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Some results on Markov processes of infinite lattice spin systems

By

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§1. **Introduction.** The Markov processes of the infinite lattice spin systems were introduced relating to statistical mechanics and many problems, such as construction of the processes, invariant measures and ergodic theorems, have been studied by many authors ([2], [4], [6]). In this paper we will present some results related to these problems.

Let Z^ν be ν-dimensional lattice space and E be the space of all spin configurations on Z^ν, that is, each element ξ of E is a map from Z^ν into $\{+1, -1\}$. E is compact in the product topology. To each finite subset X of Z^ν, a real number $\Phi(X)$ corresponds and satisfies the following conditions;

$$(1.1) \qquad \Phi(X)=\Phi(X+x) \qquad \text{for } {}^\forall x\in Z^\nu,\ {}^\forall X,$$

$$(1.2) \qquad \|\Phi\|=\sum_{X\ni 0}|\Phi(X)|<+\infty.$$

Such a function is called "*a potential function*" and we denote by $\mathbf{B}$ the family of all potential functions. Let $\Phi\in\mathbf{B}$ and introduce a function $c(x;\xi)$ on $Z^\nu\times E$ by

$$(1.3) \qquad c(x;\xi)=\exp\left[\sum_{X\ni x}\Phi(X)\sigma_X(\xi)\right] \qquad \text{where } \sigma_X(\xi)=\prod_{y\in X}\xi(y).$$

The Markov process of the infinite spin system is defined on E and its infinitesimal generator is given in the following form;

(1.4) $$Af(\xi)=\sum_{x\in Z^\nu} c(x\,;\xi)[f(\xi_x)-f(\xi)]$$

where $\xi_x\in E$ *is defined by* $\xi_x(y)=\begin{cases}\xi(y) & if\ y\neq x\\ -\xi(x) & if\ y=x.\end{cases}$

Further, let us introduce the following condition ;

(1.5) $\sum_{X\ni 0}\#(X)|\Phi(X)|<+\infty$ *where* $\#(X)$ *stands for the cardinal number of* X.

We denote by $\mathbf{B}_1$ the family of all potential functions which satisfy (1.5). If $\Phi\in\mathbf{B}_1$, a Markov process on E exists corresponding to Φ, ([2], [6]). In the present paper we will consider only potentials of $\mathbf{B}_1$, but we don't assume that potentials are of finite range. In §2 we will discuss some properties of this process. In particular it is shown that *every extremal Gibbsian measure (w. r. t. Φ) is a limiting distribution of the Markov process starting from each configuration of a dense set of E.* In §3 we will study the free energy and *prove under our assumption the remarkable results which were obtained by Holley* [4], *in the case of finite range potentials.* By these results we can conclude that a Z^ν-invariant equilibrium state by the several definitions in statistical mechanics is equivalent to a Z^ν-invariant stationary probability measure of the Markov process of the infinite lattice spin system.

§2. The Markov processes of the infinite lattice spin system

Let us define some σ-fields on E. For each subset V of Z^ν, denote by $\mathscr{F}_V$ the σ-field generated by $\{\xi(x)\}$, $x\in V$ and define $\mathscr{F}$ and $\mathscr{F}_\infty$ as follows ;

(2.1) $$\mathscr{F}=\mathscr{F}_{Z^\nu}\qquad \mathscr{F}_\infty=\bigcap_{\substack{V\subset Z^\nu\\ \#(V)<+\infty}}\mathscr{F}_{V^c}.$$

For each subset V let E_V be the spin configuration space on V. Denote by $\mathscr{P}$ the family of all probability measures on $(E, \mathscr{F})$ and each element ρ of $\mathscr{P}$ is called "state". For each ρ and each finite subset V of Z^ν, ρ_V stands for the cylindrical measure on V, that is,

$\rho_V(\eta)=\rho(S(V;\eta))$ for each $\eta\in E_V$, where

$$S(V;\eta)=\{\xi\in E\ ;\ \xi(x)=\eta(x)\ \textit{for every}\ x\in V\}.$$

Every probability measure on E_V can be regarded as a probability measure on $(E, \mathscr{F})$ by specifying a fixed configuration of $E_{Z^\nu/V}$ outside of V. If we equip with the topology of the weak convergence, then $\mathscr{P}$ is compact. Let $C(E)$ be the Banach space of all continuous functions on E with the supremum norm $\|\cdot\|$, and denote by $C_V(E)$ the family of $\mathscr{F}_V$-measurable functions of $C(E)$.

Next we will define the Gibbsian measures. Let $\Phi\in\mathbf{B}$ be fixed and define $f_V^\Phi(\eta:\xi)$, $g_{V,\xi}(\eta)$ for each finite subset $V\subset Z^\nu$ and $\eta\in E_V$, $\xi\in E$ by

$$(2.2)\qquad f_V^\Phi(\eta:\xi)=\exp\Big[-\sum_{\substack{X\subset Z^\nu\\ X\cap V\neq\phi}}\Phi(X)\sigma_X(\eta:\xi|_{V^c})\Big]$$

where $\xi|_V\in E_V$ *is the restriction of* $\xi\in E$ *on* V *and*

$$\eta\cdot\xi|_{V^c}(x)=\eta(x)\ \textit{if}\ x\in V,\ \textit{and}\ \xi(x)\ \textit{if}\ x\in V^c,$$

$$(2.3)\qquad g_{V,\xi}(\eta)=f_V^\Phi(\eta:\xi)/\sum_{\tilde\eta\in E_V}f_V^\Phi(\tilde\eta:\xi).$$

Then $g_{V,\xi}(\cdot)$ is a probability measure on E_V and $g_{V,\cdot}(\eta)$ is $\mathscr{F}_{V^c}$-measurable. $g_{V,\xi}(\cdot)$ is called *the Gibbsian measure on* V *with the boundary condition* ξ. Since $\{g_{V,\xi}(\eta)\}$ satisfies a consistency condition, the following definition is possible.

$\rho\in\mathscr{P}$ is called *a limiting Gibbsian measure w.r.t.* Φ if it satisfies that for every finite subset $V\subset Z^\nu$ and every $\eta\in E_V$

$$(2.4)\qquad \rho(S(V:\eta)|\mathscr{F}_{V^c})=g_{V,\xi}(\eta)\ \textit{for almost all}\ \xi\ \textit{of}\ S(V:\eta).$$

Denoting by $\mathscr{G}_\Phi=\mathscr{G}$ the collection of all limiting Gibbsian measures, $\mathscr{G}$ is a nonempty compact convex set and denote by $\mathscr{G}_{ex}$ all the extremal elements of $\mathscr{G}$. Generally $\mathscr{G}_{ex}$ is not finite. In fact Dobrushin proved that $\mathscr{G}_{ex}$ was infinite for a 3-dimensional Ising potential. ([3]).

Lemma 2.1. *ρ of $\mathscr{P}$ is a limiting Gibbsian measure if and only if (2.4) holds only for any one point set $V=\{x\}$, $x\in Z^\nu$.*

Now, we describe the Markov process generated by (1.4). For each finite set $V\subset Z^{\nu}$, define the operator A^{V} on $C(E)$ by

$$(2.5)\qquad A^{V}f(\xi)=\sum_{x\in V}c(x;\xi)[f(\xi_{x})-f(\xi)].$$

Then A^{V} generates a strongly continuous conservative Feller semi-group $\{T_{t}^{V}\}$, $t\geqq 0$ on $C(E)$ since A^{V} is a bounded operator on $C(E)$ because of

$$(2.6)\qquad e^{-\|\Phi\|}\leqq c(x;\xi)\leqq e^{\|\Phi\|}\ \textit{for any } x\in Z^{\nu}, \textit{and any } \xi\in E.$$

The following theorem was proved by Dobrushin [2] and Liggett [6].

Theorem 2.2. *(i) For any $\Phi\in\mathbf{B}_{1}$, there exists a strongly continuous conservative Feller semi-group $\{T_{t}\}$, $t\geqq 0$ on $C(E)$ which satisfies*

(2.7) *for any increasing sequence of finite subsets $\{V_{n}\}$, $n\geqq 1$ such that extends to Z^{ν}*

$$\lim_{V_{n}\nearrow Z^{\nu}}\sup_{0\leqq t\leqq t_{0}}\|T_{t}^{V_{n}}f-T_{t}f\|=0\qquad \textit{for } {}^{\forall}f\in C(E),\ {}^{\forall}t_{0}>0.$$

(ii) Let $\mathscr{D}(A)=\{f\in C(E);\ \sum_{x\in Z^{\nu}}\sup_{\xi}|f(\xi_{x})-f(\xi)|<+\infty\}$ and define Af by (1.4) for each $f\in\mathscr{D}(A)$. If we denote by $(\bar{A},\ \mathscr{D}(\bar{A}))$ the infinitesimal generator of $\{T_{t}\}$, $t\geqq 0$, then we have

(2.8) *$\mathscr{D}(A)\subset\mathscr{D}(\bar{A})$, $\bar{A}|_{\mathscr{D}(A)}=A$ and moreover $(\alpha-A)[\mathscr{D}(A)]$ is dense in $C(E)$ for every $\alpha>0$.*

The Markov process induced by $\{T_{t}\}$, $t\geqq 0$ in Theorem 2.2 is called the Markov process of the infinite lattice spin system corresponding to Φ.

Theorem 2.3. *For each finite set V, let us be given a system of positive bounded continuous functions on E; $\{c_{V}(x;\xi)\}$, and define $\tilde{A}^{V}$ by $\tilde{A}^{V}f(\xi)=\sum_{x\in Z^{\nu}}c_{V}(x;\xi)[f(\xi_{x})-f(\xi)]$. Suppose that*

(2.9) *$\|c_{V}(x;\cdot)\|$ is bounded in x, V*

and for each $x \in Z^{\nu}$, $\lim_{V \nearrow Z^{\nu}} \| c_V(x\,;\,\cdot) - c(x\,;\,\cdot)\| = 0$.

Then if we denote by $\{\tilde{T}_t^V\}$, $t \geqq 0$ *the strongly continuous conservative Feller semi-group generated by* A^V, *we have*

$$\text{(2.10)} \quad \lim_{V \nearrow Z^{\nu}} \sup_{0 \leqq t \leqq t_0} \|\tilde{T}_t^V f - T_t f\| = 0 \qquad \textit{for } {}^{\forall} f \in C(E),\ {}^{\forall} t_0 > 0.$$

Let $\mathscr{S}_{\phi} = \mathscr{S}$ be the family of all stationary probability measures for $\{T_t\}$, $t \geqq 0$. Then, $\mathscr{S}$ is also a non-empty compact convex set. For each $\rho \in \mathscr{S}$, $\{T_t\}$, $t \geqq 0$ can be regarded as a strongly continuous contraction semi-group of operators on $L^2(E, \rho)$.

In particular ρ of $\mathscr{S}$ is *reversible* if

$$\text{(2.11)} \quad (T_t f, g)_{L^2(E,\rho)} = (f, T_t g)_{L^2(E,\rho)} \qquad \textit{for all } f \textit{ and } g \textit{ of } C(E).$$

We will denote by $\mathscr{S}_R$ the family of all reversible stationary probability measures for $\{T_t\}$, $t \geqq 0$. The following theorem was proved by Dobrushin for finite range potentials ; however, we will give the proof under our assumtion.

Theorem 2. 4. $\mathscr{S}_R = \mathscr{G}$.

(**Proof**) For every ρ of $\mathscr{G}$, set $c_V(x\,;\,\xi) = [\rho_V(\xi_x|_V)/\rho_V(\xi|_V)]^{\frac{1}{2}}$. Then we can apply Theorem 2. 3 noting the definition of limiting Gibbsian measures, and ρ_V is a reversible stationary probability measure for $\{\tilde{T}_t^V\}$ corresponding to $\{c_V(x\,;\,\xi)\}$. Since ρ_V converges to ρ, Theorem 2. 3 implies $\rho \in \mathscr{S}_R$.
Conversely assume $\rho \in \mathscr{S}_R$. Set $c_V(x\,;\,\xi) = \rho_V(\xi|_V)^{-1} \int_{S(V:\xi|_V)} c(x\,;\,\tilde{\xi})\rho(d\tilde{\xi})$ for each finite set $V \subset Z^{\nu}$, then $c_V(x\,;\,\xi) = c_V(x\,;\,\xi|_V)$. Noting (2. 11),

$$\text{(2.12)} \qquad (Af, g)_{L^2(E,\rho)} = (f, Ag)_{L^2(E,\rho)} \qquad \textit{for } {}^{\forall} f,\ {}^{\forall} g \in C_V(E).$$

By arranging (2. 12), we can show

$$\text{(2.13)} \quad c_V(x\,;\,\eta)\rho_V(\eta) = c_V(x\,;\,\eta_x)\rho_V(\eta_x) \qquad \textit{for } {}^{\forall}\eta \in E_V,\ {}^{\forall} x \in V.$$

Further, we can easily show $c_V(x\,;\,\xi)$ converges to $c(x\,;\,\xi)$ as V tends to Z^ν. If we divide the both hand sides of (2.13) by $\rho_V(\eta)+\rho_V(\eta_x)$ and extend V to Z^ν, we obtain

(2.14) $c(x\,;\,\xi)\rho[S(x:j)\,|\,\mathscr{F}_{Z^\nu/x}]=c(x\,;\,\xi_x)\rho[S(x:-j)\,|\,\mathscr{F}_{Z^\nu/x}]$ *where* $S(x:j)=\{\xi\in E\,;\,\xi(x)=j\}$, $j=+1$ *or* -1.

Therefore $\rho\in\mathscr{G}$ by Lemma 2.1, since (2.14) implies

(2.15) $\rho[S(x:j)\,|\,\mathscr{F}_{Z^\nu/x}]=c(x\,;\,\xi_x)/[c(x\,;\,\xi)+c(x\,;\,\xi_x)]=g_{\{x\},\xi}(\{j\})$.

Next, we investigate $\{T_t\}$-invariant functions in $L^2(E,\rho)$-sense for each $\rho\in\mathscr{G}$. Here we say $f(\xi)$ *a* $\{T_t\}$*-invariant function* in $L^2(E,\rho)$-sense if f of $L^2(E,\rho)$ satisfies

(2.16) $$T_tf=f \text{ in } L^2(E,\rho) \text{ for every } t\geqq 0.$$

Theorem 2.5. *Let* $\rho\in\mathscr{G}$ *and* $f\in L^2(E,\rho)$. *Then* f *is a* $\{T_t\}$*-invariant function in* $L^2(E,\rho)$*-sense if and only if* f *is* $\mathscr{F}_\infty$*-measurable.*

In order to prove this theorem we prepare two lemmas.

Let us introduce a family of operators $\{U_x\}$, $x\in Z^\nu$ which are defined as follows;

(2.17) $U_xf(\xi)=f(\xi_x)$ *for each measurable function* f *and for* $\rho\in\mathscr{P}$, $U_x\rho\in\mathscr{P}$ *is defined by* $\int f(\xi)U_x\rho(d\xi)=\int f(\xi_x)\rho(d\xi)$ *for every* f *of* $C(E)$.

Lemma 2.6. *Let* $\rho\in\mathscr{G}$. *Then we have*

(2.18) $$\int_B c(x\,;\,\xi)\rho(d\xi)=\int_B c(x\,;\,\xi_x)U_x\rho(d\xi) \quad \text{for } {}^\forall B\in\mathscr{F},\ {}^\forall x\in Z^\nu.$$

In particular ρ *and* $U_x\rho$ *are absolutely continuous mutually for each* $x\in Z^\nu$.

(**Proof**) Let V be any finite subset of Z^ν, and $f,g\in C_V(E)$. Then since $(Af,g)_{L^2(E,\rho)}=(f,Ag)_{L^2(E,\rho)}$ holds we have

(2.19) $$\sum_{x\in V}\int c(x\,;\,\xi)f(\xi_x)g(\xi)\rho(d\xi)=\sum_{x\in V}\int c(x\,;\,\xi)g(\xi_x)f(\xi)\rho(d\xi)$$
$$=\sum_{x\in V}\int c(x\,;\,\xi_x)f(\xi_x)g(\xi)U_x\rho(d\xi).$$

Hence (2.19) implies that for every finite subset V of Z^{ν}

$$(2.20)\qquad \int_B c(x\,;\,\xi)\rho(d\xi)=\int_B c(x\,;\,\xi_x)U_x\rho(d\xi)\qquad \textit{for all } B\in\mathscr{F}_V$$

and we can attain (2.18). The latter half is trivial from (2.6). Lemma 2.6 guarantees that $U_x f$ is well-defined as an element of $L^2(E,\rho)$ for each f of $L^2(E,\rho)$. Denote by $\mathscr{D}(\bar{A}\,:\,L^2(E,\rho))$ the domain of the generator of $\{T_t\}$, $t\geqq 0$ operating on $L^2(E,\rho)$.

Lemma 2.7. *For all f and g of $\mathscr{D}(\bar{A}:L^2(E,\rho))$ we have the following representation ;*

$$(2.21)\qquad -(\bar{A}f,g)_{L^2(E,\rho)}$$
$$=\frac{1}{2}\sum_{x\in Z^{\nu}}\int c(x\,;\,\xi)(U_xf(\xi)-f(\xi))(U_xg(\xi)-g(\xi))\rho(d\xi).$$

(**Proof**) 1°. Note $\mathscr{D}(A)\subset\mathscr{D}(\bar{A})\subset\mathscr{D}(\bar{A}:L^2(E,\rho))$ by Theorem 2.2, and for all f and g of $\mathscr{D}(A)$

$$\begin{aligned}(2.22)\qquad -(Af,g)_{L^2(E,\rho)}&=\sum_{x\in Z^{\nu}}\int c(x\,;\,\xi)(f(\xi_x)-f(\xi))g(\xi)\rho(d\xi)\\&=\sum_{x\in Z^{\nu}}\int c(x\,;\,\xi_x)(f(\xi)-f(\xi_x))g(\xi_x)U_x\rho(d\xi)\\&=\sum_{x\in Z^{\nu}}\int c(x\,;\,\xi)(f(\xi)-f(\xi_x))g(\xi_x)\rho(d\xi).\end{aligned}$$

In the last equality we used Lemma 2.6. Averaging (2.22) we get

$$(2.23)\qquad -(Af,g)_{L^2(E,\rho)}$$
$$=\frac{1}{2}\sum_{x\in Z^{\nu}}\int c(x\,;\,\xi)(f(\xi_x)-f(\xi))(g(\xi_x)-g(\xi))\rho(d\xi).$$

2°. Next, let us introduce two Hilbert spaces.

Let $H^{(1)}=\{f\in L^2(E,\rho)\,;\,\sum_{x\in Z^{\nu}}\int(U_xf(\xi)-f(\xi))^2\rho(d\xi)<+\infty\}$ and define a bilinear functional $\mathscr{E}^{(1)}$ on $H^{(1)}$.

$$(2.24)\qquad \mathscr{E}^{(1)}(f,g)=\frac{1}{2}\sum_{x\in Z^{\nu}}\int c(x\,;\,\xi)(U_xf(\xi)-f(\xi))(U_xg(\xi)$$
$$-g(\xi))\rho(d\xi)+(f,g)_{L^2(E,\rho)}\qquad \textit{for } {}^{\forall}f,\ {}^{\forall}g\in H^{(1)}.$$

Then it is easy to show that $\{H^{(1)}, \mathscr{E}^{(1)}\}$ is a Hilbert space.

Now, we will define another Hilbert space. Since $-\bar{A}$ is a self-adjoint positive definite operator on $L^2(E, \rho)$ it has a spectral representation.

(2.25) $-\bar{A}=\int_0^\infty \lambda\, dE_\lambda$ *where* $\{E_\lambda\}$, $\lambda \geqq 0$ *is a resolution of the identity.*

So, we define a self-abjoint operator $\sqrt{-\bar{A}}$ by $\sqrt{-\bar{A}}=\int_0^\infty \sqrt{\lambda}\, dE_\lambda$ and set

(2.26) $$H^{(2)}=\mathscr{D}(\sqrt{-\bar{A}} : L^2(E, \rho))$$

$$\mathscr{E}^{(2)}(f, g)=(\sqrt{-\bar{A}}f, \sqrt{-\bar{A}}g)_{L^2(E, \rho)}+(f, g)_{L^2(E, \rho)}$$

for each f, g *of* $H^{(2)}$.

Then we have

(2.27) $$\mathscr{E}^{(2)}(f, g)=((I-\bar{A})f, g)_{L^2(E, \rho)}$$

for ${}^\forall f \in \mathscr{D}(\bar{A} : L^2(E, \rho))$, ${}^\forall g \in H^{(2)}$.

3°. If we can show that $\{H^{(1)}, \mathscr{E}^{(1)}\}$ and $\{H^{(2)}, \mathscr{E}^{(2)}\}$ coincide, the proof of Lemma 2. 7 is completed. Hence it suffices to show that $\mathscr{D}(A)$ is dense in both Hibert spaces and $\mathscr{E}^{(1)}(f, g)=\mathscr{E}^{(2)}(f, g)$ for all f and g of $\mathscr{D}(A)$.

Suppose that $\mathscr{E}^{(1)}(f, g)=0$ for some $f \in H^{(1)}$ and all g of $\mathscr{D}(A)$. Then,

(2.28) $$\frac{1}{2}\sum_{x \in Z^\nu}\int c(x ; \xi)(U_x f(\xi)-f(\xi))(U_x g(\xi)-g(\xi))\rho(d\xi) + (f, g)_{L^2(E, \rho)}=0.$$

This relation implies

(2.29) $(f, (I-A)g)_{L^2(E, \rho)}=0$ *for all* g *of* $\mathscr{D}(A)$.

Since $(I-A)[\mathscr{D}(A)]$ is dense in $C(E)$, it is so in $L^2(E, \rho)$. Therefore $f=0$ and $\mathscr{D}(A)$ is dense in $H^{(1)}$.

If f is an element of the orthogonal complement of $\mathscr{D}(A)$ in $\{H^{(2)}, \mathscr{E}^{(2)}\}$, $\mathscr{E}^{(2)}(f, g)=((I-\bar{A})g, f)_{L^2(E, \rho)}=0$ *for all* g *of* $\mathscr{D}(A)$.

Since $(I-\bar{A})[\mathscr{D}(A)]=(I-A)[\mathscr{D}(A)]$ is dense in $L^2(E, \rho)$, $f=0$ and $\mathscr{D}(A)$ is dense in $\{H^{(2)}, \mathscr{E}^{(2)}\}$. Finally noting (2.23), we can conclude $\{H^{(1)}, \mathscr{E}^{(1)}\}=\{H^{(2)}, \mathscr{E}^{(2)}\}$.

Proof of Theorem 2.5. If f is a $\{T_t\}$-invariant function in $L^2(E, \rho)$ sense, $f\in\mathscr{D}(\bar{A}: L^2(E, \rho))$ and $\bar{A}f=0$. Hence Lemma 2.7 implies that for almost all ξ (*w.r.t.* ρ) $f(\xi)=U_x f(\xi)$ for all $x\in Z^\nu$. From this fact it is immediate that $f(\xi)$ is $\mathscr{F}_\infty$-measurable. Conversely, if f is $\mathscr{F}_\infty$-measurable, then $f\in\{H^{(1)}, \mathscr{E}^{(1)}\}$ and $\mathscr{E}^{(1)}(f, g)=(f, g)_{L^2(E,\rho)}$ for all $g\in H^{(1)}$.

Noting $\mathscr{E}^{(1)}(f, g)=\mathscr{E}^{(2)}(f, g)$, $(\sqrt{-\bar{A}}f, \sqrt{-\bar{A}}f)_{L^2(E,\rho)}=0$ and we can easily see $f\in\mathscr{D}(\bar{A}: L^2(E, \rho))$ and $\bar{A}f=0$. Therefore f is $\{T_t\}$-invariant in $L^2(E, \rho)$-sense.

The following fact is well-konwn ([7], [8]).

Lemma 2.8. *Let $\rho\in\mathscr{G}$. ρ belongs to $\mathscr{G}_{ex}$ if and only if*

$$\mathscr{F}_\infty=\{\phi, E\} \pmod{\rho}.$$

Combining Theorem 2.5 and Lemma 2.8, we have

Corollary 2.9. *Let $\rho\in\mathscr{G}$. Every $\{T_t\}$-invariant function in $L^2(E, \rho)$ sense is a constant function if and only if $\rho\in\mathscr{G}_{ex}$.*

Theorem 2.10. *For every $\rho\in\mathscr{G}_{ex}$ there exists a dense subset E_ρ of E such that*

(2.30) (*i*) $\rho(E_\rho)=1$,

(*ii*) $\exists\lim_{t\to\infty} T_t f(\xi)=\int_E f(\tilde{\xi})\rho(d\tilde{\xi})$ *for all $f\in C(E)$ if $\xi\in E_\rho$.*

(**Proof**) $\{T_t\}$, $t\geqq 0$ is a symmetric conservative semi-group on $L^2(E, \rho)$ because of $\rho\in\mathscr{G}=\mathscr{S}_R$. Hence it is known that for every f of $L^2(E, \rho)$ $T_t f(\xi)$ converges to a $\{T_t\}$-invariant function in $L^2(E, \rho)$-sense $f^*(\xi)$ as $t\to\infty$ almost everywhere ξ (*w.r.t.* ρ), c. f.

[10]. Since $f^*(\xi)$ is constant because of $\rho \in \mathscr{G}_{ex}$ and Corollary 2. 9, we have $f^*(\xi) = f^* = \int f(\tilde{\xi})\rho(d\tilde{\xi})$. Using the separability of $C(E)$ and the countability argument, there exists a subset E_ρ of E such that $\rho(E_\rho) = 1$ and $\exists \lim_{t\to\infty} T_t f(\xi) = \int f(\tilde{\xi})\rho(d\tilde{\xi})$ *for all* $f \in C(E)$ *and* $\xi \in E_\rho$. Moreover $\rho \in \mathscr{G}$ is everywhere dense, so, E_ρ is dense in E.

Remark 2. 11. *If we denote by* $\mathscr{S}_{ex}$ *the family of all the extremal element of* $\mathscr{S}$, *Theorem 2. 10 implies* $\mathscr{G}_{ex} \subset \mathscr{S}_{ex}$.

§3. Free energy——Generalization of Holley's results

In the present section we will prove under our assumption the results concerning the free energy, which were obtained by Holley [6] for finite range potentials. Our method of the proof is essentially similar to Holley's one. However we need some technical devices and it becomes rather complicated.

Throughout this section a potential function Φ of $\mathbf{B}_1$ is fixed. Let $\{V_n\}$, $\{\tilde{V}_n\}$ be two sequences of the cubes of Z^ν, defined by

$$(3.1) \qquad V_n = [-2^n+1,\ 2^n-1]^\nu,\ \tilde{V}_n = [-2^n+n+1,\ 2^n-n-1]^\nu.$$

The free energy per site of ρ of $\mathscr{P}$ is defined by

$$(3.2) \qquad F(\rho) = \varliminf_{n\to\infty} |V_n|^{-1}[s(\rho_{V_n}) - \sum_{\eta\in E_{V_n}} \rho_{V_n}(\eta) U^{\Phi}_{V_n}(\eta)]$$

$$= -\varlimsup_{n\to\infty} |V_n|^{-1} \sum_{\eta\in E_{V_n}} \rho_{V_n}(\eta) \log \frac{\rho_{V_n}(\eta)}{\exp[-U^{\Phi}_{V_n}(\eta)]},$$

$$\textit{where} \qquad s(\rho_V) = -\sum_{\eta\in E_V} \rho_V(\eta) \log \rho_V(\eta),$$

$$U^{\Phi}_V(\eta) = U_V(\eta) = \sum_{X\subset V} \Phi(X)\sigma_X(\eta).$$

For each ρ of $\mathscr{P}$, $\rho_t \in \mathscr{P}$ is defined as follows;

$$(3.3) \qquad \int f(\xi)\rho_t(d\xi) = \int T_t f(\xi)\rho(d\xi) \qquad \text{for every } f \text{ of } C(E).$$

Theorem 3.1. *For any ρ of $\mathscr{P}$, $F(\rho_t)$ is non-decreasing in $t\geqq 0$.*

The proof will be carried out in several lemmas.

Lemma 3.2.

$$(3.4)\qquad \frac{d}{dt}\left(\sum_{\eta\in E_{V_n}}(\rho_t)_{V_n}(\eta)\log\frac{(\rho_t)_{V_n}(\eta)}{\exp[-U_{V_n}(\eta)]}\right)$$

$$=\sum_{\eta\in E_{V_n}}\sum_{x\in V_n}\left[\int_{S(V_n:\eta_x)}c(x;\xi)\rho_t(d\xi)\right.$$

$$\left.-\int_{S(V_n:\eta)}c(x;\xi)\rho_t(d\xi)\right]\frac{(\rho_t)_{V_n}(\eta)}{\exp[-U_{V_n}(\eta)]}.$$

If $(\rho_t)_{V_n}(\eta)=0$ $\frac{d}{dt}(\rho_t)_{V_n}(\eta)>0$ for some $\eta\in E_{V_n}$, (3.4) holds in the sense of $-\infty=-\infty$, and otherwise the both hand sides are finite and (3.4) holds.

Lemma 3.3. *Let $\rho\in\mathscr{P}$, $n\geqq 2$.*

$$(3.5)\qquad \sum_{\eta\in E_{V_n}}\sum_{x\in V_n\setminus \mathring{V}_n}\left[\int_{S(V_n:\eta_x)}c(x;\xi)\rho(d\xi)\right.$$

$$\left.-\int_{S(V_n:\eta)}c(x;\xi)\rho(d\xi)\right]\log\frac{\rho_{V_n}(\eta)}{\exp[-U_{V_n}(\eta)]}$$

$$\leqq 2\,e^{\|\Phi\|}(e^{-1}+2\|\Phi\|)(n\nu)(2^{n+1}-1)^{\nu-1}=o(|V_n|).$$

The proof is similar to Lemma 2.5 in Holley [4].

Let us introduce some functions:

$$(3.6)\qquad c_V(x;\eta)=\exp\left[\sum_{\substack{X\subset V\\ X\ni x}}\Phi(X)\sigma_X(\eta)\right]\qquad \text{for } x\in V,\ \eta\in E_V,$$

$$(3.7)\qquad F_0(u)=\begin{cases}u-u\log u-1 & \text{for } u>0,\\ -1 & \text{for } u=1,\end{cases}$$

$$(3.8)\quad F(\rho, V, \eta, \eta_x) = \begin{cases} F_0\left(\dfrac{\rho_V(\eta_x)}{\rho_V(\eta)} c_V(x;\eta_x)^2\right)\rho_V(\eta) & \text{if } \rho_V(\eta)>0, \\ -\infty & \text{if } \rho_V(\eta)=0,\ \rho_V(\eta_x)>0, \\ 0 & \text{if } \rho_V(\eta)=0,\ \rho_V(\eta_x)=0, \end{cases}$$

for each finite set V, $\eta\in E_V$, $\rho\in\mathscr{P}$.

Lemma 3.4. *Let* $\rho\in\mathscr{P}$, $n\geqq 2$.

$$(3.9)\quad \sum_{\eta\in E_{V_n}}\sum_{x\in \bar{V}_n}\left[\int_{S(V_n:\eta_x)} c(x;\xi)\rho(d\xi) - \int_{S(V_n:\eta)} c(x;\xi)\rho(d\xi)\right]\times$$

$$\log\frac{\rho_{V_n}(\eta)}{\exp[-U_{V_n}(\eta)]}$$

$$=\sum_{\eta\in E_{V_n}}\sum_{x\in\bar{V}_n} F(\rho, V_n, \eta, \eta_x)\tilde{c}_{V_n}(x;\eta_x)/c_{V_n}(x;\eta_x)^2 + o(|V_n|),$$

where $\tilde{c}_V(x;\eta) = \begin{cases} \rho_V(\eta)^{-1}\displaystyle\int_{S(V:\eta)} c(x;\xi)\rho(d\xi) & \text{if } \rho_V(\eta)>0, \\ c_V(x;\eta) & \text{if } \rho_V(\eta)=0. \end{cases}$

(**Proof**) First, assume $\rho_{V_n}(\eta)>0$ for every $\eta\in E_{V_n}$. By (3.8) we can easily derive the following relation.

$$(3.10)\quad \sum_{\eta\in E_{V_n}}\sum_{x\in\bar{V}_n} F(\rho, V_n, \eta, \eta_x)\tilde{c}_{V_n}(x;\eta_x)/c_{V_n}(x;\eta_x)^2$$

$$=\sum_{\eta\in E_{V_n}}\sum_{x\in\bar{V}_n}\left[\int_{S(V_n:\eta_x)} c(x;\xi)\rho(d\xi) - \int_{S(V_n:\eta)} c(x;\xi)\rho(d\xi)\right]\times$$

$$\log\frac{\rho_{V_n}(\eta)}{\exp[-U_{V_n}(\eta)]}$$

$$+\sum_{\eta\in E_{V_n}}\sum_{x\in\bar{V}_n}\rho_{V_n}(\eta_x)\tilde{c}_{V_n}(x;\eta_x) - \sum_{\eta\in E_{V_n}}\sum_{x\in\bar{V}_n}\rho_{V_n}(\eta)\frac{\tilde{c}_{V_n}(x;\eta_x)}{c_{V_n}(x;\eta_x)^2}$$

$$= I_1+I_2+I_3.$$

$$I_2+I_3=\sum_{\eta\in E_{V_n}}\sum_{x\in\bar{V}_n}\int_{S(V_n:\eta_x)}(c(x;\xi)-c_{V_n}(x;\eta_x))\rho(d\xi)$$

$$+\sum_{\eta\in E_{V_n}}\sum_{x\in\bar{V}_n}\rho_{V_n}(\eta)c_{V_n}(x;\eta)^2\rho_{V_n}(\eta_x)^{-1}\times$$

$$\int_{S(V_n:\eta_x)}(c_{V_n}(x;\eta)-c(x;\xi))\rho(d\xi).$$

In the last relation we used $c_{V_n}(x\,;\,\eta_x)=c_{V_n}(x\,;\,\eta)^{-1}$.
Therefore we can conclude $I_1+I_2=o(|V_n|)$ by means of the following estimates,

$$(3.11)\quad |c_{V_n}(x\,;\,\eta)-c(x\,;\,\xi)|\leqq e^{\|\Phi\|}[\exp\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|-1]\quad \text{if } \xi|_{V_n}=\eta,$$

where $\qquad B_n=\{x\in Z^\nu\,;\,|x|\leqq n\}$,

$$(3.12)\qquad e^{-\|\Phi\|}\leqq c_{V_n}(x\,;\,\eta)\leqq e^{\|\Phi\|}.$$

If $\rho_{V_n}(\eta)=0$, and $\rho_{V_n}(\eta_x)>0$ for some $\eta\in E_{V_n}$ and some $x\in\tilde{V}_n$ (3.9) holds in the sense of $-\infty=-\infty$. And if $\rho_{V_n}(\eta)=0$ $\rho_{V_n}(\eta_x)=0$ for some $\eta\in E_{V_n}$ and all $x\in\tilde{V}_n$, such η can be omitted in $\sum_{\eta\in E_{V_n}}$ of (3.9).

Now, the proof of Theorem 3.1 is obvious from the above lemmas, noting $F(\rho,\,V_n,\,\eta,\,\eta_x)\leqq 0$.

Next, we will detail Z^ν-invariant states by means of the free energy. For each $\xi\in E$ and $a\in Z^\nu$, define $\xi^{(a)}\in E$ by $\xi^{(a)}(x)=\xi(x+a)$. $\rho\in P$ is called a Z^ν-invariant state if

$$(3.13)\quad \int f(\xi)\rho(d\xi)=\int f(\xi^{(a)})\rho(d\xi)\qquad \text{for all } a\in Z^\nu \text{ and } f\in C(E).$$

Denote by $\mathscr{I}$ the family of all Z^ν-invariant states.

It is easy to see that if $\rho\in\mathscr{I}$, then $\rho_t=T_t\rho\in\mathscr{I}$ for all $t\geqq 0$. Functionals $\{H_n\}$ on $\mathscr{I}$ are defined by the first term of the right hand side of (3.9) in Lemma 3.4, that is,

$$(3.14)\quad H_n(\rho)=\sum_{\eta\in E_{V_n}}\sum_{x\in\tilde{V}_n}F(\rho,\,V_n,\,\eta,\,\eta_x)\tilde{c}_{V_n}(x\,;\,\eta_x)c_{V_n}(x\,;\,\eta)^2.$$

We want to discuss the limit of $|V_n|^{-1}H_n(\rho)$. So, let us introduce another functionals $\{h_n\}$ on $\mathscr{I}$ which approximate $\{H_n\}$.

$$(3.15)\quad f(\rho,\,V,\,\eta,\,\eta_x)=\begin{cases} F_0(\rho_V(\eta)^{-1}\int_{S(V:\eta_x)}c(x\,;\,\xi)^2\rho(d\xi))\rho_V(\eta) \\ \qquad\qquad \text{if } \rho_V(\eta)>0, \\ -\infty \qquad \text{if } \rho_V(\eta)=0,\ \rho_V(\eta_x)>0, \\ \ 0 \qquad\ \ \text{if } \rho_V(\eta)=0,\ \rho_V(\eta_x)=0, \end{cases}$$

for each finite subset $V\subset Z^\nu$ and $\eta\in E_V$.

(3.16) $$h_n(\rho)=\sum_{\eta\in E_{V_n}}\sum_{x\in\tilde{V}_n} f(\rho,\ V_n,\ \eta,\ \eta_x)c_{B_{n-1}(x)}(x\ ;\ \eta)$$

where $B_n(x)=B_n+x=\{y\in Z^\nu\ ;\ |y-x|\leqq n\}$.

Lemma 3.5. *Let* $\rho\in\mathscr{I}$.

(3.17) $$\exists\lim_{n\to\infty}|V_n|^{-1}h_n(\rho)\geqq-\infty.$$

If we denote by $H(\rho)$ *this limit,* $H(\rho)$ *is upper semi-continuous.*

(**Proof**) First, we will show the following inequality.

(3.18) $$h_n(\rho)\leqq 2^\nu h_{n-1}(\rho)\exp\Big[-\sum_{\substack{X\ni 0\\ X\subset B_{n-1},X\not\subset B_{n-2}}}|\Phi(X)|\Big].$$

Let $a=(-2^{n-1}+1,\ -2^{n-1}+1,\ \cdots,\ -2^{n-1}+1)\in Z^\nu$,

$\sigma=(\sigma_1,\ \sigma_2,\cdots,\ \sigma_\nu)$, $D_\sigma=\tilde{V}_{n-1}-a\cdot\sigma$,

where $\sigma_i=+1,\ -1$ and $a\cdot\sigma=(\sigma_1(-2^{n-1}+1),\cdots,\ \sigma_\nu(-2^{n-1}+1))$.

Particularly we write $\underline{\sigma}=(1,\ 1,\ \cdots,\ 1)$ and $D=D_{\underline{\sigma}}$.

Because of $f(\rho,\ V_n,\ \eta,\ \eta_x)\leqq 0$, we have

(3.19) $$h_n(\rho)\leqq\sum_\sigma\sum_{x\in D_\sigma}\sum_{\eta\in E_{V_n}} f(\rho,\ V_n,\ \eta,\ \eta_x)c_{B_{n-1}(x)}(x\ ;\ \eta).$$

Hence it suffices to show

(3.20) $$\sum_{x\in D}\sum_{\eta\in E_{V_n}} f(\rho,\ V_n,\ \eta,\ \eta_x)c_{B_{n-1}(x)}(x\ ;\ \eta)$$
$$\leq h_{n-1}(\rho)\ \exp\Big[-\sum_{\substack{X\ni 0\\ X\subset B_{n-1}\\ X\not\subset B_{n-2}}}|\Phi(X)|\Big].$$

Set $W_n=V_{n-1}-a=[0,\ 2^n-2]^\nu$ and $\psi=\eta|_{V_n/W_n}$, $\zeta=\eta|_{W_n}$ for $\eta\in E_{V_n}$. Obviously we have for $x\in D$

(3.21) $$c_{B_{n-1}(x)}(x\ ;\psi\zeta)=c_{B_{n-1}(x)}(x\ ;\zeta)\geqq c_{B_{n-2}(x)}(x\ ;\zeta)\exp\Big[-\sum_{\substack{X\ni 0\\ X\subset B_{n-1}\\ X\not\subset B_{n-2}}}|\Phi(X)|\Big],$$

$$c_{B_{n-2}(x)}(x\ ;\ \zeta)=c_{B_{n-2}(x+a)}(x+a\ ;\ \zeta^{(a)}),$$

$$c(x\ ;\ \xi)=c(x+a\ ;\ \xi^{(a)}).$$

Making use of the concavity of $F_o(u)$, Z^ν-invariance of ρ and (3.21), we get

$$(3.22)\quad \sum_{\phi\in E_{V_n/W_n}} f(\rho,\ V_n,\ \phi\zeta,\ \phi\zeta_x)$$
$$\leqq F_o(\rho_{W_n}(\zeta)^{-1}\int_{S(W_n:\zeta x)} c(x\ ;\ \xi)^2\rho(d\xi))\rho_{W_n}(\zeta),$$
$$=f(\rho,\ V_{n-1},\ \zeta^{(a)},\ \zeta^{(a)}_{x+a}).$$

$$(3.23)\quad \sum_{x\in D}\sum_{\eta\in E_{V_n}} f(\rho,\ V_n,\ \eta,\ \eta_x)c_{B_{n-1}(x)}(x\ ;\ \eta)$$
$$\leqq \sum_{\zeta\in E_{W_n}}\sum_{x\in D}\sum_{\phi\in E_{V_n/W_n}} f(\rho,\ V_n,\ \phi\zeta,\ \phi\zeta_x)c_{B_{n-2}(x)}(x\ ;\ \zeta)$$
$$\times\exp[-\sum_{\substack{X\ni 0\\ X\subset B_{n-1}\\ X\not\subset B_{n-2}}}|\Phi(X)|]$$
$$\leqq \sum_{\zeta\in E_{W_n}}\sum_{x\in D} f(\rho,\ V_{n-1},\ \zeta^{(a)},\ \zeta^{(a)}_{x+a})c_{B_{n-2}(x)}(x\ ;\ \zeta)$$
$$\times\exp[-\sum_{\substack{X\ni 0\\ X\subset B_{n-1},X\not\subset B_{n-2}}}|\Phi(X)|].$$

Noting $D+a=\tilde{V}_{n-1}$ and $E_{V_{n-1}}=(E_{W_n})^{(a)}$, the right hand side of (3.23) is equal to $h_{n-1}(\rho)\exp[-\sum_{\substack{X\ni 0\\ X\subset B_{n-1},X\not\subset B_{n-2}}}|\Phi(X)|]$.

If we set $G(n)=\prod_{k=n}^{\infty}\frac{(2^{k+2}-2)^\nu}{(2^{k+2}-1)^\nu}$, $G(n)(2^{n+1}-1)^{-\nu}h_n(\rho)\exp[\sum_{\substack{X\ni 0\\ X\subset B_{n-1}}}|\Phi(X)|]$ is non-increasing in n and we define $H(\rho)$ by

$$(3.24)\quad \lim_{n\to\infty} G(n)(2^{n+1}-1)^{-\nu}h_n(\rho)\exp[\sum_{\substack{X\ni 0\\ X\subset B_{n-1}}}|\Phi(X)|]=e^{\|\Phi\|}\cdot H(\rho).$$

$H(\rho)$ is a monotone non-increasing limit of upper semi-continuous functions because $\{h_n(\rho)\}$ are upper semi-continuous in ρ. Therefore $H(\rho)=\lim_{n\to\infty}|V_n|^{-1}\cdot h_n(\rho)$ is upper semi-continuous.

Lemma 3.6.

$$(3.25)\qquad {}^{\exists}\lim_{n\to\infty}|V_n|^{-1}\cdot H_n(\rho)=H(\rho)\quad(\geqq-\infty).$$

(**Proof**) If $\rho_{V_n}(\eta)=0$ $\rho_{V_n}(\eta_x)>0$ for some $x\in\tilde{V}_n$, $H_m(\rho)=h_m(\rho)=-\infty$ for ${}^{\forall}m\geqq n$. Therefore $\lim_{n\to\infty}|V_n|^{-1}\cdot H_n(\rho)=\lim_{n\to\infty}|V_n|^{-1}\cdot h_n(\rho)=-\infty$. Since η, such as $\rho_{V_n}(\eta)=0$ $\rho_{V_n}(\eta_x)=0$ for ${}^{\forall}x\in\tilde{V}_n$, does not contribute to $\sum_{\eta\in E_{V_n}}$ of (3. 14), we may assume $\rho_{V_n}(\eta)>0$ for all $\eta\in E_{V_n}$. Then, $H_n(\rho)$ and $h_n(\rho)$ are finite for each n.

$$(3.26)\quad H_n(\rho)-h_n(\rho)=\sum_{\eta\in E_{V_n}}\sum_{x\in\tilde{V}_n}\Bigg(F(\rho,V_n,\eta,\eta_x)\tilde{c}_{V_n}(x;\eta_x)c_{V_n}(x;\eta)^2 -f(\rho,V_n,\eta,\eta_x)\frac{\tilde{c}_{V_n}(x;\eta_x)\rho_{V_n}(\eta_x)}{\int_{S(V_n:\eta_x)}c(x;\xi)^2\rho(d\xi)}\Bigg)$$
$$+\sum_{\eta\in E_{V_n}}\sum_{x\in\tilde{V}_n}f(\rho,V_n,\eta,\eta_x)\left[\frac{\tilde{c}_{V_n}(x;\eta_x)\rho_{V_n}(\eta_x)}{\int_{S(V_n:\eta_x)}c(x;\xi)^2\rho(d\xi)}-c_{B_{n-1}(x)}(x;\eta)\right]=I+II.$$

$$(3.27)\quad |I|\leqq\sum_{\eta}\sum_{x}|\rho_{V_n}(\eta_x)\log(\rho_{V_n}(\eta)^{-1}\rho_{V_n}(\eta_x)c_{V_n}(x;\eta_x)^2)\tilde{c}_{V_n}(x;\eta_x)$$
$$-\rho_{V_n}(\eta_x)\tilde{c}_{V_n}(x;\eta_x)\log\rho_{V_n}(\eta)^{-1}\int_{S(V_n:\eta_x)}c(x;\xi)^2\rho(d\xi)|$$
$$+\sum_{\eta}\sum_{x}\left|\tilde{c}_{V_n}(x;\eta_x)c_{V_n}(x;\eta)^2-\frac{\tilde{c}_{V_n}(x;\eta_x)\rho_{V_n}(\eta_x)}{\int_{S(V_n:\eta_x)}c(x;\xi)^2\rho(d\xi)}\right|\rho_{V_n}(\eta)$$
$$=\sum_{\eta}\sum_{x}\rho_{V_n}(\eta_x)\tilde{c}_{V_n}(x;\eta_x)\left|\log\rho_{V_n}(\eta_x)^{-1}\int_{S(V_n:\eta_x)}\frac{c(x;\xi)^2\rho(d\xi)}{c_{V_n}(x;\eta_x)^2}\right|$$
$$+\sum_{\eta}\sum_{x}\rho_{V_n}(\eta)\tilde{c}_{V_n}(x;\eta_x)c_{V_n}(x;\eta)^2\times$$
$$\left(1-\left[\rho_{V_n}(\eta_x)^{-1}\int_{S(V_n:\eta_x)}\frac{c(x;\xi)^2\rho(d\xi)}{c_{V_n}(x;\eta_x)^2}\right]^{-1}\right)$$
$$\leqq e^{\|\Phi\|}\cdot 2\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|\cdot|\tilde{V}_n|+e^{5\|\Phi\|}(\exp[2\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]-1)\cdot|\tilde{V}_n|=o(|V_n|).$$

Next in order to estimate II we use the following.

$$(3.28)\quad \exp[-\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]\leqq\frac{\tilde{c}_{V_n}(x;\eta_x)}{c_{B_{n-1}(x)}(x;\eta_x)}\leqq\exp[\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|],$$

$$\exp[-2\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]\leqq\frac{1}{\rho_{V_n}(\eta_x)}\int_{S(V_n;\eta_x)}\frac{c(x;\xi)^2\rho(d\xi)}{c_{B_{n-1}(x)}(x;\eta_x)^2}$$

$$\leqq\exp[2\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|].$$

Hence we can estimate II as follows,

$$(3.29)\quad II\leqq\sum_{\eta}\sum_{x}f(\rho,V_n,\eta,\eta_x)c_{B_{n-1}(x)}(x;\eta)\,(\exp[-3\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]-1)$$

$$=h_n(\rho)\,(\exp[-3\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]-1)$$

and

$$II\geqq h_n(\rho)\,(\exp[3\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]-1).$$

Thus we get

$$(3.30)\quad h_n(\rho)\,(\exp[3\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]-1)+o(|V_n|)\leqq H_n(\rho)-h_n(\rho)$$

$$\leqq h_n(\rho)\,(\exp[-3\sum_{\substack{X\ni 0\\ X\not\subset B_{n-1}}}|\Phi(X)|]-1)+o(|V_n|).$$

Therefore (3.25) holds from (3.30) and Lemma 3.5, that is, if $H(\rho)>-\infty$, $\lim_{n\to\infty}|V_n|^{-1}\cdot H_n(\rho)=H(\rho)$, and if $H(\rho)=-\infty$ the limit is, also, $-\infty$.

By the above lemmas we can show easily the following lemma.

Lemma 3.7. *For each* $\rho_0\in\mathscr{I}$

$$(3.31)\qquad F(\rho_t)-F(\rho_0)\geqq\int_0^t-H(\rho_s)ds.$$

Lemma 3.8. *Let* $\rho\in\mathscr{I}$. *If* $H(\rho)=0$, *then* $\rho\in\mathscr{G}$.

(Proof) Noting the monotone convergence of (3.24) in the proof of Lemma 3. 5 and $h_n(\rho)\leqq 0$, we have

$$h_n(\rho)=0 \qquad \text{for all } n\geqq 2. \tag{3.32}$$

Further it is impossible that $\rho_{V_n}(\eta)=0$ and $\rho_{V_n}(\eta_x)>0$ for some $x\in\tilde{V}_n$, for otherwise $h_n(\rho)=-\infty$. Hence $\rho_{V_n}(\eta)=0$ implies $\rho_{V_n}(\eta_x)=0$ for all $x\in\tilde{V}_n$.

Next we show $\rho_{V_n}(\eta)>0$ for all $\eta\in E_{V_n}$ and $n\geqq 2$. Suppose, $\rho_{V_n}(\bar{\eta})=0$ for some $\bar{\eta}\in E_{V_n}$ and n. If we choose m such as $\tilde{V}_m\supset V_n$ then we have $\rho_{V_m}(\varphi\bar{\eta})=0$ for any $\varphi\in E_{V_m/V_n}$ and $\rho_{V_m}(\varphi\bar{\eta}_x)=0$ for any $x\in V_n\subset\tilde{V}_m$. This fact implies $\rho_{V_n}(\zeta)=0$ for all $\zeta\in E_{V_n}$. But it is absurd. Noting (3. 32), we get

$$\rho_{V_n}(\eta)^{-1}\int_{S(V_n:\eta_x)}c(x;\xi)^2\rho(d\xi)=1 \qquad \text{for all } \eta\in E_{V_n} \text{ and } n. \tag{3.33}$$

It is not difficult to show

$$\lim_{n\to\infty}\frac{\rho_{V_n}(\xi|_{V_n})}{\rho_{V_n}(\xi_x|_{V_n})}=c(x;\xi_x)^2, \tag{3.34}$$

$$\begin{aligned}\rho(S(x:j)|\mathscr{F}_{Z^\nu/x}) &=\lim_{n\to\infty}\rho(S(x:j)|\mathscr{F}_{V_n/x})\\ &=\lim_{n\to\infty}\rho_{V_n}(\xi|_{V_n})/[\rho_{V_n}(\xi|_{V_n})+\rho_{V_n}(\xi_x|_{V_n})]\\ &=c(x;\xi_x)^2/[c(x;\xi_x)^2+1]\\ &=g_{\{x\},\xi}(\{j\}) \qquad \textit{for a. e. } \xi\in S(x:j).\end{aligned} \tag{3.35}$$

Therefore the proof of Lemma 3. 8 is completed by Lemma 2. 1.

Combining the above lemmas, we have the following.

Theorem 3. 9. *Let $\rho_0\in\mathscr{I}$ and $\rho_0\notin\mathscr{G}$. Then there exists a weakly open set G_{ρ_0} containing ρ_0 and ε, $\delta>0$ such that if $\nu_0\in G_{\rho_0}$ and $0\leqq s\leqq\varepsilon$, then*

$$F(\nu_s)-F(\nu_0)\geqq\delta s. \tag{3.36}$$

Corollary 3. 10. (i) *Let $\rho_0\in\mathscr{I}$ and suppose that $t_n\to\infty$ and* *that* ρ_{t_n} *converges to* ρ. *Then* $\rho \in \mathcal{G}$.

(ii) $\mathcal{S} \cap \mathcal{I} = \mathcal{G} \cap \mathcal{I}$.

(iii) *Let* $\rho \in \mathcal{I}$. $\rho \in \mathcal{G}$ *if and only if*

$$F(\rho) = \sup_{\nu \in \mathcal{I}} F(\nu). \tag{3.37}$$

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