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Supplements and corrections to my paper: The characters of some induced representations of semisimple Lie groups

By

Takeshi HIRAI

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Introduction. The principal purpose of this paper is to give supplements and corrections to the previous paper [6]. To this end and also for later uses, we remark some extentions of Harish-Chandra's results in [5] on invariant eigendistributions and invariant integral on reductive Lie groups.

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding decomposition. Take a parabolic subgroup P of G. Let N' be the unipotent radical of P, and put $S=P\cap\theta(P)$, where θ denotes the automorphism of G extending θ on $\mathfrak{g}$. Then $P=SN'$ is a direct product. (Note that we considered in [6] the standard parabolic subgroups in the sense of [1, (5.12)], but this does not affect the generality.)

Let S_1 be a subgroup of S such that $S^0(D\cap Z_G)\subset S_1\subset S$, where S^0 is the connected component of the unit element e in S, Z_G the center of G and D the analytic subgroup corresponding to the center of $\mathfrak{k}$. Take a representation L of S_1. Under a certain condition, we can construct a representation T^L of G by extending L from S_1 to S_1N' in such a way that $L_n=1$ $(n\in N')$ and next inducing it from S_1N' to G. In [6], we have studied the condition that the characters τ and π of L and T^L both exist, and obtained a formula which expresses π by means of τ as distributions on G and on S_1 respectively (Th. 1 in [6, §4]), and more explicit formula as locally summable functions on G and S_1 respectively (Th. 2 in [6, §5]). To deduce Th. 2 from Th. 1, we assumed some additional conditions on S_1 and L as summalized in Note at the end of [6, §5]. We wish in this paper to weaken and simplify these conditions. Also some mistakes in [6] on the treatment of trace class operators are corrected.

In §1, we summalize elementary properties of a not necessarily connected reductive Lie groups which satisfy a certain condition. In §2, we extend Harish-Chandra's results in [5] on connected reductive Lie groups to the non-connected case. This is applied to guarantee that the character τ of L is essentially a locally summable function on S_1 (the requirement (II) in Note at the end of [6, §5]). In §3, we establish an integral formula analogous to the Weyl's one (the requirement (I) in Note mentioned above). Then in §4, Th. 2 in [6] is established under a simple condition. In §5, we make some corrections for [6]. In Appendix, the other results on invariant eigendistributions in [5] and on invariant integral in [4 and 5] are extended to non-connected case. It seems convenient for later uses (e. g. for [7]) to mention them here together with the results in §§1-3.

NOTATIONS. For a group H, we denote by H^0 and H_0 the connected component of e in H and the center of H respectively. If K is a subgroup of a Lie group G with Lie algebra $\mathfrak{g}$ and F a subset of G or $\mathfrak{g}$, then the centralizer of F in K is denoted by $Z_K(F)$ and the centralizer of F in $\mathfrak{g}$ is denoted by $\mathfrak{z}_{\mathfrak{g}}(F)$.

§1. Elementary properties of a reductive Lie group.

Let G be a not necessarily connected real reductive Lie group with Lie algebra $\mathfrak{g}$. Let us assume that G satisfies the following condition.

CONDITION A. The adjoint group $\mathrm{Ad}(G)$ can be canonically imbedded into $\mathrm{Ad}(G_c)$, where G_c is a connected Lie group with Lie algebra $\mathfrak{g}_c$, the complexification of $\mathfrak{g}$.

Under this condition, the group G has the following properties.

(**1°**) *Let* $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{c}$, *where* $\mathfrak{g}_1=[\mathfrak{g}, \mathfrak{g}]$ *and* $\mathfrak{c}$ *is the center of* $\mathfrak{g}$. *Then* $\mathrm{Ad}(G)$ *acts on* $\mathfrak{c}$ *trivially, and the analytic subgroup* C *of* G *corresponding to* $\mathfrak{c}$ *is a closed subgroup of the center* Z_G *of* G.

Every element X of the envelopping algebra $U(\mathfrak{g}_c)$ of $\mathfrak{g}_c$ is considered canonically as a left invariant differential operator on G.

(**2°**) *The algebras of all left invariant differential operators on* G *which are also right invariant under* G *or* G^0 *respectively, are both equal to the center* $\mathfrak{Z}$ *of* $U(\mathfrak{g}_c)$.

Put $Z'_G=Z_G(G^0)$, then $Z_G\subset Z'_G$ and $G/Z'_G\cong\mathrm{Ad}(G)$.

(**3°**) *Let* $\mathfrak{k}$ *and* F *be subsets of* $\mathfrak{g}$ *and* G *respectively, and put* $S=Z_G(\mathfrak{k})$ *and* $S'=Z_G(F)$. *Then both* S/Z'_G *and* $S'/S'\cap Z'_G$ *have only finite number of connected components.*

In fact, we see from (1°) that $G/Z'_G\cong\mathrm{Ad}(G)$ can be considered as an open subgroup of the real algebraic group $\mathrm{Aut}(\mathfrak{g}_1)$ of all automorphisms of $\mathfrak{g}_1$. On the other hand, a real algebraic group has only a finite number of connected components (see e. g. [1, §14]). This proves the assertion for $\mathfrak{k}=\phi$ or $F=\phi$. The other cases can be proved analogously considering appropriate algebraic subgroups of $\mathrm{Aut}(\mathfrak{g}_1)$.

A Cartan subgroup H of G is by definition the centralizer of a Cartan subalgebra of $\mathfrak{g}$. Then, as for (3°), it follows from the property of $\mathrm{Aut}(\mathfrak{g}_1)$ the following assertion.

(**4°**) *Let* H *be a Cartan subgroup of* G. *Then* H/Z'_G *is abelian and* $H/H^0Z'_G$ *is finite. Any semisimple element in* G *is contained in a Cartan subgroup. The centralizer* $\mathfrak{z}_{\mathfrak{g}}(a)$ *of any regular element* $a\in G$ *is a Cartan subalgebra of* $\mathfrak{g}$. *Furthermore there exists only a finite number* *of conjugate classes of Cartan subgroups of G.*

Let K_1 be a maximal compact subgroup of $\mathrm{Ad}(G)\subset \mathrm{Aut}(\mathfrak{g}_1)$ and $\mathfrak{k}_1$ the Lie algebra of K_1. There exists a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ for which $\mathfrak{k}=\mathfrak{k}_1+\mathfrak{c}$ and $\mathfrak{p}\subset\mathfrak{g}_1$. Let K be the inverse image of K_1 under the natural homomorphism of G onto $\mathrm{Ad}(G)$. Then $K\supset Z'_G\supset C$ and K/Z'_G is compact.

(**5°**) *The mapping $\phi:(k, X)\longrightarrow k\exp X$ $(k\in K, X\in\mathfrak{p})$ is an analytic diffeomorphism of $K\times\mathfrak{p}$ onto G. Moreover Lemma* 26 *and its Corollaries* 1, 2, 3 *and* 4 *in* [5, §13] *are true if $Z=Z_G$ is replaced by* Z'_G (see [1] and [5]).

If G satisfies the following condition B, it clearly satisfies the condition A but the converse is not necessarily true.

CONDITION B. There exist a connected complex Lie group $G_{1,c}$ with Lie algebra $\mathfrak{g}_{1,c}=(\mathfrak{g}_1)_c$ and an isomorphism φ of G/Z_G into $G_{1,c}$ such that φ induces canonically the natural injection of $\mathrm{Ad}(G)$ into $\mathrm{Ad}(G_{1,c})$.

Under the condition B, Z'_G/Z_G is the center of G/Z_G and is finite, because the kernel in G/Z_G of the homomorphism Ad is Z'_G/Z_G and the center of $G_{1,c}$ is finite. Thus, in this case, K/Z_G is compact and moreover (3°), (4°) and (5°) remain true even when Z'_G is replaced by Z_G. For the purpose to consider the invariant integral on G as in [4 and 5], it is convenient to assume the condition B rather than A (see §3 and Appendix below).

(**6°**) *If $\mathfrak{f}$ is a subset of a Cartan subalgebra of $\mathfrak{g}$, then $S=Z_G(\mathfrak{f})$ satisfies the condition A or B along with G.*

In fact, let G_c and $G_{1,c}$ be the connected complex Lie groups appeared in the condition A and B respectively. Then by Lem. 27 in [5], we see that $Z_{G_c}(\mathfrak{f})$ and $Z_{G_{1,c}}(\mathfrak{f})$ are connected. The assertion follows immediately from this fact.

§2. Invariant eigendistributions.

We assume in this section that G satisfies the condition A. Let

H be a subgroup and Ω a subset of G. We say that Ω is *completely invariant* with respect to H if for any compact subset F of Ω the closure $Cl(F^H)$ of $F^H=\{hgh^{-1}; g\in F, h\in H\}$ is contained in Ω. As usual, an element $g\in G$ is called semisimple if $\mathrm{Ad}(g)$ is semisimple. Then there holds the following analogy of Lem. 7 in [5].

Lemma 2.1. *Let U be a subset of G, completely invariant with respect to G^0, and V a G^0-invariant subset of U which is closed in U. Then, if V contains no semisimple element of U, V is empty.*

Proof. Define the subgroup $\mathcal{N}_G$ of G as in [5, §3], then $\mathcal{N}_G \subset G^0$ and Lem. 6 in [5] holds. Its corollary is rewritten as "*$h\in Cl(x^{G^0})$, where h is the semisimple component of $x\in G$*". Thus we get the lemma. Q. E. D.

Depending on this lemma and following the proof of Harish-Chandra step by step, we get a generalization of Th. 2 in [5, §15] in the following form.

Theorem 2.2. *Let Ω be an open subset of G which is completely invariant with respect to G , and T a distribution on Ω. Assume that*

(1) *T is invariant under G^0 (through inner automorphisms),*

(2) *there exists an ideal $\mathfrak{U}\subset\mathfrak{Z}$ such that* $\dim \mathfrak{Z}/\mathfrak{U}<+\infty$ *and $uT=0$ for all $u\in\mathfrak{U}$.*

Then T is a locally summable function on Ω which is analytic on $\Omega'=\Omega\cap G'$, where G' denotes the set of all regular elements in G.

A distribution T on G is called *invariant eigendistribution* if (1) it is invariant under G and (2) there exists a homomorphism ν of $\mathfrak{Z}$ into $\boldsymbol{C}$ such that $ZT=\nu(Z)T$ $(Z\in\mathfrak{Z})$.

Corollary. *Any invariant eigendistribution on G is a locally summable function on G which is analytic on G'.*

The above theorem says that we can treat every connected component independently. The proof itself of this theorem, completely analogous to the connected case, treats in essence every connected component independently.

Here, in addition to §1, we remark some changes in translating Harish-Chandra's proof under the condition A. The numbers of sections and lemmas below are those in [5] and the same notations as there are employed without notice if there is no fear of misunderstanding.

In §3, Lem's 5, 6 and 7 hold also under the condition A, but we apply Lem. 2. 1 above instead of Lem. 7.

In §5, note that if a differential operator D on U_G is invariant under G^0, then $\Delta(D)=\delta_a(D)$ is invariant under $Z_{G^0}(a)$.

In §6, note that Lem. 13 holds also. In fact, its proof can be reduced from G to the subgroup $\mathrm{Ad}(G)\cong G/Z'_G$ of $\mathrm{Ad}(G_c)$ because of (1°) and (2°) in the preceeding section.

In §7, Lem's 14, 15 and 16 hold also. Furthermore the following analogies of Lem. 15 and its corollary hold.

Lemma 2. 3. *Let $\Omega^0=\phi(G^0\times U)=(aU)^{G^0}$ and T a distribution on Ω^0 invariant under G^0. Then there exists unique distribution σ_T on U such that $T(f_\alpha)=\sigma_T(\beta_\alpha)$ $(\alpha\in C_0^\infty(G^0\times U))$, where $f_\alpha\in C_0^\infty(\Omega^0)$ and $\beta_\alpha\in C_0^\infty(U)$ for α are defined analogously as for Lemma 15 in [5].*

Corollary. *Let D be a differential operator on Ω^0 invariant under G^0. Then $\sigma_{DT}=\Delta\sigma_T$, where $\Delta=\delta_a(D)$.*

In §9, Th. 1 holds also, but we apply the following generalization of it.

Theorem 2. 4. *Let Ω be an open subset of G completely invariant under G^0, and D an analytic differential operator on Ω. Assume that*

(1) D is invariant under G^0,

(2) $\delta_a(D)=0$ for every regular element $a\in\Omega$.

Then $DT=0$ for every distribution T on Ω invariant under G^0.

Its proof can be carried out quite analogously as for Th. 1 by induction on the dimension of G. In fact, let a be a semisimple element in Ω. If $a \notin Z'_G = Z_G(G^0)$, then dim Ξ < dim G, where $\Xi = Z_G(a)^0$. Applying Lem. 2.3 and its corollary just above, we get a Ξ-invariant distribution σ_T on $\Omega_\Xi = a^{-1}\Omega \cap \Xi'$ such that $\Delta\sigma_T = 0$. As in §9, since Ω_Ξ is completely invariant with respect to Ξ, we come to the induction hypothesis. If $a \in Z'_G$, then a commutes with all elements in G^0. Therefore this case is reduced to the discussions on $\mathfrak{g}$ as in §§10 and 11 in [5].

In §13, the analogies of Lem's 22 and 23 hold also under the assumption that U_G is only G^0-invariant.

In §14, Lem. 24 and its corollary hold, where in the latter it is sufficient to assume that U is completely invariant with respect to G^0 and T is invariant under G^0.

At last in §15, we get the generalization of Th. 2 given above as Th. 2.2.

§3. An integral formula.

In this section, we assume that G satisfies the condition B. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ invariant under the Cartan involution given in §1, and let H be the Cartan subgroup of G corresponding to $\mathfrak{h}$. Let $\tilde{H}$ and $\tilde{H}_K$ be the normalizers of $\mathfrak{h}$ in G and in K respectively. Then $\tilde{H} = \tilde{H}_K \exp(\mathfrak{h} \cap \mathfrak{p})$ by (5°) in §1, and $\tilde{H}_K / Z_G$ is compact because so is K/Z_G. Let H_0 be the center of H and put $W_H = \tilde{H}/H_0$. Then since $H_0 \supset H^0$ and $H = H_K \exp(\mathfrak{h} \cap \mathfrak{p})$, where $H_K = H \cap K$, $W_H \cong \tilde{H}_K/(H_K)_0$ and W_H is finite.

As in [5, §20], W_H operates on the left on $\bar{G} = H_0 \backslash G$ and H as follows. Put $\bar{g} = H_0 g$ for $g \in G$. For $\omega \in W_H$, take $y \in \omega$ and put $\omega\bar{g} = \overline{(yg)}$ $(g \in G)$, $h^\omega = yhy^{-1}$ $(h \in H)$. Hence W_H operates on the left on $\bar{G} \times H$ as $\omega(\bar{g}, h) = (\omega\bar{g}, h^\omega)$. Quite analogously as in [5 and 6], we see that the mapping $\phi : (\bar{g}, h) \longrightarrow g^{-1}hg = h^g$ of $\bar{G} \times H'$ onto G_H is everywhere regular, where $H' = H \cap G'$ and G_H denotes the union of $gH'g^{-1}$ over all $g \in G$. Furthermore, analogously as for [5, Lem. 41] and [6, Lem. 3.2], the following lemma holds. For any $g \in G$, let $D(g)$ be the coefficient of t^l in the polynomial of t :

$\det(\mathrm{Ad}(g)-1_{\mathfrak{g}}-t1_{\mathfrak{g}})$, where $l=\mathrm{rank}\,\mathfrak{g}_1$ and $1_{\mathfrak{g}}$ denotes the identity mapping on $\mathfrak{g}$.

Lemma 3.1. *Assume that G satisfies the condition B. Let dg, $d\bar{g}$ and dh be invariant measures on G, $\bar{G}$ and H normalized as*

$$\int_G \psi(g)\,dg=\int_{\bar{G}}\int_{H_0}\psi(hg)\,dh\,d\bar{g} \quad (\psi\in C_0(G)),$$

where $C_0(G)$ denotes the set of all continuous functions on G with comppact supports. Then

$$w_H\int_{G_H} f(g)\,dg=\int_{H'}|D(h)|\int_{\bar{G}} f(h^{\bar{g}})\,d\bar{g}\,dh \quad (f\in C_0(G_H)),$$

where w_H denotes the order of W_H. Moreover if φ is a function on H' such that $f(h^{\bar{g}})\varphi(h)$ is integrable on $\bar{G}\times H'$, then

$$\int_{\bar{G}}\int_{H'} f(h^{\bar{g}})\ \varphi(h)\ dh\,d\bar{g}=\int_{G_H} f(g)\Big\{\sum_{\omega\in W_H}\varphi(h_g^{\omega})\Big\}|D(g)|^{-1}dg,$$

where $h_g^{\omega}=(h_g)^{\omega}$ and h_g is an element of H' such that $g=g_0h_gg_0^{-1}$ for some $g_0\in G$.

In this connection, see also [8, Th. 7.2].

§4. Supplements to the paper [6].

In this and the next sections, we assume as in [6] that G is a real connected semisimple Lie group. As in the introduction, let P be a parabolic subgroup of G, assumed as in [6] to be standard in the sense of [1, (5.12)]. Put $S=P\cap\theta(P)$ and let S_1 be a subgroup of S such that

(4.1) $$S^0(D\cap Z_G)\subset S_1\subset S.$$

Then it follows from (6°) in §1 that S_1 is a reductive Lie group which satisfies the conditions A and B.

Let L be a representation of S_1 on a separable Hilbert space E. As in [6], put $\Xi=S\cap K$, $\Xi_1=\Xi\cap S_1$. Let $\boldsymbol{\omega}$ be the set of all equivalent classes of finite-dimensional irreducible representations of Ξ_1. For $\delta\in\boldsymbol{\omega}$, denote by $d(\delta)$ its dimension and let $E(\delta)$ be the subspace of E consisting of all vectors transformed under L_ξ $(\xi \in \Xi_1)$ according to δ. Let E^∞ be the algebraic sum of all $E(\delta)$ $(\delta \in \boldsymbol{\omega})$. Denote by $\mathfrak{Z}_s$ the center of $U(\mathfrak{s}_c)$, where $\mathfrak{s}$ is the Lie algebra of S. Let us assume that L satisfies the following three conditions.

(1) For any $z \in D \cap Z_G$, L_z is a scalar multiple of the identity operator.

(2) $\dim E(\delta) \leqslant N d(\delta)^2$ $(\delta \in \boldsymbol{\omega})$, where N is a constant independent of δ.

(3) $L_Z a = \alpha(Z) a$ $(a \in E^\infty,\ Z \in \mathfrak{Z}_s)$, where α is a homomorphism of $\mathfrak{Z}_s$ into $\boldsymbol{C}$.

Then the character τ of L exists [6, §2] and is an invariant eigendistribution on S_1. Applying Cor. of Th. 2. 2, we get the following lemma which guarantees the requirement (II) in Note in [6, §5].

Lemma 4. 1. *If a representation L of S_1 satisfies the above conditions* (1), (2) *and* (3). *Then its character τ is essentially a locally summable function on S_1 which is analytic on the set of all regular elements in S_1.*

In this connection, see also [8, Th's 4. 2 and 4. 6].

Now let A be a Cartan subgroup of S_1 and A_0 its center. Since S_1 satisfies the condition B, we can apply Lem. 3. 1 to S_1 and A. Put $\hat{S}_1 = A_0 \backslash S_1$ and $\hat{s} = A_0 s$ $(s \in S_1)$. Define w_A, $S_{1,A}$ and $D_s(s)$ analogously as w_H, G_H and $D(g)$.

Lemma 4. 2. *Suppose that the invariant measures ds, $d\hat{s}$ and da on S_1, $\hat{S}_1$ and A are normalized as*

$$\int_{S_1} \psi(s)\, ds = \int_{\hat{S}_1} \int_{A_0} \psi(as)\, da\, d\hat{s} \qquad (\psi \in C_0(S_1)).$$

Then,

$$w_A \int_{S_{1,A}} f(s) ds = \int_A |D_s(a)| \int_{\hat{S}} f(s^{-1} a s)\, d\hat{s}\, da \qquad (f \in C_0(S_{1,A})).$$

This lemma shows that the requirement (I) in Note in [6, §5] is also satisfied, Thus, by the above two lemmas, we get Th. 2 in [6, §5] under the weaker condition as follows.

Theorem 4. 3. *Let S_1 be a subgroup of S which satisfies* (4. 1). *Assume that a representation L of S_1 on a separable Hilbert space E fulfills the conditions* (1), (2) *and* (3). *Then there holds the formula in* [6, Th. 2] *which expresses the character of the induced representation T^L by means of that of L.*

§5. Corrections to the paper [6].

1. A bounded operator A on a separable Hilbert space $\mathscr{H}$ is called *summable* if for a complete orthonormal system $\{v_i\}_{1\leqq i<\infty}$, of $\mathscr{H}$, $\sum_{i,j=1}^{\infty}|(Av_i, v_j)|<+\infty$, where $(\cdot,\cdot)$ denotes the inner product in $\mathscr{H}$. A is called *of trace class* if for any complete orthonormal system $\{v_i\}$, $\sum_{i=1}^{\infty}(Av_i, v_i)$ is absolutely convergent. If A is summable, then it is of trace class, but the converce is not true. A is of trace class if and only if there exist two Hilbert-Schmidt operators B and C such that $A=BC$ [2, p. 123].

Let T be a representation of G on $\mathscr{H}$. Denote by Ω the set of all finite-dimensional irreducible representations of K and define for $\mathscr{D}\in\Omega$ the subspace $\mathscr{H}(\mathscr{D})$ of $\mathscr{H}$ analogously as $E(\delta)$. In [3], Harish-Chandra proved that if T is quasi-simple irreducible, the operator

$$T_x=\int_G x(g)T_g dg$$

corresponding to $x\in C_0^\infty(G)$, is summable when the subspaces $\mathscr{H}(\mathscr{D})$ $(\mathscr{D}\in\Omega)$ are mutually orthogonal, but in general T_x is only of trace class. In this connection, Lem. 2. 1 in [6] must be corrected: "*T_x is summable if $\mathscr{H}(\mathscr{D})$'s are mutually orthogonal, but in general of trace class.*" As consequences of this, the assertion "*summable*" for the operators L_x and T_x^L, in Prop. 1 in [6, §2], is true under the assumption that the subspaces $E(\delta)$ $(\delta\in\boldsymbol{\omega})$ are mutually orthogonal, but in general must be replaced by "*of trace class*". Similar correction must be made for Lem. 2. 3 in [6, § 2].

2. To prove Th. 2 in [6], we need not to use the summability of the operator T_x. Using only the fact that it is of trace class, we can easily repair the whole proof.

In that proof, we cancell Lem's 4. 2 and 4. 3 and replace them by the following one.

Lemma 5. 1. *Let* $\mathscr{H}=L^2(\mathfrak{M}, d\mu)$ *be as in* [6, Lem. 4. 2]. *Assume that* T *is a trace class operator defined by a continuous kernel* $K(m, m')$ *as*

$$T\psi(m)=\int_{\mathfrak{M}} K(m, m')\varphi(m)\,d\mu(m') \qquad (\varphi\in\mathscr{H}).$$

Then its trace $\mathrm{Sp}(T)$ *is given by the followinvg absolutely convergent integral*:

$$\mathrm{Sp}\,(T)=\int_{\mathfrak{M}} K(m, m)d\mu(m).$$

In turn, this lemma is an immediate consequence of Cor. 10. 2 of Th. 10. 1 in [2, p. 151].

Appendix. Let us remark here some extentions of Harish-Chandra's results on invariant eigendistributions in [5] and on invariant integral in [4 and 5], to not necessarily connected reductive Lie groups. They will be necessary later (see e. g. [7]).

1. Invariant eigendistributions. Let the notations be as in §1. We say that G is *acceptable* if it satisfies the condition A and if there exist a connected complex semisimple Lie group $G_{1,c}$ with Lie algebra $\mathfrak{g}_{1,c}=(\mathfrak{g}_1)_c$ and a continuous homomorphism j of G into $G_{1,c}$ which satisfy the following conditions.

(1) j induces canonically the natural injection of $\mathrm{Ad}(G)$ into $\mathrm{Ad}(G_{1,c})$. (Note that under the condition A, $\mathrm{Ad}(G)$ is naturally imbedded into $\mathrm{Ad}(G_{1,c})$ by (1°) in §1.)

(2) Let $\mathfrak{h}_{1,c}$ be a Cartan subalgebra of $\mathfrak{g}_{1,c}$ and ρ the half-sum of all positive roots of $(\mathfrak{g}_{1,c}, \mathfrak{h}_{1,c})$ with respect to an order. Then, $\xi_\rho(\exp H) = e^{\rho(H)}$ $(H \in \mathfrak{h}_{1,c})$ defines uniquely a homomorphism of the Cartan subgroup of $G_{1,c}$ corresponding to $\mathfrak{h}_{1,c}$ into $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$.

Lemma A. 1. *Suppose that G is connected. Then G is acceptable in the above sense if and only if it is acceptable in the sense of Harish-Chandra in* [5, §18].

Proof. The "if" part is clear. Let us prove the "only if" part. Assume that G is acceptable in the above sense. Let C and G_1 be the analytic subgroups of G corresponding to $\mathfrak{c}$ and $\mathfrak{g}_1$ respectively. Then, since $C \subset Z_G$, $j(C)$ is contained in the center of $G_{1,c}$ and connected. Hence $j(C) = \{e\}$. The subgroup $Z_1 = G_1 \cap C$ is discrete in G, G_1 and C. Put $\bar{G}_1 = G_1/Z_1$ and $\bar{C} = C/Z_1$, then G/Z_1 is canonically isomorphic to $\bar{G}_1 \times \bar{C}$. Let C_c be a Lie group with Lie algebra $\mathfrak{c}_c$ which contains $\bar{C}$ canonically. Put $G_c = G_{1,c} \times C_c$. Since $j(C) = \{e\}$, j induces a homomorphism j' of $\bar{G}_1$ into $G_{1,c}$. Let π be the canonical homomorphism of G onto $\bar{G}_1 \times \bar{C}$ and i the injection of $\bar{C}$ into C_c. Put $j_0 = (j' \times i) \circ \pi$, then G_c and j_0 satisfies the acceptability condition of Harish-Chandra in [5, §18]. Q. E. D.

Let us assume that G is acceptable. Let $\mathfrak{h} = \mathfrak{c} + \mathfrak{h}_1$ where $\mathfrak{h}_1 \subset \mathfrak{g}_1$, be a Cartan subalgebra of $\mathfrak{g}$. Let A and $A_{1,c}$ be the Cartan subgroups of G and $G_{1,c}$ corresponding to $\mathfrak{h}$ and $\mathfrak{h}_{1,c} = (\mathfrak{h}_1)_c$ respectively. For an $\boldsymbol{R}$-linear function λ on $\mathfrak{h}$ into $\boldsymbol{C}$ such that $\lambda = 0$ on $\mathfrak{c}$, we define, if it exists, unique homomorphism ξ_λ of $A_{1,c}$ into $\boldsymbol{C}^*$ as $\xi_\lambda(\exp H) = e^{\lambda(H)}$ $(H \in \mathfrak{h}_{1,c})$, and denote also by ξ_λ the homomorphism $\xi_\lambda \circ j$ of A into $\boldsymbol{C}^*$. Then for any root α of $(\mathfrak{g}_c, \mathfrak{h}_c)$, ξ_α is defined.

Now let us state some extentions of the results in [5, §§18–20]. Assume that $\mathfrak{h}$ is invariant under the Cartan involution and put $M = Z_G(\mathfrak{h} \cap \mathfrak{p})$. Then Lem. 30 in [5, §18] hold also in this case. In particular, if G is acceptable, the same holds for M.

Let Ω be an open subset of G completely invariant with respect to G^0, and T a distribution on Ω invariant under G^0. Denote by F the analytic function on $\Omega'=\Omega\cap G'$ corresponding to T by Th. 2. 2. Note that we can define as in [5, §19], the open subset $A'(R)$ of A and the function Δ_A on A as

$$\Delta_A(h)=\xi_\rho(h)\prod_{\alpha>0}(1-\xi_\alpha(h)^{-1})\qquad(h\in A).$$

Then, Lem. 31 in [5, §19] holds also in this case, that is, the function $\Phi_A(h)=\Delta_A(h)F(h)\ (h\in A'\cap\Omega)$ can be extended to an analytic function on $A'(R)\cap\Omega$, where $A'=A\cap G'$. Its proof can be carried out quite analogously applying (5°) in §1.

Similarly Lem's 34 and 35 in [5, §19] remain true. Moreover Lem's 36 and 37 in [5, §20] hold also in this case.

2. Invariant integral. Let us assume here that G satisfies the condition B, and extend the results in [4]. As above let A be a Cartan subgroup of G and A_0 its center.

First of all, Th. 1 in [4] remains true in this case. In fact, put $\bar{G}=\varphi(G/Z_G)$, where φ is the injection of G/Z_G into $G_{1,c}$ in the condition B. For $a_0\in A$, put $\Xi=Z_G(a_0)$ and $\Xi_1=Z_{\bar{G}}(\bar{a}_0)$, where $\bar{a}_0=\varphi(a_0Z_G)$. Then by (6°) in §1, the natural homomorphism of G/Ξ onto $\bar{G}/\Xi_1$ is a finite covering. Therefore the proof can be reduced from G to $\bar{G}\subset G_{1,c}$. In turn, for the subgroup $\bar{G}$ of $G_{1,c}$, the proof in [4] can be carried out quite analogously.

Next, Th's 2 and 3 in [4] hold also under the condition B. (The function F_f there, must be defined by the integration over G/A_0.) Their proofs can be carried out, applying Th. 1 mentioned above, without any essential modification.

Now let us assume in addition to the condition B that G is acceptable, and extend the results in [5, §22]. Put $G^*=G/A_0$, $x^*=xA_0$ and $h^{x^*}=xhx^{-1}(x\in G,\ h\in A)$ Let dx^* be an invariant measure on G^*. Put for any $f\in C_0^\infty(G)$,

$$F_f(h)=\varepsilon_R(h)\Delta_A(h)\int_{G^*}f(h^{x^*})dx^*\qquad(h\in A'),$$

where $\varepsilon_R(h)$ is defined as in [5, §22]. Then, Th. 3 in [4] takes the following form just as in [5, §22]:

$$F_{Zf}=\gamma(Z)F_f \qquad (f\in C_0^\infty(G),\ Z\in\mathfrak{Z}).$$

Moreover Lem. 40 and its corollary in [5, §22] are also true in this case.

At last, note that the integral formula in Lem. 3.1 in this paper takes the following form: for any $f\in C_0(G)$,

$$\int_{G_A} f(g)dg=c\int_{A'} F_f(h)\varepsilon_R(h)\overline{\Delta_A(h)}dh,$$

where c is a positive constant and $\overline{\Delta_A(h)}$ denotes the complex conjugate of $\Delta_A(h)$ (cf. Lem. 41 in [5, §23]).

DEPARTMENT OF MATHEMATICS,
KYOTO UNIVERSITY

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