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On the type of graded Cohen-Macaulay rings

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1. Introduction.

In this paper a ring will always mean a commutative noetherian ring with unit.

Let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and K-dim $A=d$. We define $\mathrm{r}(A)=\dim_{A/\mathfrak{m}} \mathrm{Ext}_A^d (A/\mathfrak{m}, A)$ and call it the type of A. Various properties of the type are discussed in [2]. Here we note that if $x_1, \ldots, x_n$ is an A-regular sequence in $\mathfrak{m}$, then $\mathrm{r}(A)=\mathrm{r}(A/(x_1, \ldots, x_n))$. The global type of a Cohen-Macaulay ring A is defined to be the supremum of the types of local rings $A_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of A. A Cohen-Macaulay ring is Gorenstein if and only if the ring has global type one.

Let R be a graded ring. Recently it was proved that R is Cohen-Macaulay if and only if $R_\mathfrak{p}$ is Cohen-Macaulay for every graded prime ideal $\mathfrak{p}$. ([3] and [4])

The aim of this paper is to prove the following

Theorem. *Let $R=\sum_{n\in Z} R_n$ be a commutative graded noetherian ring. If $R_\mathfrak{p}$ is a Cohen-Macaulay local ring of type $\leq r$ for every graded*

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prime ideal $\mathfrak{p}$, *then* R *is a Cohen-Macaulay ring of global type* $\leq r$. *In particular, if* $R_{\mathfrak{p}}$ *is Gorenstein for every graded prime ideal* $\mathfrak{p}$, *then* R *is Gorenstein.*

We shall prove the theorem in more precise form in §3.

The above theorem was independently obtained by the authors.

2. Preliminaries on graded rings.

Let $R=\sum_{n\in\mathbf{Z}} R_n$ be a graded ring. For every ideal $\mathfrak{a}$ of R we denote by $\mathfrak{a}^*$ the ideal generated by all homogeneous elements of $\mathfrak{a}$. Then $\mathfrak{a}^*$ is the largest graded ideal contained in $\mathfrak{a}$. It is obvious that if $\mathfrak{p}$ is prime, then so is $\mathfrak{p}^*$. An ideal $\mathfrak{p}$ of R is said to be H-maximal if it is a maximal element of the set of all proper graded ideals of R. If $\mathfrak{p}$ is H-maximal, then $\mathfrak{p}$ is prime and $R/\mathfrak{p}$ has only two graded ideals (0) and $R/\mathfrak{p}$. Recall:

Lemma 2.1. *If* R *has only two graded ideals* (0) *and* R, *then:*

(1) *If* x *is a non-zero homogeneous element of degree* n, *then* x *is invertible and* x^{-1} *is a homogeneous element of degree* $-n$. *In particular,* R_0 *is a field.*

(2) R *is an integral domain, and* R *is a field if and only if* $R=R_0$.

(3) *If* R *is not a field, we put* $d=\min\{n>0 \mid R_n \neq (0)\}$. *Then:*

(a) $R_n\neq(0)$ *if and only if* $n\in d\mathbf{Z}$.

(b) *If* k *denotes the field* R_0, *every non-zero element* X *of* R_d *is transcendental over* k *and* $R=k[X, X^{-1}]$ *as graded rings. In particular,* R *is a principal ideal domain.*

(c) *Every finitely generated graded* R*-module is free (as a graded* R*-module).*

R is called H-local if R has unique H-maximal ideal. Let S be a multiplicative set of R consisting of homogeneous elements. Then the localization $S^{-1}R$ is a graded ring. $((S^{-1}R)_n=\{r/s \mid r$ is a homogeneous element of R, $s\in S$ and $\deg r=\deg s+n.\}$ for every $n \in \mathbf{Z}$.) If $\mathfrak{p}$ is a prime ideal and if S is the multiplicative set of all homogeneous elements of R not in $\mathfrak{p}$, $S^{-1}R$ is said to be the homogeneous localization of R at $\mathfrak{p}$ and denoted by $R_{(\mathfrak{p})}$. In this case $R_{(\mathfrak{p})}$ is an H-local ring with H-maximal ideal $\mathfrak{p}^*R_{(\mathfrak{p})}$. Hence, if $\mathfrak{p}$ is a non-graded prime ideal, it follows from Lemma 2.1 that $\mathfrak{p}R_{(\mathfrak{p})} / \mathfrak{p}^*R_{(\mathfrak{p})}$ is principal and that there is no prime ideal properly between $\mathfrak{p}^*$ and $\mathfrak{p}$.

Lemma 2.2. ([4] *Lemma 1*). *If* $\mathfrak{p}$ *is a non-graded prime ideal, then* height $\mathfrak{p}$ = height $\mathfrak{p}^* + 1$.

Lemma 2.3. ([4] *Lemma* 2). *Let* $\mathfrak{a}$ *be a graded ideal of* R *and let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *be graded prime ideals which do not contain all elements of* R *of positive degree. If the set of all homogeneous elements of* $\mathfrak{a}$ *is contained in* $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$, *then* $\mathfrak{a}$ *is contained in some* $\mathfrak{p}_i$.

Let M and N be (finitely generated) graded R-modules. Then $\mathrm{Ext}_R^i(M, N)$ is a graded R-module for every $i \geq 0$.

3. Proof of Theorem.

Let $R = \sum_{n \in \mathbf{Z}} R_n$ be a graded ring and let $\mathfrak{p}$ be a non-graded prime ideal of R. We prove the following

Theorem 3.1. *If* $R_{\mathfrak{p}^*}$ *is a Cohen-Macaulay local ring of type* r, *then so is* $R_{\mathfrak{p}}$.

Proof. Considering $R_{(\mathfrak{p})}$ instead of R, we may assume that R is an H-local ring with H-maximal ideal $\mathfrak{p}^*$. Since $R_{\mathfrak{p}^*}$ is Cohen-Macaulay, there is an R-regular sequence $x_1, \ldots, x_n$ (n = height $\mathfrak{p}^*$) in $\mathfrak{p}^*$ such that x_i is homogeneous for every i by virtue of Lemma 2.3. Put $\bar{R} = R / (x_1, \ldots, x_n)$ and $\bar{\mathfrak{p}} = \mathfrak{p} / (x_1, \ldots, x_n)$. Then $\bar{\mathfrak{p}}^* = \mathfrak{p}^* / (x_1, \ldots, x_n)$ and $\bar{R}_{\bar{\mathfrak{p}}^*}$ is a Cohen-Macaulay local ring of type r. In order to prove the theorem, it is sufficient to show that $\bar{R}_{\bar{\mathfrak{p}}}$ is a

Cohen-Macaulay local ring of type r. Hence we may assume that height $\mathfrak{p}^* = 0$. Then height $\mathfrak{p} = 1$ by Lemma 2.2 and R is Cohen-Macaulay because Ass $(R) = \{\mathfrak{p}^*\}$. Since $\mathrm{Ext}_R^1 (R/\mathfrak{p}^*, R)$ is a finitely generated graded $R/\mathfrak{p}^*$-module, $\mathrm{Ext}_R^1 (R/\mathfrak{p}^*, R)$ is a free $R/\mathfrak{p}^*$-module by Lemma 2.1, whence $\mathrm{Ext}_{R_\mathfrak{p}}^1 (R_\mathfrak{p}/\mathfrak{p}^* R_\mathfrak{p}, R_\mathfrak{p})$ is a free $R_\mathfrak{p}/\mathfrak{p}^* R_\mathfrak{p}$-module. Since $R_\mathfrak{p}/\mathfrak{p}^* R_\mathfrak{p}$ is a discrete valuation ring, the equality $\mathrm{r}(R_\mathfrak{p}) = \mathrm{r}(R_{\mathfrak{p}*})$ follows from the following

Lemma 3.2. *Let A be an one dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{p}$ be a prime ideal of A such that $A/\mathfrak{p}$ is a discrete valuation ring. Then the equality $\mathrm{r}(A) = \mathrm{r}(A_\mathfrak{p})$ holds if and only if $\mathrm{Ext}_A^1 (A/\mathfrak{p}, A)$ is a free $A/\mathfrak{p}$-module.*

Proof. By the assumption there is an element x such that $\mathfrak{m} = \mathfrak{p} + xA$. Then $0 \longrightarrow A/\mathfrak{p} \xrightarrow{x} A/\mathfrak{p} \longrightarrow A/\mathfrak{m} \longrightarrow 0$ is an exact sequence. Since $\mathrm{Hom}_A (A/\mathfrak{m}, A) = 0$, we have an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_A (A/\mathfrak{p}, A) \xrightarrow{x} \mathrm{Hom}_A (A/\mathfrak{p}, A) \longrightarrow \mathrm{Ext}_A^1 (A/\mathfrak{m}, A) \longrightarrow \mathrm{Ext}_A^1 (A/\mathfrak{p}, A) \xrightarrow{x} \mathrm{Ext}_A^1 (A/\mathfrak{p}, A).$$

This shows $\mathrm{depth}_{A/\mathfrak{p}} \mathrm{Hom}_A (A/\mathfrak{p}, A) > 0$, whence $\mathrm{Hom}_A (A/\mathfrak{p}, A)$ is a free $A/\mathfrak{p}$-module by [1] Theorem 2.3 (d) because $A/\mathfrak{p}$ is a discrete valuation ring. Therefore we have:

$$\begin{aligned} \mathrm{r}(A_\mathfrak{p}) &= \dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} \mathrm{Hom}_{A_\mathfrak{p}} (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, A_\mathfrak{p}) \\ &= \mathrm{rank}_{A/\mathfrak{p}} \mathrm{Hom}_A (A/\mathfrak{p}, A) \\ &= \dim_{A/\mathfrak{m}} \mathrm{Hom}_A (A/\mathfrak{p}, A) / \mathfrak{m} \mathrm{Hom}_A (A/\mathfrak{p}, A) \\ &= \dim_{A/\mathfrak{m}} \mathrm{Hom}_A (A/\mathfrak{p}, A) / x \mathrm{Hom}_A (A/\mathfrak{p}, A) \\ &\leq \dim_{A/\mathfrak{m}} \mathrm{Ext}_A^1 (A/\mathfrak{m}, A) = \mathrm{r}(A). \end{aligned}$$

Hence $\mathrm{r}(A) = \mathrm{r}(A_\mathfrak{p})$ holds if and only if the map $\mathrm{Hom}_A (A/\mathfrak{p}, A) \longrightarrow \mathrm{Ext}_A^1 (A/\mathfrak{m}, A)$ is surjective, i. e., the map $\mathrm{Ext}_A^1 (A/\mathfrak{p}, A) \xrightarrow{x} \mathrm{Ext}_A^1 (A/\mathfrak{p}, A)$ is injective, which is equivalent to say that $\mathrm{Ext}_A^1 (A/\mathfrak{p}, A)$ is a free $A/\mathfrak{p}$-module because $A/\mathfrak{p}$ is a discrete valuation ring.

Remark 3. 3. Under the same assumption as in Lemma 3. 2, A is Gorenstein if and only if Ext_A^1 $(A/\mathfrak{p}, A)=(0)$.

Remark 3. 4. In proving the Theorem in §1 when $R_n=(0)$ for every $n<0$, by a result in [2], we can reduce to the case where R_0 is a complete local ring, whence a homomorphic image of a regular local ring. In this case, using graded syzygies, the faithful exactness of $-\otimes_R R_M$ (M is unique graded maximal ideal) on the category of graded R-modules and the following lemma, we can prove the Theorem in §1 not considering negative degree. (cf. [3])

Lemma 3. 5. *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{a}$ be an ideal of A. Then the following conditions are equivalent:*

(a) *$A/\mathfrak{a}$ is a Cohen-Macaulay local ring of type r.*

(b) *$\mathfrak{a}$ is perfect and* $\dim_{A/\mathfrak{m}} \mathrm{Tor}_d^A$ $(A/\mathfrak{m}, A/\mathfrak{a})=r$ *where* $d=\mathrm{grade}_A\mathfrak{a}$.

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