On the homotopy type of FDiff (S³, \mathscr{F}_R)

By

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§1. Introduction.

In this article, we study some topological subgroups of the group of diffeomorphisms of a manifold.

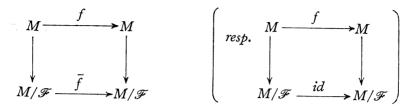
Our main result is that the topological group of foliation preserving diffeomorphisms of S^3 with respect to the Reeb foliation has the homotopy type of a torus $S^1 \times S^1$.

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§2. Spaces of foliation preserving diffeomorphisms and leaf preserving diffeomorphisms.

Let M be a smooth manifold of dimension n with a smooth foliation of codimension q.

Definition. A diffeomorphism $f: M \rightarrow M$ is called a foliation preserving diffeomorphism (resp. a leaf preserving diffeomorphism) if for each point x of M, the leaf through x is mapped into the leaf through f(x) (resp. x), that is, $f(L_*) = L_{f(*)}$ (resp. $f(L_*) = L_x$), where L_* is the leaf that contains x. Equivalently, if there is a homeomorphism \bar{f} (resp. id) of the leaf space M/\mathcal{F} , such that the diagram commutes, where vertical arrows are canonical projections (see Reeb [7]).



Let $FDiff(M, \mathcal{F})$ (resp. $LDiff(M, \mathcal{F})$) denote the space of all foliation (resp. leaf) preserving diffeomorphisms of M with respect to \mathcal{F} .

It is clear that $\mathrm{LDiff}(M, \mathscr{F}) \subset \mathrm{FDiff}(M, \mathscr{F}) \subset \mathrm{Diff}(M)$. Topologies of the spaces are induced by a relevant C^{∞} -topology of $\mathrm{Diff}(M)$. Clearly there is an exact sequence of topological groups:

$$0 \longrightarrow \text{LDiff}(M, \mathcal{F}) \longrightarrow \text{FDiff}(M, \mathcal{F}) \stackrel{\pi}{\longrightarrow} \text{Homeo}(M/\mathcal{F}),$$

where the second arrow is the inclusion map and the map π is defined as in the definition of the foliation preserving diffeomorphisms, $\pi: f \longrightarrow \bar{f}$.

Remark. In many important cases as foliations constructed by geometric manners, the homotopy types of the spaces of foliation preserving diffeomorphisms are not trivial. For example, $\pi_1(\text{FDiff}(S^5, \mathcal{F}_L))$ has non trivial element, where \mathcal{F}_L is the codimension 1 foliation constructed by Lawson [2].

§3. Reeb foliation \mathcal{F}_R .

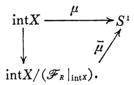
Let α be a C^{∞} -function $\alpha: [0, 1) \longrightarrow R$, such that $\alpha(0) = 0$, $\alpha'(t) > 0$ for all $t \in (0, 1)$, $\alpha^{(k)}(0) = 0$, $\lim_{t \to 1} \alpha^{(k)}(t) = \infty$ for all k. α defines a homeomorphism of [0, 1) onto $[0, \infty)$, and α restricted to (0, 1) is a diffeomorphism $(0, 1) \cong (0, \infty)$. Let $\beta = \alpha^{-1} : [0, \infty) \to [0, 1)$.

Three dimensional sphere can be decomposed into two solid tori, $S^3 = D^2 \times S^1 \cup I$, $S^1 \times D^2$, where h is a diffeomorphism $h : \partial D^2 \times S^1 \to S^1 \times \partial D^2$ defined by h(x, y) = (x, y). Let $X = D^2 \times S^1 \subset S^3$ be one of the components. X can be considered as the quotient space $I \times S^1$

 $\times S^1/\sim$, where $S^1 = \mathbf{R}/\mathbf{Z}$, I = [0, 1], the equivalence relation is defined by $(0, x, y) \sim (0, x', y)$. Express a point p of X as p = (t, x, y), $(t, x) \in D^2$, $y \in S^1$, t is the radius and x is the polar angle mod 1.

Define a foliation in X as follows: for two points $p_1 = (t_1, x_1, y_1)$, $p_2 = (t_2, x_2, y_2)$ of X, $L_{p_1} = L_{p_2}$ if and only if $t_1 = t_2 = 1$ or $\alpha(t_1) - y_1 \equiv \alpha(t_2) - y_2 \pmod{1}$. Introduce the same foliation in the other component, then these together define a foliation in S^3 . This foliation is called the *Reeb foliation*. Let us denote (S^3, \mathcal{F}_R) or simply \mathcal{F}_R .

Let $\mu: \operatorname{int} X \longrightarrow S^1$ be a map defined by $\mu(t, x, y) = \alpha(t) - y$ (mod 1). Then μ induces a diffeomorphism $\overline{\mu}: \operatorname{int} X / (\mathscr{F}_R|_{\operatorname{int} X}) \longrightarrow S$, so that the following diagram commutes,



§4. Lemmas.

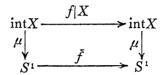
Lemma 1. There is a splitting exact sequence:

$$0 \longrightarrow LDiff(S^3, \mathscr{F}_R) \longrightarrow FDiff_0(S^3, \mathscr{F}_R) \xrightarrow{*} S^1 \times S^1 \longrightarrow 0,$$
 where $FDiff_0(S^3, \mathscr{F}_R)$ is the identity component of $FDiff(S^3, \mathscr{F}_R)$, $LDiff(S^3, \mathscr{F}_R)$ is the space of leaf preserving diffeomorphisms in $FDiff(S^3, \mathscr{F}_R)$, $S^1 \times S^1$ is the subgroup of standard rotations $SO(2) \times SO(2)$ in $Homeo(S^3/\mathscr{F}_R)$, and the global section \leftarrow is the natural lifting of standard rotations.

Proof. It is clear that the image of π contains the standard rotations $S^1 \times S^1$. Let us prove that the image of π is contained in $S^1 \times S^1$. Suppose for some f in FDiff₀(\mathscr{F}_R), $\pi f \in SO(2) \times SO(2)$. We shall deduce a contradiction from this supposition.

We may assume that the foliation preserving diffeomorphism restricted to the component X induces a homeomorphism $\bar{f}: S^1 \longrightarrow$

 S^1 by



such that $\bar{f} \notin SO(2)$.

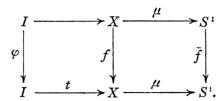
We may assume $\bar{f}(0) = 0$ and $\bar{f} \neq id$, eventually by composing some standard rotation.

Let us regard the unit interval I = [0, 1] as embedded in X by identifying $I \ni t \longleftrightarrow (t, 0, 0,) \in X$. Then f(I) and ∂X are transversal at f(1), $1 \in I \subset X$, because I and ∂X are transversal at 1 = (1, 0, 0), and f is a diffeomorphism.

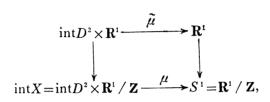
We can assume f(1)=1, eventually by composing a leaf preserving diffeomorphism induced by a vector field on ∂X (extended near ∂X). Moreover, eventually by composing a leaf preserving diffeomorphism which is the identity outside a sufficiently small neighbourhood of 1, induced by a smooth vector field tangent to the leaves, we assume that for some small ε , $\delta > 0$, f maps $(1-\varepsilon, 1]$ diffeomorphically onto $(1-\delta, 1]$. Let $\varphi = t \circ f | I : I \longrightarrow I$, where t is the radius function $t: X \longrightarrow I$. This continuous map restricted to some neighbourhood of 1 is a diffeomorphism.

Let $\Sigma = \{0, \beta(1), \beta(2), \ldots, \beta(k), \ldots\} \subset I$, and $\Sigma_n = \{\beta(n), \beta(n+1), \ldots\} \subset \Sigma$. Note that $L_0 \cap I = \Sigma$. Then φ induces a map $\bar{\varphi} = \varphi \mid \sum_{n_1} : \sum_{n_1} \cdots \sum_{n_2} \sum_{n_2} for sufficiently large numbers <math>n_1, n_2$. $\bar{\varphi}$ can be expressed as $\bar{\varphi}(\beta(k)) = \beta(k+\rho)$ for some integer ρ .

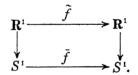
Again by composing a leaf preserving diffeomorphism which is of the form $(t, x, y) \longrightarrow (\eta(t), x, y)$ in some small tubular neighbourhood of ∂X and is the identity in the complement of some tubular neighbourhood of ∂X , induced by a vector field of ∂X rotating $\partial X - \rho$ times, mapping a neighbourhood of 1 in I into a neighbourhood of 1 in I, we may assume $\bar{\varphi} = id \sum_{\pi_1}$. Now recall the following diagram (μ is not defined on ∂X),



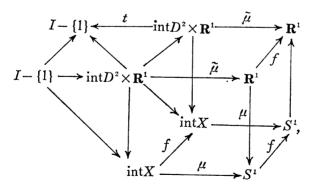
Consider a universal covering:



where horizontal arrows are canonical profections $\mathbf{R}^1 \longrightarrow S^2 = \mathbf{R}/\mathbf{Z}$, and $\tilde{\mu}$ is defined by $\tilde{\mu}(t, x, \tilde{y}) = \alpha(t) - \tilde{y}$, $(t, x) \in D^2$, $\tilde{y} \in \mathbf{R}^1$, t is the radius. Let \tilde{f} be the lifting of \tilde{f} :



Getting together the diagrams above, and defining the maps properly,



it is easy to check that $\varphi(t) = \beta \circ \tilde{f} \circ \beta^{-1}(t)$ for $t \in (1 - \varepsilon, 1)$, and $\varphi(1) = 1$. Consider the derived function $\varphi'(t) = \frac{d\varphi}{dt}(t)$. $1 \in I$ is the cluster point of Σ and φ is the identity on Σ_{n_1} , hence $\varphi'(1)$ must

be equal to 1.

On the other hand, in $(1-\varepsilon, 1)$, $\varphi'(t) = (\beta \circ \tilde{f} \circ \beta^{-1})'(t) = \beta'(\tilde{f}(y)) \circ \tilde{f}'(y)/\beta'(y)$, where $y = \beta^{-1}(t)$. We shall now deduce that $\varphi'(t)$ is not continuous at 1. This contradiction will complete our proof of lemma 1.

Note that $\tilde{f}(0) = 0$. Let $f_t = \tilde{f} \mid [0, 1] : [0, 1] \longrightarrow [0, 1]$. Then $f_t(0) = 0$, $f_t(1) = 1$, and f_t is a diffeomorphism of the unit interval. From the assumption, $f_t \neq id$, hence there is $y_0 \in I$ that satisfies following 1) or 2):

- 1) $y_0 \le f_1(y_0)$ and $f'_1(y_0) < 1$,
- 2) $y_0 \ge f_I(y_0)$ and $f'_I(y_0) > 1$.

By the way, $\lim_{t\to 1} \alpha''(t) = +\infty$, hence in some neighbourhood of 1 in I, $\alpha''(t) > 0$. Differentiating the formula $\alpha \circ \beta = id$ twice, we get $\alpha''(\beta) \cdot (\beta')^2 + \alpha'(\beta) \cdot \beta'' = 0$. Therefore there is a sufficiently large number N_0 such that for any $y \ge N_0$, $\beta''(y) < 0$.

Let $y_n = y_0 + n$. When y_0 satisfies the condition 1), then for any sufficiently large number n, $\varphi'(\beta(y_n)) = f'(y_n) \cdot \beta'(\tilde{f}(y_n)) / \beta'(y_n) \le \tilde{f}'(y_0) < 1$ (for, $\beta''(y) < 0$, hence $\beta'(\tilde{f}(y_n)) / \beta'(y_n) \le 1$).

Hence $\varphi'(\beta(y_n))$ cannot converge to 1. But $\beta(y_n)$ converge to 1. These lead to a contradiction. Analogously, when y_0 satisfies the condition 2), $\varphi'(\beta(y_n)) \ge \tilde{f}'(y_0) > 1$. Q. E. D.

Let $\mathrm{LDiff}_{\tau}(\mathscr{F}_R)$ denote the space of leaf preserving diffeomorphisms of S^3 with respect to \mathscr{F}_R such that in some tubular neighbourhood N of $T = \partial X$, the following diagram commutes:

$$N \xrightarrow{f \mid N} S^{3} - \{t = 0\}$$

$$\downarrow \qquad \qquad \downarrow$$

$$T \xrightarrow{f \mid T} T$$

where $\{t=0\}$ is the axes of two solid tori diffeomorphic to two copies of the circle S^1 , and the vertical arrows are projections which are expressed in each component as $(t, x, y) \longrightarrow (x, y)$.

Lemma 2. The inclusion map

$$\mathrm{LDiff}_{\tau}(\mathscr{F}_{R}) \longrightarrow \mathrm{LDiff}(S^{3}, \mathscr{F}_{R})$$

is a weak homotopy equivalence.

Proof Let K be any compact set and let $\Psi: K \longrightarrow LDiff(S^3, \mathscr{F}_R)$ be a continuous map. We will make a homotopy Ψ_r such that $\Psi_0 = \Psi$, $\Psi_1(k) \subset LDiff_T(\mathscr{F}_R)$. Let $\chi: [0, 1] \longrightarrow \mathbb{R}$ be a smooth function such that $\chi(x) = 0$ for $x \in [1, 1 - \frac{2}{3}\nu]$, $\chi(x) = 1$ for $x \in [1 - \frac{1}{3}\nu, 1]$, and $\chi'(x) > 0$ for $x \in (1 - \frac{2}{3}\nu, 1 - \frac{1}{3}\nu)$, where ν is a properly chosen positive number.

Let $\lambda_r : [0, 1] \longrightarrow [0, 1]$ be a family of smooth maps defined by $\lambda_r(t) = t + \tau \cdot \chi(t) \cdot (1 - t)$ for τ , $t \in [0, 1]$.

Let $f \in \text{Im}(\Psi) \subset \text{LDiff}(S^3, \mathscr{F}_R)$. Define a homotopy $f_r(t, x, y) = (\beta(f_r(\lambda_r(t), x, y) - f_r(t, x, y) + \alpha(f_r(t, x, y))), f_r(\lambda_r(t), x, y), f_r(\lambda_r(t), x, y))$, where f_r , f_r , f_r are components of f_r , f_r .

$$f(t, x, y) = (f_{*}(t, x, y), f_{*}(t, x, y), f_{*}(t, x, y)).$$

Choosing ν small enough, they define a homotopy Ψ_{\bullet} . Q. E. D.

Remark. LDiff_r(\mathscr{F}_R) and LDiff(\mathscr{F}_R) are Fréchet manifolds, hence they have homotopy types of CW-complexes ([3], [4], [5], [6]). Therefore the inclusion map is also a homotopy equivalence.

 $\mathrm{LDiff}_{\tau}(\mathscr{F}_R)$ is included in $\mathrm{FDiff}_0(S^3, \mathscr{F}_R)$, hence the restrictions to T belong to the identity component $\mathrm{Diff}_0(T)$ of $\mathrm{Diff}(T)$. Let $r: \mathrm{LDiff}_{\tau}(\mathscr{F}_R) \longrightarrow \mathrm{Diff}_0(T)$ be the restriction map, i.e., r(f) = f|T.

Lemma 3. There is an exact sequence:

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathrm{LDiff}_r(\mathscr{F}_R) \stackrel{r}{\longrightarrow} \mathrm{Diff}_0(T) \longrightarrow 0,$$

where \mathcal{I} is the kernel of r, and r is a locally trivial fibration.

Proof First we show r is surjective. For any φ in $\mathrm{Diff}_0(T)$, take a smooth path φ , from identity to φ in $\mathrm{Diff}_0(T)$, $\varphi_0=id_\tau$, $\varphi_1=\varphi$. Consider a vector field defined by $\frac{\partial \varphi}{\partial s}$ on $T\times I$ in $S^3\times I$. Take small tubular neighbourhoods N_1 , N_2 of $T\times I$ in $S^3\times I$, $N_1\supset$

 \overline{N}_2 , extend the vector field on $S^3 \times I$ so that it vanishes outside N_1 , tangent to the leaves, and that in N_2 it commutes with the differential map of the projection to T, next, modify the s-component of the vector field to be the unit vector field $\frac{\partial}{\partial s}$. Then integrating the vector field, we obtain an element of $\mathrm{LDiff}_T(\mathcal{F}_R)$ which is mapped to φ by the restriction r.

Next, define a local section as follows. Let $\theta \in \mathrm{LDiff}_{\tau}(\mathscr{F}_R)$, $\bar{\theta} = r(\theta)$. Note that a neighbourhood of θ in LDiff (S^3, \mathscr{F}_R) is homeomorphic to a neighbourhood of the zero section of the space of smooth sections of $\tau\mathscr{F}_R$, the sub-bundle of the tangent bundle consisting of tangent planes of leaves, and that the neighbourhood of $\bar{\theta}$ in Diff₀(T) is homeomorphic to that of $\tau(T)$, tangent bundle of the torus, which is canonically embedded in $\tau\mathscr{F}_R$. Extend the sections of $\tau(T)$ to sections of $\tau\mathscr{F}_R$ so that it commutes with the projection to T in some tubular neighbourhood of T.

We can give such extensions simultaneously, hence it gives a continuous local section of r. Q. E. D.

Let $\widetilde{\mathrm{Diff}}(S^2, D_+^2)$ denote the space of diffeomorphisms of the two dimensional sphere S^2 , which fix some neighbourhood of the north hemisphere D_+^2 . Let $L^d(\widetilde{\mathrm{Diff}}(S^2, D_+^2))$ denote the space of differentiable loops in $\widetilde{\mathrm{Diff}}(S^2, D_+^2)$.

Lemma 4. There is a splitting exact sequence:

$$0 \longrightarrow L^{4}(\widetilde{\operatorname{Diff}}(S^{2}, \ D^{2}_{+})) \oplus L^{4}(\widetilde{\operatorname{Diff}}(S^{2}, \ D^{2}_{+})) \longrightarrow \mathscr{I} \overset{\rho \oplus \rho}{\longleftrightarrow} \mathbf{Z} \oplus \mathbf{Z} \longrightarrow 0,$$

where ρ is defined as in the proof of lemma 1 in each component, **Z** is the group of integers.

Proof The global section is given for $(\rho_1, \rho_2) \in \mathbf{Z} \oplus \mathbf{Z}$, by integrating a vector field on $S^3 \times I$ associated to the standard rotation of (ρ_1, ρ_2) times, as in the proof of the surjectivity of r in lemma 3.

The kernel of this map $\rho \oplus \rho$ is the space of leaf preserving diffeomorphisms which fix some neighbourhood of T. The restriction

tion on each leaf may be regarded as an element of $\widetilde{\mathrm{Diff}}(S^2, D_+^2)$. Hence the kernel is homeomorphic to $L^d(\widetilde{\mathrm{Diff}}(S^2, D_+^2)) \oplus L^d(\widetilde{\mathrm{Diff}}(S^2, D_+^2))$. Q. E. D.

Lemma 5. The inclusion map:

$$\widetilde{\operatorname{Diff}}(S^2, D^2_+) \longrightarrow \operatorname{Diff}(S^2, D^2_+)$$

is a homotopy equivalence, where Diff (S^2, D_+^2) denotes the space of diffeomorphisms of S^2 which fix D_+^2 .

Proof Let q_i be a smooth path in the space of diffeomorphisms of S^2 , such that $q_0 = id$, and q_1 shifts the equator towards the north pole. Define a homotopy $q_i^* : \text{Diff}(S^2, D_+^2) \longrightarrow \text{Diff}(S^2, D_+^2)$ by $q_i^* : (f) = q_i^{-1} \circ f \circ q_i$. Then $q_1^* : \text{Diff}(S^2, D_+^2) \longrightarrow \widetilde{\text{Diff}}(S^2, D_+^2)$ defines a homotopy inverse of the inclusion map. Q. E. D.

§5. Theorem.

Theorem. The space $\mathrm{FDiff}_0(S^3, \mathcal{F}_R)$ has the homotopy type of a torus $S^1 \times S^1$.

Proof By a result of S. Smale [8], Diff (S^2, D_+^2) is contractible. By lemma 5, $\widetilde{\text{Diff}}(S^2, D_+^2)$ is also contractible. Therefore $L^d(\widetilde{\text{Diff}}(S^2, D_+^2)) \oplus L^d(\widetilde{\text{Diff}}(S^2, D_+^2))$ is contractible. By lemma 4, $\mathscr I$ is homotopy equivalent to $\mathbf Z \oplus \mathbf Z$. Consider the homotopy exact sequence of the fibration r in Lemma 3,

$$\cdots \longrightarrow \pi_2(\mathrm{Diff}_0(T)) \longrightarrow \pi_1(\mathscr{I}) \longrightarrow \pi_1(\mathrm{LDiff}_T(\mathscr{F}_R))$$
$$\longrightarrow \pi_1(\mathrm{Diff}_0(T)) \longrightarrow \pi_0(\mathscr{I}) \longrightarrow \pi_0(\mathrm{LDiff}_T(\mathscr{F}_R)) \longrightarrow 0.$$

Taking in considerations that Diff(T) has the homotopy type of a torus [9], and that $LDiff_{\tau}(\mathcal{F}_{R})$ is connected, we can deduce that all the homotopy groups of $LDiff_{\tau}(\mathcal{F}_{R})$ vanish. By lemma 2, all the homotopy groups of $LDiff(S^{3}, \mathcal{F}_{R})$ vanish (i. e., contractible). Hence by lemma 1, $FDiff_{0}(S^{3}, \mathcal{F}_{R})$ is homotopy equivalent to $S^{1} \times S^{1}$.

O. E. D.

Corollary. Let $(S^3, \mathcal{F}_{R'})$ be the codimension one foliation defined

as follows. Decompose $S^3 = S^1 \times D^2 \cup S^1 \times S^1 \times I \cup D^2 \times S^1$, and introduce the Reeb foliation in two solid tori, and the bundle foliation induced by the submersion onto I defined by the projection in $S^1 \times S^1 \times I$. Then $\mathrm{FDiff}_0(S^3, \mathcal{F}_R)$ has the homotopy type of a torus $S^1 \times S^1$.

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