{12}]\#\cdots\#[w_{k0}]$$

and the set of admissible homotopy classes by $\pi_2^{ad}(\mathcal{E};\vec{p})$.

We call such a tuple $\mathcal{E}$ an *anchored Lagrangian chain.*

REMARK 3.6

We remark that we denote by $\mathfrak{L}$ a chain $(L_0,\ldots,L_k)$ of Lagrangian submanifolds and by $\mathcal{E}$ that of anchored Lagrangian submanifolds.

When the collection $\mathcal{E} = \{(L_i,\gamma_i)\}_{0\le i\le k}$ is given, we note that not all homotopy classes in $\pi_2(\mathcal{L};\vec{p})$ are admissible. But we have the following basic lemma, which is enough for the construction of a Fukaya category, whose proof is easy and so is omitted.

LEMMA 3.3

Let $w_{i(i+1)}$ be given for $i = 1,\ldots,k$, and let $B \in \pi_2(\mathcal{L};\vec{p})$. Then they canonically define a class $[w_{k0}] \in \pi_1(\ell_{k0}, p_{k0})$ by

$$[w_{k0}] := [\overline{w}_{01}]\#\cdots\#[\overline{w}_{(k-1)k}]\#B.$$

The following basic identity immediately follows from the definitions.

PROPOSITION 3.4

Suppose that $B \in \pi_2(\mathcal{E},\vec{p})$ is given as in Lemma 3.3, and provide the analytic coordinates at the puncture $z_j \in \partial D^2 \subset D^2$ so that all z_j are outgoing. Then we have $\omega(B) = \sum_{i=0}^k \mathcal{A}([p_i,w_i])$ and

$$\omega(B) = \sum_{i=0}^{k} \mathcal{A}_{\ell_{i(i+1)}}([p_i,w_i]), \tag{3.8}$$

$$\mu(\mathcal{E},\vec{v};B) = \sum_{i=0}^{k} \mu([p_i,w_i];\lambda_{i(i+1)}). \tag{3.9}$$

In particular, the sums in the right-hand sides do not depend on the choice of $\lambda_i \subset \gamma_i^* TM$.

Here the index $\mu([p_i,w_i];\lambda_{i(i+1)})$ is the Maslov-Morse index relative to the path $\lambda_{i(i+1)}$ of Lagrangian planes as defined in [FO+1], which provides a coherent grading $\mu : \mathrm{Crit}\,\mathcal{A} \to \mathbb{Z}$.

The action functional provides a canonical $\mathbb{R}$-filtration on the set

$$\mathcal{A} : \mathrm{Crit}\,\mathcal{A} \to \mathbb{R}.$$

In addition, inside the collection of anchored Lagrangian submanifolds (L,γ) we are given a coherent system of single-valued action functionals

$$\mathcal{A}:\widetilde{\Omega_0}(L_i,L_j;\overline{\gamma}_i\#\gamma_j)\to\mathbb{R}$$

at one stroke for any pair from the given collection $\mathcal{E}$ of anchored Lagrangian submanifolds.

3.4. Relation to the graded Lagrangian submanifolds

Now we go back to the collection of Lagrangian submanifolds with vanishing Maslov class on a Calabi-Yau manifold. In this case, we are able to obtain a canonical Lagrangian path λ along a given anchor γ of L to y. We call a pair (γ,λ) a *graded anchor* of L.

Let J be a compatible almost complex structure. The assumption that $2c_1(M)=0$ implies that the bundle $\Delta=\Lambda^n(TM,J)^{\otimes 2}$ is trivial. Choose a section Θ of Δ^* that has length one everywhere in terms of the metric $g=\omega(\cdot,J\cdot)$. This determines a map $\det^2_\Theta:\Lambda(M,\omega)\to S^1$ and then an ∞-fold Maslov covering by

$$\Lambda_\infty=\big\{(\Lambda,t)\in\Lambda\times\mathbb{R}\,\big|\,\det{}^2_\Theta(\Lambda)=e^{2\pi it}\big\}, \tag{3.10}$$

where $s_L:L\to\Lambda(M,\omega)|_L$ is the natural section defined by the Gauss map $s_L(x)=T_xL$. An Λ_∞-grading of a Lagrangian submanifold $L\subset(M,\omega)$ according to Seidel [Se2] is just a lift to $\mathbb{R}$ of the map

$$\det{}^2_\Theta\circ s_L:L\to S^1.$$

First of all, the condition on $\mu_L=0$ implies that there is such a lifting to Λ_∞. A grading of L is a lift $\widetilde{s}_L$ of s_L.

We now explain how we give a coherent grading to anchored Lagrangian submanifolds with vanishing Maslov class.

First we go from a graded anchored Lagrangian submanifold to a graded Lagrangian submanifold. We first pick a Lagrangian subspace $V_y\in T_yM$ such that

$$\det{}^2_\Theta(y)(V_y)=1.$$

We consider a graded anchored Lagrangian submanifold (L,γ,λ) with $\lambda(0)=V_y$. We lift λ to a section of $\gamma^*\widetilde{\mathrm{Lag}}(TM)$ so that $\widetilde{\lambda}(0)=(V_y,0)$. Then $\widetilde{\lambda}(1)$ is a lifting of $\lambda(1)$ in $\widetilde{\mathrm{Lag}}(T_{\gamma(1)}M)$. Since the lifting of $\widetilde{\lambda}$ of λ is homotopically unique, $\widetilde{\lambda}(1)$ depends only on (L,γ) and the fixed V_y. Therefore if $\mu_L=0$, then this determines a unique grading $\widetilde{s}$ of L with $\widetilde{s}(\gamma(1))=\widetilde{\lambda}(1)$.

DEFINITION 3.7

Let (M,ω) be such that $2c_1(M)=0$, fix a base point $y\in M$, and let $\Lambda_y\in\Lambda(M,\omega)_y$ be a Lagrangian subspace. Let (L,γ) be a Calabi-Yau anchored Lagrangian. We denote the above common lifting by $\alpha_{(L,(x_L,\gamma_L))}:L\to\mathbb{R}$ and call it the *canonical grading* of (L,γ) relative to (y,Λ_y).

Next we go from a lifting $\widetilde{s}_L$ to λ. Let $\widetilde{s}_L$ be a grading of L. Consider any anchored Lagrangian submanifold (L,γ) with $\mu_L = 0$. We take a section $\widetilde{\lambda}$ of the pullback $\gamma^*(\widetilde{\mathrm{Lag}}(TM)) \to [0,1]$ such that

$$\widetilde{\lambda}(0) = (V_y, 0), \qquad \widetilde{\lambda}(1) = \widetilde{s}_L\big(\lambda_i(1)\big).$$

Such a path is unique up to homotopy because $[0,1]$ is contractible, and so $\gamma_i^* \widetilde{\mathrm{Lag}}(TM)$ is simply connected. We push it out and obtain a section λ in $\gamma^* \mathrm{Lag}(TM)$. In this way, a graded Lagrangian submanifold $(L, \widetilde{s})$ canonically determines a grading λ of an anchored Lagrangian submanifold (L,γ).

We remark that the path λ_{01} induced by these graded anchors lifts to $\widetilde{\lambda}_{01}$ joining $s_0(\ell_{01}(0))$ to $s_1(\ell_{01}(1))$.

We then define $\mu([p,w];\ell_{01})$ using this path λ_{01} as in Section 3.2.

LEMMA 3.5

$\mu([p,w];\ell_{01})$ *is independent of* w.

Proof

Independence of the degree of w is a consequence of our assumption that Maslov indices of L_0, L_1 are zero. We omit the detail. □

We refer to [FO+2, Section 9.2] for a more detailed explanation on the above discussion.

Because of this presence of canonical grading associated to (L,γ), we drop λ from our notation $\mathcal{L} = (L,\gamma,\lambda)$ when we consider Calabi-Yau Lagrangian branes later in this paper.

Adapting to the convention from [K1] and [Se2], we denote $\widetilde{L}[0] = (L, \alpha_{(L,\gamma)})$ and

$$\widetilde{L}[k] = (L, \alpha_{(L,\gamma)} - k).$$

4. Calabi-Yau Lagrangian branes and Dehn twists

We restrict ourselves to the case of (M,ω) with $2c_1(M,\omega) = 0$ and $L \subset M$ whose Maslov class vanishes from now on. We give a precise definition of Calabi-Yau Lagrangian branes in this section. This is the case that is most relevant to mirror symmetry and to the extension of Seidel's long exact sequence of $\mathbb{Z}$-graded symplectic Floer cohomology.

Now we introduce a class of decorated Lagrangian submanifolds on Calabi-Yau manifold (M,ω) which we call *Calabi-Yau Lagrangian branes*.

DEFINITION 4.1

Let $y \in M$ be a base point, and let $\Lambda_y \subset T_y M$ be a fixed Lagrangian subspace. Suppose that Θ is a quadratic complex volume form on (M,ω,J). Let $\mathcal{E}^{\mathrm{CY}}$ be the Calabi-Yau Lagrangian collection of (M,ω). We consider the triple $(\mathcal{L}, s, [b])$,

$\mathcal{L} = (L, \gamma)$, which we call an *anchored Calabi-Yau Lagrangian brane*, that satisfies the following data:

(1) L is a Lagrangian submanifold of M such that the Maslov index of L is zero and $[\omega] \in H^2(M, L; \mathbb{Z})$; we also enhance L with a flat complex line bundle;
(2) γ is an anchor of L to y;
(3) s is a spin structure of L;
(4) $[b] \in \mathcal{M}(L)$ is a bounding cochain described in Section 8.2.

We denote the collection of Calabi-Yau (CY) Lagrangian branes by $\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$ and denote the Fukaya category generated by them by $\mathrm{Fuk}(\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}})$.

The first simplification arising from considering the CY Lagrangian collection is that we have only to use the Novikov ring of the form

$$\Lambda^{(0)}_{\mathrm{nov}} = \Big\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,\Big|\, a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lambda_i \le \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \Big\}, \tag{4.1}$$

which becomes a field. We also consider the subring

$$\Lambda^{(0)}_{0,\mathrm{nov}} = \Big\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda_{\mathrm{nov}} \,\Big|\, \lambda_i \ge 0 \Big\}. \tag{4.2}$$

This is because the Maslov index satisfies $\mu(w) = \mu_L(\partial w) = 0$ for any disc map $w : (D^2, \partial D^2) \to (M, L)$, where $\mu_L \in H^1(L; \mathbb{Z})$ is the Maslov class of L.

REMARK 4.2

Furthermore, as we mentioned in Section 3.4, the anchor provides a canonical graded structure on a CY Lagrangian brane. Therefore it provides a canonical $\mathbb{R}$-filtration and a $\mathbb{Z}$-grading on

$$CF(\mathcal{L}_0, \mathcal{L}_1) = CF\big((L_0, \gamma_0), (L_1, \gamma_1)\big) := CF(L_0, L_1; \overline{\gamma}_0 \# \gamma_1)$$

for any pair $(\mathcal{L}_0, \mathcal{L}_1)$ of CY Lagrangian branes, and hence on its cohomology $HF(\mathcal{L}_0, \mathcal{L}_1)$. See Section 8 for a related discussion.

Now we examine the effect of Dehn twists on the CY Lagrangian collection.

PROPOSITION 4.1

Let $\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$ be the associated collection of anchored CY Lagrangian branes. Then $\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$ is closed under the action of τ_L's for all framed Lagrangian spheres $(L, [f])$ and so induces an autoequivalence of $\mathrm{Fuk}(\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}})$.

Proof

We note that the Dehn twist τ_L is a symplectic automorphism. Therefore it pushes the spin structure of L_0 to the image $\tau_L(L_0)$ and pulls back the Maslov class. Therefore the Maslov class of $\tau_L(L_0)$ is also zero. Similarly we can push forward the anchor of L to $\tau_L(L_0)$. Finally, [FO+1, Theorem B (B.2)] states that the bounding cochain can also be canonically pushed out under the symplectic automorphism and hence under the action of τ_L. This finishes the proof. $\square$

Therefore we can ask the question about how the Floer cohomology changes under the Dehn twist along a Lagrangian sphere. The answer is supposed to come from a long exact sequence that Seidel introduced in [Se3] for the context of exact Lagrangian submanifolds. The rest of the paper is occupied with the construction of this long exact sequence for the CY Lagrangian branes on Calabi-Yau manifolds.

5. Lefschetz Hamiltonian fibration and coupling form

In this section, we first recall the basics on *smooth* Hamiltonian fibrations presented in [GLS] and [En] and extend our discussion to fibrations with Lefschetz-type singular fibers. Especially we generalize the notion of *coupling form* to the current singular fibration and prove the uniqueness of the coupling form on a given Lefschetz Hamiltonian fibration when the fibration $\pi : E \to \Sigma$ is proper, that is, when the fiber of E is compact.

The notion of Hamiltonian fibrations introduced by Guillemin, Lerman, and Sternberg [GLS] is the family of symplectic manifolds of a fixed isomorphism type, which could be twisted on the parameter space Σ. On the other hand, Seidel introduced the notion of exact Lefschetz fibrations which could have a finite number of singular fibers of type A_1-singularity.

Combining [GLS] and [Se3], we give the following definition.

DEFINITION 5.1 (LEFSCHETZ HAMILTONIAN FIBRATION)

A Lefschetz Hamiltonian fibration over a compact surface Σ with boundary $\partial\Sigma$ consists of the data $(E, \pi, \Omega, J_0, j_0)$ as follows.

(1) We have $\partial E = \pi^{-1}(\partial\Sigma)$, and $\pi|_{\partial E} \to \partial\Sigma$ forms a smooth fiber bundle.

(2) The projection $\pi : E \to \Sigma$ can have at most finitely many critical points, and no two may lie on the same fiber. Denote $E^{\mathrm{crit}} \subset E$ and $\Sigma^{\mathrm{crit}} \subset \Sigma$.

(3) J_0 is a complex structure on a neighborhood of E^{crit}, j_0 is a positively oriented complex structure on a neighborhood of Σ^{crit}, and π is (J_0, j_0)-holomorphic near E^{crit}. And the Hessian $D^2\pi$ at any critical point is nondegenerate as a complex quadratic form.

(4) Ω is a closed 2-form on E which must be nondegenerate on $\ker D\pi_x$ for each $x \in E$, and a Kähler form for J_0 in some neighborhood of E^{crit}.

We say that the fibration $\pi : E \to \Sigma$ is (symplectically) *Calabi-Yau* if $c_1(E^v_z) = 0$ at all $z \notin \Sigma^{\mathrm{crit}}$.

We remark that one may allow more than one critical point in the same fiber, which could be useful for the study of a family of Lefschetz Hamiltonian fibrations.

REMARK 5.2

We wish highlight that at a critical point $x \in E$ of π, Definition 5.1(4) implies that the 2-form Ω_x is required to be nondegenerate on the whole tangent space $T_e E$ since $\ker D\pi_x = T_x E$, while at a regular point Ω_x it is required to be so only on the vertical tangent space $T_e^v E$ as $\ker D\pi_x = T_x^v E$.

When a generic fiber E_z with $z \in \Sigma \setminus \Sigma^{\mathrm{crit}}$ is compact, it is proved in [GLS] for a smooth Hamiltonian fibration that the choice of Ω is uniquely determined by the following additional requirement:

$$\pi_* \Omega^{n+1} = 0, \tag{5.1}$$

where π_* is the integration along the fiber. Now we prove the following analogue to this result for the case of Lefschetz Hamiltonian fibrations.

THEOREM 5.1

Let $(E, \pi, \Omega, J_0, j_0)$ be a Lefschetz Hamiltonian fibration as in Definition 5.1. Then there exists a closed 2-form Ω' smooth on $E \setminus E^{\mathrm{crit}}$ that satisfies the following:

(1) $\Omega'|_{T^v E} = \Omega|_{T^v E}$ *at all* $e \in E \setminus E^{\mathrm{crit}}$,
(2) *it satisfies (5.1) on* $E \setminus \pi^{-1}(\Sigma^{\mathrm{crit}})$.

Furthermore, such a form Ω' is unique. We call such a form the coupling form *of* $(E, \pi, \Omega, J_0, j_0)$.

Proof

We first consider the subset $E \setminus E^{\mathrm{crit}}$. Then the form Ω induces a splitting

$$\Gamma : T_x E = T_x E^v \oplus T_x^h E$$

at any regular point $x \in E \setminus E^{\mathrm{crit}}$, where the horizontal space is given by

$$T_x^h E = \left\{ \eta \in T_x E \;\middle|\; \Omega(\eta, \xi) = 0, \forall \xi \in T_x^v E \right\}$$

and hence induces a natural (Ehresman) connection on $E \setminus E^{\mathrm{crit}}$ whose monodromy is symplectic.

Because of the closedness of Ω, the connection is Hamiltonian (see [GLS]) in that its curvature $\mathrm{curv}(\Gamma)$ has its values contained in $\mathrm{Ham}(E_{\pi(e)})$, the set of Hamiltonian vector fields of the fiber $E_{\pi(e)}$, and hence the restriction $\pi : E \setminus E^{\mathrm{crit}} \to \Sigma \setminus \Sigma^{\mathrm{crit}}$ is a smooth Hamiltonian fibration in the sense of [GLS]. In particular, if we restrict to $E \setminus \pi^{-1}(\Sigma^{\mathrm{crit}}) \to \Sigma \setminus \Sigma^{\mathrm{crit}}$, its fibers are all compact, and so we can construct a closed 2-form $\Omega' = \Omega + \pi^* \alpha$ on $E \setminus \pi^{-1}(\Sigma^{\mathrm{crit}})$ for some closed 2-form α on $\Sigma \setminus \Sigma^{\mathrm{crit}}$ that satisfies (5.1) thereon. In fact, Ω' (and so $d\beta$) can be explicitly constructed by requiring

$$\Omega'(\eta_1^\sharp, \eta_2^\sharp) = H_{\eta_1, \eta_2}, \tag{5.2}$$

where H_{η_1, η_2} is the smooth function whose restriction to each fiber over a point in $\Sigma \setminus \Sigma^{\mathrm{crit}}$ is uniquely determined by the following two requirements:

(1) H_{η_1,η_2} generates the Lie algebra element $\mathrm{curv}_\Gamma(\eta_1,\eta_2)$ of $\mathrm{Ham}(E_z,\omega_z)$ which is a Hamiltonian vector field,

(2) it satisfies the normalization condition

$$\int_{E_z} H_{\eta_1,\eta_2}\omega_z^n = 0, \quad \omega_z = \Omega|_{E_z},$$

for all $z \in \Sigma \setminus \Sigma^{\mathrm{crit}}$.

This finishes the proof. □

DEFINITION 5.3 (COUPLING FORM)

We call the above unique closed 2-form constructed in Theorem 5.1 the *coupling form* of the Lefschetz Hamiltonian fibration $E \to \Sigma$.

The following result was essentially proved by Seidel in [Se3, Lemma 1.6]. Seidel proved this for the context of exact Lefschetz fibrations, but the same proof applies if one ignores his consideration of generating functions of Q therein.

LEMMA 5.2 (SEE [Se3, LEMMA 1.6])

Let (E,π,Ω,J_0,j_0) be a Lefschetz Hamiltonian fibration, and let x_0 be a critical point of π. Then there are smooth families $\Omega^\mu \in \Omega^2(E)$, $0 \le \mu \le 1$, such that

(1) *$\Omega^0 = \Omega$;*

(2) *for all μ, $\Omega^\mu = \Omega^0$ outside a small neighborhood of x_0;*

(3) *each $(E,\pi,\Omega^\mu,J_0,j_0)$ is a Lefschetz Hamiltonian fibration;*

(4) *there is a holomorphic Morse chart (ξ,Ξ) around x_0 with $\Xi : V \subset \mathbb{C}^{n+1} \to E$ such that $\Xi^*\Omega^1$ agree near the origin with the standard forms $\omega_{\mathbb{C}^{n+1}} = (i/2) \times \sum dx_k \wedge d\overline{x}_k$.*

In fact, if near E^{crit} we are given a 1-form Θ with $\Omega = d\Theta$ as in the exact cases, we can also deform the 1-form to Θ^μ so that $\Xi^*\Theta^1$ becomes the standard 1-form

$$\theta_{\mathbb{C}^{n+1}} = \frac{i}{4}\Big(\sum x_k\, d\overline{x}_k - \overline{x}_k\, dx_k\Big)$$

(see [Se3, Lemma 1.6]).

6. Exact Lagrangian boundary condition

Now we consider a subbundle $i_Q : Q \to \partial\Sigma$ of the symplectic fiber bundle $(E|_{\partial\Sigma},\Omega|_{\partial\Sigma})$ whose fiber Q_z is a Lagrangian submanifold of Ω_z for each $z \in \partial\Sigma$. We call such a Q a *fiberwise Lagrangian submanifold* of $(E|_{\partial\Sigma},\Omega|_{\partial\Sigma})$.

We start with the notion of exact Lagrangian boundary over $\partial\Sigma$.

DEFINITION 6.1
We call a fiberwise Lagrangian submanifold $Q \subset \partial E$ an *exact Lagrangian boundary over* $\partial\Sigma$ if there exists a 1-form κ_Q on Q such that

$$\kappa_Q|_{T(\partial E)^v} \equiv 0 \quad \text{and} \quad i_Q^*\Omega = d\kappa_Q,$$

where $i_Q : Q \to \partial E$ is the inclusion map and $T^v(\partial E)$ is the vertical tangent space of ∂E.

We note that when Σ is oriented and the boundary orientation on $\partial\Sigma$ is provided by an orientation 1-form, denoted by $d\theta$ with $\theta \in \partial\Sigma$, the connection induced by the form $\Omega|_{\partial E}$ enables us to express any such 1-form κ_Q as

$$\kappa_Q(\theta, x) = h_i(\theta, x)\, d\theta$$

for $(\theta, x) \in \partial E$ with $\theta \in \partial_i\Sigma$ and $h_i : \partial E \to \mathbb{R}$, where $\partial\Sigma = \coprod \partial_i\Sigma$. The function h_i is unique up to the addition of the function $c_i : \partial\Sigma_i \to \mathbb{R}$.

DEFINITION 6.2
We define $\|\kappa_Q\|_{(1,\infty)}$ by

$$\|\kappa_Q\|_{(1,\infty)} = \int_{\partial\Sigma} \operatorname{osc}(h_\theta)\, d\theta$$

with $\operatorname{osc}(h_\theta) := \max_{x\in E_x} h(\theta, x) - \min_{x\in E_x} h(\theta, x)$ and call it the $L^{(1,\infty)}$-*norm* of κ_Q.

We remark that $\|\kappa_Q\|_{(1,\infty)}$ does not depend on the choice of the function h.

To give readers some insight on these definitions, we compare this with the classical notion of *exact Lagrangian isotopy* (see [Gr]).

EXAMPLE 6.3
Let $E = (\mathbb{R} \times [0,1]) \times (M, \omega)$ with $\pi : (\mathbb{R} \times [0,1]) \times (M, \omega) \to (M, \omega)$ be the projection. Consider the 2-form $\Omega = \pi^*\omega$. Let $L_i \subset (M, \omega)$ be a Lagrangian submanifold, and let $\psi_i : [0,1] \times L \to M$ be a Lagrangian isotopy for $i = 1, 2$. The isotopy ψ_i is an exact Lagrangian isotopy if there is a smooth function $h_i : [0,1] \times L \to \mathbb{R}$ such that $\psi_i^*\omega = dh_i \wedge dt = d(h_i dt)$ (see [Gr]). This definition is a special case of Definition 6.1: just consider the embedding

$$(h_i, \psi_i) : \mathbb{R} \times L_i \to \mathbb{R} \times [0,1] \times M,$$

set $Q_i = \operatorname{im}(h_i, \psi_i)$ as the Lagrangian suspension, and let $\kappa_{Q_i}(t, x) = h_i(t, x)\, dt$.

We now study the structure of $\pi_2(E, Q)$. We start with the following relative version of *section class*. A class $B \in \pi_2(E, Q)$ is a section class if $\pi_* B \in \pi_2(\Sigma, \partial\Sigma)$ is the positive generator with respect to the given orientation of Σ. We say that B is a *fiber class* if it is in the image of $\pi_2(M, L) \to \pi_2(E, Q)$ induced from the inclusion of the fiber. The following is proved in [HL, Lemma 2.2] for the smooth

Hamiltonian fibration but the same proof applies to the current case with singular fibers. We refer readers to [HL] for its proof.

LEMMA 6.1

The following sequence of homotopy groups is exact at the middle term:

$$\pi_2(M,L) \to \pi_2(E,Q) \to \pi_2(\Sigma,\partial\Sigma).$$

The following proposition is the reason why the notion of exact Lagrangian boundary is relevant to the study of pseudoholomorphic curves with boundary later. Similar estimates were previously obtained in [En], [Oh3], and [Se3] in somewhat different contexts.

PROPOSITION 6.2

Suppose that Σ is oriented, and denote by $d\theta$ a given orientation 1-form on $\partial\Sigma$. Let $Q \subset \partial E$ over $\partial\Sigma$ be an exact Lagrangian boundary of E, and let κ_Q be a corresponding Hamiltonian 1*-form. Consider a section $s : \Sigma \to E \setminus E^{\mathrm{crit}}$ with $s(\partial\Sigma) \subset Q \subset \partial E$. Then for each given section class $[s,\partial s] \in \pi_2(E,Q;\mathbb{Z})$, there exists a constant $C = C(\kappa_Q,[s,\partial s]) > 0$ such that the integral bound*

$$\Big|\int_\Sigma s^*\Omega\Big| \le C = C(B)$$

holds for any section s in a fixed class $B = [s,\partial s] \in H_2(E,Q;\mathbb{Z})$. In fact, we have

$$\Big|\int_\Sigma s_2^*\Omega - \int_\Sigma s_1^*\Omega\Big| \le \|\kappa_Q\|_{(1,\infty)} \tag{6.1}$$

for any two such sections with $[s_1,\partial s_1] = [s_2,\partial s_2]$.

Proof

Recall that $i_Q^*\Omega = d\kappa_Q$ for a 1-form κ_Q which exists by definition of exact Lagrangian boundary Q. Let s_i, $i = 1,2$, be two sections of E with $s_i(\partial\Sigma) \subset Q$ and $[s_1,\partial s_1] = [s_2,\partial s_2]$. Then we have a geometric chain (S,C) with

$$\partial S = s_1 \coprod s_2 \coprod C,$$

and we have $\partial C = \partial s_2 - \partial s_1$ as a chain in Q.

By Stokes's formula and closedness of Ω, we have

$$0 = \int S^*(d\Omega) = \int_{\partial S} \Omega = \int_\Sigma s_2^*\Omega - \int_\Sigma s_1^*\Omega - \int C^*\Omega$$

and hence

$$\int_\Sigma s_2^*\Omega - \int_\Sigma s_1^*\Omega = \int_C \Omega.$$

But by the exactness of the fiberwise Lagrangian subbundle Q and since C has its image in Q, we obtain

$$\int_C \Omega = \int_C d\kappa_Q = \int_C \kappa_Q = \int_{\partial s_2} \kappa_Q - \int_{\partial s_1} \kappa_Q.$$

Therefore we have obtained

$$\int_\Sigma s_2^*\Omega - \int_\Sigma s_1^*\Omega = \int_{\partial s_2} \kappa_Q - \int_{\partial s_1} \kappa_Q = \int_{\partial\Sigma} (h\circ s_2 - h\circ s_1)\, d\theta,$$

and so

$$\Big|\int_\Sigma s_2^*\Omega - \int_\Sigma s_1^*\Omega\Big| \le \int_{\partial\Sigma} \Big(\max_{x\in E_x} h_\theta(x) - \min_{x\in E_x} h_\theta(x)\Big)\, d\theta \le \|\kappa_Q\|_{(1,\infty)}.$$

Since $\|\kappa_Q\|_{(1,\infty)}$ does not depend on s, this finishes the proof. $\square$

Next we consider the topological index associated to the section $(s,\partial s)$ for s which does not pass through critical points $E^{\rm crit}$. By the pullback $s^*(TE^v)$, it defines a symplectic bundle pair $(s^*TE^v, (\partial s)^*TQ^v)$, where $TQ^v = TQ^v = TQ\cap TE^v|_{\partial\Sigma}$. Therefore we can associate the Maslov index, which we denote by $\mu([s,\partial s])$ (see [KL], [FO+1]).

Now we examine the topological dependence of $\mu([s,\partial s])$. Note that each section $(s,\partial s)$ defines an element in $\pi_2(E,Q)$. We denote the corresponding class by $s_*([\Sigma,\partial\Sigma])$, where $[\Sigma,\partial\Sigma]$ is the fundamental class which is a generator of $H_2(\Sigma,\partial\Sigma;\mathbb{Z})\cong\mathbb{Z}$. The following lemma immediately follows from the definition of the Maslov index for the bundle pair.

LEMMA 6.3
Suppose that $(s_1)_*([\Sigma,\partial\Sigma]) = (s_2)_*([\Sigma,\partial\Sigma])$. Then we have

$$\mu([s_1,\partial s_1]) = \mu([s_1,\partial s_2]).$$

DEFINITION 6.4
We denote by $\pi_2^{\rm sec}(E,Q)\subset\pi_2(E,Q)$ the subset of section classes $[s,\partial s]$ in $\pi_2(E, Q)$. We say that two section classes B_1, B_2 are Γ-*equivalent* if they satisfy

$$\Omega(B_1) = \Omega(B_2), \qquad \mu(B_1) = \mu(B_2)$$

and denote by $\Pi(E,Q)$ the quotient group

$$\Pi(E,Q) = \pi_2^{\rm sec}(E,Q)/\sim.$$

For the Calabi-Yau Lefschetz fibrations, one can proceed to the study of Maslov indices following the exposition given in [Se4]. Consider the bundle of relative quadratic volume forms

$$\mathcal{K}^2_{E/\Sigma} = \pi^*{\rm det}^2_{\mathbb{C}}(T\Sigma)\otimes{\rm det}_{\mathbb{C}}^{\otimes -2}.$$

By definition, if $E\to\Sigma$ is Calabi-Yau, we have nowhere a zero section $\eta^2_{E/B}$ of this on $E\setminus E^{\rm crit}$. Furthermore, we can require $\eta^2_{E/B}$ to satisfy

$$\eta^2_{E/B} = \frac{(dz_1\wedge\cdots\wedge dz_{n+1})^2}{(2z_1dz_1+\cdots+2z_{n+1}dz_{n+1})^{\wedge 2}}$$

in a neighborhood $U\subset E$ of each critical point under the given identification $\pi: U\setminus\{x\}\to\pi(U\setminus\{x\})$ with $q:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{C}\setminus 0$.

DEFINITION 6.5

We say that a fiberwise Lagrangian submanifold $Q \subset E$ is (relatively) *graded* if there exists a function

$$\alpha : Q \to \mathbb{R}$$

such that

$$\exp\big(2\pi\sqrt{-1}\alpha(x)\big) = \eta^2_{E/B}(T^v_x Q), \quad x \in Q.$$

We call α an $\mathcal{L}^\infty$-grading of Q.

For a given pair of a fiberwise Lagrangian submanifold $Q_1, Q_2 \subset E$ intersecting transversely, we can associate a natural $\mathbb{Z}$-grading on the intersection $Q_1 \cap Q_2$ in the following way.

We consider the 2-form

$$\omega_{E,\lambda} = \Omega + \lambda\pi^*\omega_\Sigma$$

which is nondegenerate on $E \setminus E^{\mathrm{crit}}$. The existence of such $\lambda > 0$ is easy to check. The next lemma also immediately follows. We leave the proof to the readers.

LEMMA 6.4

Any fiberwise Lagrangian submanifold $Q \subset E$ is Lagrangian for $\omega_{E,\lambda}$.

Finally, we define the bundle analogue to the relative spin structure introduced in [FO+1].

DEFINITION 6.6

Let $E \to \Sigma$ be a Lefschetz Hamiltonian fibration, and let $Q \subset \partial E \to \partial\Sigma$ with $Q \subset \partial E$ be a Lagrangian boundary condition such that the vertical tangent bundle $T^v Q \to \partial\Sigma$ is orientable. A fiberwise relative spin structure of $Q \to \partial\Sigma$ is a pair of a vector bundle $V \to \Sigma$ and a spin structure on $T^v Q \oplus i^*_\Sigma V$ for the inclusion map $i_\Sigma : \partial\Sigma \to \Sigma$.

Any relative spin structure on Q can be used to induce an orientation on the moduli space $\mathcal{M}(E, Q; \vec{p}; B)$ of pseudoholomorphic sections whose explanation is now in order. But in the present paper, we restrict our attention to the case of spin structures.

7. Pseudoholomorphic sections

In this section, we perform various studies of geometry and analysis of pseudoholomorphic sections with Lagrangian boundary condition in the setting of Lefschetz Hamiltonian fibrations.

Let $(E, \pi, \Omega, J_0, j_0)$ be a Lefschetz Hamiltonian fibration, and let $x_0 \in E^{\mathrm{crit}}$. Denote by (ξ, Ξ) a *holomorphic Morse chart* at x_0, that is, a j_0-holomorphic coordinates $\xi : U \to S$ with $\xi(0) = z_0 = \pi(x_0)$, where $U \subset \mathbb{C}$ is a neighborhood of the origin and $\Xi : W \to E$ is a J_0-holomorphic chart with $W \subset \mathbb{C}^{n+1}$ a neighborhood of the origin in $\mathbb{C}^{n+1}$ with $\Xi(0) = x_0$ such that

$$(\xi^{-1} \circ \pi \circ \Xi)(x) = x_1^2 + \cdots + x_{n+1}^2. \tag{7.1}$$

With (ξ, Ξ) fixed, we denote the model Lefschetz fibration by $q : \mathbb{C}^{n+1} \to \mathbb{C}$ defined by

$$q(z_1, \ldots, z_n) = z_1^2 + \cdots + z_{n+1}^2.$$

Now for the given $(E, \pi, \Omega, J_0, j_0)$ and the holomorphic Morse charts (ξ, Ξ) at each of the critical points of E, we consider an almost complex structure J on E that satisfies the following:

(1) $J = J_0$ in a neighborhood of E^{crit};
(2) $d\pi \circ J = j \circ d\pi$;
(3) $\Omega(\cdot, J\cdot)|_{TE_x^v}$ is symmetric and positive definite for any $x \in E$.

Following Seidel [Se3], we call such J *compatible relative to* j. An immediate consequence of the definition is the following lemma.

LEMMA 7.1

Let J be compatible relative to j. Then for any given area form ω_Σ on Σ with $\int_\Sigma \omega_\Sigma = 1$, the 2-form $\Omega + \lambda \pi^* \omega_\Sigma$ tames J for all sufficiently large λ.

We refer to [Se3] for a more complete explanation of the structure of J's compatible to j.

7.1. Energy estimates and Hamiltonian curvature

Next we study the energy estimates of pseudoholomorphic sections in terms of the topological action $\int u^*\Omega$ and the contribution coming from the curvature integral of $\int_\Sigma u^*\Omega$ of a canonical symplectic connection of the Hamiltonian fibration $E \to \Sigma$ associated to the coupling form Ω defined in Definition 5.3. This kind of estimate has been studied in [Se3], [Oh3], and [MS].

Using the connection associated to Ω, we decompose $Du = (Du)^v + (Du)^h$ into vertical and horizontal components. Now we consider the symplectic form

$$\omega_E = \Omega + \lambda \pi^* \omega_\Sigma$$

with ω_Σ an area form on Σ with $\int_\Sigma \omega_\Sigma = 1$. We like to remark that an almost complex structure J compatible to j it is *not compatible* in the usual sense in that the bilinear form

$$\Omega(\cdot, J\cdot)$$

may not be symmetric. However, if λ is sufficiently large, it is tame to ω_E (see [Se3, Lemma 2.1]). Therefore we can symmetrize this bilinear form and define the associate metric g_J by

$$(7.2) \qquad \langle V, W\rangle = g_J(V,W) := \frac{1}{2}\big(\Omega(V,JW) + \Omega(W,JV)\big).$$

We call (7.2) the metric associated to J and denote

$$|V|^2 = |V|_J^2 = g_J(V,V).$$

With respect to this metric, we still have the following basic identity, whose proof we omit.

LEMMA 7.2
Let $s:\Sigma\to E$ be any smooth section v. Then we have

$$(7.3) \qquad \frac{1}{2}\int |Ds|^2 = \int s^*\omega_E + \int |\overline{\partial}_J s|^2.$$

In particular, if s is J-holomorphic, then

$$\frac{1}{2}\int |Ds|^2 = \int s^*\omega_E.$$

To examine some positivity property of J-holomorphic sections, we now decompose $Ds = (Ds)^v + (Ds)^h$ into vertical and horizontal parts and write

$$|Ds|^2 = |(Ds)^v|^2 + |(Ds)^h|^2 + 2\langle (Ds)^v, (Ds)^h\rangle.$$

Then it is straightforward to prove

$$(7.4) \qquad |(Ds)^h|^2\omega_\Sigma = 2(s^*\Omega + \lambda\omega_\Sigma)$$

by the identity

$$\begin{aligned}
\sum_{i=1}^2 |(Ds)^h(e_i)|^2 &= \sum_{i=1}^2 \omega_{E,\lambda}\big((Ds)^h(e_i), J(Ds)^h(e_i)\big)\\
&= \sum_{i=1}^2 (\Omega + \lambda\pi^*\omega_\Sigma)\big((Ds)^h(e_i), J(Ds)^h(e_i)\big)\\
&= 2\big(\omega_E((Ds)^h(e_1),(Ds)^h(e_2)) + \lambda\omega_\Sigma(e_1,e_2)\big)
\end{aligned}$$

for an orthonormal frame $\{e_1,e_2\}$ and then apply the curvature identity

$$d\big(\Omega(e_1^\#, e_2^\#)\big) = -[e_1^\#, e_2^\#]\rfloor\Omega: \text{ fiberwise.}$$

Here $e_i^\#$ is the horizontal lift of e_i, and we have

$$\mathrm{curv}_\Gamma(e_1,e_2) = \Omega(e_1^\#, e_2^\#)$$

by the definition of coupling form (see [GLS, (1.12)] but with caution on the sign convention). The following is an immediate corollary of (7.4).

PROPOSITION 7.3
Let Ω be the coupling form of E. Suppose that $\Omega + \lambda\pi^*\omega_\Sigma$ for $\lambda > 0$ is positive,

that is, symplectic for an area form ω_Σ *on* Σ. *Then we have the inequality as a* 2*-form*

$$s^*\Omega + \lambda\omega_\Sigma \geq 0 \tag{7.5}$$

for any J*-holomorphic section* s. *In other words, if we write* $s^*\Omega = f(s)\omega_\Sigma$ *for a function* $f : \Sigma \to \mathbb{R}$, *then we have* $f(s) + \lambda \geq 0$.

Proof
Just choose a complex structure j on Σ and a j-compatible J on E. Positivity (7.5) immediately follows from (7.4). ☐

7.2. Gromov-Floer moduli space of J-holomorphic sections

We first translate the anchor introduced in Section 3 in the setting of pointed Lagrangian boundary conditions in Hamiltonian fibrations over the surface Σ with boundary $\partial\Sigma \neq \emptyset$. This is needed to study morphisms between two Floer chain modules constructed via the moduli space of pseudoholomorphic sections of Lefschetz fibrations over Σ.

Let $y \in M$ be a base point, and let $(L,\gamma),(L',\gamma')$ be two anchored Lagrangian submanifolds of (M,ω) which intersect transversely. Let γ and γ' be the paths $\gamma(0) = \gamma'(0) = y$ and $\gamma(1) \in L, \gamma'(1) = L'$ given as in the anchor data. We denote by $\widetilde{\gamma}$ the time reversal of γ, that is, the path defined by

$$\widetilde{\gamma}(t) = \gamma(1-t).$$

Now to each intersection $p \in L \cap L'$ we associate a Hamiltonian fibration over $[0,1]^2$. The paths γ and γ' provide a path in M along $\{0\} \times [0,1]$ via the obvious concatenation of $\widetilde{\gamma}$ and γ' with the midpoint given by y.

We take the trivial fibrations $E = [0,1]^2 \times (M,\omega) \to [0,1]^2$. Then for each given pair $[p,w]$ with $p \in L \cap L'$ and with a bounding strip $w : [0,1]^2 \to M$ such that

$$\begin{aligned} &w(0,t) = \widetilde{\gamma}\#\gamma'(t), \quad w(1,t) \equiv p, \\ &w(s,0) \in L_0, \quad w(s,1) \in L_1, \end{aligned} \tag{7.6}$$

we can associate a section of $s_{[p,w]} : [0,1]^2 \to E$ by

$$(s,t) \mapsto \big((s,t), w(s,t)\big)[0,1]^2 \to [0,1]^2 \times (M,\omega).$$

We call this fibration over $[0,1]^2$ with a section $s_{[p,w]}$ an *anchor cap* associated to $[p,w]$ relative to the given anchor. For notational convenience, we denote the corresponding fibration with the fiberwise Lagrangian submanifolds by

$$(E_{[p,w]}; [0,1] \times \{0\} \times L_0, [0,1] \times \{1\} \times L_1; s_{[p,w]})$$

or simply as $(E_{[p,w]}; s_{[p,w]})$.

Note that the set of homotopy classes $[p,w]$ of w relative to the boundary condition (7.6) has one-to-one correspondence with the homotopy class of sections with the obvious corresponding boundary condition on $\partial[0,1]^2$.

For a given compact surface Σ with marked points $\vec{\zeta} = \{\zeta_0, \dots, \zeta_k\}$, we consider the corresponding surface $\dot{\Sigma} = \Sigma \setminus \vec{\zeta}$ with punctures. We denote the given preferred holomorphic chart $\varphi_\zeta : D_\zeta \subset \Sigma \to D^+$ of the half-disc $D^+ = D \cap \{\mathrm{im}(z) \geq 0\}$ with $\varphi_\zeta(\zeta) = 0$. We also have a local trivialization

$$\Phi_\zeta : E|_{D_\zeta \setminus \{\zeta\}} \to D^+ \setminus \{0\} \times M$$

lying over φ_ζ. When $Q \subset E$ is a Lagrangian boundary condition, we have a unique pair $L_{\zeta,\pm}$ of Lagrangian submanifolds of M such that

$$\Phi_\zeta(Q) = [-1, 0) \times L_{\zeta,-} \cup (0, 1] \times L_{\zeta,+}.$$

For the given ordered chain of Lagrangian boundaries $\mathcal{Q} = (Q_0, Q_1, \dots, Q_k)$, denote $Q = \bigcup_i Q_i$. Then we require the unique pair $L_{\zeta_i,\pm}$ at ζ_i to be

$$\Phi_{\zeta_i}(Q) = [-1, 0) \times L_i \cup (0, 1] \times L_{i+1}$$

at each ζ_i. In this way, for each given $(\pi : E \to \Sigma; \mathcal{Q})$ and a chain of intersection points $\vec{p} = (p_0, \dots, p_k)$ with $p_i \in L_i \cap L_{i+1}$, we consider the moduli space

$$\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)$$

of *smooth* J-holomorphic sections for each section class $B \in \pi_2(E, \mathcal{Q}; \vec{p})$.

The following lemma immediately follows from the definition.

LEMMA 7.4

For $[p_i, w_i]$ with $i = 0, \dots, k$ realizing $B = ([\widetilde{w}_0])\#([w_1]\# \cdots \#[w_k])$, we have the identity

$$\int w_0^*\omega = \sum_{i=1}^k \int w_i^*\omega - \Omega(B) \tag{7.7}$$

or equivalently

$$\Omega(B) = \sum_{i=1}^k \int w_i^*\omega - \int w_0^*\omega. \tag{7.8}$$

Since any section s in class $[s, \partial s] = B$ that satisfies the asymptotic condition $s(z_i) = x_i$, with $x_i \in Q$ with $x_i = (p_i, u(p_i))$ in the trivialization, satisfies $\int s^*\Omega = \Omega(B)$, Lemmas 7.2 and 7.4 imply

$$\frac{1}{2}\int \|Ds\|^2 = \sum_{i=1}^k \int w_i^*\omega - \int w_0^*\omega + \lambda < \infty. \tag{7.9}$$

Once we have this energy estimate, it follows by a standard compactness argument that there are only finitely many section classes B (and so $\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)$ is nonempty) such that $\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)$.

Finally, we state the Gromov-Floer type compactness of $\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)$ in a precise way for later use. For this we introduce additional marked points in the interior of Σ and on the boundary $\partial\Sigma$ besides the punctures $\vec{\zeta}$.

DEFINITION 7.1
A *configuration* on $\dot\Sigma$ is the set of finite points consisting of

$$(x_j) \in \operatorname{Int}\Sigma \quad \text{for } j = 1, \dots, m,$$
$$(y_j^{(i)}) \in \partial_i \dot\Sigma \quad \text{for } j = 1, \dots, n_i \text{ and for } i = 0, \dots, k.$$

We denote by $\widetilde{C}_{m;\vec n}$ the set of such configurations.

We denote such a configuration by

$$C = \big(\{(x_j)\}_{1\le j\le m}; \{(y_j^{(i)})\}_{1\le j\le n_i}, 0 \le i \le k\big)$$

in general. We note that there are no nontrivial holomorphic automorphisms of C except the following cases:

- $\#(\vec\zeta) = 0$ and $2m + \sum_{i=0}^k n_i \le 2$,
- $\#(\vec\zeta) = 1$, $m = 0$ and $k = 1, n_1 = 1$.

We consider the pairs

$$(s, C) \in \widetilde{\mathcal M}_J(E, \mathcal Q; \vec p; B) \times \widetilde{C}_{(m;\vec n)}, \quad \vec n = (n_0, n_1, \dots, n_k).$$

There is a natural $\operatorname{Aut}(\dot\Sigma)$-action on the product $\widetilde{\mathcal M}_J(E, \mathcal Q; \vec p; B) \times \widetilde{C}_{(m;\vec n)}$ defined by

$$(s, C) \mapsto \big(s \circ \phi^{-1}, \phi(C)\big),$$

where $\phi \in \operatorname{Aut}(\dot\Sigma)$. We define $\mathcal M_{J,(m;\vec n)}(E, \mathcal Q; \vec p; B)$ to be the quotient

$$\mathcal M_{J,(m;\vec n)}(E, \mathcal Q; \vec p; B) = \widetilde{\mathcal M}_J(E, \mathcal Q; \vec p; B) \times \widetilde{C}_{(m;\vec n)} / \operatorname{Aut}(\dot\Sigma).$$

We note that when $C \ne \emptyset$, we have the natural evaluation maps

$$\mathrm{ev} : \mathcal M_{(m;\vec n)}(E, \mathcal Q; \vec p; B) \to E^m \times \prod_{i=0}^k Q_i^{n_i},$$

which respect the above-mentioned $\operatorname{Aut}(\dot\Sigma)$-action and so are well defined.

We denote by $\overline{\mathcal M}_1(E_z, J_z; \alpha_z)$ the stable maps of genus zero with one marked point and by $\overline{\mathcal M}_1(E_z, Q_z, J_z; \beta_z)$ the set of bordered stable maps with one marked point at a boundary of the disc. We consider the fiber product

$$\begin{aligned}\widetilde{\mathcal M}_{(m;\vec n)}\big(E, \mathcal Q; \vec p; B_0; \{\alpha_i\}, \{\beta_j^i\}\big) &:= \widetilde{\mathcal M}_{(m;\vec n)}(E, \mathcal Q; \vec p; B_0) \\ {}_{\mathrm{ev}}\times_{\mathrm{ev}} \Big(\prod_i \mathcal M(E_{z_i}, J_{z_i}; \alpha_i) &\times \prod_{i=0}^k \prod_{j=1}^{n_i} \mathcal M(E_{z_i}, Q_{z_i}, J_{z_i}; \beta_j^{(i)})\Big)\end{aligned} \tag{7.10}$$

with respect to the obvious evaluation maps.

7.3. Bubble may hit critical points

In this subsection, we analyze the failure of convergence of a sequence of *smooth* pseudoholomorphic sections $\mathcal M_J(E, \mathcal Q; \vec p; B)$ with

$$\operatorname{vir\,dim} \mathcal M_J(E, \mathcal Q; \vec p; B) = 0 \tag{7.11}$$

of symplectic Lefschetz fibrations $E \subset \Sigma$ with the asymptotic condition provided by $\vec{p}$. We are especially interested in bubble components passing through critical points.

By definition, the pullback $\Xi^* J$ is the standard complex structure on $\mathbb{C}^{n+1}$, $\xi_* j$ is the standard one on $\mathbb{C}$, and $\xi^{-1} \circ \pi \circ \Xi = q$ on the neighborhoods $W \subset \mathbb{C}^{n+1}$, $U \subset \mathbb{C}$ of the origins, corresponding, respectively, to the given holomorphic Morse chart (ξ, Ξ) at each $x_0 \in E^{\mathrm{crit}}$. We denote by $W_{x_0} \subset E$, $U_{x_0} \subset \Sigma$ the corresponding neighborhoods of x_0 and $\pi(x_0)$, respectively. We also denote by $B^{2n+2}(r)$ the ball of radius $r > 0$ in $\mathbb{C}^{n+1}$ with its center at the origin, and by $B_{x_0}(r)$ its image under Ξ at $x_0 \in E^{\mathrm{crit}}$, and similarly for $B^2(\varepsilon)$ and $B_{z_0}(\varepsilon)$.

Since we assume that there are finitely many critical points, we can choose constants $\varepsilon, r > 0$ and the above-mentioned balls $B_{x_0}(r)$ and $B^2(\varepsilon)$ so that

$$B_z(\varepsilon) \cap B_{z'}(\varepsilon) = \emptyset \quad \text{for } z \neq z' \text{ with } z = \pi(x),\ z' = \pi(x') \tag{7.12}$$

for $x, x' \in E^{\mathrm{crit}}$ and

$$\pi\big(B_{x_0}(r)\big) \supset B_{z_0}(\varepsilon) \quad \text{for all } x_0 \in E^{\mathrm{crit}} \cap E_{z_0}. \tag{7.13}$$

LEMMA 7.5

The graph of any differentiable *section does not intersect* E^{crit}.

Proof

Since s is a section, we have $\pi \circ s = \mathrm{id}$. By differentiating this, we obtain $d\pi \circ ds = \mathrm{Id}$. In particular, $d\pi$ is surjective at any point $s(z)$; that is, $s(z)$ must be a regular point of π, and hence $s(z) \in E \setminus E^{\mathrm{crit}}$. □

Obviously we have the inequality

$$\mathrm{dist}\big(s(z), E^{\mathrm{crit}}\big) \geq C > 0 \quad \text{for } z \in \Sigma \setminus \bigcup_{x \in E^{\mathrm{crit}}} B_{\pi(x)}(\varepsilon), \tag{7.14}$$

where $C = C(\varepsilon, (\xi, \Xi))$ is a constant depending only on ε and the holomorphic Morse chart (ξ, Ξ) independent of s. Now we consider the restriction of s on $\bigcup_{x \in E^{\mathrm{crit}}} B_{\pi(x)}(\varepsilon)$. On this neighborhood, we can identify the section s to the holomorphic map

$$f : B^2(\varepsilon) \subset \mathbb{C} \to B^{2n+2}(r) \subset \mathbb{C}^{n+1}$$

satisfying $q \circ f(z) = z$ for all $z \in B^2(\varepsilon)$; that is,

$$f_1^2(z) + \cdots + f_{n+1}^2(z) = z.$$

In particular, we have

$$|f(z)|^2 = \sum_{j=1}^{n+1} |f_j^2(z)| \geq |f_1^2(z) + \cdots + f_{n+1}^2(z)| = |z|$$

for all $z \in B^2(\varepsilon)$. Therefore we obtain

$$|f(z)| \geq \sqrt{\varepsilon} \quad \text{for } z \in \partial B^2(\varepsilon). \tag{7.15}$$

The following theorem is the main theorem proved in this section.

THEOREM 7.6
Suppose that the Lefschetz Hamiltonian fibration with Lagrangian boundary $Q \subset E|_{\partial\Sigma}$ *such that* E *is relative Calabi-Yau and* Q *has vanishing fiberwise Maslov class. Then there exists a dense subset of* j*-compatible* J*'s such that for any such* J*, there exists a constant* $C > 0$ *depending only on* (E, Q, J, j)*, the section class* $[s]$*, and* $\varepsilon > 0$ *such that we have*

$$\operatorname{dist}(\operatorname{Im} s, E^{\text{crit}}) \geq C \tag{7.16}$$

for any smooth *section* $s : \Sigma \to E$*.*

We prove Theorem 7.6 by contradiction. Let J be j-compatible and suppose that there is a sequence s_i of smooth J-holomorphic sections such that

$$\min_{z\in\Sigma} \operatorname{dist}\big(s_i(z), E^{\text{crit}}\big) \to 0. \tag{7.17}$$

Since E^{crit} is a finite set and by (7.12), we may choose a critical point $x_0 \in E^{\text{crit}}$ and a sequence $z_i \in \Sigma$ such that

$$\operatorname{dist}\big(s_i(z_i), x_0\big) \to 0. \tag{7.18}$$

By choosing a subsequence of z_i if necessary, we may assume that $z_i \to z_0$, and so

$$z_i \in B_{z_0}(\varepsilon) \tag{7.19}$$

for all i. By the Gromov-Floer convergence applied to J-holomorphic curves $s_i : (\Sigma, j) \to (E, J)$, which are also J-holomorphic sections, there exists a subsequence that converges to

$$s_\infty = s_0 + \text{bubble components},$$

where s_0 is a smooth section of $E \to \Sigma$ and where each bubble must be either a fiberwise pseudoholomorphic sphere or a fiberwise pseudoholomorphic disc. And each disc bubble has its boundary lying in the given Lagrangian boundary condition.

Due to property (7.19), at least one bubble must pass through the critical point x_0 whose image is contained in E_{z_0}. By the connectedness of the image of the limit, this bubble is contained in a bubble tree rooted at a point $z_1 \in \Sigma$ in the principal component (s_0, Σ). The image of this bubble tree itself must be contained in the same fiber E_{z_0}. Denote this bubble tree by $(v, (C, z))$, $z \in C$, which is a stable map in E such that

$$v(z) = s_0(z_1) \in E.$$

However, since E_{z_0} contains a singularity x_0 and so is not a smooth manifold, we need further clarification on the bubble component passing through x_0. Since $\pi : E \to \Sigma$ is isomorphic to the standard Lefschetz fibration

$$q(z_1, \dots, z_{n+1}) = z_1^2 + \cdots + z_{n+1}^2$$

near x_0, there is a well-defined multiplicity of the component at the critical point x_0. We make this statement precise in the next subsection.

7.4. Proper holomorphic curves in $E_{z_0} \setminus \{x_0\}$

Using the holomorphic Morse chart (ξ, Ξ) at the critical point $x_0 \in E_{z_0} \subset E$, we consider the decomposition

$$E = B_{x_0}(\delta) \coprod \big(E \setminus B_{x_0}(\delta)\big),$$

where $B_{x_0}(\delta) = \Xi^{-1}(B^{2(n+1)}(\delta)) \cap E_{z_0}$ for $0 < \delta < \varepsilon$.

Now we consider the hypersurface

$$q^{-1}(0) = \big\{(x_1, \dots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 0\big\}.$$

The only singularity of this hypersurface is $0 \in \mathbb{C}^{n+1}$, and so $q^{-1}(0) \setminus \{0\}$ is a smooth complex hypersurface of $\mathbb{C}^{n+1} \setminus 0$. We denote by $\theta_{\mathbb{C}^{n+1}}$ the 1-form

$$\theta_{\mathbb{C}^{n+1}} = \frac{i}{4}\Big(\sum x_k \, d\overline{x}_k - \overline{x}_k \, dx_k\Big)$$

and the standard Kähler form

$$\omega_{\mathbb{C}^{n+1}} = -d\theta_{\mathbb{C}^{n+1}} \Big(= \sum dq_k \wedge dp_k\Big),$$

where $x_k = q_k + ip_k$. We denote by θ and ω the restriction of these to $q^{-1}(0) \setminus \{0\}$.

Following [Se3], we denote $T = T^*S^n$, and $T(0)$ equals the zero section of T, and by θ_T and $\omega_T = -d\theta_T$, respectively, we denote the standard Liouville 1-form and the standard symplectic form on the cotangent bundle T. We identify T with the subset

$$\big\{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \bigm| \langle u, v\rangle = 0, \|v\| = 1\big\}$$

and then consider the map

$$\Phi : q^{-1}(0) \setminus \{0\} \to T \setminus T(0)$$

defined by

$$\Phi(x) = \big(\mathrm{im}(\widehat{x}) \| \mathrm{re}(\widehat{x})\|, \mathrm{re}(\widehat{x}) \| \mathrm{re}(\widehat{x})\|^{-1}\big), \tag{7.20}$$

where $se^{i\alpha}$ are polar coordinates on the base of $q : \mathbb{C}^{n+1} \to \mathbb{C}$, and $\widehat{x} = e^{-i\alpha/2}$. We note that this map is equivariant with respect to the canonical $O(n+1)$-actions on $q^{-1} \setminus \{0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$ and $T = T^*S^n$.

The following is a consequence of a straightforward calculation, which is a restriction of the identity in [Se3, p. 1014].

LEMMA 7.7

Φ is a diffeomorphism such that

$$\Phi^*\theta_T = \theta,$$

and so $\Phi^*\omega_T = \omega$. In particular, the symplectic manifold $(q^{-1}(0) \setminus \{0\}, \omega)$ is symplectomorphic to the cotangent bundle $T = T^*S^n$.

This lemma shows that $E_{z_0} \setminus \{x_0\}$ is a symplectic manifold with *negative* cylindrical end whose asymptotic boundary is symplectomorphic to the unit cosphere bundle $S^1(T^*S^n)$. Furthermore, complex structure on $\Xi^{-1}(W)$ is required to be induced from the standard complex structure from $q^{-1}(0) \setminus \{0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$. Therefore any j-compatible J provides a translational invariant almost complex structure with respect to the cylindrical structure, and any J-holomorphic curve is genuinely holomorphic near the end of $q^{-1}(0) \setminus \{0\}$.

In particular, any such curve converges to a Reeb orbit of $S^1(T^*S^n)$ with finite multiplicity. This motivates us to study the moduli problem of proper J-holomorphic curves from $\mathbb{C} \cong \mathbb{C}P^1 \setminus \{N\}$ with the asymptotic boundary condition given by (γ, k), where γ is a simple Reeb orbit of $S^1(T^*S^n)$ and a multiplicity $k \in \mathbb{Z}_+$. We denote by $\widetilde{\mathcal{R}}_1(S^1(T^*S^n))$ the set of parameterized Reeb orbits on $S^1(T^*S^n)$ with period 2π and by $\mathcal{R}_1(S^1(T^*S^n))$ the quotient by the natural S^1-action, that is, the set of unparameterized Reeb orbits.

We denote by (s, Θ) the cylindrical coordinates of $T \setminus T(0)$, where s and Θ are defined by

$$s(q,p) = \|p\|, \qquad \Theta(q,p) = \left(q, \frac{p}{|p|}\right).$$

Note that all geodesics on S^n have the same period, which implies that the contact manifold $S^1(T^*S^n)$ is foliated by Reeb orbits, all of which have the same period. We denote the corresponding cylindrical coordinates on $B_{x_0}(\varepsilon) \setminus \{x_0\} \subset E \setminus \{x_0\}$ by the same letters (s, Θ). By a suitable translation of s-coordinates, we may assume the identification

$$(s, \Theta) : B_{x_0}(\varepsilon) \setminus \{x_0\} \to (-\infty, 0] \times S^1(T^*S^n).$$

Now we are ready to define the moduli space of our interest. For simplicity of notation, we denote

$$E_{z_0} \setminus \{x_0\} = E^*_{z_0},$$

and $\dot S = S \setminus \{z_0\}$ is either an open Riemann surface isomorphic to $\mathbb{C}$ or an open Riemann surface with boundary isomorphic to $\mathbb{C} \setminus D^1(1)$. We fix an analytic coordinate $z = e^{\tau + it}$ near $z_0 \in S$. Let $u : \dot S \to E_{z_0} \setminus \{x_0\}$ be a pseudoholomorphic curve with Lagrangian boundary condition

$$u(\partial S) \subset Q_{z_0} \subset E_{z_0} \setminus \{x_0\}.$$

Since the treatment of the latter is essentially similar to the former, we focus on the former case in the following exposition. We briefly mention the latter case at the end of our discussion.

By the properness and exponential convergence property of u as $\tau \to -\infty$, we have

$$\operatorname{Im} u|_{(-\infty,-R] \times S^1} \subset B_{x_0}(\varepsilon)$$

for a sufficiently large $R > 0$. It is proved in [H] that

$$T = \lim_{\tau \to -\infty} \int (\Theta \circ u_\tau)^* \lambda$$

with $T = 2\pi k$ for some integer $k \geq 1$, where $u_\tau(t) := u(\tau, t)$. Then it is proved in [H] and [Bo] that there exist constants $C, \delta > 0$ depending only on $(S^1(T^*S^n), \lambda)$ such that

$$\lim_{\tau - \infty} \operatorname{dist}\big(u(\tau/T, t/T), u_\gamma(\tau + \tau_0, t + t_0)\big) \leq Ce^{-\delta|\tau|}$$

for some simple Reeb orbit γ of $S^1(T^*S^n)$ and $\tau_0 \in \mathbb{R}$ and $t_0 \in S^1$. (See also [FO+4] for a proof of a similar exponential convergence result in the relative context.) Here $u_\gamma : [0, \infty) \times S^1 \to (-\infty, 0] \times S^1(T^*S^n)$ denotes the trivial cylinder map $u_\gamma(t) = (\tau, \gamma(t))$.

By the above discussion, we can now define the following moduli spaces for each given integer $k \geq 1$ and a homotopy class A.

DEFINITION 7.2

Let $\gamma \in \mathcal{R}_1(S^1(T^*S^n))$ and $k \in \mathbb{Z}_+$. For each given (γ, k), we define

$$\begin{aligned}\mathcal{M}^{\mathrm{SFT}}_{z_0}(E^*, J_0, \gamma; A, k) = \Big\{u : \dot{\Sigma} \to E^*_{z_0} \;\Big|\; \overline{\partial}_{J_0} u = 0, \int u^*\Omega < \infty, \\ \lim_{\tau \to -\infty} u(\tau/2\pi k, t/2\pi k) = u_\gamma(t), [u] = A\Big\}.\end{aligned}$$

We then define

$$\mathcal{M}^{\mathrm{SFT}}_{z_0}(E^*, J_0; A, k) = \bigcup_{\gamma \in \mathcal{R}_1(S^1(T^*S^n))} \mathcal{M}^{\mathrm{SFT}}_{z_0}(E^*, J_0, \gamma; A, k).$$

The following general index formula can be derived from [Bo, Corollary 5.4]. In this regard, we note that the dimension of the space $\mathcal{R}_1(S^1(T^*S^n))$ of simple Reeb orbits of $S^1(T^*S^n)$ is n.

PROPOSITION 7.8

Let $u \in \mathcal{M}^{\mathrm{SFT}}_{z_0}(E^*, J_0; A, k)$. Then we have

$$\operatorname{Index} D_u \overline{\partial}_J = \Big(-\mu_{CZ}(\gamma) + \frac{n}{2}\Big) + (n - 3) + 2c_1(u; \phi_\gamma), \tag{7.21}$$

where μ_{CZ} is the generalized Conley-Zehnder index defined by Robbin and Salamon [RS].

For the reader's convenience, we provide the precise definitions of $\mu_{CZ}(\gamma)$ and $c_1(u; \phi_\gamma)$ in the appendix. Once the definitions are made precise, its proof follows from that of [Bo].

The following transversality result is an easy consequence of a standard argument whose proof is omitted. We first note that we have

$$\mu_{CZ}(\gamma) = \mathrm{Morse}(\gamma) + \frac{\dim \mathcal{R}_{\mathrm{sim}}}{2}$$

for the Reeb orbit at the *negative* end (see, e.g., [Mo], [EGH, Corollary 1.7.4] for such a formula). We denote by

$$\mathcal{M}^{\mathrm{SFT}}_{\mathrm{inj}}(E^*, J_0; A, k) \subset \mathcal{M}^{\mathrm{SFT}}(E^*, J_0; A, k)$$

the open subset consisting of somewhere injective curves.

PROPOSITION 7.9

Let $x_0 \in E^{\mathrm{crit}}$. There exists a dense subset $\mathcal{J}^{\mathrm{tr}}(x_0)$ of the set $\mathcal{J}(j)$ of Ω-compatible almost complex structures such that $\mathcal{M}^{\mathrm{SFT}}_{\mathrm{inj}}(E^*, J_0; A, k)$ is Fredholm-regular and so becomes a smooth manifold of dimension

$$-\operatorname{Morse}(\gamma) + (n-3) + 2c_1(u; \phi_\gamma) \tag{7.22}$$

for any $k \geq 1$.

An immediate corollary of this proposition is the following vanishing result.

COROLLARY 7.10

Let $x_0 \in E^{\mathrm{crit}}$ and $E^* = E \setminus \{x_0\}$. Suppose that the relative Maslov class of $E \to \Sigma$ is zero. Then for any $J_0 \in \mathcal{J}^{\mathrm{tr}}(x_0)$, $\mathcal{M}^{\mathrm{SFT}}(E^*, J_0; A, k) = \emptyset$ for all A and k.

Proof

Since any element of $\mathcal{M}^{\mathrm{SFT}}(E^*_{z_0}, J_0; A, k)$ is a composition of a somewhere injective curve and a branched covering of the domain, it is enough to prove that

$$\mathcal{M}^{\mathrm{SFT}}_{\mathrm{inj}}(E^*, J_0; A, k) = \emptyset$$

for a generic choice of J_0.

By the assumption, we have $c_1(A) = 0$ for all $A \in \pi_2(E^*_{z_0}; z_0)$. Furthermore, the Morse index for the simple closed geodesic of S^n is given by $n-1$ and is greater than $n-1$ for multiple geodesics. Therefore we derive

$$\dim \mathcal{M}^{\mathrm{SFT}}_{\mathrm{inj}}(E^*_{z_0}, J_0; A, k) \leq -(n-1) + n - 3 = -2$$

for all $k \geq 1$ from (7.22) and hence the proof. $\square$

Finally, we briefly mention the case in which $\dot{S} \cong \mathbb{C} \setminus D^2(1)$ and $u(\partial S) \subset Q_{z_0}$. Since we assume that Q has vanishing fiberwise Maslov class, the corresponding moduli space

$$\mathcal{M}^{\mathrm{SFT}}(E^*_{z_0}, Q_{z_0}, J_0; A, k)$$

has its dimension exactly the same as that of $\mathcal{M}^{\mathrm{SFT}}(E^*_{z_0}, J_0; A, k)$. Therefore the same dimension-counting argument applies in exactly the same way (see, e.g., [Mo] for a discussion on the dimension formula in the context of Lagrangian boundary).

We note that the above dimension-counting argument strongly relies on the fact that E is (fiberwise) Calabi-Yau so that $c_1(u; \phi_\gamma) = 0$ and Q has vanishing fiberwise Maslov class. It may be interesting to investigate how the long exact sequence will be transformed in other contexts, such as in the Fano case.

7.5. Bubble does not hit critical points

In this subsection, we restrict ourselves to the case of vanishing relative first Chern class $c_1(E) = 0$.

We prove Theorem 7.6 in this subsection. In fact, it is enough to take

$$\mathcal{J}^{\mathrm{tr}} := \bigcap_{x_0 \in E^{\mathrm{crit}}} \mathcal{J}^{\mathrm{tr}}(x_0)$$

for the dense subset of j-compatible J's.

Proof of Theorem 7.6
Let $J \in \mathcal{J}^{\mathrm{tr}}$ be defined as above.

We have derived before by the Gromov-Floer compactness applied to $\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)$ in E that there exists a subsequence which converges to

$$s_\infty = s_0 + \text{bubble components},$$

where s_0 is a smooth section of $E \to \Sigma$ and each bubble tree is contained in a fiber which consists of either fiberwise pseudoholomorphic spheres or discs. By Corollary 7.10, there cannot be any bubble tree passing through a critical point of E and so contained in $E \setminus E^{\mathrm{crit}}$. Since the principal component s_0, which is smooth, cannot pass through a critical point by Lemma 7.5, we have proved that the Gromov-Floer limit of $s_i : \Sigma \to E \setminus E^{\mathrm{crit}}$ does not pass any critical point of E. Therefore the compactification $\overline{\mathcal{M}}_J(E, \mathcal{Q}; \vec{p}; B)$ of $\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)$ has image contained in $E \setminus E^{\mathrm{crit}}$.

Once we have achieved this, the rest of the proof follows, in the same way as in the case of *smooth* Hamiltonian fibration, by the standard dimension-counting argument from the fact that $E \to \Sigma$ is a Hamiltonian Lefschetz fibration with vanishing relative Maslov class. We consider the evaluation maps

$$\mathrm{ev} : \mathcal{M}_{(m;\vec{n})}(E, \mathcal{Q}; \vec{p}; B) \to E^m \times \prod_{i=0}^{k} Q_i^{n_i}$$

and

$$\mathrm{ev}_1 : \mathcal{M}_1(E_z, J_z; \alpha_z) \to E_z, \qquad \mathrm{ev}_1 : \mathcal{M}_1(E_z, Q_z, J_z; \beta_z) \to E_z$$

for $z \in \Sigma \setminus \Sigma^{\mathrm{crit}}$ and consider the fiber product

$$\widetilde{\mathcal{M}}_{(m;\vec{n})}\big(E, \mathcal{Q}; \vec{p}; B_0; \{\alpha_i\}, \{\beta^i_j\}\big) := \widetilde{\mathcal{M}}_{(m;\vec{n})}(E, \mathcal{Q}; \vec{p}; B_0)$$

$$ {}_{\mathrm{ev}} \times_{\mathrm{ev}_1} \Big(\prod_i \mathcal{M}(E_{z_i}, J_{z_i}; \alpha_i) \times \prod_{i=0}^{k} \prod_{j=1}^{n_i} \mathcal{M}(E_{z_i}, Q_{z_i}, J_{z_i}; \beta^{(i)}_j)\Big) \tag{7.23}$$

with respect to the obvious evaluation maps.

Note that the dimension of the moduli space of a holomorphic sphere in any class in a fiber $E_z \setminus E^{\mathrm{crit}}$ has virtual dimension given by $2n - 6$, and so

$$\mathrm{vir.dim} \bigcup_{z \in \Sigma} \mathcal{M}_1(E_z, J_z; \alpha_z) = 2n - 4.$$

In particular, for a generic choice of J all somewhere injective holomorphic spheres are regular, and hence we have the condition $c_1 = 0$, which implies semi-positivity; the standard argument from [RT] implies that the evaluation image of the moduli space in each fiber has at least codimension 4, which obviously avoids the images of pseudoholomorphic sections in the classes whose associated moduli space has dimension zero. This proves that there cannot be any bubble passing through critical points of E in the limit, and hence the above fiber product, for a generic choice of J, becomes empty. The relevant Fredholm theory needed to perform this kind of dimension-counting argument is by now standard. We refer to [Se3, Section 2] for an elegant exposition on this Fredholm theory in the context of exact Lefschetz fibrations, which applies to the current context of Lefschetz Hamiltonian fibrations without change. This proves that there exist a dense subset $\mathcal{J}^{\mathrm{reg,tr}} \subset \mathcal{J}^{\mathrm{tr}}$ such that for any given $(\vec{p}; B)$ the moduli spaces $\mathcal{M}_{(m;\vec{n})}(E, \mathcal{Q}, J; \vec{p}; B)$ is compact for $J \in \mathcal{J}^{\mathrm{reg,tr}}$.

In particular, there exists a constant $C = C(E, Q, \vec{p}; B) > 0$:

$$\min_{z\in\Sigma} \operatorname{dist}\big(s_\infty(z), E^{\mathrm{crit}}\big) > C.$$

But this together with Hausdorff convergence of the image of s_i to that of s_∞ contradicts the assumption that $s_i(z_i) \to x_0 \in E^{\mathrm{crit}}$. This finishes the proof. $\square$

This proposition shows that as far as compactness property of the set of smooth pseudoholomorphic sections is concerned, we can ignore the presence of critical points of the fibration $E \to \Sigma$.

7.6. Gromov-Floer moduli space of J-holomorphic trajectories

With Theorem 7.6 at our disposal, we can safely ignore the critical points in the study of *smooth* J-holomorphic sections and their degenerations for the Lefschetz Hamiltonian fibrations $\pi : E \to \Sigma$. This, together with the energy estimates derived from Section 6, makes the study essentially the same as for the case of usual smooth Hamiltonian fibrations (without critical points) as studied in [En] and [MS].

We formulate the definition of Fukaya, Oh, Ohta, and Ono's $\mathfrak{m}_k$-maps in the setting of fibrations. Obviously this discussion can be extended to the Lefschetz Hamiltonian fibration $\pi : E \to \Sigma$ with punctures $\vec{\zeta} = \{\zeta_0, \ldots, \zeta_k\} \subset \partial\dot{\Sigma}$ and with a chain of Lagrangian boundary conditions

$$\mathcal{Q} = (Q_0, \ldots, Q_k)$$

by considering the anchor caps attached to $[p_{ij}, w_{ij}]$ with $p \in L_i \cap L_j$. Here Q_j is the parallel fiberwise Lagrangian submanifold corresponding to L_j. We decompose

$$\partial\dot{\Sigma} = \coprod_{j=0}^{k} \partial_j \Sigma,$$

where $\partial_j\Sigma$ is the jth connected component of $\partial\dot{\Sigma}$. We briefly add some necessary modification from [FO+1] to accommodate the possible critical point in the fibration $\pi : E \to \Sigma$ following the notation from [Se3].

For a given compact surface Σ with marked points $\vec{\zeta}$, we consider the corresponding surface $\dot{\Sigma} = \Sigma \setminus \vec{\zeta}$ with punctures. We fix a given preferred holomorphic chart $\varphi_\zeta : D_\zeta \subset \Sigma \to D^+$ of the half-disc $D^+ = D \cap \{\mathrm{im}(z) \geq 0\}$ with $\varphi_\zeta(\zeta) = 0$. We consider the moduli space

$$\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)$$

as in Section 7.2 for all the section class $B \in \pi_2(E, \mathcal{Q}; \vec{p})$ with

$$\text{vit.dim}\, \mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B) = 0.$$

More specifically, the map is supposed to be given by

$$\Phi_0^{\mathrm{rel}}(E, \pi; \mathcal{Q})\Big(\bigotimes_{j=1}^{k}[p_j, w_j]\Big) = \sum_k \#\big(\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B)\big)[p_k, w_k]$$

with a suitable definition of the coefficient $\#(\mathcal{M}_J(E, \mathcal{Q}; \vec{p}; B))$.

Finally, we recall the notion of broken Floer trajectory moduli spaces; that is, the case corresponds to $k + 1 = 2$ (see [FOn] for the corresponding definition for the closed case).

DEFINITION 7.3

Let $J = \{J_t\}_{0 \leq t \leq 1}$, and let $x, y \in L_+ \cap L_-$. A *stable broken Floer trajectory* from p to q is a triple

$$u = \big((u_1, \ldots, u_a); (\sigma_1, \ldots, \sigma_m), (\gamma_1^0, \ldots, \gamma_{n_0}^0), (\gamma_1^1, \ldots, \gamma_{n_1}^1); o\big)$$

that satisfies the following.

(1) For $i = 1, \ldots, a-1$, $u_i \in \mathcal{M}(x_i, x_{i+1})$ and satisfies

$$u_1(-\infty) = p, u_a(\infty) = q,$$
$$u_i(\infty) = u_{i+1}(-\infty) \quad \text{for } i = 1, \ldots, a-1. \tag{7.24}$$

We call (7.24) the *matching condition*, and we say that a pair (u, u') of Floer trajectories is *gluable* if it satisfies the matching condition.

(2) We have $\sigma_i \in \overline{\mathcal{M}}_1(J_{t_i}; \alpha_i)$ for $i = 1, \ldots, m$.

(3) We have $\gamma_j^0 \in \overline{\mathcal{M}}_1(L_0, J_0; \beta_j^0)$ for $j = 1, \ldots, n_0$ and $\gamma_k^1 \in \overline{\mathcal{M}}_1(L_1, J_1; \beta_k^1)$ for $k = 1, \ldots, n_1$.

(4) For each $\ell = 1, \ldots, a$, either the map u_ℓ is nonstationary or $\Theta_\ell \cap \mathrm{Im}\, o \neq \emptyset$.

We denote the domain of u simply by Θ_u, which is the product of a broken configuration. This is the union of finite copies of

$$\mathbb{R} \times [0, 1],$$

the *principal components*, and the prestable curves of closed or bordered Riemann surfaces of genus zero are the *bubble components* with their roots attached to the principal components of Θ_u. Each broken Floer trajectory $u : \Theta_u \to M$ can be regarded as a broken Floer trajectory into the fiber M_ζ for a $\zeta \in \vec{\zeta}$. We denote by $\widetilde{\mathcal{M}}(p,q;B_0;\{\alpha_i\},\{\beta_{0,j}\},\{\beta_{1,k}\})$ the set of stable broken Floer trajectories in the prescribed topological types. The group $\mathrm{Aut}(\dot\Sigma)$ acts on the moduli space $\widetilde{\mathcal{M}}(p,q;B_0;\{\alpha_i\},\{\beta_{0,j}\},\{\beta_{1,k}\})$ by the simultaneous translation of the roots of the bubbles attached to the principal component. Then we denote by $\mathcal{M}(p,q;B)$ the set of all stable broken trajectories in class $B \in \pi_2(p,q)$.

We then define

$$\overline{\mathcal{M}}(E,\mathcal{Q};\vec{p};B) = \coprod \mathcal{M}_{(m;\vec{n})}(E,\mathcal{Q};\vec{p}\,';B';\{\alpha_i\},\{\beta^i_j\})\#\Big(\prod_i \mathcal{M}(p'_i,q_i;B_i)\Big)$$

for all choices of $B_0, \{\alpha_i\}, \{\beta^i_j\}$, and B_i's satisfying

$$B = B_0 + \sum_i \alpha_i + \sum_{i=0}^{k}\Big(\sum_{j=1}^{n_i}\beta_j^{(i)}\Big) + \sum B_i$$

and provide it with a topology of stable maps. We denote the corresponding decomposition of maps by

$$s = s^0\#\Big(\prod_i v_i\Big)\#\Big(\#_{i=0}^k\Big(\prod_j w_j^{(i)}\Big)\Big)\#(u_i),$$

and we call s^0 the *principal component* and other fiberwise curves the *bubble components.*

At this stage, we emphasize that this compactification is defined as a topological space for *any* choice of (j,J) for a transversal chain $(L_0,\ldots,L_k)$ of Lagrangian submanifolds. The topological space $\overline{\mathcal{M}}(E,\mathcal{Q};\vec{p};B)$ will not be a smooth "manifold" even for a generic choice of J, but will be a space with Kuranishi structure (see [FOn]).

7.7. Orientation and signs

Recall that our main object of concern is the collection $\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}}$ consisting of anchored CY Lagrangian branes and are the CY Lagrangian branes, that is, $((L,\gamma),s,[b])$. We also recall that the action of the Dehn twist τ_L preserves this collection.

In particular, we assume that L is orientable and spin. Because of this we ignore the background class *st* in the definition of relative spin structures; that is, we restrict ourselves to $st = 0$. The existence of compatible orientations on the ordinary Floer moduli spaces

$$\mathcal{M}_{k+1}(\mathcal{L},\vec{p};B),$$

together with some enhancement on their fiber products with chains in Lagrangian submanifolds, is already established in [FO+2, Section 6] based on the analysis of compatible orientations given in [FO+1, Chapter 6]. This establishes all the necessary ingredients needed for the construction of A_∞-structures on the Fukaya category $\mathfrak{Fuk}(\mathcal{E}^{\mathrm{CY}}_{\mathrm{brane}})$.

For the purpose of constructing Seidel-type exact sequences over $\mathbb{Z}$-coefficients, not just $\mathbb{Z}_2$-coefficients, in addition to the study performed in [FO+1] and [FO+2], we need to study orientation of the moduli space of pseudoholomorphic sections

$$\mathcal{M}(E, \mathcal{Q}, J; \vec{p}; B)$$

and establish the morphism $\Phi_0^{\mathrm{rel}}(E, \pi; \mathcal{Q})$ over $\mathbb{Z}$. Recall that Seidel considered this morphism over $\mathbb{Z}_2$ in [Se3] but extended his construction over $\mathbb{Z}$ by incorporating the study of orientations in [Se4] in the exact framework.

For the reader's convenience, we first borrow the exposition of orientations from [FO+2] in this subsection. We examine the orientation on $\mathcal{M}(E, \mathcal{Q}, J; \vec{p}; B)$ later in Section 9.3 in relation to the construction of Seidel's map $\mathfrak{c}$ which is induced by the Lefschetz Hamiltonian fibration associated to the Dehn twist.

Let $p, q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$. We consider $u : \mathbb{R} \times [0,1] \to M$ such that

$$\frac{du}{d\tau} + J\frac{du}{dt} = 0, \tag{7.25a}$$

$$u(\tau, 0) \in L_0, \qquad u(\tau, 1) \in L_1, \qquad \int u^*\omega < \infty, \tag{7.25b}$$

$$u(-\infty, \cdot) \equiv p, \qquad u(\infty, \cdot) \equiv q. \tag{7.25c}$$

It induces a continuous map $\overline{u} : [0,1]^2 \to M$ with $\overline{u}(0,t) \equiv p, u(1,t) \equiv q$ in an obvious way. With an abuse of notation, we denote by $[u]$ the homotopy class of the map $\overline{u}$ in $\pi_2(p,q)$. We denote by $\widetilde{\mathcal{M}}^\circ(p,q;B)$ the moduli space consisting of the maps u satisfying (7.25) and compactify $\widetilde{\mathcal{M}}^\circ(p,q;B)/\mathbb{R}$ its quotient by the τ-translations by using an appropriate notion of stable maps as in [FO+1, Section 2.1].

If (L_0, L_1) is a relative spin pair, then $\mathcal{M}(p,q;B)$ is orientable. Furthermore, a choice of relative spin structures gives rise to a compatible system of orientations for $\mathcal{M}(p,q;B)$ for all pairs $p, q \in L_0 \cap L_1$ and $B \in \pi_2(p,q)$. For the sake of completeness, we now recall from [FO+1, Section 8.1]) how the relative spin structure gives rise to a system of coherent orientations.

Let $p \in L_0 \cap L_1$ and $[w] \in \pi_2(\ell_{01}; p)$ for a relevant map $w : [0,1]^2 \to M$ as before. We denote by $\mathrm{Map}(\ell_{01}; p; L_0, L_1; \alpha)$ the set of such maps $[0,1]^2 \to M$ and denote its homotopy class by $[w] = \alpha$ in $\pi_2(\ell_{01}; p)$. Let $w \in \mathrm{Map}(\ell_{01}; p; L_0, L_1; \alpha)$. Let $\Phi : w^*TM \to [0,1]^2 \times T_pM$ be a (homotopically unique) symplectic trivialization as before. The trivialization Φ, together with the boundary condition $w(0,t) = \ell_{01}(t)$ and the Lagrangian path λ_{01} along ℓ_{01}, defines a Lagrangian path

$$\lambda^\Phi = \lambda^\Phi_{([p,w];\lambda_{01})} : [0,1] \to T_pM$$

satisfying $\lambda^\Phi(0) = T_pL_0, \lambda^\Phi(1) = T_pL_1$. The homotopy class of this path does not depend on the trivialization Φ but depends only on $[p,w]$ and the homotopy class of λ_{01}. Hereafter we omit Φ from the notation.

We remark that relative spin structure determines a trivialization of $V_{\lambda_{01}(0)} \oplus T_{\lambda_{01}(0)}L_0 = V_{\lambda_{01}(0)} \oplus \lambda_{01}(0)$ and $V_{\lambda_{01}(1)} \oplus T_pL_1 = V_{\lambda_{01}(1)} \oplus \lambda_{01}(1)$. We take and fix away to extend this trivialization to the family $\ell_{01}^*V \oplus \lambda_{01}$ on $[0,1]$.

We consider the following boundary-valued problem for the section ξ of w^*TM on $\mathbb{R}_{\geq 0} \times [0,1]$ of a $W^{1,p}$-class such that

$$D_w\overline{\partial}(\xi) = 0, \tag{7.26a}$$

$$\xi(0,t) \in \lambda_{01}(t), \qquad \xi(\tau,0) \in T_pL_0, \qquad \xi(\tau,1) \in T_pL_1. \tag{7.26b}$$

Here $D_w\overline{\partial}$ is the linearization operator of the Cauchy-Riemann equation.

We define $W^{1,p}(\mathbb{R}_{\geq 0} \times [0,1], T_pM; \lambda_{01})$ to be the set of sections ξ of w^*TM on $\mathbb{R}_{\geq 0} \times [0,1]$ of a $W^{1,p}$-class satisfying (7.26b). Then (7.26a) induces an operator

$$D_w\overline{\partial} : W^{1,p}(\mathbb{R}_{\geq 0} \times [0,1], T_pM; \lambda) \to L^p(\mathbb{R}_{\geq 0} \times [0,1], T_pM \otimes \Lambda^{0,1}),$$

which we denote by $\overline{\partial}_{([p,w];\lambda_{01})}$. The following proposition was proved in [FO+1, Lemma 3.7.69].

PROPOSITION 7.11

We have

$$\operatorname{Index} \overline{\partial}_{([p,w];\lambda_{01})} = \mu([p,w];\lambda_{01}). \tag{7.27}$$

We denote its determinant line by

$$\det \overline{\partial}_{([p,w];\lambda_{01})}.$$

By varying w in its homotopy class $\alpha \in \pi_2(\ell_{01};p) = \pi_2(\ell_{01};p;L_0,L_1)$, these lines define a line bundle

$$\det \overline{\partial}_{([p,w];\lambda_{01})} \to \operatorname{Map}(\ell_{01};p;L_0,L_1;\alpha). \tag{7.28}$$

The bundle (7.28) is trivial if (L_0,L_1) is a relatively spin pair (see [FO+1, Section 8.1]).

We need to find a systematic way to orient (7.28) for various $\alpha \in \pi_2(\ell_{01};p)$ simultaneously. Following [FO+1, Section 8.1.3], we proceed as follows. Let $\lambda_p : [0,1] \to T_pM$ be a path connecting from T_pL_0 to T_pL_1 in $\operatorname{Lag}^+(T_pM,\omega)$. The relative spin structure determines a trivialization of $V_p \oplus T_pL_0 = V_p \oplus \lambda_p(0)$ and of $V_p \oplus T_pL_1 = V_p \oplus \lambda_p(1)$. We fix an extension of this trivialization of the $[0,1]$-parameterized family of vector spaces $V_p \oplus \lambda_p$. We define

$$Z_+ = \{(\tau,t) \in \mathbb{R}^2 \mid \tau \leq 0, 0 \leq t \leq 1\} \cup \{(\tau,t) \mid \tau^2 + (t-1/2)^2 \leq 1/4\}. \tag{7.29}$$

We consider maps $\xi : Z_+ \to T_pM$ of the $W^{1,p}$-class and study the linear differential equation

$$\overline{\partial}\xi = 0, \tag{7.30a}$$

$$\xi(e^{\pi i(t-1/2)}/2 + i/2) \in \lambda_p(t), \qquad \xi(\tau,0) \in T_pL_0, \qquad \xi(\tau,1) \in T_pL_1. \tag{7.30b}$$

It defines an operator

$$W^{1,p}(Z_+, T_pM; \lambda_p) \to L^p(Z_+; T_pM \otimes \Lambda^{0,1}),$$

which we denote by $\overline{\partial}_{\lambda_p}$. Let $\mathrm{Index}\,\overline{\partial}_{\lambda_p}$ be its index, which is a virtual vector space. The following theorem is proved in [FO+2, Theorem 6.5] whose proof in turn follows that of [FO+1, Chapter 6].

THEOREM 7.12

Let (L_0, L_1) be a relatively spin pair of oriented Lagrangian submanifolds. Then for each fixed α, the bundle (7.28) is trivial.

If we fix a choice of systems of orientations o_p on $\mathrm{Index}\,\overline{\partial}_{\lambda_p}$ for each p, then it determines orientations on (7.28), which we denote by $o_{[p,w]}$.

Moreover o_p, $o_{[p,w]}$ determine an orientation of $\mathcal{M}(p,q;B)$ denoted by $o(p,q;B)$ by the gluing rule

$$o_{[q,w\#B]} = o_{[p,w]}\#o(p,q;B) \tag{7.31}$$

for all $p,q \in L_0 \cap L_1$ and $B \in \pi_2(p,q)$ so that they satisfy the gluing formulae

$$\partial o(p,r;B) = o(p,q;B_1)\#o(q,r;B_2)$$

whenever the virtual dimension of $\mathcal{M}(p,r;B)$ is 1. Here $\partial o(p,r;B)$ is the induced boundary orientation of the boundary $\partial\mathcal{M}(p,r;B)$ and $B = B_1\#B_2$, and $\mathcal{M}(p,q;B_1)\#\mathcal{M}(q,r;B_2)$ appears as a component of the boundary $\partial\mathcal{M}(p,r;B)$.

REMARK 7.4

For the definition of the orientation of the moduli spaces for the filtered bimodule structure, see [FO+1, Sections 8.7, 8.8, Definition 8.8.11].

One can generalize the above discussion to the moduli space of pseudoholomorphic polygons, which we describe below.

Consider a disc D^2 with $k+1$ marked points $z_{0k}, z_{k(k-1)}, \ldots, z_{10} \subset \partial D^2$ respecting the counterclockwise cyclic order of ∂D^2. We take a neighborhood U_i of $z_{i(i-1)}$ and a conformal diffeomorphism $\varphi_i : U_i \setminus \{z_{i(i-1)}\} \subset D^2 \cong (-\infty, 0] \times [0,1]$ of each $z_{i(i-1)}$. For any smooth map

$$w : D^2 \to M; w(z_{i(i-1)}) = p_{i(i-1)}, w(\overline{z_{(i+1)i}z_{i(i-1)}}) \subset L_i,$$

we deform w so that it becomes constant on $\varphi_i^{-1}((-\infty,-1] \times [0,1]) \subset U_i$; that is, $w(z) \equiv p_{i(i-1)}$ for all $z \in \varphi^{-1}((-\infty,-1]\times[0,1])$. So assume that this holds for w from now on. We now consider the Cauchy-Riemann equation

$$D_w\overline{\partial}(\xi) = 0, \tag{7.32a}$$

$$\xi(\theta) \in T_{w(t)}L_i \quad \text{for } \theta \in \overline{z_{(i+1)i}z_{i(i-1)}} \subset \partial D^2. \tag{7.32b}$$

We remark that on $U_i = (-\infty, 0] \times [0,1]$ the boundary condition (7.32b) becomes

$$\xi(s,0) \in L_{i-1}, \qquad \xi(s,1) \in L_i. \tag{7.33}$$

Equation (7.32) induces a Fredholm operator, which we denote by

$$\overline{\partial}_{w;\mathfrak{L}} : W^{1,p}(D^2; w^*TM; \mathfrak{L}) \to L^p(D^2; w^* \otimes \Lambda^{0,1}). \tag{7.34}$$

Moving w, we obtain a family of Fredholm operators $\overline{\partial}_{(\mathfrak{L};\vec{p};B)}$ parameterized by a suitable completion of $\mathcal{F}(\vec{p};\mathfrak{L};B)$ for $B \in \pi_2(\vec{p};\mathfrak{L})$. Therefore we have a well-defined determinant line bundle

$$\det \overline{\partial}_{(\mathfrak{L};\vec{p};B)} \to \mathcal{F}(\mathfrak{L};\vec{p};B). \tag{7.35}$$

The following theorem is an extension of Theorem 7.12, which is [FO+2, Theorem 6.7], whose proof we refer readers thereto.

THEOREM 7.13

Suppose that $\mathfrak{L} = (L_0, \ldots, L_k)$ *is a relatively spin Lagrangian chain. Then each* $\det \overline{\partial}_{(\vec{p};\mathfrak{L};B)}$ *is trivial.*

Moreover, we have the following. If we fix orientations $o_{p_{ij}}$ *on* $\operatorname{Index} \overline{\partial}_{\lambda_{p_{ij}}}$ *as in Theorem 7.12 for all* $p_{ij} \in L_i \cap L_j$*, with* L_i *transversal to* L_j*, then we have a system of orientations, denoted by* $o_{k+1}(\vec{p};\mathfrak{L};B)$*, on the bundles (7.35), so that it is compatible with the gluing map in an obvious sense.*

We can prove that the orientation of $\overline{\partial}_{(\mathfrak{L};\vec{p};B)}$ depends on the choice of $o_{p_{(i+1)i}}$ (and so on λ_p) with $i = 0, \ldots, k$, but is independent of the choice of $w^+_{(i+1)i}$, and so on. This is a consequence of the proof of Theorem 7.12. Therefore the orientation in Theorem 7.13 is independent of the choice of anchors.

REMARK 7.5

To give an orientation of $\mathcal{M}(\mathfrak{L};\vec{p};B)$, we have to take the moduli parameters of marked points and the action of the automorphism group into account. We also treat the intersection point $p_{i(i-1)}$ as if it is a chain of codimension $\mu([p_{i(i-1)}, w^+_{i(i-1)}];\lambda_{(i-1)i})$ in a way similar to [FO+1, Chapter 8, Section 8.5].

8. Anchored Floer cohomology: Review

In the first subsection, we recall the exposition from [FO+2] on the Lagrangian Floer theory of anchored Lagrangian submanifolds.

8.1. Floer chain complex

Let (L_0, L_1) be a pair with L_0 intersecting L_1 transversely.

Let (L_i, γ_i), $i = 0, 1$, be anchored Lagrangian submanifolds. Let $p, q \in L_0 \cap L_1$ be admissible intersection points. We defined the set $\pi_2(p,q) = \pi_2((L_0, L_1), (p,q))$ in Section 3. We also defined $\pi_2(\ell_{01};p)$ there. We now define the following.

DEFINITION 8.1

Let R be the underlying coefficient field. We define $CF(\mathcal{L}_1, \mathcal{L}_0) = CF((L_1, \gamma_1), (L_0, \gamma_0))$ to be a free R-module over the basis $[p, w]$, where $p \in \mathcal{L}_0 \cap \mathcal{L}_1$ is an admissible intersection point and w is a map from $[0,1]^2 \to M$ connecting ℓ_{01} and $\widehat{p}$.

Here R is a ground ring such as $\mathbb{Q}$, $\mathbb{C}$, or $\mathbb{R}$.

REMARK 8.2

We remark that the set $[p,w]$, where p is the admissible intersection point, is identified with the set of the critical points of the action functional $\mathcal{A}$ defined on the Novikov covering space of $\Omega(L_0, L_1; \ell_{01})$. The group $\Pi(\mathcal{L}_0, \mathcal{L}_1)$ defined in Section 3.2 acts freely on it so that the quotient space is the set of admissible intersection points.

We next take a grading λ_i to (L_i, γ_i) as in Section 3.3. It induces a grading of $[p,w]$ given by $\mu([p,w];\lambda_{01})$, which gives the graded structure on $CF(\mathcal{L}_1, \mathcal{L}_0)$

$$CF(\mathcal{L}_1, \mathcal{L}_0) = \bigoplus_k CF^k(\mathcal{L}_1, \mathcal{L}_0; \lambda_{01}),$$

where $CF^k(\mathcal{L}_1, \mathcal{L}_0; \lambda_{01}) = \mathrm{span}_R\{[p,w] \in CF(\mathcal{L}_1, \mathcal{L}_0) \mid \mu([p,w];\lambda_{01}) = k\}$.

For given $B \in \pi_2(p,q)$, we denote by $\mathrm{Map}(p,q;B)$ the set of such w's in class B.

We summarize the extra structures added in the discussion of Floer homology for the anchored Lagrangian submanifolds in the following.

(1) We assume that (L_0, L_1) is a relatively spin pair. We consider a pair (L_0, γ_0), (L_1, γ_1) of anchored Lagrangian submanifolds and the base path $\ell_{01} = \overline{\gamma}_0 * \gamma_1$.

(2) We fix a grading λ_i of γ_i for $i = 0, 1$, which in turn induce a grading of ℓ_{01}, $\lambda_{01} = \overline{\lambda_0} * \lambda_1$.

(3) We fix an orientation o_p of $\mathrm{Index}\,\overline{\partial}_{\lambda_p}$ for each $p \in L_0 \cap L_1$ as in [FO+2].

Under these conditions, orientation of the Floer moduli space $\mathcal{M}(p,q;B)$ is induced. Using the virtual fundamental chain technique (see [FOn], [FO+1, Appendix A.1]), we can take a system of multisections and obtain a system of *rational* numbers $n(p,q;B) = \#(\mathcal{M}(p,q;B))$ whenever the virtual dimension of $\mathcal{M}(p,q;B)$ is zero. Finally, we define the Floer "boundary" map $\partial : CF(\mathcal{L}_1, \mathcal{L}_0) \to CF(\mathcal{L}_1, \mathcal{L}_0)$, defined in [Fl] and [Oh1], by the sum

$$\partial([p,w]) = \sum_{q \in L_0 \cap L_1} \sum_{B \in \pi_2(p,q)} n(p,q;B)[q, w\#B]. \tag{8.1}$$

By Remark 8.2, $CF(\mathcal{L}_1, \mathcal{L}_0)$ carries a natural $\Lambda(\mathcal{L}_0, \mathcal{L}_1)$-module structure and $CF^k(\mathcal{L}_1, \mathcal{L}_0; \lambda_{01})$ carries a $\Lambda^{(0)}(\mathcal{L}_0, \mathcal{L}_1)$-module structure, where

$$\Lambda^{(0)}(\mathcal{L}_0, \mathcal{L}_1) = \Big\{ \sum a_g[g] \in \Lambda(\mathcal{L}_0, \mathcal{L}_1) \;\Big|\; \mu([g]) = 0 \Big\}.$$

We define

$$C(\mathcal{L}_1, \mathcal{L}_0) = CF(\mathcal{L}_1, \mathcal{L}_0) \otimes_{\Lambda(\mathcal{L}_0, \mathcal{L}_1)} \Lambda_{\mathrm{nov}}, \tag{8.2}$$

where we use the embedding $\Lambda(\mathcal{L}_0, \mathcal{L}_1) \to \Lambda_{\mathrm{nov}}$ given in (3.2).

We write the Λ_{nov}-module (8.2) also as

$$C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{\mathrm{nov}}).$$

DEFINITION 8.3

We define the *energy filtration* $F^\lambda CF(\mathcal{L}_1, \mathcal{L}_0)$ of the Floer chain complex $CF(L_1, \gamma_1), (L_0, \gamma_0))$ (here $\lambda \in \mathbb{R}$) such that $[p,w]$ is in $F^\lambda CF(\mathcal{L}_1, \mathcal{L}_0)$ if and only if $\mathcal{A}([p,w]) \geq \lambda$.

This filtration also induces a filtration on (8.2).

REMARK 8.4

We remark that this filtration not only depends on the homotopy class but also on γ_i's themselves. A different choice of γ_i's induces a global uniform shift of filtration levels.

It is easy to see the following from the definition of ∂ above.

LEMMA 8.1

We have

$$\partial\big(F^\lambda CF((L_1, \gamma_1), (L_0, \gamma_0)) \subseteq F^\lambda CF((L_1, \gamma_1), (L_0, \gamma_0))\big).$$

According to definition (8.1) of the map ∂, we have the formula for its matrix coefficients

$$\langle \partial\partial[p,w], [r, w\#B]\rangle = \sum_{q \in L_0 \cap L_1} \sum_{B = B_1 \# B_2 \in \pi_2(p,r)} n(p,q;B_1) n(q,r;B_2) T^{\omega(B)}, \tag{8.3}$$

where $B_1 \in \pi_2(p,q)$ and $B_2 \in \pi_2(q,r)$.

To prove that $\partial\partial = 0$, one needs to prove that $\langle \partial\partial[p,w], [r, w\#B]\rangle = 0$ for all pairs $[p,w], [r, w\#B]$. On the other hand, it follows from definition that each summand

$$n(p,q;B_1) n(q,r;B_2) T^{\omega(B)} = n(p,q;B_1) T^{\omega(B_1)} n(q,r;B_2) T^{\omega(B_2)}$$

and the coefficient $n(p,q;B_1)n(q,r;B_2)$ is nothing but the number of broken trajectories lying in $\mathcal{M}(p,q;B_1)\#\mathcal{M}(q,r;B_2)$. This number is nonzero in the general situation in which we work.

To handle the problem of obstruction to $\partial \circ \partial = 0$ and of bubbling-off discs in general, a structure of a filtered A_∞-algebra $(C, \mathfrak{m})$ with *nonzero* $\mathfrak{m}_0$*-term* is associated to each Lagrangian submanifold L (see [FO+3], [FO+1]).

8.2. A_∞-algebra

In this subsection, we review the notion and construction of a filtered A_∞-algebra associated to a Lagrangian submanifold. To make the construction consistent with the one in Section 8.1, where $\Lambda(\mathcal{L}_0, \mathcal{L}_1)$ is used for the coefficient ring rather than the universal Novikov ring, we rewrite them using the smaller Novikov ring $\Lambda(L)$ which we define below. Let L be a relatively spin Lagrangian submanifold. We have a homomorphism

$$(E, \mu): H_2(M, L; \mathbb{Z}) \to \mathbb{R} \times \mathbb{Z},$$

where $E(\beta) = \beta \cap [\omega]$ and where μ is the Maslov index homomorphism. We put $g \sim g'$ for $g, g' \in H_2(M, L; : \mathbb{Z})$ if $E(g) = E(g')$ and $\mu(g) = \mu(g')$. We write $\Pi(L)$ as the quotient with respect to this equivalence relation. It is a subgroup of $\mathbb{R} \times \mathbb{Z}$. We define

$$\Lambda(L) = \Big\{ \sum c_g[g] \;\Big|\; g \in \Pi(L), c_g \in R, E(g) \geq 0, \\ \forall E_0 \# \{g \mid c_g \neq 0, E(g) \leq E_0\} < \infty \Big\}.$$

We have the natural embedding $\Lambda(L) \to \Lambda_{0,\mathrm{nov}}$ similarly as in (3.2).

Let $\overline{C}$ be a graded R-module, and let $CF = \overline{C} \widehat{\otimes}_R \Lambda(L)$. From now on we use the symbol CF for the modules over $\Lambda(L)$ or $\Lambda(L_0, L_1)$ and C for the modules over the universal Novikov ring.

We denote by $CF[1]$ its suspension defined by $CF[1]^k = CF^{k+1}$. We denote by $\deg(x) = |x|$ the degree of $x \in C$ before the shift and by $\deg'(x) = |x|'$ that after the degree shift, that is, $|x|' = |x| - 1$. Define the *bar complex* $B(CF[1])$ by

$$B_k(CF[1]) = (CF[1])^{k\otimes}, \qquad B(CF[1]) = \bigoplus_{k=0}^{\infty} B_k(CF[1]).$$

Here $B_0(CF[1]) = R$ by definition. The tensor product is taken over $\Lambda(L)$. We provide the degree of elements of $B(CF[1])$ by the rule

$$|x_1 \otimes \cdots \otimes x_k|' := \sum_{i=1}^{k} |x_i|' = \sum_{i=1}^{k} |x_i| - k, \tag{8.4}$$

where $|\cdot|'$ is the shifted degree. The ring $B(CF[1])$ has the structure of a *graded coalgebra.*

DEFINITION 8.5

A filtered A_∞-algebra over $\Lambda(L)$ is a sequence of $\Lambda(L)$ module homomorphisms

$$\mathfrak{m}_k : B_k(CF[1]) \to CF[1], \quad k = 0, 1, 2, \ldots$$

of degree $+1$ such that the coderivation $d = \sum_{k=0}^{\infty} \widehat{\mathfrak{m}}_k$ satisfies $dd = 0$, which is called the A_∞*-relation.* Here we denote by $\widehat{\mathfrak{m}}_k : B(CF[1]) \to B(CF[1])$ the unique extension of $\mathfrak{m}_k$ as a coderivation on $B(CF[1])$. A *filtered* A_∞*-algebra* is an A_∞-algebra with a filtration for which $\mathfrak{m}_k$ are continuous with respect to the induced non-Archimedean topology.

If we have $\mathfrak{m}_1\mathfrak{m}_1 = 0$, it defines a complex $(CF, \mathfrak{m}_1)$. We define the $\mathfrak{m}_1$-cohomology by

$$H(CF, \mathfrak{m}_1) = \operatorname{Ker} \mathfrak{m}_1 / \operatorname{Im} \mathfrak{m}_1. \tag{8.5}$$

The first two terms of the A_∞-relation for a A_∞-algebra are given as

$$\mathfrak{m}_1(\mathfrak{m}_0(1)) = 0, \tag{8.6}$$

$$\mathfrak{m}_1\mathfrak{m}_1(x) + (-1)^{|x|'}\mathfrak{m}_2\big(x, \mathfrak{m}_0(1)\big) + \mathfrak{m}_2\big(\mathfrak{m}_0(1), x\big) = 0. \tag{8.7}$$

In particular, for the case when $\mathfrak{m}_0(1)$ is nonzero, $\mathfrak{m}_1$ does not necessarily satisfy the boundary property; that is, $\mathfrak{m}_1\mathfrak{m}_1 \neq 0$ in general.

We now describe the A_∞-operators $\mathfrak{m}_k$ in the context of A_∞-algebra of Lagrangian submanifolds. For a given compatible almost complex structure J, consider the moduli space of stable maps of genus zero,

$$\mathcal{M}_{k+1}(\beta;L) = \left\{ \left(w,(z_0,z_1,\dots,z_k)\right) \mid \overline{\partial} w = 0, z_i \in \partial D^2, [w]=\beta \text{ in } \pi_2(M,L)\right\}/\sim,$$

where $\sim$ is the conformal reparameterization of the disc D^2. We require that $z_0,\dots,z_k$ respect the counterclockwise cyclic order of S^1. (We wrote this moduli space $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta;L)$ in [FO+1, Section 2.1]. The symbol "main" indicates the compatibility of $z_0,\dots,z_k$ with counterclockwise cyclic order. We omit the symbol in this paper since we always assume it.)

$\mathcal{M}_{k+1}(\beta;L)$ has a Kuranishi structure, and its dimension is given by

$$n+\mu(\beta)-3+(k+1) = n+\mu(\beta)+k-2. \tag{8.8}$$

Now let $[P_1,f_1],\dots,[P_k,f_k] \in C_*(L;\mathbb{Q})$ be k-smooth singular simplices of L. (Here we denote by $C(L;\mathbb{Q})$ a *suitably chosen countably generated* cochain complex of smooth singular chains of L.) We put the cohomological grading $\deg P_i = n - \dim P_i$ and consider the fiber product

$$\mathrm{ev}_0 : \mathcal{M}_{k+1}(\beta;L) \times_{(\mathrm{ev}_1,\dots,\mathrm{ev}_k)} (P_1 \times \cdots \times P_k) \to L.$$

A simple calculation shows that the expected dimension of this chain is given by $n+\mu(\beta)-2+\sum_{j=1}^{k}(\dim P_j + 1 - n)$, or equivalently, we have the degree

$$\deg[\mathcal{M}_{k+1}(\beta;L) \times_{(\mathrm{ev}_1,\dots,\mathrm{ev}_k)} (P_1 \times \cdots \times P_k), \mathrm{ev}_0] = \sum_{j=1}^{n}(\deg P_j - 1) + 2 - \mu(\beta).$$

For each given $\beta \in \pi_2(M,L)$ and $k=0,\dots,$ we define $\mathfrak{m}_{1,0}(P) = \pm\partial P$ and

$$\begin{aligned} \mathfrak{m}_{k,\beta}(P_1,\dots,P_k) &= [\mathcal{M}_{k+1}(\beta;L) \times_{(\mathrm{ev}_1,\dots,\mathrm{ev}_k)} (P_1 \times \cdots \times P_k), \mathrm{ev}_0] \\ &\in C(L;\mathbb{Q}). \end{aligned} \tag{8.9}$$

(More precisely, we regard the right-hand side of (8.9) as a smooth singular chain by taking the appropriate multivalued perturbation (multisection) and choosing a simplicial decomposition of its zero set.)

We put

$$CF(L) = C(L;\mathbb{Q}) \,\widehat{\otimes}_{\mathbb{Q}}\, \Lambda(L).$$

We define $\mathfrak{m}_k : B_kCF(L)[1] \to B_kCF[1]$ by

$$\mathfrak{m}_k = \sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{k,\beta} \otimes [\beta].$$

Then it follows that the map $\mathfrak{m}_k : B_kCF(L)[1] \to CF(L)[1]$ is well defined, has degree 1, and is continuous with respect to non-Archimedean topology. We extend $\mathfrak{m}_k$ as a coderivation $\widehat{\mathfrak{m}}_k : BCF[1] \to BCF[1]$, where $BCF(L)[1]$ is the completion of the direct sum $\bigoplus_{k=0}^{\infty} B_kCF(L)[1]$, where $B_kCF(L)[1]$ itself is the completion of $CF(L)[1]^{\otimes k}$. $BCF(L)[1]$ has a natural filtration defined similarly as in Definition 8.3. Finally, we take the sum

$$\widehat{d} = \sum_{k=0}^{\infty} \widehat{\mathfrak{m}}_k : BCF(L)[1] \to BCF(L)[1].$$

We then have the following coboundary property.

THEOREM 8.2

Let L be an arbitrary compact relatively spin Lagrangian submanifold of an arbitrary tame symplectic manifold (M, ω). The coderivation $\widehat{d}$ is a continuous map that satisfies the A_∞-relation $\widehat{d}\widehat{d} = 0$, and so $(CF(L), \mathfrak{m})$ is a filtered A_∞-algebra over $\Lambda(L)$.

We put

$$C(L; \Lambda_{0,\mathrm{nov}}) = CF(L) \widehat{\otimes}_{\Lambda(L)} \Lambda_{0,\mathrm{nov}},$$

on which a filtered A_∞-structure on $C(L; \Lambda_{0,\mathrm{nov}})$ (over the ring $\Lambda_{0,\mathrm{nov}}$) is induced. This is the filtered A_∞-structure given in [FO+1, Theorem A].

In the presence of $\mathfrak{m}_0$, $\widehat{\mathfrak{m}}_1 \widehat{\mathfrak{m}}_1 = 0$ no longer holds in general. This leads us to consider deforming Floer's original definition by a bounding cochain of the obstruction cycle arising from bubbling-off discs. One can always deform the given (filtered) A_∞-algebra $(CF(L), \mathfrak{m})$ by an element $b \in CF(L)[1]^0$ by redefining the A_∞-operators as

$$\mathfrak{m}_k^b(x_1, \dots, x_k) = \mathfrak{m}(e^b, x_1, e^b, x_2, e^b, x_3, \dots, x_k, e^b)$$

and taking the sum $\widehat{d}^b = \sum_{k=0}^{\infty} \widehat{\mathfrak{m}}_k^b$. This defines a new filtered A_∞-algebra in general. Here we simplify notation by writing

$$e^b = 1 + b + b \otimes b + \cdots + b \otimes \cdots \otimes b + \cdots .$$

Note that each summand in this infinite sum has degree zero in $CF(L)[1]$ and converges in the non-Archimedean topology if b has positive valuation, that is, $\mathfrak{v}(b) > 0$ (see Section 3.2 for the definition of $\mathfrak{v}$).

PROPOSITION 8.3

For the A_∞-algebra $(CF(L), \mathfrak{m}_k^b)$, $\mathfrak{m}_0^b = 0$ if and only if b satisfies

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \dots, b) = 0. \tag{8.10}$$

This equation is a version of the Maurer-Cartan equation *for the filtered A_∞-algebra.*

DEFINITION 8.6

Let $(CF(L), \mathfrak{m})$ be a filtered A_∞-algebra in general, and let $BCF(L)[1]$ be its bar complex. An element $b \in CF(L)[1]^0 = CF(L)^1$ is called a *bounding cochain* if it satisfies (8.10) and $v(b) > \mathfrak{v}(b)$. We denote by $\widetilde{\mathcal{M}}(L;\Lambda(L))$ the set of bounding cochains.

In general, a given A_∞-algebra may or may not have a solution to (8.10). In our case we define the following.

DEFINITION 8.7
A filtered A_∞-algebra $(CF(L),\mathfrak{m})$ is called *unobstructed over* $\Lambda(L)$ if (8.10) has a solution $b \in CF(L)[1]^0 = CF(L)^1$ with $v(b) > \mathfrak{v}(b)$.

One can define the notion of homotopy equivalence between two bounding cochains as described in [FO+1, Chapter 4]. We denote by $\mathcal{M}(L;\Lambda(L))$ the set of equivalence classes of bounding cochains of L.

REMARK 8.8
In Definition 8.6, we consider the bounding cochain contained in $CF(L) \subset C(L;\Lambda_0)$ only. This is the reason why we write $\widetilde{\mathcal{M}}(L;\Lambda(L))$ in place of $\widetilde{\mathcal{M}}(L)$. (The latter is used in [FO+1].)

8.3. A_∞-bimodule

Once the A_∞-algebra is attached to each Lagrangian submanifold L, we then construct a structure of filtered A_∞*-bimodules* on the module $CF(\mathcal{L}_1,\mathcal{L}_0) = CF((L_1,\gamma_1),(L_0,\gamma_0))$, which was introduced in Section 8.1 as follows. This filtered A_∞-bimodule structure is by definition a family of operators

$$\begin{aligned}\mathfrak{n}_{k_1,k_0} : B_{k_1}\big(CF(L_1)[1]\big)\,\widehat{\otimes}_{\Lambda(L_1)}\,CF\big((L_1,\gamma_1),(L_0,\gamma_0)\big)\,\widehat{\otimes}_{\Lambda(L_0)}\,B_{k_0}\big(CF(L')[1]\big)\\ \to CF\big((L_1,\gamma_1),(L_0,\gamma_0)\big)\end{aligned}$$

for $k_0,k_1 \geq 0$. Here the left-hand side is defined as follows. It is easy to see that there are embeddings $\Lambda(L_0) \to \Lambda(\mathcal{L}_0,\mathcal{L}_1)$, $\Lambda(L_1) \to \Lambda(\mathcal{L}_0,\mathcal{L}_1)$. Therefore a $\Lambda(\mathcal{L}_0,\mathcal{L}_1)$-module $CF((L_1,\gamma_1),(L_0,\gamma_0))$ can be regarded both as a $\Lambda(L_0)$-module and a $\Lambda(L_1)$-module. Hence we can take tensor product in the left-hand side ($\widehat{\bigotimes}_{\Lambda(L_i)}$ is the completion of this algebraic tensor product). The left-hand side then becomes a $\Lambda(\mathcal{L}_0,\mathcal{L}_1)$-module since the rings involved are all commutative.

We briefly describe the definition of $\mathfrak{n}_{k_1,k_0}$. A typical element of the tensor product

$$B_{k_1}\big(CF(L_1)[1]\big)\,\widehat{\otimes}_{\Lambda(L_1)}\,CF\big((L_1,\gamma_1),(L_0,\gamma_0)\big)\,\widehat{\otimes}_{\Lambda(L_0)}\,B_{k_0}\big(CF(L_0)[1]\big)$$

has the form

$$P_{1,1}\otimes\cdots\otimes P_{1,k_1}\otimes[p,w]\otimes P_{0,1}\otimes\cdots\otimes P_{0,k_0}$$

with $p \in L_0 \cap L_1$ being an admissible intersection point. Then the image $\mathfrak{n}_{k_0,k_1}$ thereof is given by

$$\sum_{q,B} T^{\omega(B)} e^{\mu(B)/2} \#\big(\mathcal{M}(p,q;B;P_{1,1},\ldots,P_{1,k_1};P_{0,0},\ldots,P_{0,k_0})\big)[q,B\#w].$$

Here B denotes homotopy class of Floer trajectories connecting p and q, the summation is taken over all $[q,B]$ with

$$\dim \mathcal{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{0,1},\dots,P_{0,k_0}) = 0,$$

and $\#(\mathcal{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{0,1},\dots,P_{0,k_0}))$ is the "number" of elements in the "zero"-dimensional moduli space $\mathcal{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{0,1},\dots,P_{0,k_0})$. Here the moduli space $\mathcal{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{0,1},\dots,P_{0,k_0})$ is the Floer moduli space $\mathcal{M}(p,q;B)$ cut down by intersecting with the given chains $P_{1,i} \subset L_1$ and $P_{0,j} \subset L_0$ (see [FO+1, Section 3.7]). An orientation on this moduli space can be given in [FO+1] and [FO+2].

THEOREM 8.4

Let $(\mathcal{L}_0,\mathcal{L}_1)$ be a pair of anchored Lagrangian submanifolds. Then the family $\{\mathfrak{n}_{k_1,k_0}\}$ defines a left $(CF(L_1),\mathfrak{m})$ and right $(CF(L_0),\mathfrak{m})$ filtered A_∞-bimodule structure on $CF(\mathcal{L}_1,\mathcal{L}_0)$.

See [FO+1, Section 3.7] and [FO+2] for the definition of filtered A_∞-bimodules. (In [FO+1] the case of the universal Novikov ring as a coefficient is considered. It is easy to modify it to our case of a $\Lambda(L_0,L_1)$-coefficient.)

In the case where both L_0, L_1 are unobstructed, we can perform this deformation of $\mathfrak{n}$ using bounding cochains b_0 and b_1 of $CF(L_0)$ and $CF(L_1)$, respectively, in a way similar to $\mathfrak{m}^b$. Namely, we define $\delta_{b_1,b_0} : CF(\mathcal{L}_1,\mathcal{L}_0) \to CF(\mathcal{L}_1,\mathcal{L}_0)$ by

$$\delta_{b_1,b_0}(x) = \sum_{k_1,k_0} \mathfrak{n}_{k_1,k_0}(b_1^{\otimes k_1} \otimes x \otimes b_0^{\otimes k_0}) = \widehat{\mathfrak{n}}(e^{b_1}, x, e^{b_0}).$$

We can generalize the story to the case where L_0 has clean intersection with L_1, especially to the case $L_0 = L_1$. In the case $L_0 = L_1$ we have $\mathfrak{n}_{k_1,k_0} = \mathfrak{m}_{k_0+k_1+1}$. So in this case, we have $\delta_{b_1,b_0}(x) = \mathfrak{m}(e^{b_1}, x, e^{b_0})$.

We define the Floer cohomology of the pair $\mathcal{L}_0 = (L_0,\gamma_0,\lambda_0)$, $\mathcal{L}_1 = (L_1,\gamma_1,\lambda_1)$ by

$$HF\big((\mathcal{L}_1,b_1),(\mathcal{L}_0,b_0)\big) = \operatorname{Ker}\delta_{b_1,b_0} / \operatorname{Im}\delta_{b_1,b_0}.$$

This is a module over $\Lambda(\mathcal{L}_0,\mathcal{L}_1)$.

THEOREM 8.5

$HF((\mathcal{L}_1,b_1),(\mathcal{L}_0,b_0)) \otimes_{\Lambda(\mathcal{L}_0,\mathcal{L}_1)} \Lambda_{\mathrm{nov}}$ is invariant under the Hamiltonian isotopies of $\mathcal{L}_0$ and $\mathcal{L}_1$ and under the gauge equivalence of bounding cochains b_0, b_1.

We refer to [FO+1, Section 4.3] for the definition of gauge equivalence and to [FO+1, Theorem 4.1.5] for the proof of this theorem.

9. Definitions of Seidel's maps $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{h}$

In this section, we recall the definition of Seidel's cochain maps $\mathfrak{b}, \mathfrak{c}$, and the homotopy $\mathfrak{h}$ and give the definition of the analogues thereof in our general setting.

They are the maps

$$\mathfrak{b} : CF(L, L_1) \otimes CF(\tau_L(L_0), L) \to CF(\tau_L(L_0), L_1),$$
$$\mathfrak{c} : CF(\tau_L(L_0), L_1) \to CF(L_0, L_1)$$

and the homotopy $\mathfrak{h} : CF(L, L_1) \otimes CF(\tau_L(L_0), L) \to CF(L_0, L_1)$ between the composition $\mathfrak{c} \circ \mathfrak{b}$ and the zero map.

We generalize these maps to our nonexact case and describe all the necessary properties of the maps in this section. We consider a quadruple of anchored Lagrangian submanifolds

$$\begin{aligned} &\mathcal{L} = (L, \gamma), \qquad \mathcal{L}_0 = (L_0, \gamma_0), \qquad \mathcal{L}_1 = (L_1, \gamma_1), \\ &\tau_* \mathcal{L} = \big(\tau_L(L_0), \tau_L(\gamma_0)\big). \end{aligned} \tag{9.1}$$

For simplicity of notation in this section, we denote the action functional associated to the pair (L, γ) and (L', γ') just by $\mathcal{A}_{\mathcal{L}\mathcal{L}'}$.

9.1. A simple invariant and its vanishing theorem

Let $\Phi_1(E, \pi, Q)$ be the invariant

$$\Phi_1(E, \pi, Q) = (\mathrm{ev}_\zeta)_* [\mathcal{M}_J] \in H_*(Q_\zeta; \mathbb{Z})$$

for the Calabi-Yau Lefschetz fibration. The following proposition replaces a similar proposition, [Se3, Proposition 2.13], for the present Calabi-Yau Lefschetz fibration setting.

For each given section class $A \in \pi_2^{\mathrm{sec}}(E, Q)$, we define the moduli space $\mathcal{M}(E, Q, J; A)$ of J-holomorphic section $s : D \to E$ with $[s, \partial s] = A$ and define an invariant

$$\Phi_1(E, \pi, Q) = \sum_{A \in \pi_2^{\mathrm{sec}}(E, Q)} (\mathrm{ev}_\zeta)_* \big(\mathcal{M}_J(A)\big) T^{\Omega(A)} \in H_*(Q_\zeta; \Lambda_{0,\mathrm{nov}}).$$

We start with the following slight generalization of Proposition 2.2 in the general context of Lagrangian spheres in general symplectic manifolds.

PROPOSITION 9.1

Let $(L, [f])$ be a framed Lagrangian sphere in M. There is a 1-parameter family of Lefschetz Hamiltonian fibrations $(E_r^L, \pi_r^L) \to D(r)$ together with an isomorphism $\phi_r^L : E_r^L \to M$ of symplectic manifolds, such that we have the following.

(1) *Consider the rescaling map $\lambda_r : D(r) \to D(1)$ defined by $z \mapsto z/r$. Then*

$$(\lambda_r)^* E_1^L = E_r^L.$$

(2) *If ρ_r^L is the symplectic monodromy around $\partial \overline{D}(r)$, then $\phi_r^L \circ \rho^L \circ (\phi_r^L)^{-1}$ is a Dehn twist along $(L, [f])$.*

We denote any of these maps by τ_L, as before.

Proof
The proof follows from Seidel's proof of [Se3, Proposition 1.11] stripping all the things related to the exactness requirement in the proof. In fact, the proof is easier because we do not have to concern ourselves with the exactness requirement in the construction. □

We also recall that each fiber $E^L_z, z \neq 0$, of the fibration E^L contains a distinguished Lagrangian sphere Σ^L_z. We call this fibration a *standard Calabi-Yau Lefschetz fibration*. The following is a crucial proposition needed in Seidel's construction of the long exact sequence in [Se3], whose proof goes through in our current context with orientation consideration incorporated.

PROPOSITION 9.2
Let $L \subset M$ be any CY Lagrangian brane, and let $\pi^L : E^L \to D(r)$ be the associated standard Calabi-Yau Lefschetz fibration. Then we have

$$\Phi_1(E^L, \pi^L, Q^L) = 0.$$

Proof
The proof of this proposition follows verbatim that of [Se3, Proposition 2.13] except for the consideration of orientations. The relevant consideration of orientation of the moduli space $\mathcal{M}(E, Q, J; A)$ is given in [Se4, Section 17i] and so is omitted. □

9.2. The map $\mathfrak{b}$

Let Σ be a compact surface with boundary marked points $\vec{\zeta} = \{\zeta_0, \zeta_1, \zeta_2\}$. We denote $\dot\Sigma = \Sigma \setminus \vec{\zeta}$ and $\partial\dot\Sigma = \bigcup_{i=0}^2 \partial_i \dot\Sigma$. We consider the three anchored Lagrangian submanifolds (L_0, γ_0), (L, γ), and (L_1, γ_1). Take the trivial Hamiltonian fibration $\pi : E = \dot\Sigma \times M \to \dot\Sigma$ with the 2-form Ω equal to the 2-form pulled back from ω in M. Equip this with the Lagrangian boundary condition

$$Q = \big(\partial_0\dot\Sigma \times \tau_L(L_0)\big) \cup (\partial_1\dot\Sigma \times L) \cup (\partial_3\dot\Sigma \times L_1).$$

Q is an exact Lagrangian boundary with $\kappa_Q = 0$. We note that E has the trivial connection given by $K \equiv 0$ and hence has zero curvature.

LEMMA 9.3
Suppose that $\mathcal{L} = (L, \gamma), \mathcal{L}_0 = (L_0, \gamma_0), \mathcal{L}_1 = (L_1, \gamma_1)$ are given anchors of the type (9.1), and suppose that $\tau_*\mathcal{L} = (\tau_L(L_0), \tau_L(\gamma_0))$. Let $s : \dot\Sigma \to \dot\Sigma \times M$ be a section with the given exact Lagrangian boundary condition Q as above. Let $[p_0, w_0] \in$ $\mathrm{Crit}\,\mathcal{A}_{\tau_L(\mathcal{L}_0)\mathcal{L}_1}$, $[p, w] \in \mathrm{Crit}\,\mathcal{A}_{\mathcal{L}\mathcal{L}_1}$, and $[p_1, w_1] \in \mathrm{Crit}\,\mathcal{A}_{\tau_L(\mathcal{L}_0)\mathcal{L}}$. Suppose that the homotopy class $[s, \partial s]$ is admissible and satisfies

$$[w_0] = [s, \partial s] * [w] * [w_1] \quad \text{in } \pi_2\big(\tau_L(L_0), L_1; \tau_L(\overline{\gamma}_0) * \gamma_1\big). \tag{9.2}$$

Then we have

$$\int s^*\Omega = \mathcal{A}_{\tau_L(\mathcal{L}_0)\mathcal{L}_1}([p_0,w_0]) - \mathcal{A}_{\mathcal{L}\mathcal{L}_1}([p,w]) - \mathcal{A}_{\tau_L(\mathcal{L}_0)\mathcal{L}}([p_1,w_1]). \tag{9.3}$$

Proof

This immediately follows from Proposition 3.4. □

DEFINITION 9.1

Let $L, L' \subset M$ be a pair of Lagrangian submanifolds. We say that J lies in $\mathcal{J}^{\mathrm{reg}}(M;L,L')$ if all Floer trajectories associated to J for the pair (L,L') are Fredholm regular.

At this point, we fix

$$J^{(1)} \in \mathcal{J}^{\mathrm{reg}}\big(M;\tau_L(L_0),L\big),$$
$$J^{(2)} \in \mathcal{J}^{\mathrm{reg}}(M;L,L_1),$$
$$J^{(3)} \in \mathcal{J}^{\mathrm{reg}}\big(M;\tau_L(L_0),L_1\big).$$

By choosing a horizontal $J \in \mathcal{J}(E,\pi,j,J^{(1)},J^{(2)},J^{(3)})$, that is, J satisfying $J_x(TE^h_x) = TE^h_x$ for all $x \in E \setminus E^{\mathrm{crit}}$ with the above trivial fibration $E = \dot\Sigma \times M$, this gives rise to the standard pants product map

$$CF(\mathcal{L},\mathcal{L}_1) \otimes CF\big(\tau_L(\mathcal{L}_0),\mathcal{L}\big) \to CF\big(\tau_L(\mathcal{L}_0),\mathcal{L}_1\big) \tag{9.4}$$

in the cochain level. Seidel's map $\mathfrak{b}$ in the cochain level is nothing but this pants product map.

By choosing bounding cochains b_0, b_1 of L_0, L_1, respectively, and considering $(\tau_L)_* b_0$ on $\tau_L(L_0)$, we define the deformed $\mathfrak{m}_2$:

$$\mathfrak{m}_2^{\vec b} : CF(\mathcal{L},\mathcal{L}_1) \otimes CF\big(\tau_L(\mathcal{L}_0),\mathcal{L}\big) \to CF\big(\tau_L(\mathcal{L}_0),\mathcal{L}_1\big),$$

where $\vec b = (0,(\tau_L)_* b_0, b_1)$ and $\mathfrak{m}_i^{\vec b}$ is defined as in [FO+2, (8.15)]. Here we use $\mathfrak{m}_2^{\vec b}$ instead of $\mathfrak{b}$ to be consistent with the notation from [FO+1], [Fu], and [FO+2] and to highlight the point that Seidel's map $\mathfrak{b}$ is nothing but the special case of the $\mathfrak{m}_2$-map therefrom.

According to [FO+1], [Fu], and [FO+2], this induces a cochain map up to the higher homotopy map $\mathfrak{m}_3^{\vec b}$ and induces a homomorphism in cohomology. For the reader's convenience, we summarize the construction of this product map $\mathfrak{m}_k^{\vec b}$ in the appendix.

9.3. The map $\mathfrak{c}$

Let (E^L, π^L) be the standard fibration over a disc $D(1/2)$ whose monodromy around $\partial D(1/2)$ is τ_L, as defined in Section 2, by choosing r small. Let $0 < r < 1/2$ be given, and choose a function g with $g(t) = t$ for small t, $g(t) \equiv r$ for $t \geq r$, and $g'(t) \geq 0$ everywhere. We consider the map $p : D(1/2) \to D(1/2)$ defined by

$p(z) = g(|z|)z/|z|$, and consider the pullback fibration

$$(E^p, \pi^p) = p^*(E^L, \pi^L).$$

This is flat on the annulus $D(1/2) \setminus \mathrm{Int}(D(r))$.

Now take the surface

$$\Sigma^f = \mathbb{R} \times [-1,1] \setminus \mathrm{Int}\, D(1/2) \subset \mathbb{R}^2$$

with coordinates (s,t), and divide it into two parts $\Sigma^{f,\pm} = \Sigma^f \cap \{t \in \mathbb{R}_\pm\}$ so that

$$\Sigma^{f,+} \cap \Sigma^{f,-} = \Big(\Big(-\infty, -\frac{1}{2}\Big] \cup \Big[\frac{1}{2}, \infty\Big)\Big) \times \{0\}.$$

Consider trivial fibrations

$$\pi^{f,\pm} : E^{f,\pm} = \Sigma^{f,\pm} \times M \to \Sigma^{f,\pm}$$

over the two parts, and equip them with 2-forms $\Omega^{f,\pm}$ the pullback of ω. We define a fibration (E^f, π^f) over Σ^f by identifying the fibers $E^{f,+}_{(s,0)} \to E^{f,-}_{(s,0)}$ via id_M for $s \geq 1/2$ and via τ_L for $s \leq -1/2$. Because τ_L is symplectic, this defines a flat Hamiltonian fibration.

Using the fact that two fibrations E^p and E^f are flat close to the curve $|z| = 1/2$, we now paste them along the curve. Denote the resulting fibration over

$$\Sigma^c := \Sigma^p \cup \Sigma^f = \mathbb{R} \times [-1,1]$$

by (E^c, π). Equip (E^c, π) with the Lagrangian boundary condition

$$Q^c = \begin{cases} \mathbb{R} \times \{1\} \times L_1 \subset E^{f,+}, \\ \mathbb{R} \times \{-1\} \times \tau_L(L_0) \subset E^{f,-}. \end{cases} \tag{9.5}$$

This defines an exact Lagrangian boundary with the 1-form $\kappa_{Q^c} = 0$. As explained in [Se3, Section 3.3], (E^c, π) is modeled on $(\tau_L(L_0), L_1)$ over the positive end of Σ, and over the negative end, it is modeled on (L_0, L_1) due to the monodromy effect around the critical value $(0,0) \in \Sigma$.

Let j be some complex structure on Σ, standard over the ends. Take some $J^{(3)} \in \mathcal{J}^{\mathrm{reg}}(M, \tau_L(L_0), L_1)$ as in Section 9.2, and choose an additional $J^{(5)} \in \mathcal{J}^{\mathrm{reg}}(M; L_0, L_1)$. Using a regular $J^{(6)} \in \mathcal{J}(E^c, \pi, Q^c, j, J^{(3)}, J^{(5)})$, we define a map

$$\mathfrak{c} = C\Phi_0^{\mathrm{rel}}(E^c, \pi, Q^c, J^{(6)}) : CF\big(\tau_L(\mathcal{L}_0), \mathcal{L}_1\big) \to CF(\mathcal{L}_1, \mathcal{L}_0)$$

as defined in [Se3, Section 3.3]. However, since we need to establish this morphism in $\mathbb{Z}$-coefficients, we need to incorporate the study of orientation in its definition. Furthermore, we also need to involve the bounding cochains since we need to deform Seidel's definition just as we do for the Floer coboundary map.

9.3.1. Seidel map $\mathfrak{c}$ over $\mathbb{Z}$

For each given $p \in L_1 \cap \tau_L(L_0)$ and $q \in L_1 \cap L_0$ and a section class $B \in \pi_2(E^c, Q^c; p, q)$, we consider the moduli space $\mathcal{M}^c(p, q; B)$ consisting of the $J^{(6)}$-holomorphic sections $s : \Sigma^c \to E^c$ satisfying the conditions

$$[s] = B, \quad s(-\infty) = \widetilde{p}, s(\infty) = \widetilde{q}.$$

Here $\widetilde{p}, \widetilde{q}$ are horizontal sections of $(E^c|_{\pm\infty}, \partial E^c_{\pm\infty})$ which are trivialized to

$$(E^c|_{-\infty}; \partial E^c|_{-\infty}) \cong \big([-1,1] \times M; \big\{\{-1\} \times \tau_L(L_0)\big\} \cup \big\{\{1\} \times L_1\big\}\big)$$

and

$$(E^c|_{+\infty}; \partial E^c|_{+\infty}) \cong \big([-1,1] \times M; \big\{\{-1\} \times L_1\big\} \cup \big\{\{1\} \times L_0\big\}\big)$$

over $([0,1], \{0,1\})$ from the definition of the fibration $E^c \to \Sigma^c$ above.

We need to equip $\mathcal{M}^c(p,q;B)$ with an orientation denoted by $o^c(p,q;B)$ that is compatible with the orientations $o_{[p,w]}$ provided at $[p,w]$ defined in Theorem 7.12. This amounts to saying that we establish the existence of a relative spin structure on $(E, \mathcal{Q}) \to (\Sigma, \partial\Sigma)$ such that Σ has one incoming end and one outgoing end of the type (9.5) and satisfies the following requirement.

Let $Z_\pm$ be the space given in (7.29), and consider the bundle pairs

$$E_\pm = Z_\pm \times \mathbb{C}^n, \quad \lambda_\pm,$$

defined right above (7.29). Then we consider the gluing bundle pair

$$(E_-, \lambda_-) \# \big(s^* T^v E, (\partial s)^* T^v Q\big) \# (E_+, \lambda_+).$$

This is a bundle pair over a disc. By construction, this bundle pair is homotopic to the model bundle pair around the critical value $(0,0)$ induced by the model fibration (E^L_r, π^L_r) with model Lagrangian boundary condition $Q^L_r \subset E^L_r$ given in Proposition 9.1.

We denote by $o(\cdot)$ the orientation given in the relevant real determinant bundle $(\cdot)$. Then the requirement for the choice of

$$o^c(p,q;B) := o\big(\det(s^* T^v E, (\partial s)^* T^v Q)\big)$$

is the gluing rule

$$\begin{aligned} &o(\det D\overline{\partial}_{(E^L_r, \pi^L_r)}) \\ &\quad = o(\det D\overline{\partial}_{(E_-,\lambda_-)}) \# o^c(p,q;B) \# o(\det D\overline{\partial}_{(E_+,\lambda_+)}). \end{aligned} \tag{9.6}$$

This coherent choice of orientations is already discussed in [Se4, Section 17] in a more general context of relative spin structure and so is omitted by referring readers thereto. The outcome then gives rise to the Seidel map

$$\mathfrak{c} = C\Phi_0^{\mathrm{rel}}(E^c, \pi, Q^c, J^{(6)}) : CF\big(\tau_L(\mathcal{L}_0), \mathcal{L}_1\big) \to CF(\mathcal{L}_1, \mathcal{L}_0)$$

over $\mathbb{Z}$ coefficients and so over arbitrary coefficients. However, this map may not satisfy the chain property for the same reason that the Floer map δ might not satisfy the coboundary property, which leads us to consider the deformed version of $\mathfrak{c}$ whose explanation is now in order.

9.3.2. Deformed Seidel map $\mathfrak{c}$

Since we use the deformed Floer complex by bounding cochains, we need to construct the deformed version of the map $\mathfrak{c}$. The construction will resemble that of an A_∞-bimodule structure on $CF(\mathcal{L}_1, \mathcal{L}_0)$ performed in Section 8.3. Some details of construction are in order.

We first define a family of operators

$$\mathfrak{c}_{k_1,k_0} : B_{k_1}\big(CF(L_1)[1]\big)\,\widehat{\otimes}_{\Lambda(L_1)}\,CF\big(\mathcal{L}_1, \tau_L(\mathcal{L}_0)\big)\,\widehat{\otimes}_{\Lambda(L_0)}\,B_{k_0}\big(CF(L_0)[1]\big) \\ \to CF(\mathcal{L}_1, \mathcal{L}_0)$$

for $k_0, k_1 \ge 0$. A typical element of the tensor product

$$B_{k_1}\big(CF(L_1)[1]\big)\,\widehat{\otimes}_{\Lambda(L_1)}\,CF(\mathcal{L}_1, \tau_L(\mathcal{L}_0))\,\widehat{\otimes}_{\Lambda(L_0)}\,B_{k_0}\big(CF(L_0)[1]\big)$$

has the form

$$P_{1,1} \otimes \cdots \otimes P_{1,k_1} \otimes [p,w] \otimes P_{0,1} \otimes \cdots \otimes P_{0,k_0}$$

with $p \in \tau_L(\mathcal{L}_0) \cap \mathcal{L}_1$ being an admissible intersection point. Then the image $\mathfrak{c}_{k_0,k_1}$ thereof is given by

$$\sum_{q,B} T^{\omega(B)} e^{\mu(B)/2} \#\big(\mathcal{M}^c(p,q;B;P_{1,1},\ldots,P_{1,k_1};P_{0,0},\ldots,P_{0,k_0})\big)[q, B\#w].$$

Here B denotes the section class of J-holomorphic sections connecting p and q, the summation is taken over all $[q,B]$ with

$$\dim \mathcal{M}^c(p,q;B;P_{1,1},\ldots,P_{1,k_1};P_{0,1},\ldots,P_{0,k_0}) = 0,$$

and $\#(\mathcal{M}^c(p,q;B;P_{1,1},\ldots,P_{1,k_1};P_{0,1},\ldots,P_{0,k_0}))$ is the "number" of elements in the "zero"-dimensional moduli space $\mathcal{M}^c(p,q;B;P_{1,1},\ldots,P_{1,k_1};P_{0,1},\ldots,P_{0,k_0})$. Here the moduli space $\mathcal{M}^c(p,q;B;P_{1,1},\ldots,P_{1,k_1};P_{0,1},\ldots,P_{0,k_0})$ is the moduli space $\mathcal{M}^c(p,q;B)$ of J-holomorphic sections of $\pi : E^c \to \Sigma$ cut down by intersecting with the given chains $P_{1,i} \subset L_1$ and $P_{0,j} \subset L_0$.

THEOREM 9.4

Let $(\mathcal{L}_0, \mathcal{L}_1) = ((L_0,\gamma_0),(L_1,\gamma_1))$ be a pair of anchored Lagrangian submanifolds. Then the family $\{\mathfrak{c}_{k_1,k_0}\}$ defines a left $(CF(L_1),\mathfrak{m})$ and right $(CF(L_0),\mathfrak{m})$ filtered A_∞-bimodule homomorphism from $CF(\mathcal{L}_1,\tau_L(\mathcal{L}_0))$ to $CF(\mathcal{L}_1,\mathcal{L}_0)$.

The proof of Theorem 9.4 is similar to that of Theorem 8.4 given in the proof of [FO+1, Theorem 3.7.21].

When both L_0, L_1 are unobstructed, we can perform this deformation of $\mathfrak{c}$ using bounding cochains b_0 and b_1 of $CF(L_0)$ and $CF(L_1)$, respectively, in a way similar to $\mathfrak{n}^{b_0,b_1}$. Namely, we define $\mathfrak{c}^{b_1,b_0} : CF(\mathcal{L}_1,\tau_L(\mathcal{L}_0)) \to CF(\mathcal{L}_1,\mathcal{L}_0))$ by

$$\mathfrak{c}^{b_1,b_0}(x) = \sum_{k_1,k_0} \mathfrak{c}_{k_1,k_0}\big(b_1^{\otimes k_1} \otimes x \otimes b_0^{\otimes k_0}\big) = \widehat{\mathfrak{c}}(e^{b_1}, x, e^{b_0}).$$

The following proposition is all we need to add to our context for the construction of Seidel's map $\mathfrak{c}^{b_1,b_0}$ in [Se3].

PROPOSITION 9.5

Let b_0, b_1 be bounding cochains of L_0, L_1, respectively. Then $\mathfrak{c}$ defines a chain map

$$\mathfrak{c}^{b_1,b_0} = C\Phi_0^{\mathrm{rel}}(E,\pi,Q,J^{(6)}) : \big(CF(\tau_L(\mathcal{L}_0),\mathcal{L}_1),\mathfrak{m}_1^{(\tau_L)_*(b_0)}\big) \to \big(CF(\mathcal{L}_1,\mathcal{L}_0),\mathfrak{m}_1^{b_1}\big)$$

and hence induces a homomorphism

$$\mathfrak{c}^{b_1,b_0} : HF\big(\tau_L(\mathcal{L}_0),(\tau_L)_*b_0,(\mathcal{L}_1,b_1)\big) \to HF\big((\mathcal{L}_1,b_1),(\mathcal{L}_0,b_0)\big).$$

Proof

The proof proceeds in the same way as that of Theorem 8.4. The only difference from the latter is that the moduli space $\mathcal{M}^c(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{0,1},\dots,P_{0,k_0})$ does not have $\mathbb{R}$-action anymore, and so we consider the moduli space of sections without considering the quotient. □

9.4. The gluing $\mathfrak{b}\#_\rho\mathfrak{c}$, the composition $\mathfrak{c}\circ\mathfrak{b}$, and the homotopy $\mathfrak{h}$

We denote by (E^b,π^b), Σ^b, and Q^b the fibration, surface, and boundary condition associated to the map b and by (E^c,π^c), Σ^c, and Q^c those associated to the map c constructed in Sections 9.1–9.3. We glue the positive end of E^c and the negative end of E^b to obtain $(E^{bc}_\rho,\pi^{bc}_\rho)$, Σ^{bc}_ρ and Q^{bc}_ρ, for a sufficiently large gluing parameter ρ in the glued ends. This fibration provides a cochain map

$$\mathfrak{b}\#_\rho\mathfrak{c} : CF(\mathcal{L},\mathcal{L}_1)\otimes CF\big(\tau_L(\mathcal{L}_0),\mathcal{L}\big) \to CF(\mathcal{L}_1,\mathcal{L}_0).$$

From now on we denote

$$\mathfrak{b} = \mathfrak{m}_2^{\vec{b}}, \qquad \mathfrak{c} = \mathfrak{c}^{b_1,b_0}$$

for notational simplicity.

LEMMA 9.6

Let (j,J) be such that j is a complex structure on Σ^{bc}_ρ which is standard on its ends, and let $J \in \mathcal{J}(E^{bc}_\rho,\pi^{bc}_\rho,Q^{bc}_\rho,j,J^{(0)},J^{(2)},J^{(5)})$. The cochain map $\mathfrak{b}\#_\rho\mathfrak{c}$ coincides with $\mathfrak{c}\circ\mathfrak{b}$ for any $\rho \geq \rho_0$ with ρ_0 sufficiently large.

Proof

We note that the composition $\mathfrak{c}\circ\mathfrak{b}$ is defined by counting the elements of the fiber product

$$\mathcal{M}(E^b,\mathcal{Q}^b;\vec{p}^{\,b};B^b)_{\mathrm{ev}_+}\#_{\mathrm{ev}_-}\mathcal{M}(E^c,\mathcal{Q}^c;\vec{p}^{\,c};B^c),$$

while the map $\mathfrak{b}\#_\rho\mathfrak{c}$ is defined by counting those in the moduli space

$$\mathcal{M}(E^{bc}_\rho,\mathcal{Q}^{bc}_\rho;\vec{p}^{\,bc}_\rho;B^{bc}_\rho),$$

where $B^{bc}_\rho = B^b\#_\rho B^c$ is the obvious glued homotopy class. By a gluing theorem in the Floer complex (see, e.g., [FO+1]), the two moduli spaces are diffeomorphic to each other, and hence we have the proof. □

Finally, we can construct a homotopy from $\mathfrak{b}\#_\rho\mathfrak{c}$ to the zero map following Seidel's argument from [Se3] verbatim (see [Se3, Figures 9, 12] in particular). We omit details of the construction.

10. Dichotomy of tiny and big pseudoholomorphic polygons

In this section, we study the decomposition of contributions in the cochain maps $\mathfrak{b},\mathfrak{c}$, and the homotopy $\mathfrak{h}$ arising from pseudoholomorphic polygons of *very small energy* and of *not small energy.* We call the polygons of the first type *tiny* and the other *big.*

For this purpose, we first note that the Dehn twist $\tau_L^{-1}: M \to M$ acts by $(L_i,\gamma_i) \to (\tau_L^{-1}(L_i), \tau_L^{-1}(\gamma_i))$ and induces a one-to-one correspondence

$$\tau_L(L_0)\cap L \to L_0 \cap L, \qquad x \mapsto \tau_L^{-1}(x).$$

Since $\tau_L|_L = \mathrm{id}|_L$, this lifts to a diffeomorphism

$$\widetilde{\Omega}\big(\tau_L(\mathcal{L}_0),\mathcal{L}\big) \to \widetilde{\Omega}(\mathcal{L}_0,\mathcal{L}), \qquad [p,w]\mapsto [\tau_L^{-1}(p),\tau_L^{-1}(w)].$$

This latter diffeomorphism induces a filtration-preserving isomorphism

$$(\tau_L^{-1})_*: CF\big(\tau_L(\mathcal{L}_0),\mathcal{L}\big) \to CF(\mathcal{L}_0,\mathcal{L}),$$

that is, satisfies

$$\mathcal{A}_{\tau_L(\mathcal{L}_0),\mathcal{L}}([p,w]) = \mathcal{A}_{\mathcal{L}_0,\mathcal{L}}\big([\tau_L^{-1}(p),\tau_L^{-1}(w)]\big).$$

By perturbing L_0 and L_1 if necessary and choosing $\varepsilon > 0$ sufficiently small, we may assume that

(1) $L\cap L_0$, $L\cap L_1$, and $L_0\cap L_1$ are transverse intersections, and $L\cap L_0\cap L_1 = \emptyset$;

(2) each $\iota^{-1}(L_k)\subset T(r)$ is a union of fibers; one can write this as

$$\iota^{-1}(L_k) = \bigcup_{y\in\iota^{-1}(L\cap L_k)} T(r)_y;$$

(3) R satisfies $0\ge 2\pi R(0) > -\varepsilon$ and is such that τ_L is δ-wobbly.

The following lemma is a part of [Se3, Lemma 3.2]. For the reader's convenience, we provide its proof.

LEMMA 10.1

Suppose that $L_0\cap L\cap L_1=\emptyset$. Then we can choose $\operatorname{supp}\tau_L$ *so close to L that $L_0\cap L_1 \subset M\setminus \operatorname{im}\tau_L$ and $\tau_L(L_0)$, L_1 intersect transversally, and there are injective maps*

$$p: \big(\tau_L(L_0)\cap L\big)\times (L\cap L_1) \to \tau_L(L_0)\cap L_1, \tag{10.1}$$

$$q: L_0\cap L_1 \to \tau_L(L_0)\cap L_1, \tag{10.2}$$

such that $\tau_L(L_0)\cap L_1$ is the disjoint union of their images.

Proof
Conditions (1) and (2) right before Lemma 10.1 imply that $L_0 \cap L_1 \cap U = \emptyset$. Since τ_L is the identity outside U, one has $L_0 \cap L_1 = (\tau_L(L_0) \cap L_1) \setminus U$, so that q can indeed be defined to be the inclusion. There is a bijective correspondence between pairs $(\widetilde{x}_0, x_1) \in (\tau_L(L_0) \cap L) \times (L \cap L_1)$ and $(y_0, y_1) \in \iota^{-1}(L_0 \cap L) \times \iota^{-1}(L \cap L_1)$, given by setting $y_0 = \iota^{-1}(\tau_L^{-1}(\widetilde{x}_0)), y_1 = \iota^{-1}(x_1)$. As a consequence of condition (3) above,

$$\iota^{-1}\big(\tau_L(L_0) \cap L_1\big) = \bigcup_{y_0, y_1} \tau(T(\lambda)_{y_0}) \cap T(\lambda)_{y_1}. \tag{10.3}$$

It is clear from their definitions that p, q are injective. A point of $\tau_L(L_0) \cap L_1$ falls into $\mathrm{im}(q)$ or $\mathrm{im}(p)$ depending on whether it lies inside or outside $\mathrm{im}(\iota)$; hence the two images are disjoint and cover $\tau_L(L_0) \cap L_1$. The transversality follows from the definition of τ_L for $\mathrm{im}(p)$ and from that of $L_0 \cap L_1$ for $\mathrm{im}(q)$. $\square$

We consider the triple

$$\mathcal{L} = (L, \gamma), \qquad \mathcal{L}_0 = (L_0, \tau_L^{-1} \circ \gamma), \qquad \mathcal{L}_1 = (L_1, \gamma_1).$$

We note that $\tau_L(\mathcal{L}_0) = (\tau_L(L_0), \gamma_0) = (\tau_L(L_0), \gamma)$. To make our discussion nontrivial, we may assume

$$\mathcal{L} \cap \mathcal{L}_0 \neq \emptyset, \qquad \mathcal{L} \cap \mathcal{L}_1 \neq \emptyset. \tag{10.4}$$

We fix elements $x_1 \in \mathcal{L} \cap \mathcal{L}_1$ and $\widetilde{x}_0 \in \tau_L(\mathcal{L}_0) \cap \mathcal{L}$ and $\widetilde{z}_0 = p(\widetilde{x}_0, x_1)$, where p is the injective map given in Lemma 10.1.

LEMMA 10.2
Suppose that $x_1 \in \mathcal{L} \cap \mathcal{L}_1$ and $\widetilde{x}_0 \in \tau_L(\mathcal{L}_0) \cap \mathcal{L}$ and $\widetilde{z}_0 = p(\widetilde{x}_0, x_1)$. Then we have $\widetilde{z}_0 \in \mathcal{L}_1 \cap \tau_L(\mathcal{L}_0)$.

Proof
We first note that there is a canonical homotopy class $B_{\mathrm{can}} = B(\widetilde{x}_0, x_1, \widetilde{z})$ spanned by a tiny triangle contained in $U = \mathrm{im}\, \iota$. Choose a path w_1 from $\overline{\gamma} * \gamma_1$ and w_0 from $\overline{\gamma}_0 * \gamma$. Then it follows that we can choose a path w from $\overline{\gamma}_1 * \gamma_0$ defined by $u \# w_1 \# w_0$, where u is the above tiny triangle, that is, any w such that

$$[w] = B_{\mathrm{can}} \# [w_1] \# [w_0].$$

This finishes the proof. $\square$

Now we state the following lemma which is a variation of [Se3, Lemma 3.2] in our context.

PROPOSITION 10.3
Let L_0, L_1, and L be as in Lemma 10.1, and consider the maps p, q defined therein. Then we can choose $\mathrm{im}\, \tau_L$ close to L so that $\tau_L(L_0) \cap L_1$ satisfy the following properties in addition.

(1) *The map q is the inclusion $q(x)=x$. Moreover, for any $z\in\tau_L(\mathcal{L}_0)\cap\mathcal{L}_1$ and $z\in\mathcal{L}_0\cap\mathcal{L}_1$ with $z\neq q(x)$, one has*

$$\mathcal{A}_{\mathcal{L}_0\mathcal{L}_1}([x,w])-\mathcal{A}_{\mathcal{L}_0\mathcal{L}_1}([z,w'])\notin[0;3\varepsilon)$$

whenever the corresponding Floer moduli space $\mathcal{M}(x,z;[\overline{w}\#w'])$ is nonempty.

(2) *Set $\widetilde{z}=p(\widetilde{x}_0,x_1)$. Then there is a canonical homotopy class $B_{\rm can}=B(\widetilde{x}_0,x_1,\widetilde{z})$ spanned by a tiny triangle contained in $U=\operatorname{im}\iota$. And we have*

$$|\Omega(B_{\rm can})|<\varepsilon. \tag{10.5}$$

(3) *For any $z\in\tau_L(\mathcal{L}_0)\cap\mathcal{L}_1$ and $(\widetilde{x}_0,x_1)\in(\tau_L(\mathcal{L}_0)\cap\mathcal{L})\times(\mathcal{L}\cap\mathcal{L}_1)$ with $z\neq p(\widetilde{x}_0,x_1)$, or for $z=\widetilde{x}=p(\widetilde{x}_0,x_1)$ with $B\neq B_{\rm can}$, we have*

$$\Omega(B)\geq C=C(\mathcal{E};J) \tag{10.6}$$

independent of $\varepsilon>0$ whenever $\mathcal{M}_J(\widetilde{x}_0,x_1,z;B)\neq\emptyset$ for some J.

(4) *Suppose that there are $x_k\in\mathcal{L}\cap\mathcal{L}_k$, $k=0,1$ whose preimages $y_k=\iota^{-1}(x_k)$ are antipodes on S^n. Since $\tau|_{S^n}$ is the antipodal map, $\widetilde{x}_0=\tau_L(x_0)$ is equal to x_1. (Hence $x_1\in\tau_L(\mathcal{L}_0)\cap\mathcal{L}\cap\mathcal{L}_1$, and these are all such triple intersection points.) In that case $p(\widetilde{x}_0,x_1)=\widetilde{x}_0=x_1$.*

Proof

The proof is a slight modification of that of [Se3, Lemma 3.2]. Since we need to strip all the exact Lagrangian setting away from Seidel's proof thereof and incorporate contributions coming from different choices of homotopy classes B, we give a complete proof of the proposition.

Now let $s(z)=(z,u(z))$ be the section of $E\to\dot{\Sigma}$ in class $B_{\rm can}$ satisfying the Lagrangian boundary condition. By definition of $B_{\rm can}$, we can choose u so that its image is contained in $\operatorname{im}\iota$. Then (10.5) follows from the identity

$$\Omega(B_{\rm can})=\pi^*\omega(B_{\rm can})=\omega_0(\iota^{-1}\circ u)$$

which can be made as small as we want by choosing r small in the definition of the Dehn twist τ_L.

We now turn to (3). First, we recall that since E is trivial (and so of zero curvature), we have

$$\Omega(B)=\frac{1}{2}\int_{\dot{\Sigma}}\|(Du)^v\|_J^2,$$

and hence whenever $\mathcal{M}_J(\widetilde{x}_0,x_1,z;B)\neq\emptyset$ for some J, we have $\Omega(B)\geq 0$. Define

$$\begin{aligned}C(\mathcal{E};J)=\inf_u\big\{\omega(u)\bigm| u\not\equiv\text{const}, u\in\mathcal{M}_J(\widetilde{x}_0,x_1,z;B),\\ B\neq B_{\rm can}\text{ in }\Pi(E,Q)\big\}.\end{aligned}$$

PROPOSITION 10.4

Let B be an admissible class, and let $u\in\mathcal{M}_J(\widetilde{x}_0,x_1,z;B)$ with $\mu(B)=0$ such that

$$\operatorname{im}u\subset U.$$

Then we have $B = B_{\mathrm{can}}$ *in* $\Pi(E,Q)$*. In particular, we have*

$$C(\mathcal{E};J) > 0. \tag{10.7}$$

Proof
Since we know that $\mu(B_{\mathrm{can}}) = 0$, it is enough to prove that $\omega(B) = \omega(B_{\mathrm{can}})$ by definition of $\Pi(E,Q)$. For this purpose, we compare the action for the paths whose image is contained in the Darboux neighborhood $U = \iota(V)$ with $\iota : (V,\omega_0) \hookrightarrow (M,\omega)$ and whose end points lie either on L, on the fibers F of the cotangent bundle $T^*L \cap V$, or in the Dehn twists $\tau_L(F)$. We recall that the model Dehn twist τ is a Hamiltonian diffeomorphism that satisfies

$$\tau^*\theta_T - \theta_T = dK_\tau$$

for $K_\tau = 2\pi(\mu R'(\mu) - R(\mu))$.

On the cotangent bundle T^*T, the action functionals $\mathcal{A}_{o_T F_1}, \mathcal{A}_{\tau(F_0)o_T}$, and $\mathcal{A}_{\tau(F_0)F_1}$ are defined by

$$\begin{aligned}
\mathcal{A}_{o_T F_1}(z) &= \int z^*\theta_T, \\
\mathcal{A}_{\tau(F_0)o_T}(z) &= \int z^*\theta_T + K_\tau \circ \tau^{-1}\big(z(0)\big), \\
\mathcal{A}_{\tau(F_0)F_1}(z) &= \int z^*\theta_T + K_\tau \circ \tau^{-1}\big(z(0)\big),
\end{aligned}$$

for a path $z : [0,1] \to T^*L$. Here we use the fact that

$$\theta_T|_F \equiv 0 \equiv \theta_T|_{o_T}, \qquad \theta_T|_{\tau(F)} = d(K_\tau \circ \tau^{-1}|_{\tau(F)})$$

(see [Oh2, (2.28)] or [Se3, (1.1)]). Since it is easy to realize such $B = [u]$ as the gluing $-[w_0]\#[w]\#[w_1]$ so that all w_i, w have their images contained in U, we can write

$$\omega(B) = \mathcal{A}_{o_T F_1}(\widehat{p}_0) - \mathcal{A}_{\tau(F_0)o_T}(\widehat{p}) - \mathcal{A}_{\tau_L(F_0)F_1}(\widehat{p}_1) = \omega(B_{\mathrm{can}}).$$

Here the “hat” denotes the constant path; for example, $\widehat{p}$ is the constant path $\widehat{p}(t) \equiv p$. This proves $B = B_{\mathrm{can}}$ in $\Pi(E,Q)$.

For the proof of (10.7), it follows from the first part of the proof that for any $u \in \mathcal{M}_J(\widetilde{x}_0, x_1, z; B)$ with $B \neq B_{\mathrm{can}}$, the image of u must go out of U. Now a simple application of the Gromov-type compactness theorem implies (10.7). This finishes the proof. □

This finishes the proof of (3). The statements in (4) are obvious from definition. □

Now we consider a Darboux neighborhood $U = \operatorname{im}\iota$ of L such that $\operatorname{supp}\tau_L \subset U$ and Lemma 2.1 and Proposition 2.2 hold. We denote by $\mathcal{M}_J(\tau_L(L_0), L, L_1)$ the moduli space of J-holomorphic sections of the fibration (E^b, π^b, Q^b), where J is chosen as before. Then the arguments in [Se3, Section 3.2] give rise to the decomposition

$$\mathfrak{b} = \beta + (\mathfrak{b} - \beta) \tag{10.8}$$

that satisfies β of order $[0;\varepsilon)$, while $(\mathfrak{b} - \beta)$ has order $[3\varepsilon;\infty)$. Furthermore, b is precisely the class induced by the canonical "small" map continued from the constant map p.

Similar consideration of [Se3, Section 3.3] also gives rise to the decomposition

$$\mathfrak{c} = \gamma + (\mathfrak{c} - \gamma) \tag{10.9}$$

such that γ has order zero while $(\mathfrak{c} - \gamma)$ has order $[3\varepsilon;\infty)$. We refer to [Se3] for the detailed proofs of the decomposition results.

Finally, we can construct a homotopy from $\mathfrak{b}\#_\rho\mathfrak{c}$ to the zero map following Seidel's argument from [Se3]. For this purpose we use Proposition 9.2, which is a slight generalization of [Se3, Proposition 2.2] in the general context of Lagrangian spheres in general symplectic manifolds. We omit the details of this construction, referring readers to [Se3] for the details.

11. Construction of long exact sequence

In this section, we combine all the results obtained in Section 10 to construct the required long exact sequence. We first recall two basic lemmas that Seidel used in his construction of long exact sequences for the *exact* case.

LEMMA 11.1

Let D be an $\mathbb{R}$-graded vector space with a differential d_D of order $[0;\infty)$. Suppose that D has gap $[\varepsilon;2\varepsilon)$ for some $\varepsilon > 0$. One can write $d_D = \delta + (d_D - \delta)$ with δ of order $[0;\varepsilon)$, satisfying $\delta^2 = 0$, and $(d_D - \delta)$ of order $[2\varepsilon;\infty)$. Suppose in addition that $H(D,\delta) = 0$; then $H(D,d_D) = 0$.

Seidel then applied this lemma to the direct sum

$$D = C' \oplus C \oplus C''$$

with the differentials given by

$$d_D = \begin{pmatrix} d_{C'} & 0 & 0 \\ b & d_C & 0 \\ h & c & d_{C''} \end{pmatrix}, \qquad \delta = \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix},$$

where

$$C' = CF(\mathcal{L},\mathcal{L}_1) \otimes CF\big(\tau_L(\mathcal{L}_0),\mathcal{L}\big),$$
$$C = CF\big(\tau_L(L_0),L_1\big), \qquad C'' = CF(\mathcal{L}_1,\mathcal{L}_0),$$

and the entries of the matrices are given as stated in the lemma below.

LEMMA 11.2 ([Se3, LEMMA 2.32])

Take three $\mathbb{R}$-graded vector spaces C', C, C'', each of them with a differential of *order* $(0;\infty)$. *Suppose that we have the differential maps* $b : C' \to C$, $c : C \to C''$ *and a homotopy* $h : C' \to C''$ *between* $c \circ b$ *and the zero map such that the following conditions are satisfied for some* $\varepsilon > 0$.

(1) C', C'' *have gap* $(0, 3\varepsilon)$, *and* C *has gap* $(0, 2\varepsilon)$.
(2) *For all* $r \in \operatorname{supp}(C')$ *and* $s \in \operatorname{supp}(C'')$, $|r - s| \geq 4\varepsilon$.
(3) *One can write*

$$b = \beta + (b - \beta), \qquad c = \gamma + (c - \gamma)$$

with β *of order* $[0;\varepsilon)$ *and* $(b-\beta)$ *of order* $[2\varepsilon;\infty)$ *and with the same properties for* γ *and* $(c - \gamma)$. *The lower order parts (which do not need to be differential maps) fit into a short exact sequence of modules*

$$0 \to C' \xrightarrow{\beta} C \xrightarrow{\gamma} C'' \to 0. \tag{11.1}$$

(4) *The map* h *is of order* $[0;\infty)$.

Then the maps on cohomology induced by b, c *fit into a long exact sequence*

$$\cdots \to H(C'; d_{C'}) \xrightarrow{b_*} H(C; d_C) \xrightarrow{c_*} H(C''; d_{C''}) \xrightarrow{\delta} H(C'; d_{C'}) \to \cdots .$$

The proofs of both lemmas rely on an argument involving spectral sequences. For the exact case, all the complexes involved are finite-dimensional vector spaces with *bounded* filtration and *gap* and so existence of spectral sequences for such a complex is easy.

On the other hand, for the case of our current interest, the Floer complex as a $\mathbb{Q}$-vector space is infinite-dimensional with *unbounded* filtration and *without gap* on the vector space itself in general. The existence of a spectral sequence in this case is much more nontrivial (this has been studied by Fukaya, Oh, Ohta, and Ono in [FO+3], [FO+1]). A crucial algebraic model of such a spectral sequence is the notion of a *differential graded completed free filtered* $\Lambda^{(0)}_{0,\mathrm{nov}}$*-module*, abbreviated as d.g.c.f.z. The upshot of this definition is that the $\Lambda_{0,\mathrm{nov}}$-module $C(L, L'; \Lambda_{0,\mathrm{nov}})$ naturally carries this structure. We summarize this construction of spectral sequences in Section A.2.

In the meantime, we remark that the proof of exactness of (11.1) is exactly the same as that of [Se3] based on Lemmas 10.1, 10.2, and on some uniqueness result on a small pseudoholomorphic triangle. (See Section 3.2 [Se3] for details.)

11.1. $CF(\mathcal{L}, \mathcal{L}')$ versus $C(\mathcal{L}, \mathcal{L}'; \Lambda_{0,\mathrm{nov}})$

We first recall basic results on the structure of the Floer cochain group $CF(\mathcal{L}, \mathcal{L}')$ as a module of the Novikov ring $\Lambda(\mathcal{L}, \mathcal{L}')$. We note that in the current study of CY Lagrangian branes we can use the Novikov ring

$$\Lambda^{(0)}_{0,\mathrm{nov}}$$

as the coefficient ring, where $\Lambda^{(0)}_{0,\mathrm{nov}}$ is the degree zero part of $\Lambda_{0,\mathrm{nov}}$ which is a field. Recall the definition from (4.2).

Having this in mind, we first recall the basic construction on the spectral sequence of the $\Lambda_{0,\mathrm{nov}}^{(0)}$-module $C(L,L';\Lambda_{0,\mathrm{nov}})$ from [FO+1, Chapter 6] (or [FO+3, Appendix]) restricting to the *finitely generated case*.

Under the given condition on L_0, L_1 and the given embedding $f: S^n \to L \subset M$, $CF(\tau_L(\mathcal{L}_0),\mathcal{L}), CF(\mathcal{L},\mathcal{L}_1)$ and $CF(\mathcal{L}_1,\mathcal{L}_0)$ are all finitely generated over Λ_{nov}. Here we note that there is a natural injective homomorphism

$$I_{\mathcal{L},\mathcal{L}'} : \Lambda(\mathcal{L},\mathcal{L}') \to \Lambda_{0,\mathrm{nov}}$$

and so we naturally extend their coefficient rings to $\Lambda_{0,\mathrm{nov}}$.

The tiny-big decompositions of the maps $\mathfrak{b}$ and $\mathfrak{c}$ given in Section 10 imply that both maps are *gapped* in the sense of Definitions A.3 and A.5. However, these vector spaces, as they are, do not quite manifest the structure of *d.g.c.f.z.* yet. Because of this, we follow the procedure given in [FO+1, Section 12.4] turning these into a *d.g.c.f.z.* For the reader's convenience, we collect the definition of *d.g.c.f.z.* and the construction of spectral sequence given in [FO+1, Appendix].

Let $(\mathcal{L}_0,\mathcal{L}_1)$ be a general relatively spin pair of anchored Lagrangian submanifolds of M. We first construct a $\Lambda_{0,\mathrm{nov}}$-module $C(\mathcal{L}_1,\mathcal{L}_0;\Lambda_{0,\mathrm{nov}})$ which has a filtered A_∞-bimodule structure over $(C(L_0;\Lambda_{0,\mathrm{nov}}),\mathfrak{m}^{(0)})$ and $(C(L_1;\Lambda_{0,\mathrm{nov}}),\mathfrak{m}^{(1)})$, where the latter are the filtered A_∞-algebras defined in [FO+1]. We consider the intersection $\mathcal{L}_1\cap\mathcal{L}_0$ and the $\mathbb{R}$-filtered set

$$\widehat{I}(\mathcal{L}_0,\mathcal{L}_1) := \big\{T^\lambda e^\mu[p,w] \;\big|\; p\in\mathcal{L}_0\cap\mathcal{L}_1, [p,w]\in\widetilde{\Omega}(\mathcal{L}_0,\mathcal{L}_1), \lambda\in\mathbb{R}, \mu\in\mathbb{Z}\big\}.$$

This is an $\mathbb{R}\times\mathbb{Z}$ principal bundle over $\mathcal{L}_0\cap\mathcal{L}_1$.

We define an equivalence relation $\sim$ on $\widehat{I}(L_0,L_1)$ as follows. We say that

$$T^\lambda e^\mu[p,w] \sim T^{\lambda'} e^{\mu'}[p',w']$$

for $T^\lambda e^\mu[p,w], T^{\lambda'} e^{\mu'}[p',w'] \in \widehat{I}(\mathcal{L}_0,\mathcal{L}_1)$ if and only if the following conditions are satisfied:

$$\begin{gathered} p = p', \\ \lambda + \int w^*\omega = \lambda' + \int (w')^*\omega, \\ 2\mu + \mu([p,w];\lambda_{01}) = 2\mu' + \mu([p',w';\lambda_{01}]). \end{gathered}$$

Here $\mu([p,w];\lambda_{01})$ is the Maslov-Morse index. It is easy to see that this relation is compatible with the conditions of the Γ-equivalence given, and so $\sim$ defines an equivalence relation on $\widehat{I}(\mathcal{L}_0,\mathcal{L}_1)$. Furthermore, we define the *action level* on $E:\widehat{I}(\mathcal{L}_0,\mathcal{L}_1)\to\mathbb{R}$ by

$$E(T^\lambda e^\mu[p,w]) = \lambda + \int w^*\omega$$

and the associated filtration on the set by setting

$$T^\lambda e^\mu[p,w] \in F^{\lambda'}\big(\widehat{I}(\mathcal{L}_0,\mathcal{L}_1)\big)$$

if

$$\lambda + \int w^*\omega \geq \lambda'.$$

We now define

$$I(\mathcal{L}_0, \mathcal{L}_1) = \widehat{I}(\mathcal{L}_0, \mathcal{L}_1)/\sim$$

and somewhat ambiguously denote an element thereof still by $T^\lambda e^{\mu/2}[p,w]$ as long as no danger of confusion arises. The above-mentioned filtration on $\widehat{I}(\mathcal{L}_0, \mathcal{L}_1)$ obviously induces on the quotient $I(\mathcal{L}_0, \mathcal{L}_1)$. We now define

$$I_{\geq 0}(\mathcal{L}_0, \mathcal{L}_1) := \Big\{T^\lambda e^{\mu/2}[p,w] \in I(\mathcal{L}_0, \mathcal{L}_1) \;\Big|\; \lambda + \int w^*\omega \geq 0\Big\}.$$

Consider the formal sum

$$\alpha = \sum_{\lambda,\mu,[p,w]} a_{\lambda,\mu,[p,w]} T^\lambda e^{\mu/2}[p,w]$$

for $\lambda \in \mathbb{R}, \mu \in \mathbb{Z}$ and $[p,w] \in \operatorname{Crit}\mathcal{A}$, and define $\operatorname{supp}\alpha$ to be

$$\operatorname{supp}\alpha = \big\{T^\lambda e^{\mu/2}[p,w] \in I(\mathcal{L}_0, \mathcal{L}_1) \;\big|\; a_{\lambda,\mu,[p,w]} \neq 0\big\}.$$

DEFINITION 11.1

We define by $C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\mathrm{nov}})$ the $\Lambda_{0,\mathrm{nov}}$-module

$$\begin{aligned} &C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\mathrm{nov}}) \\ &\quad := \big\{\alpha \;\big|\; E(\alpha) \geq 0, \#\big(\operatorname{supp}\alpha \cap E^{-1}((-\infty,\lambda])\big) < \infty \text{ for all } \lambda \in \mathbb{R}\big\}. \end{aligned}$$

Obviously $C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\mathrm{nov}})$ has the structure of a $\Lambda_{0,\mathrm{nov}}$-module. In addition, we have the following.

PROPOSITION 11.3

$C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\mathrm{nov}})$ is a d.g.c.f.z.

Given the grading of an element $T^\lambda e^\mu[p,w]$ $2\mu + \mu([p,w];\lambda_{01})$, it becomes a filtered graded free $\Lambda_{0,\mathrm{nov}}$-module. Following [FO+1, Section 5.1.3], we write

$$\langle p\rangle = \begin{cases} T^{-\mathcal{A}_{\mathcal{L}_0\mathcal{L}_1}([p,w])} e^{-\mu([p,w];\lambda_{01})}[p,w] & \text{if } \mu([p,w];\lambda_{01}) \text{ is even,} \\ T^{-\mathcal{A}_{\mathcal{L}_0\mathcal{L}_1}([p,w])} e^{-(\mu([p,w];\lambda_{01})-1)/2}[p,w] & \text{if } \mu([p,w];\lambda_{01}) \text{ is odd.} \end{cases}$$

Thus we have $E(\langle p\rangle) = 0$, and $\deg(\langle p\rangle)$ is either zero or 1 depending on the parity of $\mu([p,w];\lambda_{01})$.

It is easy to see that $C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\mathrm{nov}})$ is isomorphic to the completion (with respect to the filtration on $\Lambda_{0,\mathrm{nov}}$) of the free $\Lambda_{0,\mathrm{nov}}$-module generated by $\langle p\rangle$ for the intersection points $p \in \mathcal{L}_0 \cap \mathcal{L}_1$. Namely, we have a canonical isomorphism

$$C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\mathrm{nov}}) \cong \widehat{\bigoplus_{p \in \mathcal{L}_0 \cap \mathcal{L}_1}} \Lambda_{0,\mathrm{nov}}\langle p\rangle$$

as a ($\mathbb{Z}_2$-graded) $\Lambda_{0,\mathrm{nov}}$-module.

Recall the definition of the Floer cochain module. By definition, we have an inclusion

$$CF^*(\mathcal{L}_1, \mathcal{L}_0) \to C^*(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{\mathrm{nov}})$$

defined by

$$[p, w] \mapsto e^{(\mu([p,w]) - \mu(\langle p \rangle))/2} T^{\rho A_{\ell_0}([p,w])} \langle p \rangle. \tag{11.2}$$

It is compatible with the obvious inclusion $I_{\mathcal{L}_0, \mathcal{L}_1} : \Lambda(\mathcal{L}_0, \mathcal{L}_1) \to \Lambda_{\mathrm{nov}}$.

We take the coefficient $R = \mathbb{Q}$ and recall that Floer cohomology

$$HF(\mathcal{L}^{(1)}, \mathcal{L}^{(0)}) = \operatorname{Ker} \delta^{\vec{b}} / \operatorname{Im} \delta^{\vec{b}}, \quad \vec{b} = (b_1, b_0),$$

is defined as a $\Lambda(\mathcal{L}^{(0)}, \mathcal{L}^{(1)})$-module.

We remark that

$$CF(\mathcal{L}_1, \mathcal{L}_0) \cong \Lambda(\mathcal{L}_1, \mathcal{L}_0)^{\#(\mathcal{L}_0 \cap \mathcal{L}_1)},$$

where $\#\mathcal{L}_0 \cap \mathcal{L}_1$ is finite by the transversality hypothesis. Therefore we have the isomorphism

$$C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{\mathrm{nov}}) \cong CF(\mathcal{L}_1, \mathcal{L}_0; \ell_{01}) \otimes_{\Lambda(\mathcal{L}_1, \mathcal{L}_0)} \Lambda_{\mathrm{nov}}.$$

On the other hand, the Novikov ring $\Lambda(\mathcal{L}_1, \mathcal{L}^{(0)})$ is a field if the ground ring is $\mathbb{Q}$. Therefore this leads to the isomorphism

$$HF\big((\mathcal{L}_1, b_1), (\mathcal{L}_0, b_0); \Lambda_{\mathrm{nov}}\big) \cong HF(\mathcal{L}_1, \mathcal{L}_0) \otimes_{\Lambda(\mathcal{L}_1, \mathcal{L}_0)} \Lambda_{\mathrm{nov}}. \tag{11.3}$$

Finally, we explain how we combine the above-discussed anchored versions into a single nonanchored version of Floer cohomology following [FO+1, Section 5.1.3].

We first note that the filtered $\Lambda_{0,\mathrm{nov}}$-module structure of $C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\mathrm{nov}})$ depends only on the homotopy class ℓ_{01}. So we form the completed direct sum

$$C(L_1, L_0; \Lambda_{0,\mathrm{nov}}) = \widehat{\bigoplus_{[\ell_0] \in \pi_0(\Omega(L_0, L_1))}} C(L_1, L_0; \ell_0; \Lambda_{0,\mathrm{nov}}).$$

We note that we have the natural inclusion map

$$C(\mathcal{L}_1, \mathcal{L}_0) \to C(L_1, L_0; \overline{\gamma}_0 * \gamma_1; \Lambda_{0,\mathrm{nov}}) \subset C(L_1, L_0; \Lambda_{0,\mathrm{nov}})$$

defined as (11.2). We define the corresponding Floer cohomology by

$$HF(L_1, L_0; \Lambda_{0,\mathrm{nov}}) := \operatorname{Ker} \delta^{\vec{b}} / \operatorname{Im} \delta^{\vec{b}}.$$

Then we have

$$HF(L_1, L_0; \Lambda_{0,\mathrm{nov}}) \cong \bigoplus_{[\ell_0] \in \pi_0(\Omega(L_0, L_1))} HF(\mathcal{L}_1, \mathcal{L}_0) \otimes_{\Lambda(L_1, L_0; \ell_0)} \Lambda_{\mathrm{nov}}$$

(see [FO+1, Proposition 5.1.17]).

11.2. Wrapping it up

Now combining all the discussions in the previous sections, we are ready to prove the main theorem, which we now restate here.

THEOREM 11.4

Let (M,ω) be compact (symplectically) Calabi-Yau. Let L be a Lagrangian sphere in M together with a preferred diffeomorphism $f : S^2 \to L$. Denote by $\tau_L = \tau_{(L,[f])}$ the Dehn twist associated to $(L,[f])$.

Consider any CY Lagrangian branes L_0, L_1. Then for any pair (b_0, b_1) of the Maurer-Cartan-solutions $b_0 \in \mathcal{M}(L_0;\Lambda(L_0))$, $b_1 \in \mathcal{M}(L_1;\Lambda(L_1))$, there is a long exact sequence of $\mathbb{Z}$-graded Floer cohomologies

$$\begin{aligned}&\longrightarrow HF\big((\tau_L(\mathcal{L}_0),(\tau_L)_*(b_0)),(\mathcal{L}_1,b_1)\big) \longrightarrow HF\big((\mathcal{L}_0,b_0),(\mathcal{L}_1,b_1)\big)\\ &\longrightarrow HF\big((\mathcal{L},0),(\mathcal{L}_1,b_1)\big)\otimes HF\big((\mathcal{L}_0,b_0),(\mathcal{L}_1,b_1)\big) \longrightarrow\end{aligned} \tag{11.4}$$

as a Λ_{nov}-module where the Floer cohomologies involved are the deformed Floer cohomologies constructed in [FO+3] and [FO+1]. We also have the nonanchored version of the exact sequence.

The same exact sequence still holds for any orientable relatively spin pair (L_0, L_1) if they are just unobstructed, whose Maslov classes do not necessarily vanish.

To highlight the main points of the construction, let us first assume that $b_0 = b_1 = 0$ are Maurer-Cartan-solutions. In this case, the Floer cohomology is the standard one which uses the Floer boundary map δ. We first state the following lemma, which is a consequence of Corollary A.8 and a variation of Lemma 11.2

LEMMA 11.5

Let D be an $[0;\infty)$-graded vector space with a differential d_D of order $[0;\infty)$, which is not necessarily finite-dimensional but forms a d.g.c.f.z. in the sense of Definition A.2. Suppose that one can then write $d_D = \delta + (d_D - \delta)$ with δ of order $[0;\varepsilon)$, satisfying $\delta^2 = 0$, and $(d_D - \delta)$ of order $[2\varepsilon;\infty)$. Suppose in addition that $H(D,\delta) = 0$; then $H(D, d_D) = 0$.

Now we apply this lemma to the direct sum

$$D = C' \oplus C \oplus C''$$

with the differentials given by

$$d_D = \begin{pmatrix} d_{C'} & 0 & 0 \\ b & d_C & 0 \\ h & c & d_{C''} \end{pmatrix}, \qquad \delta = \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix},$$

where this time we consider the $\Lambda_{0,\text{nov}}$-modules

$$\begin{aligned} C' &= C(\mathcal{L},\mathcal{L}_1) \otimes C\big(\tau_L(\mathcal{L}_0),\mathcal{L}\big),\\ C &= C\big(\tau_L(\mathcal{L}_0),\mathcal{L}_1\big), \quad C'' = C(\mathcal{L}_0,\mathcal{L}_1), \end{aligned}$$

and the entries of the matrices are given as stated before.

Now Lemma 10.3 and the tiny-big decomposition results in Section 10 give rise to a long exact sequence

$$\begin{aligned}&\longrightarrow HF(\tau_L(\mathcal{L}_0),\mathcal{L}_1;\Lambda_{0,\mathrm{nov}})\longrightarrow HF(\mathcal{L}_0,\mathcal{L}_1;\Lambda_{0,\mathrm{nov}})\\ &\longrightarrow HF(\mathcal{L},\mathcal{L}_1;\Lambda_{0,\mathrm{nov}})\otimes HF(\mathcal{L}_0,\mathcal{L}_1;\Lambda_{0,\mathrm{nov}})\longrightarrow\end{aligned}\tag{11.5}$$

for $\Lambda_{0,\mathrm{nov}}^{(0)}$-modules. Since we have

$$HF(\mathcal{L},\mathcal{L}';\Lambda_{\mathrm{nov}})\cong HF(\mathcal{L},\mathcal{L}';\Lambda_{0,\mathrm{nov}})\otimes_{\Lambda_{0,\mathrm{nov}}}\Lambda_{\mathrm{nov}}$$

from (11.3) and since Λ_{nov} is a field, tensoring (11.5) with Λ_{nov} produces the exact sequence (11.4) for the case $b_0 = b_1 = 0$.

Now the same reasoning as for the case $b_i = 0$ induces the long exact sequence (11.4). This finishes the proof of Theorem 11.4.

Appendix

A.1. Index formula for $E_{z_0}\setminus\{x_0\}$

In this section, we prove the index formula (7.27). There is an index formula stated in the various literature in terms of the "capping surfaces" stated as in [EGH] and [Bo], which, however, does not fit our need. For this reason, we give a complete proof of (7.27).

In fact, we consider the following general setup. Consider a symplectic manifold W with a contact-type boundary of the type

$$\partial W \cong S^1(T^*N)$$

with *negative* end for an oriented compact manifold N. We attach the cylinder $\mathbb{R}_+\times\partial W$ and also denote by W the completed manifold. We denote $(r(x),\Theta(x))$ for a point $x\in\mathbb{R}^+\times S^1(T^*N)$. Composing this with the diffeomorphism

$$(s,\Theta)\mapsto(e^s,\Theta),\qquad \mathbb{R}\times\partial W\to\mathbb{R}_+\times\partial W,$$

we put a translational invariant almost complex structure J on the end.

Next let γ be a Reeb orbit of $S^1(T^*N)$ with period T. We note that the symplectic vector bundle $\gamma^*T(T^*N)$ carries a splitting

$$\gamma^*T(T^*N)=\mathbb{C}\oplus\gamma^*\xi_N,$$

where ξ_N is the contact distribution of $S^1(T^*N)$. Furthermore, we fix a Riemannian metric g on N and consider the canonical almost complex structure J_g on T^*N. The projection of γ to N is nothing but a geodesic on N with respect to g. Denote by $c = c_\gamma$ the associated geodesic on N. Since we assume that N is oriented, we can take a trivialization $\gamma^*T(S^1(T^*N))$ which is tangent to the vertical fibers of $T(T^*N)$. Using this, we can define the Conley-Zehnder index of γ when γ is nondegenerate, which we denote by $\mu_{CZ}(\gamma)$. For the Bott-Morse case, one uses the generalized Conley-Zehnder index defined by Robbin and Salamon [RS].

Next, this choice of trivialization of $\gamma^*\big(T(S^1(T^*N))\big)=\gamma^*T(\partial W)$) also allows one to define a *relative Chern number* of a map $u:\dot\Sigma\to W$ with the asymptotic condition

$$\lim_{\tau\to\infty}\Theta\circ u(\tau,t)=\gamma(t),\qquad \lim_{\tau\to\infty} s\circ u(\tau,t)=-\infty. \tag{A.1}$$

Denote by $\overline{u}:(\widehat{\dot\Sigma},\partial\widehat{\dot\Sigma})\to(W,\gamma)$ the obvious compactified map.

Then $\overline{u}^*(TW)$ is a symplectic vector bundle with a trivialization $\phi_\gamma : \gamma^*(T(\partial W))\to S^1\times\mathbb{C}^{n-1}$ constructed above.

This gives rise to the main definition.

DEFINITION A.1

We define the relative Chern number, denoted by $c_1(u;\gamma)$, by

$$c_1(u;\gamma)=c_1(u^*TW;\phi_\gamma).$$

Once we have made the notions of relative Chern number and Conley-Zehnder index precise, the following index formula can be derived from the formula in [Bo, Corollary 5.4].

THEOREM A.1

The expected dimension of $\mathcal{M}(W,J;\gamma;A)$ *is given by*

$$-\mu_{CZ}(\gamma)+(n-3)+2c_1(u;\gamma),\quad [u]=A,$$

for a nondegenerate geodesic.

For the Morse-Bott case in which $\mathcal{R}_{\rm sim}$ *forms a smooth manifold, the expected dimension of the moduli space* $\mathcal{M}(W,J;A;1)$ *consisting of* J*-holomorphic* u*'s with asymptotics*

$$\lim_{\tau\to\inf}\Theta\circ u(e^{2\pi(\tau+it)})=\gamma(t),$$

where γ *is a simple Reeb orbit, is given by*

$$-\mu_{CZ}(\gamma)+\frac{\dim\mathcal{R}_{\rm sim}}{2}+(n-3)+2c_1(u;\gamma),$$

where μ_{CZ} *is the generalized Conley-Zehnder index of* γ*.*

A.2. d.g.c.f.z. and spectral sequence

We first start from the following situation. Let $V=(\Lambda_{0,\rm nov}^{(0)})^{\oplus I}$ be a free $\Lambda_{0,\rm nov}^{(0)}$-module with $\#(I)$ finite. We define a filtration on V in the obvious way, which induces a topology on V. Let $\widehat V$ be the completion of V. We call such $\widehat V$ a *completed free filtered* $\Lambda_{0,\rm nov}^{(0)}$*-module generated by energy zero elements* or, in short, *c.f.z.* If V is finitely generated (as a $\Lambda_{0,\rm nov}^{(0)}$-module) in addition, we say that it is a *finite* c.f.z. We define a function, which we call the *(action) level*,

$$E:\widehat V\setminus\{0\}\to\mathbb{R}_{\ge 0}$$

such that

$$\mathfrak{v}\in F^{E(\mathfrak{v})}V,\qquad \mathfrak{v}\notin F^{\lambda}V\quad\text{if }\lambda>E(\mathfrak{v}).$$

Let $\overline{V} = V/\Lambda^{+,(0)}_{0,\mathrm{nov}}V \cong R^I$. *We always take an embedding (splitting)*

$$\overline{V} \subset V$$

as the energy zero part of V so that its composition with the projection $V \to \overline{V}$ is the identity map.

Let $\mathfrak{v} \in V$. We put

$$\mathfrak{v} = \sum T^{\lambda_i} v_i,$$

where $v_i \in \overline{V}$, $\lambda_i < \lambda_{i+1}$, $\lim_{i\to\infty} \lambda_i = \infty$, and $v_i \neq 0$. We call $T^{\lambda_i} v_i$ the *components* of $\mathfrak{v}$, with $T^{\lambda_1} v_1$ the *leading component* and v_1 the *leading coefficient* of $\mathfrak{v}$. We denote the leading coefficient v_1 of $\mathfrak{v}$ by $\sigma(\mathfrak{v})$. We also define the leading component and the leading coefficient of an element of $\Lambda^{(0)}_{0,\mathrm{nov}}$ in the same way.

Now we consider the case of graded $\Lambda_{0,\mathrm{nov}}$-modules.

DEFINITION A.2 ([FO+1, DEFINITION 6.3.8])

Let $\widehat{C}$ be a graded $\Lambda_{0,\mathrm{nov}}$-module. We assume that $\widehat{C}^k$ is a c.f.z. for each k. A *differential graded c.f.z.* (abbreviated as d.g.c.f.z.) is a pair $(\widehat{C}, \delta)$ with a degree 1 operator $\delta : \widehat{C} \to \widehat{C}$ such that

$$\delta \circ \delta = 0, \qquad \delta(F^\lambda \widehat{C}) \subseteq F^\lambda \widehat{C}.$$

We call the pair a *finite* d.g.c.f.z. if each $\widehat{C}^k$ is a finite c.f.z.

The following proposition is essential for the proof of some convergence properties of the spectral sequence.

PROPOSITION A.2 ([FO+1, PROPOSITION 6.3.9])

Let W be a finitely generated $\Lambda^{(0)}_{0,\mathrm{nov}}$-submodule of $\widehat{C}^k$. Then there exists a constant c depending only on W but independent of λ such that

$$\delta(W) \cap F^\lambda \widehat{C}^{k+1} \subset \delta(W \cap F^{\lambda - c} \widehat{C}^k).$$

Now let $(\widehat{C}, \delta)$ be a d.g.c.f.z., and let $\widehat{C}^k$ be a completion of C^k. We assume that C^k is *free over* $\Lambda^{(0)}_{0,\mathrm{nov}}$. We put

$$\overline{C} = C/\Lambda^{+(0)}_{0,\mathrm{nov}} C \cong \widehat{C}/\Lambda^{+(0)}_{0,\mathrm{nov}} \widehat{C}$$

and let $\overline{\delta}$ be the induced derivation on $\overline{C}$. We again embed $\overline{C} \subseteq C \subseteq \widehat{C}$ as the energy zero part. In general $\overline{C}$ is *not* a differential graded subalgebra of $\widehat{C}$. Let $\{\mathfrak{e}_i\}$ be a basis of C (over $\Lambda^{(0)}_{0,\mathrm{nov}}$), and let $\overline{\mathfrak{e}}_i$ be the corresponding basis of $\overline{C}$ over $R = \Lambda^{(0)}_{0,\mathrm{nov}}/\Lambda^{+(0)}_{0,\mathrm{nov}}$. We put

$$\overline{\delta}(\overline{\mathfrak{e}}_i) = \sum \delta_{0,ij} \overline{\mathfrak{e}}_j$$

and define $\delta_0 : \widehat{C} \to \widehat{C}$ by $\delta_0 = \overline{\delta} \otimes 1$, that is, by

$$\delta_0 \mathfrak{e}_i = \sum \delta_{0,ij} \mathfrak{e}_j.$$

DEFINITION A.3

We say that $(\widehat{C}, \delta)$ satisfies the *gapped condition* if $(\delta - \delta_0)$ has order $[\lambda'', \infty)$, that is, if there exists $\lambda'' > 0$ such that for any λ we have

$$\delta \mathfrak{v} - \delta_0 \mathfrak{v} \in F^{\lambda + \lambda''} \widehat{C}$$

for all $\mathfrak{v} \in F^{\lambda} \widehat{C}$.

Under the gapped condition, we take a constant λ_0 with $0 < \lambda_0 < \lambda''$ and define a filtration on $\widehat{C}$ by

$$F^n \widehat{C} = F^{n\lambda_0} \widehat{C};$$

[FO+1] then uses this filtration to define a spectral sequence.

LEMMA A.3 ([FO+1, LEMMA 6.3.20])

Denote

$$\Lambda^{(0)}(\lambda_0) = \Lambda^{(0)}_{0,\mathrm{nov}} / F^{\lambda_0} \Lambda^{(0)}_{0,\mathrm{nov}}.$$

Then there exists a $\Lambda^{(0)}(\lambda_0)$*-module homomorphism*

$$\delta_r^{p,q} : E_r^{p,q}(\widehat{C}) \to E_r^{p+1,q+r-1}(\widehat{C})$$

such that

(1) $\delta_r^{p+1,q+r-1} \circ \delta_r^{p,q} = 0,$
(2) $\mathrm{Ker}(\delta_r^{p,q}) / \mathrm{Im}(\delta_r^{p-1,q-r+1}) \cong E_{r+1}^{p,q}(\widehat{C}),$
(3) $e^{\pm 1} \circ \delta_r^{p,q} = \delta_r^{p \pm 2,q} \circ e^{\pm 1}.$

Of course, the construction of $E_r^{p,q}(\widehat{C})$ is quite standard. One difference from the standard case is that the filtration used here is not bounded. Namely, we do *not* have $F^n \widehat{C} = 0$ for large n. Hence the convergence property of our spectral sequence is far from being trivial in general. However, it is stable from below in that $F^0 \widehat{C} = \widehat{C}$. As a consequence, we have the following.

LEMMA A.4 ([FO+1, LEMMA 6.3.22])

There exists an injection

$$E_{r+1}^{p,q}(\widehat{C}) \to E_r^{p,q}(\widehat{C})$$

if $q - r + 2 \leq 0$.

An immediate consequence of Lemma A.4 is the following convergence result.

PROPOSITION A.5

The projective limit

$$E_\infty^{p,q}(\widehat{C}) := \varprojlim E_r^{p,q}(\widehat{C})$$

exists.

Furthermore, from the construction, we have the description of the E_2-term of the associated spectral sequence.

LEMMA A.6 ([FO+1, LEMMA 6.3.24])

We have an isomorphism

$$E_2^{*,*}(\widehat{C}) \cong H(\overline{C};\overline{\delta}) \otimes_R \mathrm{gr}_*(F\Lambda_{0,\mathrm{nov}})$$

as $\mathrm{gr}_*(F\Lambda_{0,\mathrm{nov}})$*-modules.*

Proof

By definition, we have

$$E_1^{*,*}(\widehat{C}) \cong \overline{C} \otimes_R \mathrm{gr}_*(F\Lambda_{0,\mathrm{nov}}).$$

It follows from the gapped condition $\delta_1 = \overline{\delta}$. Hence it finishes the proof. $\square$

DEFINITION A.4

We define $F^q H(\widehat{C},\delta)$ to be the image of $H(F^q\widehat{C},\delta)$ in $H(\widehat{C},\delta)$.

To relate the limit $E_\infty^{p,q}$ of the spectral sequence and $F^q H(\widehat{C},\delta)$, we need some finiteness assumptions which we now describe. Let (C,δ) and (C',δ') be d.g.c.f.z.'s satisfying the gap condition. Let $\varphi : C \to C'$ be a map such that $\varphi\delta = \delta'\varphi$, and let $\overline{\varphi} : \overline{C} \to \overline{C}'$ be the map induced on $\overline{C} = C/\Lambda_{0,\mathrm{nov}}^{+(0)}C$ and $\overline{C}' = C'/\Lambda_{0,\mathrm{nov}}^{+(0)}C'$, respectively. The induced map $\overline{\varphi}$ lifts to $\varphi_0 = \overline{\varphi} \otimes 1 : C \to C'$.

DEFINITION A.5 ([FO+1, DEFINITION 6.3.26])

Under the situation above, we say that $\varphi : C \to C'$ satisfies *a gapped condition*, or is *a gapped cochain map*, if there exists λ'' such that

$$(\varphi - \varphi_0)(F^\lambda \widehat{C}) \subset F^{\lambda+\lambda''}\widehat{C}.$$

Using these definitions, [FO+1] proves the following.

THEOREM A.7 ([FO+1, THEOREM 6.3.28])

If C is finite, then there exists r_0 such that

$$E_{r_0}^{p,q}(\widehat{C}) \cong E_{r_0+1}^{p,q}(\widehat{C}) \cong \cdots \cong E_\infty^{p,q}(\widehat{C}) \cong F^q H^p(\widehat{C},\delta)/F^{q+1}H^p(\widehat{C},\delta)$$

as $\Lambda^{(0)}(\lambda_0) = \Lambda_{0,\mathrm{nov}}^{(0)}/F^{\lambda_0}\Lambda_{0,\mathrm{nov}}^{(0)}$*-modules.*

We summarize the above discussion into the following vanishing result which is crucial in our spectral sequence arguments.

COROLLARY A.8

Let C be a finite d.g.c.f.z. If $H(\overline{C},\overline{\delta}) = 0$, then $H(C,\delta) = 0$.

A.3. Products

In this subsection, we recall the description of the deformed products $\mathfrak{m}_k^{\vec{b}}$ from [FO+2]. We refer to [Fu] and [FO+2] for the relevant proofs of the statements we make without proofs here.

Let $\mathfrak{L} = (L_0, L_1, \dots, L_k)$ be a chain of compact Lagrangian submanifolds in (M, ω) that intersect pairwise transversely without triple intersections.

Let $\vec{z} = (z_{0k}, z_{k(k-1)}, \dots, z_{10})$ be a set of distinct points on $\partial D^2 = \{z \in \mathbb{C} \mid |z| = 1\}$. We assume that they respect the counterclockwise cyclic order of ∂D^2. The group $\mathrm{PSL}(2;\mathbb{R}) \cong \mathrm{Aut}(D^2)$ acts on the set in an obvious way. We denote by $\mathcal{M}_{k+1}^{\mathrm{main},\circ}$ the set of $\mathrm{PSL}(2;\mathbb{R})$-orbits of $(D^2, \vec{z})$.

In this subsection, we consider only the case $k \geq 2$ since the case $k = 1$ is already discussed Section A.2. In this case there is no automorphism on the domain $(D^2, \vec{z})$; that is, $\mathrm{PSL}(2;\mathbb{R})$ acts freely on the set of such $(D^2, \vec{z})$'s.

Let $p_{j(j-1)} \in L_j \cap L_{j-1}$ $(j = 0, \dots, k)$, be a set of intersection points.

We consider the pair $(w; \vec{z})$, where $w : D^2 \to M$ is a pseudoholomorphic map that satisfies the boundary condition

$$w(\overline{z_{j(j-1)} z_{(j+1)j}}) \subset L_j, \tag{A.2a}$$

$$w(z_{(j+1)j}) = p_{(j+1)j} \in L_j \cap L_{j+1}. \tag{A.2b}$$

We denote by $\widetilde{\mathcal{M}}^\circ(\mathfrak{L}, \vec{p})$ the set of such $((D^2, \vec{z}), w)$.

We identify two elements $((D^2, \vec{z}), w)$, $((D^2, \vec{z}'), w')$ if there exists $\psi \in \mathrm{PSL}(2;\mathbb{R})$ such that $w \circ \psi = w'$ and $\psi(z'_{j(j-1)}) = z_{j(j-1)}$. Let $\mathcal{M}^\circ(\mathfrak{L}, \vec{p})$ be the set of equivalence classes. We compactify it by including the configurations with disc or sphere bubbles attached and denote it by $\mathcal{M}(\mathfrak{L}, \vec{p})$. Its element is denoted by $((\Sigma, \vec{z}), w)$, where Σ is a genus zero bordered Riemann surface with one boundary component, $\vec{z}$ are boundary marked points, and $w : (\Sigma, \partial\Sigma) \to (M, L)$ is a bordered stable map.

We can decompose $\mathcal{M}(\mathfrak{L}, \vec{p})$ according to the homotopy class $B \in \pi_2(\mathfrak{L}, \vec{p})$ of continuous maps satisfying (A.2a), (A.2b) into the union

$$\mathcal{M}(\mathfrak{L}, \vec{p}) = \bigcup_{B \in \pi_2(\mathfrak{L};\vec{p})} \mathcal{M}(\mathfrak{L}, \vec{p}; B).$$

In the case when we fix an anchor γ_i to each L_i and put $\mathcal{E} = ((L_0, \gamma_0), \dots, (L_k, \gamma_k))$, we consider only admissible classes B and put

$$\mathcal{M}(\mathcal{E}, \vec{p}) = \bigcup_{B \in \pi_2^{ad}(\mathcal{E};\vec{p})} \mathcal{M}(\mathcal{E}, \vec{p}; B).$$

THEOREM A.9

Let $\mathfrak{L} = (L_0, \dots, L_k)$ be a chain of Lagrangian submanifolds, and let $B \in \pi_2(\mathfrak{L}; \vec{p})$. Then $\mathcal{M}(\mathfrak{L}, \vec{p}; B)$ has an oriented Kuranishi structure (with boundary and corners). Its (virtual) dimension satisfies

$$\dim \mathcal{M}(\mathfrak{L}, \vec{p}; B) = \mu(\mathfrak{L}, \vec{p}; B) + n + k - 2, \tag{A.3}$$

where $\mu(\mathfrak{L}, \vec{p}; B)$ is the polygonal Maslov index of B.

We next take graded anchors (γ_i, λ_i) to each L_i. We assume that B is admissible and write $B = [w_{01}^-]\#[w_{12}^-]\#\cdots\#[w_{k0}^-]$ as in Definition 3.5. We put $w_{(i+1)i}^+(s,t) = w_{i(i+1)}^-(1-s,t)$. We also put $w_{k0}^+(s,t) = w_{0k}^+(s,1-t)$ $([w_{k0}^+] \in \pi_1(\ell_{k0}; p_{k0}))$. We also put $\lambda_{k0}(t) = \lambda_{0k}(1-t)$.

LEMMA A.10 ([FO+2, LEMMA 8.11])
If $\dim \mathcal{M}(\mathfrak{L}, \vec{p}; B) = 0$, *then we have*

$$\big(\mu([p_{k0}, w_{k0}^+]; \lambda_{0k}) - 1\big) = 1 + \sum_{i=1}^{k} \big(\mu([p_{i(i-1)}, w_{i(i-1)}^+]; \lambda_{(i-1)i}) - 1\big). \tag{A.4}$$

Using the case $\dim \mathcal{M}(\mathfrak{L}, \vec{p}; B) = 0$, we define the k-linear operator

$$\begin{aligned}\mathfrak{m}_k : CF\big((L_k,\gamma_k),(L_{k-1},\gamma_{k-1})\big) \otimes \cdots \otimes CF\big((L_1,\gamma_1),(L_0,\gamma_0)\big)\\ \to CF\big((L_k,\gamma_k),(L_0,\gamma_0)\big)\end{aligned}$$

as follows:

$$\begin{aligned}&\mathfrak{m}_k([p_{k(k-1)}, w_{k(k-1)}^+], [p_{(k-1)(k-2)}, w_{(k-1)(k-2)}^+], \dots, [p_{10}, w_{10}^+])\\ &\quad = \sum \#\big(\mathcal{M}_{k+1}(\mathfrak{L}; \vec{p}; B)\big)[p_{k0}, w_{k0}^+].\end{aligned} \tag{A.5}$$

Here the sum is over the basis $[p_{k0}, w_{k0}^+]$ of $CF((L_k,\gamma_k),(L_0,\gamma_0))$, where $\vec{p} = (p_{0k}, p_{k(k-1)}, \dots, p_{10})$, B is as in Definition 3.5, and $w_{(i+1)i}^+(s,t) = w_{i(i+1)}^-(1-s,t)$.

Formula (A.4) implies that $\mathfrak{m}_k$ above has degree 1.

In general, the operator $\mathfrak{m}_k$ above does *not* satisfy the A_∞-relation for the same reason as that of the case of boundary operators (see Section 8.1). We need to use bounding cochains b_i of L_i to deform $\mathfrak{m}_k$ in the same way as in the case of A_∞-bimodules (see Section 8.3), whose explanation is now in order.

Let $m_0, \dots, m_k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{M}_{m_0,\dots,m_k}(\mathfrak{L}, \vec{p}; B)$ be the moduli space obtained from the set of $((D^2, \vec{z}), (\vec{z}^{(0)}, \dots, \vec{z}^{(k)}), w))$ by taking the quotient by $\mathrm{PSL}(2,\mathbb{R})$-action and then by taking the stable map compactification as before. Here $z^{(i)} = (z_1^{(i)}, \dots, z_{k_i}^{(i)})$ and $z_j^{(i)} \in \overline{z_{(i+1)i} z_{i(i-1)}}$ such that $z_{(i+1)i}, z_1^{(i)}, \dots, z_{k_i}^{(i)}, z_{i(i-1)}$ respects the counterclockwise cyclic ordering;

$$\big((D^2, \vec{z}), (\vec{z}^{(0)}, \dots, \vec{z}^{(k)}), w\big) \mapsto \big(w(z_1^{(0)}), \dots, w(z_{m_k}^{(k)})\big)$$

induces an evaluation map

$$\mathrm{ev} = (\mathrm{ev}^{(0)}, \dots, \mathrm{ev}^{(k)}) : \mathcal{M}_{m_0,\dots,m_k}(\mathfrak{L}, \vec{p}; B) \to \prod_{i=0}^{k} L_i^{m_i}.$$

Let $P_j^{(i)}$ be smooth singular chains of L_i, and put

$$\vec{P}^{(i)} = (P_1^{(i)}, \dots, P_{m_i}^{(i)}), \qquad \vec{\vec{P}} = (\vec{P}^{(0)}, \dots, \vec{P}^{(k)}).$$

We then take the fiber product to obtain

$$\mathcal{M}_{m_0,\dots,m_k}(\mathfrak{L}, \vec{p}; \vec{\vec{P}}; B) = \mathcal{M}_{m_0,\dots,m_k}(\mathfrak{L}, \vec{p}; B) \times_{\mathrm{ev}} \vec{\vec{P}}.$$

We use this to define

$$\mathfrak{m}_{k;m_0,\dots,m_k} : B_{m_k}\big(CF(L_k)\big) \otimes CF\big((L_k,\gamma_k),(L_{k-1},\gamma_{k-1})\big)$$
$$\otimes \cdots \otimes CF\big((L_1,\gamma_1),(L_0,\gamma_0)\big) \otimes B_{m_0}\big(CF(L_0)\big) \to CF\big((L_k,\gamma_k),(L_0,\gamma_0)\big)$$

by

$$\mathfrak{m}_{k;m_0,\dots,m_k}(\vec{P}^{(k)}, [p_{k(k-1)}, w^+_{k(k-1)}], \dots, [p_{10}, w^+_{10}], \vec{P}^{(0)})$$
$$= \sum \#\big(\mathcal{M}_{k+1}(\mathfrak{L}; \vec{p}; \vec{\vec{P}}; B)\big)[p_{k0}, w_{k0}].$$

Finally, for each given $b_i \in CF(L_i)[1]^0$ $(b_i \equiv 0 \mod \Lambda_+)$, $\vec{b} = (b_0, \dots, b_k)$, and $x_i \in CF((L_i,\gamma_i),(L_{i-1},\gamma_{i-1}))$, we put

(A.6) $$\mathfrak{m}^{\vec{b}}_k(x_k, \dots, x_1) = \sum_{m_0,\dots,m_k} \mathfrak{m}_{k;m_0,\dots,m_k}(b_k^{m_k}, x_k, b_{k-1}^{m_{k-1}}, \dots, x_1, b_0^{m_0}).$$

THEOREM A.11

If b_i satisfies the Maurer-Cartan equation (8.10), then $\mathfrak{m}^{\vec{b}}_k$ in (A.6) satisfies the A_∞-relation

(A.7) $$\sum_{k_1,k_2,i} (-1)^* \mathfrak{m}^{\vec{b}}_{k_1}\big(x_k, \dots, \mathfrak{m}^{\vec{b}}_{k_2}(x_{k-i-1}, \dots, x_{k-i-k_2}), \dots, x_1\big) = 0,$$

where we take the sum over $k_1 + k_2 = k + 1$, $i = -1, \dots, k - k_2$. (We write $\mathfrak{m}_k$ in place of $\mathfrak{m}^{\vec{b}}_k$ in (A.7).) The sign $*$ is $* = i + \deg x_k + \cdots + \deg x_{k-i}$.

We summarize the above discussion as follows.

THEOREM A.12

We can associate a filtered A_∞-category to a symplectic manifold (M,ω) such that

(1) *its object is $(\mathcal{L}, b, sp)$, where $\mathcal{L} = (L, \gamma, \lambda)$ is a graded anchored Lagrangian submanifold, $[b] \in \mathcal{M}(CF(L))$ is a bounding cochain, and* sp *is a spin structure of L;*

(2) *the set of morphisms is $CF((L_1,\gamma_1),(L_0,\gamma_0))$;*

(3) *$\mathfrak{m}^{\vec{b}}_k$ are the operations defined in (A.6).*

REMARK A.6

Here we spell out the choice of orientations; o_p of $\mathrm{Index}\,\overline{\partial}_{\lambda_p}$ is included. This choice in fact does not affect the module structure $CF((L_1,\gamma_1),(L_0,\gamma_0))$ up to isomorphism: if we take an alternative choice o'_p at p, then all the signs appearing in the operations $\mathfrak{m}_k$ that involve $[p,w]$ for some w are reversed. Therefore $[p,w] \mapsto -[p,w]$ gives the required isomorphism.

The operations $\mathfrak{m}_k$ are compatible with the filtration. Namely, we have the following.

PROPOSITION A.13
If $x_i \in F^{\lambda_i} CF((L_i,\gamma_i),(L_{i-1},\gamma_{i-1}))$*, then*

$$\mathfrak{m}_k^{\vec{b}}(x_k,\dots,x_1) \in F^{\lambda} CF\big((L_k,\gamma_k),(L_0,\gamma_0)\big),$$

where $\lambda = \sum_{i=1}^{k} \lambda_i$.

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