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Mixed problems for hyperbolic equations II

Existence Theorems with Zero Initial Datas and
Energy Inequalities with Initial Datas.

By

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§0. Introduction

This paper has two aims. One of them is to obtain existence theorems for hyperbolic mixed problems in a quadrant:

$$(P_0)\quad \begin{cases} Au = f & \text{in } t>0,\ x>0,\ y\in R^{n-1}, \\ B_j u = g_j & (j=1, 2, \cdots, \mu) \quad \text{on } x=0,\ t>0,\ y\in R^{n-1}, \\ D_t^j u = 0 & (j=0, 1, \cdots, m-1) \quad \text{on } t=0,\ x>0,\ y\in R^{n-1}. \end{cases}$$

The other is to obtain energy inequalities for

$$(P)\quad \begin{cases} Au = f & \text{in } t>0,\ x>0,\ y\in R^{n-1}, \\ B_j u = g_j & (j=1, 2, \cdots, \mu) \quad \text{on } x=0,\ t>0,\ y\in R^{n-1}, \\ D_t^j u = u_j & (j=0, 1, \cdots, m-1) \quad \text{on } t=0,\ x>0,\ y\in R^{n-1}. \end{cases}$$

The assumption (A) stated in part I is also assumed throughout in this paper.

Our main results are as follows:

Theorem 1. *There exists a positive number γ_0 as follows. Let $f\in\mathcal{H}_{0,\gamma}((0, \infty)\times R^n_+)$, $g_j\in\mathcal{H}_{m-1-r_j,\gamma}((0, \infty)\times R^{n-1})$ $(\gamma\geqslant\gamma_0)$ be given, then there exists a unique solution u of (P_0), which belongs to $\mathcal{H}_{m-1,\gamma}((0, \infty)\times R)$.*

Theorem 2. *There exist positive numbers C and γ_0, such that it holds for $\gamma\geqslant\gamma_0$*

$$\gamma|u|^2_{m-1,\gamma,+}+\sum_{j=0}^{m-1}\langle D_x^j u\rangle^2_{m-1-j,\gamma,+}$$
$$\leqslant C\left\{\frac{1}{\gamma}|f|^2_{0,\gamma,+}+\sum_{j=1}^{\mu}\langle g_j\rangle^2_{m-1-r_j,\gamma,+}+\sum_{j=0}^{m-1}[u_j]^2_{m-1-j,\gamma}\right\}$$

for $u \in \mathcal{H}_{m,\gamma}((0, \infty) \times R_+^n)$, *satisfying* (P).

Notations will be explained in the following sections.

§ 1. Existence theorems with zero initial datas

1.1. Adjoint systems.

Let us denote

$$X = (t_0, 0, y_0 ; \tau, \eta),$$
$$\bar{X} = (t_0, 0, y_0 ; \bar{\tau}, \bar{\eta}).$$

Let

$$C_j(\xi) = \xi^{r_j} \qquad j = \mu+1, \cdots, m,$$

where $\{r_j\}_{j=\mu+1,\cdots,m}$ is the complement of $\{r_j\}_{j=1,\cdots,\mu}$ in $\{0, 1, \cdots, m-1\}$, then

$$[\{B_j(X ; \xi)\}_{j=1,\cdots,\mu}, \{C_j(\xi)\}_{j=\mu+1,\cdots,m}]$$

becomes a Dirichlet set, since $\{B_j(X : \xi)\}_{j=1,\cdots,\mu}$ is a normal set.

Lemma 1.1. *There exist differential operators*

$$\{C'(t, y ; D_t, D_x, D_y)\}_{j=1,\cdots,\mu}, \ \{B'(t, y ; D_t, D_x, D_y)\}_{j=\mu+1,\cdots,m},$$

with constant coefficients outside of K_0, *satisfying*

$$(Au, v) - (u, A^*v) = \sum_{j=1}^{\mu} \langle B_j u, C'_j v\rangle + \sum_{j=\mu+1}^{m} \langle C_j u, B'_j v\rangle$$

for $u \in \mathcal{H}_{m,\gamma}(R_+^{n+1})$ *and* $v \in \mathcal{H}_{m,-\gamma}(R_+^{n+1})$.

Remark. Let us denote the principal part of $\{A^*, B'_j, C'_j\}$ by $\{A_0^*, B'_{j0}, C'_{j0}\}$ $(A_0^* = A)$. Then $[\{C'_{j0}(X ; \xi)\}_{j=1,\cdots,\mu}, \{B'_{j0}(X ; \xi)\}_{j=\mu+1,\cdots,m}]$ is a Dirichlet set, and

$$\int_0^\infty A(X ; D_x)u(x)\overline{v(x)}dx - \int_0^\infty u(x)\overline{A(\bar{X} ; D_x)v(x)}dx$$
$$= \sum_{j=1}^{\mu} B_j(X ; D_x)u(0)\overline{C'_{j0}(\bar{X} ; D_x)v(0)} + \sum_{j=\mu+1}^{m} C_j(D_x)u(0)\overline{B'_{j0}(\bar{X} ; D_x)v(0)}$$

for $u, v \in H^m(0, \infty)$.

We say that $\{A^*, \{B'_j\}_{j=\mu+1,\cdots,m}\}$ is an adjoint system for $\{A, \{B_j\}_{j=1,\cdots,\mu}\}$.

Proof. Since the statement is well known for $u, v \in \mathcal{D}(\overline{R_+^{n+1}})$, we only remark that $\mathcal{D}(\overline{R_+^{n+1}})$ is dense in $\mathcal{H}_{m,\gamma}(R_+^{n+1})$ as well as in $\mathcal{H}_{m,-\gamma}(R_+^{n+1})$.

Let us denote

$$\tilde{A}^* = A^*(-t, x, y; -D_t, D_x, D_y),$$
$$\tilde{B}'_j = B'_j(-t, y; -D_t, D_x, D_y),$$

then we have

Lemma 1.2. *Let* $\{A, \{B_j\}_{j=1,\cdots,\mu}\}$ *satisfy the assumption* (A), *then* $\{\tilde{A}_0^*, \{\tilde{B}'_{j0}\}_{j=\mu+1,\cdots,m}\}$ *satisfies also* (A).

Proof. Let

$$L^+ = \{(\tau, \eta); \tau = \sigma - i\gamma,\ \gamma > 0,\ \sigma \in R^1,\ \eta \in R^{n-1},\ \sigma^2 + \gamma^2 + |\eta|^2 = 1\}.$$

There exists $\{D'_j(X; \xi)\}_{j=\mu+1,\cdots,m}$, which is equivalent to $\{B'_{j0}(\bar{X}; \xi)\}_{j=\mu+1,\cdots,m}$ and $|Lop\{\bar{A}_-(X; \xi); D'_{\mu+1}(X; \xi), \cdots, D'_m(X; \xi)\}| > d > 0$ in $R^n \times L^+$ (c.f. proposition 3.2, chapter I, [6]), therefore we have $|Lop\{\bar{A}_-(X; \xi); B'_{\mu+1,0}\bar{X}; \xi), \cdots, B'_{m_0}(\bar{X}; \xi)\}| > d_1 > 0$ in $R^n \times L^+$ (c.f. lemma 3.3, chapter I, [6]). Since $\bar{A}_-(X; \xi) = A_-(\bar{X}; \xi)$, we have

$$|Lop\{A_-(\bar{X}; \xi); B'_{\mu+1}(\bar{X}; \xi), \cdots, B'_m(\bar{X}; \xi)\}| > d_1 > 0 \quad \text{in} \quad R^n \times L^+.$$

Here we have

Corollary.

$$\gamma |v|^2_{m-1,-\gamma} + \sum_{j=0}^{m-1} \langle D_x^j v \rangle^2_{m-1-j,-\gamma}$$
$$\leqslant C\left\{\frac{1}{\gamma}|A^* v|^2_{0,-\gamma} + \sum_{j=\mu+1}^{m} \langle B'_j v \rangle^2_{m-1-r'_j,-\gamma}\right\}$$

for $v \in \mathcal{H}_{m,-\gamma}$, $\gamma \geqslant \gamma_0$.

Proof. From the theorem in I, we have

$$\gamma |u|^2_{m-1,\gamma} + \sum_{j=0}^{m-1} \langle D_x^j u \rangle^2_{m-1-j,\gamma}$$
$$\leqslant C\left\{\frac{1}{\gamma}|\tilde{A}^* u|^2_{0,\gamma} + \sum_{j=\mu+1}^{m} \langle \tilde{B}'_j u \rangle^2_{m-1-r'_j,\gamma}\right\}$$

for $u \in \mathcal{H}_{m,\gamma}$, $\gamma \geqslant \gamma_0$. We only remark that $v \in \mathcal{H}_{m,-\gamma}$ means $u(t, x, y) = v(-t, x, y) \in \mathcal{H}_{m,\gamma}$ and that

$$\begin{cases} (\tilde{A}^*u)(t, x, y) = (A^*v)(-t, x, y), \\ (\tilde{B}'_j u)(t, x, y) = (B'_j v)(-t, x, y). \end{cases}$$

1.2. Existence theorems without initial conditions.

We consider the problem

$$(P') \qquad \begin{cases} Au = f & \text{in } x>0,\ (t, y)\in R^n, \\ B_j u = g_j & j=1, 2, \cdots, \mu \quad \text{on} \quad x=0,\ (t, y)\in R^n. \end{cases}$$

Let us denote for $v\in\mathcal{H}_{s,-\gamma}(R^n)$

$$\Lambda'(\gamma, D_t, D_y)v = e^{-\gamma t}\mathcal{F}^{-1}[\Lambda(\gamma, \sigma, \eta)\mathcal{F}[e^{\gamma t}v]]\in\mathcal{H}_{s-1,-\gamma}(R^n).$$

Then we have from the corollary of Lemma 1.2, for $\Lambda'^s v\in\mathcal{H}_{m,-\gamma}$ and $\gamma\geqslant\gamma_s$,

$$\gamma|\Lambda'^s v|^2_{m-1,-\gamma}+\sum_{j=0}^{m-1}\langle D_x^j v\rangle^2_{m-1-j+s,-\gamma}$$
$$\leqslant C_s\left\{\frac{1}{\gamma}|\Lambda'^s A^*v|^2_{0,-\gamma}+\sum_{j=\mu+1}^{m}\langle B'_j v\rangle^2_{m-1-r'_j+s,-\gamma}\right\}.$$

Let $\mathfrak{H}_{s,-\gamma}$ be a Hilbert space defined by the completion of

$$\{v\,;\ \Lambda'^s v\in\mathcal{H}_{m,-\gamma}(R^{n+1}_+)\}$$

with the norm

$$|v|^2_{\mathfrak{H}_{s,-\gamma}}=|\Lambda'^s A^*v|^2_{0,-\gamma}+\sum_{j=\mu+1}^{m}\langle B'_j v\rangle^2_{m-1-r'_j+s,-\gamma},$$

then we have

$$\gamma|\Lambda'^s v|^2_{m-1,-\gamma}+\sum_{j=0}^{m-1}\langle D_x^j v\rangle^2_{m-1-j+s,-\gamma}\leqslant C|v|^2_{\mathfrak{H}_{s,-\gamma}} \qquad (\gamma\geqslant\gamma_s).$$

Proposition 1.1. *Let $f\in\mathcal{H}_{k,\gamma}(R^{n+1}_+)$, and $g_j\in\mathcal{H}_{m-1-r_j+k,\gamma}(R^n)$, then there exists a solution u of (P') in $\mathcal{H}_{m-1+k,\gamma}(R^{n+1}_+)$, where $\gamma\geqslant\gamma_k$ and $k=0, 1, 2, \cdots$.*

Proof. Let us denote $\mathfrak{H}=\mathfrak{H}_{-(m-1+k),-\gamma}$ for simplicity.

1) Since

$$|(f, v)|=|(\Lambda^k f, \Lambda'^{-k}v)|\leqslant|f|_{k,\gamma}|\Lambda'^{-(m-1+k)}v|_{m-1,-\gamma}\leqslant C|f|_{k,\gamma}|v|_{\mathfrak{H}},$$
$$|\langle g_j, C'_j v\rangle|=|\langle\Lambda^{m-1-r_j+k}g_j, \Lambda'^{-(m-1-r_j+k)}C'_j v\rangle|$$
$$\leqslant C\langle g_j\rangle_{m-1-r_j+k,\gamma}\sum_{i=0}^{m-1}\langle D_x^i v\rangle_{-k-i,-\gamma}\leqslant C\langle g_j\rangle_{m-1-r_j+k,\gamma}|v|_{\mathfrak{H}},$$

there exists $w\in\mathfrak{H}$ such that

$$(w, v)_{\mathfrak{H}} = (f, v) - \sum_{j=1}^{\mu} \langle g_j, C'_j v \rangle$$

for all $v \in \mathfrak{H}$, where

$$(w, v)_{\mathfrak{H}} = (\Lambda'^{-(m-1+k)} A^* w, \Lambda'^{-(m-1+k)} A^* v)_{0,-\gamma} + \sum_{j=\mu+1}^{m} \langle B'_j w, B'_j v \rangle_{-k-r'_j,-\gamma}.$$

Let

$$u = \Lambda^{-(m-1+k)} e^{2\gamma t} \Lambda'^{-(m-1+k)} A^* w$$

then we have $\Lambda^{m-1+k} u \in \mathcal{H}_{0,\gamma}$ and $Au = f$ in distribution sense.

2) Let us denote

$$\begin{aligned} Au &= A\Lambda^{-(m-1+k)} e^{2\gamma t} \Lambda'^{-(m-1+k)} A^* v = \sum_{l=0}^{2m} D_x^l \Lambda^{2(1-k)-l} \alpha_l v \\ &= \Lambda^{-k+2-m} D_x^m \{ \Lambda^{-(m-1+k)-1} \sum_{l=0}^{m} \Lambda^{m-l} D_x^l \alpha_{l+m} v \} \\ &\quad + \Lambda^{-k+2} \{ \Lambda^{-(m-1+k)} \sum_{l=0}^{m-1} \Lambda^{m-1-l} D_x^l \alpha_l v \} \\ &= \Lambda^{-k+2-m} D_x^m \varphi + \Lambda^{-k+2} \psi , \end{aligned}$$

where

$$|\alpha_l v|_{s,\gamma} \leqslant C_s |v|_{s,-\gamma} \qquad (\alpha_{2m} = e^{2\gamma t})$$

and

$$|\varphi|_{0,\gamma}, \ |\psi|_{0,\gamma} \leqslant C |v|_{\mathfrak{H}}.$$

Therefore we have from the interpolation theorem

$$\Lambda^{-m} \varphi \in \mathcal{H}_{m,\gamma} .$$

Since

$$\begin{aligned} \varphi &= \Lambda^{-(m-1+k)-1} \sum_{l=0}^{m} \Lambda^{m-l} D_x^l \alpha_{l+m} v \\ &= \Lambda^{-(m-1+k)-1} D_x^m \alpha_{2m} v + \Lambda^{-(m-1+k)-1} \sum_{l=0}^{m-1} \Lambda^{m-l} D_x^l \alpha_{l+m} v , \end{aligned}$$

we have

$$\Lambda'^{-m-(m-1+k)-1} v \in \mathcal{H}_{2m,-\gamma} ,$$

then

$$\Lambda^{k-2} u \in \mathcal{H}_{m,\gamma} .$$

Moreover, since

$$f = Au = D_x^m u + \sum_{l=0}^{m-1} a_l D_x^l u ,$$

we have

$$u \in \mathcal{H}_{m+k-2,\gamma}.$$

3) Here we assume $k \geqslant 2$, then we have $u \in \mathcal{H}_{m,\gamma}$ and $Au = f$. Now we have from lemma 1.1,

$$\begin{aligned} 0 &= (f, v) - \sum_{j=1}^{\mu} \langle g_j, C'_j v \rangle - (u, A^* v) \\ &= \sum_{j=1}^{\mu} \langle B_j u, C'_j v \rangle - \sum_{j=1}^{\mu} \langle g_j, C'_j v \rangle \end{aligned}$$

for $v \in \mathcal{H}_{m,-\gamma}$ satisfying $B'_j v \mid_{x=0} = 0$. Since $C'_j v \mid_{x=0}$ are arbitrary, we have

$$B_j u \mid_{x=0} = g_j \qquad j = 1, \cdots, \mu.$$

4) If $f \in \mathcal{H}_{k,\gamma}(R^{n+1}_+)$, $g_j \in \mathcal{H}_{m-1-r_j+k,\gamma}(R^n)$ $(k = 0, 1)$, there exist $f_l \in \mathcal{D}(\overline{R^{n+1}_+})$, $g_{jl} \in \mathcal{D}(\overline{R^n})$ such that

$$\begin{cases} f_l \xrightarrow[l \to \infty]{} f & \text{in } \mathcal{H}_{k,\gamma}(R^{n+1}_+), \\ g_{jl} \xrightarrow[l \to \infty]{} g_j & \text{in } \mathcal{H}_{m-1-r_j+k,\gamma}(R^n), \end{cases}$$

and then there exists the solution u_l of

$$\begin{cases} Au_l = f_l, \\ B_j u_l \mid_{x=0} = g_{jl} \qquad (j = 1, 2, \cdots, \mu), \end{cases}$$

where $u_l \in \mathcal{H}_{m-1+k,\gamma}$. From the corollary 2 of the theorem in part I, we have

$$u_l \xrightarrow[l \to \infty]{} u \qquad \text{in } \mathcal{H}_{m-1+k,\gamma},$$

and u becomes the solution of (P').

1.3. Uniqueness theorems without initial conditions.

We say that u is a weak solution of (P') in $\mathcal{H}_{0,\gamma}$ if and only if $u \in \mathcal{H}_{0,\gamma}$, $f \in \mathcal{H}_{0,\gamma}$, $g_j \in \mathcal{H}_{0,\gamma}(R^n)$ and it holds

$$(f, v) - (u, A^* v) = \sum_{j=1}^{\mu} \langle g_j, C'_j v \rangle$$

for all v belonging to

$$\{v \in \mathcal{H}_{m,-\gamma};\ B'_j v \mid_{x=0} = 0 \qquad j = \mu+1, \mu+2, \cdots, m\}.$$

Proposition 1.2. *Weak solution of (P') is unique in $\mathcal{H}_{0,\gamma}$ $(\gamma \geqslant \gamma_0)$.*

Proof. For given $\varphi \in \mathcal{D}(\overline{R^{n+1}_+})$, there exists v in $\mathcal{H}_{m,\gamma}$ such that

$$\begin{cases} A^*v = \varphi\,, \\ B'_j v\,|_{x=0} = 0 \qquad (j=\mu+1,\ \mu+2,\ \cdots,\ m)\,, \end{cases}$$

from proposition 1.1. Now let u be a weak solution of (P') in $\mathcal{H}_{0,\gamma}$ with zero datas, then we have

$$(u,\ \varphi) = 0 \qquad \text{for} \quad \varphi \in \mathcal{D}(\overline{R^{n+1}_+}),$$

therefore $u=0$.

We are going to show the uniqueness in a larger class $\mathcal{H}_{0,\gamma,\gamma'}$ $(\gamma'>\gamma)$ than $\mathcal{H}_{0,\gamma}$, where

$$\mathcal{H}_{k,\gamma,\gamma'} = \{u \in H^k_{\text{loc.}}\,;\ u_- \in \mathcal{H}_{k,\gamma,-}\,,\ u_+ \in \mathcal{H}_{k,\gamma',+}\}\,,$$

where $u_-(u_+)$ is the restriction of u in $t<0(t>0)$ and

$$\mathcal{H}_{k,\gamma,-} = \{u : e^{-\gamma t}u \in H^k((-\infty,\ 0)\times R^n_+)\}\,,$$
$$\mathcal{H}_{k,\gamma,+} = \{u\,;\ e^{-\gamma t}u \in H^k((0,\ \infty)\times R^n_+)\}\,.$$

In the similar way to the above, we say that u is a weak solution of (P') in $\mathcal{H}_{0,\gamma,\gamma'}$ if and only if $u \in \mathcal{H}_{0,\gamma,\gamma'}$, $f \in \mathcal{H}_{0,\gamma,\gamma'}$, $g_j \in \mathcal{H}_{0,\gamma,\gamma'}(R^n)$ and it holds

$$(f,\ v) - (u,\ A^*v) = \sum_{j=1}^{\mu} \langle g_j,\ C'_j v\rangle$$

for all v belonging to

$$\{v \in \mathcal{H}_{m,-\gamma,-\gamma'}\,;\ B'_j v\,|_{x=0} = 0\,, \qquad j=\mu+1,\ \mu+2,\ \cdots,\ m\}\,.$$

Proposition 1.3. *Let δ be positive, then there exists γ'_δ such that weak solution of (P') is unique in $\mathcal{H}_{0,\gamma,\gamma-\delta}$ $(\gamma \geqslant \gamma'_\delta)$.*

Proof. Let

$$\varphi_\delta(t) = \alpha(t)e^{-\delta t} + (1-\alpha(t))$$

where

$$\begin{cases} \alpha(t) = 0 & \text{in} \quad t \leqslant 0, \\ 0<\alpha(t)<1 & \text{in} \quad 0<t<1, \\ \alpha(t) = 1 & \text{in} \quad t \geqslant 1, \end{cases}$$

then

$$\begin{cases} \varphi_\delta(t) = 1 & \text{in} \quad t \leqslant 0, \\ e^{-\delta} \leqslant \varphi_\delta(t) \leqslant 1 & \text{in} \quad 0<t<1, \\ \varphi_\delta(t) = e^{-\gamma t} & \text{in} \quad t \geqslant 1. \end{cases}$$

Since

$$\varphi_\delta(t)D_t^k u(t) = \left(D_t + i\frac{\varphi_\delta'(t)}{\varphi_\delta(t)}\right)^k(\varphi_\delta u)$$

and

$$\left|D_t^k\left(\frac{\varphi_\delta'(t)}{\varphi_\delta(t)}\right)\right| \leqslant C_{\delta,k} \qquad (k=0, 1, 2, \cdots),$$

we have

$$\varphi_\delta Au = A^{(\delta)}(\varphi_\delta u),$$

where the principal part of $A^{(\delta)}$ is A and coefficients of lower order terms in $A^{(\delta)}$ belong to $\mathcal{B}^\infty$. Then proposition 1.3 follows from proposition 1.2.

1.4. Existence theorems with zero initial datas.

Proposition 1.4. *There exists γ_k'' as follows: Let*

$$f \in \bigcap_{\gamma \geqslant \gamma_k''} \mathcal{H}_{k,\gamma}(R_+^{n+1}), \quad g_j \in \bigcap_{\gamma \geqslant \gamma_k''} \mathcal{H}_{m-1-r_j+k,\gamma}(R^n),$$

then there exists a unique solution u of (P') in $\bigcap_{\gamma \geqslant \gamma_k''} \mathcal{H}_{m-1+k,\gamma}(R_+^{n+1})$.

Proof. Let γ_1' be given in proposition 1.3, by taking $\delta=1$. From proposition 1.1, there exists $u_\gamma \in \mathcal{H}_{m-1+k,\gamma}$, a solution of (P'), for every $\gamma \geqslant \gamma_k$. Here we denote $\gamma_k'' = \max(\gamma_k, \gamma_1')$. Now let $\gamma \geqslant \gamma_k''$ and $0 < \delta \leqslant 1$, then $u_\gamma - u_{\gamma+\delta} \in \mathcal{H}_{m-1+k,\gamma,\gamma+\delta} \subset \mathcal{H}_{m-1+k,\gamma,\gamma+1}$, therefore we have from proposition 1.3 that

$$u_\gamma = u_{\gamma+\delta}.$$

Hence it follows that u_γ is independent of γ for $\gamma \geqslant \gamma_k''$.

Lemma 1.3. *Let $f \in \mathcal{H}_{k,\gamma_0}(R_+^{n+1})$ $(\gamma_0 > 0)$ and* $\mathrm{supp}[f] \subset [0, \infty) \times \overline{R_+^n}$, *then*

i) $f \in \bigcap_{\gamma \geqslant \gamma_0} \mathcal{H}_{k,\gamma}(R_+^{n+1})$,

ii) $|f|_{k,\gamma} \leqslant C_k \gamma^k |f|_{k,\gamma_0}$.

The converse is also true.

Proof. Let us prove the converse, only in the case when $k=0$. Let us denote $f=f(t)$,

$$\|f(t)\|^2 = \int_{R_+^n} |f(t, x, y)|^2 dxdy,$$

$$\tilde{f}(\sigma-i\gamma) = \int_{-\infty}^{\infty} e^{-i\sigma t-\gamma t} f(t)dt\,,$$

then

$$\int_{-\infty}^{\infty} \|\tilde{f}(\sigma-i\gamma)\|^2 d\sigma = C\int_{-\infty}^{\infty} \|e^{-\gamma t}f(t)\|^2 dt$$
$$= C|f|^2_{0,\gamma} \leqslant C'|f|^2_{0,\gamma_0}\,.$$

Hence we have $\operatorname{supp}_t[f(t)] \subset [0, \infty)$.

Now we have, from proposition 1.4 and lemma 1.3,

Theorem 1. *Let* $f \in \mathcal{H}_{k,\gamma}(R^{n+1}_+)$, $g_j \in \mathcal{H}_{m-1-r_j-k,\gamma}(R^n)$ $(k \geqslant 0,\ \gamma \geqslant \gamma_k'')$, *and* $\operatorname{supp}_t[f]$, $\operatorname{supp}_t[g_j] \subset [0, \infty)$, *then there exists* $u \in \mathcal{H}_{m-1+k,\gamma}(R^{n+1}_+)$, *which satisfies*

$$(P') \qquad \begin{cases} Au = f\,, \\ B_j u\big|_{x=0} = g_j \qquad (j=1, 2, \cdots, \mu) \end{cases}$$

and $\operatorname{supp}_t[u] \subset [0, \infty)$.

§ 2. Energy inequalities with initial datas

2.1. Green's formulas.

At first we consider quadrants

$$R^1_+ \times R^n_+ = \{t>0,\ (x, y) \in R^n_+\}\,,$$

and

$$R^1_- \times R^n_+ = \{t<0,\ (x, y) \in R^n_+\}\,.$$

Let us denote

$$\mathcal{H}_{m,\gamma,+} = \mathcal{H}_{m,\gamma}(R^1_+ \times R^n_+)\,,$$
$$\mathcal{H}_{m,\gamma,-} = \mathcal{H}_{m,\gamma}(R^1_- \times R^n_+)\,,$$
$$\Lambda_0 = \Lambda_0(\gamma, D_y)\,, \quad (\Lambda_0 u)(y) = \mathcal{F}^{-1}[\sqrt{\gamma^2+|\eta|^2}\,\mathcal{F}[u(y)](\eta)]\,,$$
$$[u]^2_{m,\gamma} = \sum_{i-j-|\nu|=m} \int_{R^n_+} |\gamma^i D_x^j D_y^\nu u(x, y)|^2 dx\,dy\,.$$

Lemma 2.1. *Let* (k, ν, ν') *satisfy*

$$0 \leqslant k \leqslant m-1\,,$$
$$0 \leqslant \nu_j \leqslant \min(m-1-r_j, k) \qquad (j=1, 2, \cdots, m),$$
$$\nu_j + \nu_j' = k \qquad (j=1, 2, \cdots, m).$$

Then for $u\in\mathcal{H}_{m,\gamma,+}$ *and* $v\in\mathcal{H}_{m,-\gamma,-}$, *we have*

$$\begin{aligned}&(Au,\ \Lambda_0^{m-1-k}D_t^k v)_+-(\Lambda_0^{m-1-k}D_t^k u,\ A^*v)_+\\&=\sum_{j=1}^{\mu}\langle\Lambda_0^{m-1-r_j-\nu_j}D_t^{\nu_j}B_j u,\ \Lambda_0^{r_j-\nu'_j}D_t^{\nu'_j}C'_j v\rangle_+\\&\quad+\sum_{j=\mu+1}^{m}\langle\Lambda_0^{m-1-r_j-\nu_j}D_t^{\nu_j}C_j u,\ \Lambda_0^{r_j-\nu'_j}D_t^{\nu'_j}B'_j v\rangle_+\\&\quad+G^{\nu\nu'}[u,\ v]+R^{\nu\nu'}(u,\ v)_++R_0^{\nu\nu'}\langle u,\ v\rangle_+\,,\end{aligned}$$

where

$$\begin{aligned}&|G^{\nu\nu'}[u,\ v]|\leqslant C\sum_{j=0}^{m-1}[D_t^j u]_{m-1-j,\gamma}\sum_{j=0}^{m-1}[D_t^j v]_{m-1-j,\gamma}\,,\\&|R^{\nu\nu'}(u,\ v)_+|\leqslant C|u|_{m-1,\gamma,+}|v|_{m-1,-\gamma,+}\,,\\&|R_0^{\nu\nu'}\langle u,\ v\rangle_+|\leqslant C\langle u\rangle_{m-2,\gamma,+}\langle v\rangle_{m-1,\gamma,+}\,.\end{aligned}$$

Proof. We only remark that

$$\begin{aligned}&\langle D_t^k u,\ \Lambda_0^k v\rangle_+-\langle\Lambda_0^k u,\ D_t^k v\rangle_+\\&=\sum_{l=1}^{k}\{[\Lambda_0^{k-l}D_t^{l-1}D_x u,\ \Lambda_0^l D_t^{k-l}v]-[\Lambda_0^l D_t^{k-l}u,\ \Lambda_0^{k-l}D_t^{l-1}D_x u]\}\,.\end{aligned}$$

In fact, both sides become the same integral on $\{t=0,\ x=0,\ y\in R^{n-1}\}$, using integral by parts.

Next we consider a time interval

$$I=(T_1,\ T_2)\,.$$

Let us denote

$$\begin{aligned}(u,\ v)_{0,\gamma,I}&=\int_{T_1}^{T_2}e^{-2\gamma t}\left(\int_{R_+^n}u(t,\ x,\ y)\overline{v(t,\ x,\ y)}dx\,dy\right)dt\\&=\int_{T_1}^{T_2}e^{-2\gamma t}[u(tx,\ y),\ v(t,\ x,\ y)]\,dt\,,\\\langle u,\ v\rangle_{0,\gamma,I}&=\int_{T_1}^{T_2}e^{-2\gamma t}\left(\int_{R^{n-1}}u(t,\ 0,\ y)\overline{v(t,\ 0,\ y)}dy)\right)dt\\&=\int_{T_1}^{T_2}e^{-2\gamma t}\langle\!\langle u(t,\ 0,\ y),\ v(t,\ 0,\ y)\rangle\!\rangle dt\,.\end{aligned}$$

Lemma 2.2. *Let* $A'(t,\ x,\ y\,;\ \tau,\ \xi,\ \eta)=\dfrac{\partial}{\partial\tau}A(t,\ x,\ y\,;\ \tau,\ \xi,\ \eta)$. *Then we have*

$$\begin{aligned}&(Au,\ A'u)_{0,\gamma,I}-(A'u,\ Au)_{0,\gamma,I}\\&=iG_x\langle u,\ u\rangle_{0,\gamma,I}-iG_t[u,\ u]_{0,\gamma,T_2}+iG_t[u,\ u]_{0,\gamma,T_1}\\&\quad-2i\gamma G_t(u,\ u)_{0,\gamma,I}+R(u,\ u)_{0,\gamma,I}\end{aligned}$$

where

$$G_x\langle u, u\rangle_{0,\gamma,I} = \int_{T_1}^{T_2} e^{-2\gamma t} \sum_{|\nu|,|\nu'|\leqslant m-1} \langle\!\langle g_{x\nu\nu'}(t, y)D^\nu u(t, 0, y), D^{\nu'}u(t, 0, y)\rangle\!\rangle dt\,,$$

$$G_t[u, u]_{0,\gamma,t} = e^{-2\gamma t} \sum_{|\nu|,|\nu'|\leqslant m-1} [g_{t\nu\nu'}(t, x, y)D^\nu u(t, x, y), D^{\nu'}u(t, x, y)]\,,$$

$$G_t(u, u)_{0,\gamma,I} = \int_{T_1}^{T_2} G_t[u, u]_{0,\gamma,t}\,dt\,,$$

$$R(u, u)_{0,\gamma,I} = \int_{T_1}^{T_2} e^{-2\gamma t} \sum_{|\nu|,|\nu'|\leqslant m-1} [r_{\nu\nu'}(t, x, y)D^\nu u(t, x, y), D^{\nu'}u(t, x, y)]dt\,,$$

$(D = (D_t, D_x, D_y),\ g_{x\nu\nu'},\ g_{t\nu\nu'},\ r_{\nu\nu'} \in \mathscr{B}^\infty)$.

Moreover since

$$\operatorname{Re} G_t[u, u]_{0,\gamma,t} \geqslant ce^{-2\gamma t} \sum_{j=0}^{m-1} [D_t^j u(t, x, y)]^2_{m-1-j,\gamma}$$
$$- Ce^{-2\gamma t} \sum_{i+j=0}^{m-2} \langle\!\langle D_t^i D_x^j u(t, 0, y)\rangle\!\rangle^2_{m-1-i-j-1/2,\gamma}\,,$$

where c and C are positive constants independent of t and γ, we have

$$\gamma |u|^2_{m-1,\gamma,I} + e^{-2\gamma T_2}[u]^2_{m-1,\gamma,t=T_2}$$
$$\leqslant C\left\{\frac{1}{\gamma}|Au|^2_{0,\gamma,I} + \sum_{j=0}^{m-1}\langle D_x^j u\rangle^2_{m-1-j,\gamma,I} + e^{-2\gamma T_1}[u]^2_{m-1,\gamma,t=T_1}\right\} \qquad (\gamma \geqslant \gamma_0)\,.$$

Proof. This lemma is a modified form of lemma 1.5 in I, using the property of G_t stated in the remark 2 in I.

Corollary.

i) *Let $u \in \mathscr{H}_{m,\gamma,+}$, then*

$$\gamma |u|^2_{m-1,\gamma,-} \leqslant C\left\{\frac{1}{\gamma}|Au|^2_{0,\gamma,-} + \sum_{j=0}^{m-1}\langle D_x^j u\rangle^2_{m-1-j,\gamma,-} + \sum_{j=0}^{m-1}[D_t^j u]^2_{m-1-j,\gamma}\right\}.$$

ii) *Let $u \in \mathscr{H}_{m,\gamma,-}$, then*

$$\gamma |u|^2_{m-1,\gamma,-} + \sum_{j=0}^{m-1}[D_t^j u]^2_{m-1-j,\gamma} \leqslant C\left\{\frac{1}{\gamma}|Au|^2_{0,\gamma,-} + \sum_{j=0}^{m-1}\langle D_x^j u\rangle^2_{m-1-j,\gamma,-}\right\}.$$

2.2. Energy inequalities in $t<0$.

Let $\gamma \geqslant \gamma_0$, then

$$(P') \qquad \begin{cases} Au = f \in \mathscr{H}_{0,\gamma}(R^{n+1}_+)\,. \\ B_j u\,|_{x=0} = g_j \in \mathscr{H}_{m-1-r_j,\gamma}(R^n) \qquad (j = 1, 2, \cdots, \mu) \end{cases}$$

has a unique solution $u \in \mathscr{H}_{m-1,\gamma}$ from proposition 1.1.

Now let us decompose $\{f, g_j\}$ to $\{f_0+f_1, g_{j0}+g_{j1}\}$ as follows:

$$|f_0|_{0,\gamma}\leqslant C|f|_{0,\gamma,-},$$
$$\operatorname*{supp}_t[f_1]\subset[0,\infty),\ |f_1|_{0,\gamma}\leqslant C|f|_{0,\gamma},$$
$$\langle g_{j0}\rangle_{m-1-r_j,\gamma}\leqslant C\langle g_j\rangle_{m-1-r_j,\gamma,-},$$
$$\operatorname*{supp}_t[g_{j1}]\subset[0,\infty),\ \langle g_{j1}\rangle_{m-1-r_j,\gamma}\leqslant C\langle g_j\rangle_{m-1-r_j,\gamma}.$$

Let $u_0, u_1\in\mathcal{H}_{m-1,\gamma}$ such that

$$\begin{cases} Au_0 = f_0, \\ B_j u_0|_{x=0} = g_{j0} \qquad (j=1,\cdots,\mu), \end{cases} \qquad \begin{cases} Au_1 = f_1, \\ B_j u_1|_{x=0} = g_{j1} \qquad (j=1,\cdots,\mu), \end{cases}$$

then

$$u = u_0+u_1.$$

From theorem 1, we have

$$u_1(t) = 0 \qquad \text{for} \quad t<0,$$

that is

$$u(t) = u_0(t) \qquad \text{for} \quad t<0.$$

Hence applying the theorem in I to u_0, we have

$$\begin{aligned}
&\gamma|u|^2_{m-1,\gamma,-}+\sum_{j=0}^{m-1}\langle D_x^j u\rangle^2_{m-1-j,\gamma,-}\\
&\leqslant\gamma|u_0|^2_{m-1,\gamma}+\sum_{j=0}^{m-1}\langle D_x^j u_0\rangle^2_{m-1-j,\gamma}\\
&\leqslant C\left\{\frac{1}{\gamma}|f_0|^2_{0,\gamma}+\sum_{j=1}^{\mu}\langle g_{j0}\rangle^2_{m-1-r_j,\gamma}\right\}\\
&\leqslant C\left\{\frac{1}{\gamma}|f|^2_{0,\gamma,-}+\sum_{j=1}^{\mu}\langle g_j\rangle^2_{m-1-r_j,\gamma,-}\right\}.
\end{aligned}$$

From this inequality and corollary of lemma 2.2, it follows

Proposition 2.1.

$$\begin{aligned}
&\gamma|u|^2_{m-1,\gamma,-}+\sum_{j=0}^{m-1}\langle D_x^j u\rangle^2_{m-1-j,\gamma,-}+\sum_{j=0}^{m-1}[D_t^j u]^2_{m-1-j,\gamma}\\
&\leqslant C\left\{\frac{1}{\gamma}|Au|^2_{0,\gamma,-}+\sum_{j=1}^{\mu}\langle B_j u\rangle^2_{m-1-r_j,\gamma,-}\right\}
\end{aligned}$$

for $u\in\mathcal{H}_{m,\gamma,-}$, $\gamma\geqslant\gamma_0$.

2.3. Energy inequalities with initial datas.

Let us consider energy inequalities for

$$(P)\qquad \begin{cases} Au = f, & \\ B_j u\,|_{x=0} = g_j & (j=1, 2, \cdots, \mu), \\ D_t^j u\,|_{t=0} = u_j & (j=0, 1, \cdots, m-1), \end{cases}$$

where $u \in \mathcal{H}_{m,\gamma,+}$.

At first we remark that, for every given $\varphi \in \mathcal{D}(\overline{R_+^{n+1}})$, there exists $v \in \mathcal{H}_{m,-\gamma}$ such that

$$\begin{cases} A^* v = 0, \\ B'_{j_0} v\,|_{x=0} = \varphi, \\ B'_j v\,|_{x=0} = 0 \qquad (j \neq j_0), \end{cases}$$

and

$$\gamma |v|^2_{m-1,-\gamma,+} + \sum_{j=0}^{m-1} \langle D_x^j v\rangle^2_{m-1-j,-\gamma,+} + \sum_{j=0}^{m-1} [D_t^j v]^2_{m-1-j,\gamma} \leqslant C\langle\varphi\rangle^2_{m-1-r'_{j_0},-\gamma,+}$$
$$= C\langle\varphi\rangle^2_{r_{j_0},-\gamma,+}$$

for $\gamma \geqslant \gamma_0$.

In fact, we only apply proposition 1.1 and proposition 2.1 to $\{\tilde{A}^*, \tilde{B}'_j\}$, remarking lemma 1.2.

Let us apply lemma 2.1 to u and v in the above, then we have for every (ν_{j_0}, ν'_{j_0}), satisfying $0 \leqslant \nu_{j_0} \leqslant m-1-r_{j_0}$, $0 \leqslant \nu'_{j_0} \leqslant r_{j_0}$,

$$\begin{aligned} &|\langle \Lambda_0^{m-1-r_{j_0}-\nu_{j_0}} D_t^{\nu_{j_0}} C_{j_0} u,\ \Lambda_0^{r_{j_0}-\nu'_{j_0}} D_t^{\nu'_{j_0}} \varphi\rangle_+| \\ &\leqslant C\{|f|_{0,\gamma,+}|v|_{m-1,-\gamma,+} + \sum_{j=1}^{\mu} \langle g_j\rangle_{m-1-r_j,\gamma,+} \sum_{j=0}^{m-1} \langle D_t^j v\rangle_{m-1-j,-\gamma,+} \\ &\quad + \sum_{j=0}^{m-1} [u_j]_{m-1-j,\gamma} \sum_{j=0}^{m-1} [D_t^j v]_{m-1-j,\gamma} + |u|_{m-1,\gamma,+}|v|_{m-1,-\gamma,+} \\ &\quad + \sum_{j=0}^{m-2} \langle D_x^j u\rangle_{m-2-j,\gamma,+} \sum_{j=0}^{m-1} \langle D_x^j v\rangle_{m-1-j,-\gamma,+}\} \\ &\leqslant C\{F + |u|_{m-1,\gamma,+}\}\langle\varphi\rangle_{r_{j_0},-\gamma,+}, \end{aligned}$$

where

$$F^2 = \frac{1}{\gamma}|f|^2_{0,\gamma,+} + \sum_{j=1}^{\mu} \langle g_j\rangle^2_{m-1-r_j,\gamma,+} + \sum_{j=0}^{m-1} [u_j]^2_{m-1-j,\gamma}.$$

On the other hand, we have

$$\langle u\rangle_{k,\gamma,+} \leqslant \sum_{k'=0}^{k} \sum_{h'=0}^{h} \sup_{\varphi \in \mathcal{H}_{h,-\gamma,+}} \frac{|\langle \Lambda_0^{k-k'} D_t^{k'} u,\ \Lambda_0^{h-h'} D_t^{h'} \varphi\rangle_+|}{\langle\varphi\rangle_{h,-\gamma,+}}$$

(k, h: non-negative integers).

In fact, since there exists $\varphi\in\mathcal{H}_{h,-\gamma,-}$ for $v\in\mathcal{H}_{0,-\gamma,+}$ such that

$$(D_t+i\Lambda_0)^h e^{\gamma t}\varphi = e^{\gamma t} v$$

and

$$\langle\varphi\rangle_{h,-\gamma,+}\leqslant C\langle v\rangle_{0,-\gamma,+},$$

we have

$$\begin{aligned}\langle u\rangle_{k,\gamma,+}&\cong\sum_{k'=0}^{k}\langle D_0^{k-k'}D_t^{k'}u\rangle_{0,\gamma,+}\\&\cong\sum_{k'=0}^{k}\sup_{v\in\mathcal{H}_{0,-\gamma,+}}\frac{|\langle\Lambda_0^{k-k'}D_t^{k'}u,\ v\rangle_+|}{\langle e^{\gamma t}v\rangle_-}\\&\leqslant\sum_{k'=0}^{k}\sup_{\varphi\in\mathcal{H}_{h,-\gamma,-}}\frac{|\langle\Lambda_0^{k-k'}D_t^{k'}u,\ e^{-\gamma t}(D_t+i\Lambda_0)^h e^{\gamma t}\varphi\rangle_+|}{\langle(D_t+i\Lambda_0)^h e^{\gamma t}\varphi\rangle_+}\\&\leqslant C\sum_{k'=0}^{k}\sum_{k''=0}^{h}\sup_{\varphi\in\mathcal{H}_{h,-\gamma,+}}\frac{|\langle\Lambda_0^{k-k'}D_t^{k'}u,\ \Lambda_0^{h-k''}D_t^{k''}\varphi\rangle_+|}{\langle\varphi\rangle_{h,-\gamma,+}}.\end{aligned}$$

Therefore we have

$$\langle C_j u\rangle_{m-1-r_j,\gamma,+}\leqslant C\{F+|u|_{m-1,\gamma,+}\},$$

Since $\{B_j\}_{j=1,2,\cdots,\mu}$ is a normal set, we have

$$\sum_{j=0}^{m-1}\langle D_x^j u\rangle_{m-1-j,\gamma,+}\leqslant C\{\sum_{j=1}^{\mu}\langle B_j u\rangle_{m-1-r_j,\gamma,+}+\sum_{j=\mu+1}^{m}\langle C_j u\rangle_{m-1-r_j,\gamma,-}\}.$$

Hence we have

Lemma 2. 3.

$$\sum_{j=0}^{m-1}\langle D_x^j u\rangle_{m-1-j,\gamma,+}^2\leqslant C\left\{\frac{1}{\gamma}|f|_{0,\gamma,+}^2+\sum_{j=1}^{\mu}\langle g_j\rangle_{m-1-r_j,\gamma,+}^2+\sum_{j=0}^{m-1}[u_j]_{m-1-j,\gamma}^2+|u|_{m-1,\gamma,+}^2\right\}.$$

From lemma 2. 3 and corollary (i) of lemma 2. 2, we have

Theorem 2.

$$\begin{aligned}&\gamma|u|_{m-1,\gamma,+}^2+\sum_{j=0}^{m-1}\langle D_x^j u\rangle_{m-1-j,\gamma,+}^2\\&\leqslant C\left\{\frac{1}{\gamma}|f|_{0,\gamma,+}^2+\sum_{j=1}^{\mu}\langle g_j\rangle_{m-1-r_j,\gamma,+}^2+\sum_{j=0}^{m-1}[u_j]_{m-1-j,\gamma}^2\right\}\end{aligned}$$

for $u\in\mathcal{H}_{m,\gamma,+}$, $\gamma\geqslant\gamma_0$.

Making use of lemma 2. 2, in the same way as in the proof of proposition 2. 1, we have

Corollary. *Let* $u \in H^m((0, T) \times R)$, *then*

$$|u|^2_{m-1,(0,T)} + \sum_{j=0}^{m-1} \langle D_x^j u \rangle^2_{m-1-j,(0,T)} + \sum_{j=0}^{m-1} [D_t^j u]^2_{m-1-j,t=T}$$
$$\leqslant C_T \{ |f|^2_{0,(0,T)} + \sum_{j=1}^{\mu} \langle g_j \rangle^2_{m-1-r_j,(0,T)} + \sum_{j=0}^{m-1} [u_j]^2_{m-1-j} \} .$$

Finally we remark that the statements of theorem 2 and its corollary are extended to the corresponding ones, in case when $u \in \mathcal{H}_{m+k,\gamma,+}$ $(k=0, 1, 2, \cdots)$.

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