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A decomposition of meromorphic differentials and its applications

By

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Introduction

The purpose of the present paper is to give some results concerned with the theory of Abelian differentials on open Riemann surfaces with certain null boundaries. So far almost all the theories of Abelian differentials on open Riemann surfaces have dealt with the meromorphic differentials which are square integrable outside of compact subsets. For instance Riemann-Roch's theorem and Abel's theorem are formulated in terms of those meromorphic differentials and their integrals with certain boundary behaviors. (cf. L. Ahlfors [1], Y. Kusunoki [2] [3], M. Shiba [5] and M. Yoshida [7], etc.)

Recently Y. Sainouchi [4] has introduced some metric conditions on open Riemann surfaces and meromorphic differentials, and succeeded in a systematic treatment of meromorphic differentials with an infinite number of polar singularities under these metric conditions. On the other hand M. Shiba has generalized the notion of the divisors on open Riemann surfaces by making use of the notion of behavior spaces introduced by himself in [5] and proved a duality theorem [6]. This generalized notion of divisors makes possible to deal with certain infinite divisors. However Sainouchi's treatment and Shiba's one for infinite divisors are different and it is desirable to unify two approaches.

In the present paper we give a generalization of the notion of divisors on open Riemann surfaces with certain null boundaries and prove a duality theorem (Theorem I) which includes the Sainouchi's duality theorem and also, in the case of our surfaces, a duality theorem analogous to that of Shiba. We also prove an interpolation theorem for multiplicative meromorphic functions.

§1 contains the preliminary facts and the definition of the generalized divisors. In §2 we define a decomposition of meromorphic differentials. This decomposition will play fundamental roles in §§3 and 4. §3 is devoted to prove a duality theorem. The special cases of this duality theorem will be discussed also in §3. Finally in §4 we shall be concerned with an interpolation theorem from which we derive a theorem of Abel type for our divisors.

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§1. Preliminaries and the definitions of divisors

1.1. *Riemann surfaces with certain null boundaries and elementary differentials.*

Let W be an open Riemann surface of genus g $(1 \leqq g \leqq \infty)$ and $\{W_n\}_{n=1}^{\infty}$ be a canonical exhaustion of W. We denote by $g(n)$ the genus of W_n and $\Xi = \{A_j, B_j\}_{j=1}^{g}$ a canonical homology basis of W whose restriction to $W_{n+1} - \overline{W}_n$ forms a canonical homology basis modulo boundary of $W_{n+1} - \overline{W}_n$. Let $\partial W_n = \bigcup_{i=1}^{l(n)} \gamma_n^i$ be the decomposition of ∂W_n into its connected components. We take a ring domain D_{ni} containing γ_n^i so that $D_{ni} \cap D_{nj} = \emptyset$ for $i \neq j$ and we put $D_n = \bigcup_{i=1}^{l(n)} D_{ni}$. We assume that $D_n \cap D_m = \emptyset$ for $n \neq m$. Let υ_n^i and υ_n be the harmonic moduli of D_{ni} and D_n respectively, that is to say,

$$\upsilon_n = \frac{2\pi}{d_n} \qquad d_n = \int_{(\partial D_n) \cap W_n} {}^*du,$$

where u_n is the harmonic function on D_n such that it vanishes identically on $(\partial D_n) \cap W_n$ and is identically equal to 1 on $\partial D_n - (\partial D_n) \cap W_n$. v_n^i is defined analogously. We put $D = \bigcup_{n=1}^{\infty} D_n$ and define a function u on D so that

$$u = \sum_{i=1}^{n-1} v_i + v_n u_n \quad \text{on} \quad D_n.$$

If we denote by v the conjugate harmonic function of u, then $u + iv$ maps conformally the domain D with suitable slits onto the domain $D' = \{(u, v) | 0 < u < R = \sum_{i=1}^{\infty} v_i,\ 0 < v < 2\pi\}$ with suitable slits. In the following we only consider an open Riemann surface satisfying the condition

$$\text{(A)} \qquad \inf_{n} \min_{1 \leqq i \leqq l(n)} v_n^i > 0$$

for suitable choices of $\{W_n\}_{n=1}^{\infty}$ and $\{D_{ni}\}$. On such a surface we fix $\{W_n\}_{n=1}^{\infty}$ and $\{D_{ni}\}$ satisfying the condition (A). There are elementary differentials with the following properties uniquely on such a surface, Sainouchi [4], § 1:

(I) dw_j $(1 \leqq j \leqq g)$: the square integrable semi-exact holomorphic differential such that

$$\int_{A_i} dw_j = \delta_{ij}, \qquad i = 1, 2, \dots .$$

(II) $dY_{p,n}$ $(n \geqq 1)$: the semi-exact meromorphic differential, holomorphic except at p and square integrable outside each neighborhood of p, such that

$$\text{(1)} \qquad \int_{A_j} dY_{p,n} = 0 \quad \text{with} \quad 1 \leqq j \leqq g,$$

$$\text{(2)} \qquad dY_{p,n} = \left(-\frac{n}{z^{n+1}} + reg.\ term \right) dz$$

for some local coordinate z about p $(z = 0 \leftrightarrow p)$.

(III) $d\Pi_{p,q}$: the meromorphic differential, holomorphic except at p and q, where $d\Pi_{p,q}$ has simple poles with residues $+1$ at p and -1 at q respectively. Furthermore $d\Pi_{p,q}$ is square integrable outside each neighborhood of $\{p\} \cup \{q\}$, semi-exact in $W - C$, where C is a path from

p to q, and

$$\int_{A_j} d\Pi_{p,q}=0 \quad \text{with} \quad 1\leqq j\leqq g.$$

There hold some relations between the above differentials, Sainouchi [4], propositions 2, 3 and 4:

Proposition 1.1.

(1)
$$\int_{B_k} d\Pi_{p,q}=-2\pi i\int_q^p dw_k,$$

where the path of integration from q *to* p *is chosen in* $W_0=W-\Xi$.

(2)
$$\int_{B_k} dY_{p,n}=\frac{2\pi i}{(n-1)!}w_k^{(n)}(P),$$

(3)
$$\int_q^p d\Pi_{r,s}=\int_s^r d\Pi_{p,q},$$

where the two paths of integration are chosen in W_0 *and do not intersect with each other.*

(4) *If we put*

$$Y_{p,n}^{s,t}=Y_{p,n}(s)-Y_{p,n}(t)=\int_t^s dY_{p,n},$$

where the path of integration is chosen in $W_0-\{p\}$, *then*

$$\frac{1}{(n-1)!}\frac{\partial^n \Pi_{s,t}^{p,q}}{\partial p^n}=\int_s^t dY_{p,n},$$

$$\frac{1}{(m-1)!}\frac{d^m Y_{p,n}}{dq^m}(q)=\frac{1}{(n-1)!}\frac{d^n Y_{q,m}}{dp^n}(p).$$

From [4], corollary 1 to theorem 4 we have also the following lemma.

Lemma 1.1. *If* dv *is a semi-exact holomorphic differential on* W *such that it is square integrable and has no non zero* A*-periods, then* dv *vanishes identically.*

1.2. *Definitions of divisors.* Let W^* be the Kerékjártó — Stoïlow compactification of W and we put $\partial W = W^* - W$. We denote by P a regular partition of ∂W. Since a regular partition P is induced by a consistent system $\{P_n\}$ of partitions $P_n(W_n)$ of $W - W_n$, P induces a partition of ∂W_n for each n (cf. Ahlfors — Sario [1], chap. 1, §6). Let $\partial W_n = \bigcup_{j=1}^{k(n)} \beta_{nj}$ be a partition of ∂W_n induced by P. In particular we denote by Q the canonical partition of ∂W and the induced partition of ∂W_n by Q is assumed to be the partition of ∂W_n into its connected components. Further we denote by $\varepsilon(W)$ the set of all canonical ends, that is, the set of complements of closures of canonical regions of W. We associate with P and $U \in \varepsilon(W)$ a complex vector space denoted by $m(P, U)$ such that each element ϕ of $m(P, U)$ is a meromorphic differential defined on U and satisfies the period conditions

$$\int_{\beta_{nj}} \phi = 0 \quad \text{with} \quad 1 \leqq j \leqq k(n)$$

if ∂W_n is contained in U. We put

$$A_0(P, U) = \{\phi \in m(P, U) | \text{(i) } \phi \text{ is holomorphic (ii) } \phi \text{ has no non-zero } A\text{-periods}\},$$

$$A_d(P, U) = \{\phi \in A_0(P, U) | \, \|\phi\|_U < \infty\},$$

where $\|\phi\|_U$ is the Dirichlet norm of ϕ on U. $\{m(P, U)\}_{U \in \varepsilon(W)}$, $\{A_0(P, U)\}_{U \in \varepsilon(W)}$ and $\{A_d(P, U)\}_{U \in \varepsilon(W)}$ becomes inductive systems in an obvious manner. We put their inductive limits

$$(P)m = \varinjlim m(P, U), \quad (P)A_0 = \varinjlim A_0(P, U), \quad (P)A_d = \varinjlim A_d(P, U).$$

In particular we put $(Q)A_d = A_d$. Then evidently $(P)m \supset (P)A_0 \supset A_d$. $\mathscr{D}_P = (P)m/A_d$ and $\mathscr{D}'_P = (P)A_0/A_d$ are complex vector spaces and $\mathscr{D}'_P$ is a subspace of $\mathscr{D}_P$.

Let V be a subspace of $\mathscr{D}'_P$ and put $\mathscr{D}_P/V = \mathscr{D}_P(V)$. Let

$$\eta_P\colon (P)m \longrightarrow \mathscr{D}_P, \quad \eta_P^V\colon \mathscr{D}_P \longrightarrow \mathscr{D}_P(V)$$

be the respective natural mapping. To each element ϕ of $m(P, U)$

there corresponds an element $\tilde{\phi}$ of $(P)m$, and hence an element $\eta_P^V \circ \eta_P(\tilde{\phi})$ of $\mathscr{D}_P(V)$. For simplicity we put $\eta_P^V \circ \eta_P(\tilde{\phi}) = <\phi>_V$. Since V is a subspace of $\mathscr{D}'_P$, there is a subspace $\tilde{V}$ of $(P)A_0$ such that $\tilde{V}$ contains A_d and $V = \tilde{V}/A_d$. It is easy to see that $\phi_1 \in m(P, U_1)$ and $\phi_2 \in m(P, U_2)$ determine the same element $<\phi_1>_V = <\phi_2>_V$ if and only if there are a suitable canonical end U, an element $d\tilde{v}$ of $A_0(P, U)$ which determines an element of $\tilde{V}$, and an element λ of $A_d(Q, U)$ such that

$$\phi_1 = \phi_2 + d\tilde{v} + \lambda \qquad \text{on} \quad U.$$

From now on we represent elements of $(P)m$, $(P)A_0$ and A_d by their representatives.

Definition 1.1. We call a subspace V of $\mathscr{D}'_P$ a (P)-*divisor at boundary*. An element σ of $\mathscr{D}_P(V)$ is called a P_V-*singularity* if and only if there exist a $U \in \varepsilon(W)$ and $\phi \in m(Q, U)$ such that $\|\phi\|_{(\cup D_n) \cap U} < \infty$ and $\sigma = <\phi>_V$. The subspace of $\mathscr{D}_P(V)$ consisting of all P_V-singularities is called the space of P_V-singularities and denoted by $\mathscr{S}(P_V)$. To distinguish a (P)-divisor at boundary and a usual divisor (a finite or infinite linear combination of points of W with integer coefficients), we call a usual divisor an inner divisor. The inner divisor δ we shall consider in the following has the support $|\delta|$ contained in $W_0 - D = W - \Xi - \bigcup_{n=1}^{\infty} D_n$.

Definition 1.2. Let δ be an inner divisor and ∂ be a subspace of $\mathscr{S}(P_V)$. We call the couple $\Delta = (\delta, \partial)$ a P_V-*divisor*. A multiplicative meromorphic function f is said to be *multiple* of $\Delta = (\delta, \partial)$ if and only if $(f) \geqq \delta$ and $<df>_V \in \partial$. A meromorphic differential ϕ is said to be a multiple of $\Delta = (\delta, \partial)$ if and only if $(\phi) \geqq \delta$ and $<\phi>_V \in \partial$. We use the notation $\Delta | f$ to show that f is a multiple of Δ. $\Delta | \phi$ also means that ϕ is a multiple of Δ. If δ is a positive inner divisor and f is an additive meromorphic function, then f is said to be a multiple of $\Delta = (-\delta, \partial)$ if and only if $(f) \geqq -\delta$ and $<df>_V \in \partial$. We denote by $\Delta | f$ this relation.

Let V be a (P)-divisor at boundary and dv be an element of V.

Let q be a point of ∂W. If there is an open subset F of W^* containing q and dv has a representative $d\tilde{v}$ in $\tilde{V}$, i.e. $\eta_P(d\tilde{v})=dv$, such that the restriction $d\tilde{v}|F\cap U$ of $d\tilde{v}$ is semi-exact and square integrable, then we say that dv is regular at q. Here we have assumed that $d\tilde{v}$ belongs to $A_0(P, U)$. This definition does not depend on a choice of a representative $d\tilde{v}$. Indeed if $d\tilde{v}'$ is another representative of dv in $\tilde{V}$, then there are a suitable canonical end $U'\subset U$ and an element λ of $A_d(Q, U')$ such that $d\tilde{v}'=d\tilde{v}+\lambda$ on U'. Thus $d\tilde{v}'$ is semi-exact on $U'\cap F$ and

$$\|d\tilde{v}'\|_{U'\cap F}\leqq\|d\tilde{v}\|_{U\cap F}+\|\lambda\|_{U'\cap F}<\infty.$$

The support of dv is, by definition, the set

$$S(dv)=\{q\in\partial W|\, dv \text{ is not regular}\}.$$

This is a closed subset of W.

Definition 1.3. $\operatorname{Supp.} V=\overline{\bigcup_{dv\in V} S(dv)}$,
where the closure is considered in W^*.

Since ∂W is closed in W^* and $S(dv)$ is a subset of ∂W, Supp. V is contained in ∂W. For a subset B of W^* $\bar{B}$ means the closure of B and $Int\, B$ the interior of B.

Proposition 1.2. *Let V be a (P)-divisor at boundary and E be a closed subset of ∂W such that $E\cap \operatorname{Supp.} V=\emptyset$. Then for a given representative $d\tilde{v}$ in $\tilde{V}$ of $dv\in V$, there exists an integer n_0 with the following property: Suppose $\{W_n\}_{n=1}^{\infty}$ is a canonical exhaustion of W and $W-W_n=\bigcup_{i=1}^{l(n)} U_i^{(n)}$ is the decomposition of $W-W_n$ into its connected components. Let $U^{(n)}$ be the union of $U_j^{(n)}$ with $\overline{U_j^{(n)}}\cap E\neq\emptyset$, then $d\tilde{v}|U^{(n)}$ is semi-exact and square integrable for $n\geqq n_0$.*

Proof. Let q be a point of E. Then $q\notin \operatorname{Supp.} V=\overline{\bigcup_{dv\in V} S(dv)}$ and this means that $q\notin S(dv)$ for all dv in V. Therefore for a given dv there exist an open subset F_q of W^* and a representative $d\tilde{v}$ in $\tilde{V}$ of dv such that $d\tilde{v}|F_q\cap U$ is semi-exact and square integrable, where

$d\tilde{v}$ is assumed to belong to $A_0(P, U)$. We may assume that $F_q \subset U$ and thus $F_q \cap U = F_q$. The set $F = \bigcup_{q \in E} F_q$ is an open neighborhood of E. Let q be a given point of E. To each positive integer n there is a component $U_j^{(n)}$ of $W - W_n$ such that q is contained in $\overline{U_j^{(n)}}$. We put $U_j^{(n)} = U^{(n)}(q)$. Then evidently $U^{(n)}(q) \subset U^{(m)}(q)$ for $n \geqq m$. Thus $\{U^{(n)}(q)\}$ determines a boundary component which defines the point q. Hence there is an integer $n(q)$ such that $U^{(n)}(q) \subset F$ if $n > n(q)$. $\{Int\, \overline{U^{(n(q))}(q)}\}_{q \in E}$ is an open covering of E. Since E is compact there is a finite number of points $q_1, \dots, q_s$ of E such that $\{Int\, \overline{U^{(ni)}}\}_{i=1}^{s}$ is an open covering of E, where we put $U^{(ni)} = U^{(n(q_i))}(q_i)$. Now let n_0 be $\max_{1 \leqq i \leqq s} n(q_i)$. If $n > n_0$ and $Int\, \overline{U_j^{(n)}} \cap E \neq \emptyset$, then there is an i with $1 \leqq i \leqq s$ such that $Int\, \overline{U_j^{(n)}} \cap Int\, \overline{U^{(ni)}} \neq \emptyset$, and thus we have $U_j^{(n)} \subset U^{(ni)}$. Therefore $\cup\, U_j^{(n)} = U^{(n)}$ is contained in F. Since $U_j^{(n)} \subset U^{(ni)}$ we may assume that $U_j^{(n)}$ is contained in F_{q_i}. Thus $d\tilde{v}|U_j^{(n)}$ is semi-exact and square integrable, and so is $d\tilde{v}|U^{(n)}$. q.e.d.

§2. A decomposition of meromorphic differentials

In this section we fix an open Riemann surface W satisfying the condition (A).

2.1. *Some lemmas.* The following lemma 2.1 is easily proved by (4) of proposition 1.1.

Lemma 2.1. (1) *If we put*

$$h(p, q) = \frac{dY_{p,1}}{dq}(q) = \frac{dY_{q,1}}{dp}(p),$$

then $h(p, q) = h(q, p)$ *and* $h(p, q)dpdq$ *is a double differential.* $h(p, q)dp$ *(resp.* $h(p, q)dq$*) has a finite norm outside of each neighborhood of* q *(resp.* p*).*

(2) $$d\Pi_{s,t}^{p,q} = \left(\frac{\partial}{\partial p}\Pi_{s,t}^{p,q}\right)dp = \left\{\int_s^t h(r, p)dr\right\}dp,$$

where the path of integration is chosen in $W_0 - \{p\}$.

Lemma 2.2. *Let ϕ and ψ be meromorphic differentials defined in a neighborhood of ∂W such that their poles do not lie in $D=\bigcup_{n=1}^{\infty} D_n$ and*

$$\int_{\gamma_n^i}\phi=0, \quad \int_{\gamma_n^i}\psi=0$$

for all γ_n^i contained in the common domain of ϕ and ψ. If D_n is contained in the common domain of ϕ and ψ, then we have an inequality

$$\left|\int_{\partial W_n}\left(\int\phi\right)\psi\right| \leqq 2\pi\frac{\|\phi\|_{D_n}\|\psi\|_{D_n}}{\min\limits_{1\leqq i\leqq l(n)} v_n^i} \tag{2.1}$$

The proof of lemma 2.2 is contained in the proof of [4], lemma 3. Next we show a similar inequality as in lemma 2.2. We put $\partial D_n \cap W_n=\partial D_n^{(i)}$ and $\partial D_n^{(e)}=\partial D_n-\partial D_n^{(i)}$.

Lemma 2.3. *Under the same conditions as in lemma 2.2, we have*

$$\left|\int_{\partial D_n^{(e)}}\left(\int\phi\right)\bar{\psi}\right| \leqq 2\pi\frac{\|\phi\|_{D_n}\|\psi\|_{D_n}}{\min\limits_{1\leqq i\leqq l(n)} v_n^i}+\|\phi\|_{D_n}\|\psi\|_{D_n}.$$

Proof. Let u be the harmonic function on D_n such that $u|\partial D_n^{(i)}=0$ and $u|\partial D_n^{(e)}=v_n$. We denote by v the conjugate harmonic function of u. The function $u+iv$ maps D_n with slits conformally onto the plane domain $\{(u, v)|0<u<v_n, 0<v<2\pi\}$ with slits. We put $C(r)=\{p\in D_n|u(p)=r\}$ for r with $0<r<v_n$. $C(r)$ is a union of closed curves in D_n and the component of $C(r)$ contained in D_{n_i} is homologous to γ_n^i. The part $D_n(r)$ which is surrounded by $\partial D_n^{(e)}$ and $C(r)$ is a union of ring domains in D_n. By the Stokes' theorem

$$\int_{\partial D_n^{(e)}}\left(\int\phi\right)\bar{\psi}=\int_{C(r)}\left(\int\phi\right)\bar{\psi}+\int_{D_n(r)}\phi\bar{\psi}.$$

$$\left|\int_{\partial D_n^{(e)}}\left(\int\phi\right)\bar{\psi}\right| \leqq \left|\int_{C(r)}\left(\int\phi\right)\bar{\psi}\right|+\left|\int_{D_n(r)}\phi\bar{\psi}\right|$$

$$\leqq \left|\int_{C(r)}\left(\int\phi\right)\bar{\psi}\right| + \|\phi\|_{D_n(r)}\|\psi\|_{D_n(r)}$$

$$\leqq \left|\int_{C(r)}\left(\int\phi\right)\bar{\psi}\right| + \|\phi\|_{D_n}\|\psi\|_{D_n}$$

In exactly the same way as the inequality (2.1) we can show that

$$\inf_r \left|\int_{C(r)}\left(\int\phi\right)\bar{\psi}\right| \leqq 2\pi\frac{\|\phi\|_{D_n}\|\psi\|_{D_n}}{\min\limits_{1\leqq i\leqq 1(n)} v_n^i}.$$

This completes a proof.

Lemma 2.4. *Let Ω be a relatively compact subregion of W with $\Omega = W_{n_0}$ for some n_0. Then*

$$\|d\Pi_{p,q}\|^2_{W-\Omega} = \frac{i}{2}\int_{W-\Omega} |\partial\Pi^{s,t}_{p,q}/\partial s|^2 ds\overline{ds}$$

is continuous function on $(\Omega \cap W_0)\times(\Omega \cap W_0)$. Moreover if we put

$$H_\Omega(p) = \frac{i}{2}\int_{W-\Omega} |h(p,q)|^2 dq\overline{dq}$$

then $\sqrt{H_\Omega(p)}\,|dp|$ is a continuous invariant form on Ω.

Proof. Since proofs for the cases of $d\Pi_{p,q}$ and $H_\Omega(p)|dp|$ are same, we give a proof for the latter. We put $\tilde{W}_n = W_n \cup D_n$. The boundary of $\tilde{W}_n$ is $\partial D_n^{(e)}$. If n is sufficiently large so that $W_n \supset \Omega$, then by the Stokes' theorem

$$\frac{i}{2}\int_{\tilde{W}_n-\Omega} |h(p,q)|^2 dq\overline{dq} = \frac{i}{2}\int_{\partial D_n^{(e)}}\left(\int h(p,q)dq\right)\overline{h(p,q)dq}$$

$$-\frac{i}{2}\int_{\partial\Omega}\left(\int h(p,q)dq\right)\overline{h(p,q)dq}.$$

By lemma 2.3 the first term of the right hand side tends to 0 for $n\to\infty$. Thus we see

$$H_\Omega(p) = -\frac{i}{2}\int_{\partial\Omega}\left(\int h(p,q)dq\right)\overline{h(p,q)dq}.$$

Since the integrand is continuous with respect to p and $\partial\Omega$ is compact we see that $H_\Omega(p)$ is continuous with respect to p. This completes a proof.

Lemma 2.5. *Let γ be a dividing cycle and σ be a continuous differential form on γ. Then for a fixed branch of $\Pi_{s,t}^{p;q}$ on γ*

$$dF(t)=d_t\int_\gamma \Pi_{s,t}^{p;q}\sigma(p)$$

is a differential on $W-\gamma$ and has a finite norm outside of a sufficiently large compact subset of W, where d_t means the exterior differential operator with respect to t. Here the points q and s are assumed to be fixed in W_1 and do not lie on γ.

Proof. We choose W_n so large that W_n contains γ. We show that the norm of $dF(t)$ on $U=W-\bar{W}_n$ is finite. We divide γ into small arcs γ_i $(1\leqq i\leqq l)$ so that each γ_i is contained in a coordinate neighborhood. If we put

$$dF_i(t)=d_t\int_{\gamma_i}\Pi_{s,t}^{p;q}\sigma(p),$$

then we have $dF(t)=\sum_{i=1}^{l} dF_i(t)$. Hence it suffices to show that each $dF_i(t)$ is of finite norm on U. Since $\gamma_i\cap\bar{U}=\emptyset$ we obtain

$$dF_i(t)=\left\{\int_{\gamma_i}(\partial\Pi_{s,t}^{p;q}/\partial t)\sigma(p)\right\}dt=\left\{\int_{\gamma_i}\sigma(p)\int_q^p h(r,\ t)dr\right\}dt,$$

where the second equality holds by lemma 2.1. Let $f(r)=\int^r\sigma(p)$ and p_i, q_i the end points of γ_i, then by the integration by parts

$$dF_i(t)=-\left\{\int_{\gamma_i}f(p)h(p,\ t)dp\right\}dt+\left\{f(p_i)\int_q^{p_i}h(r,\ t)dr-\right.$$

$$\left.-f(q_i)\int_q^{q_i}h(r,\ t)dr\right\}dt.$$

We have only to show that the first term on the right hand side is of finite norm on U, since we can show in exactly the same way that the second term is so. We put

$$dG_i(t)=\left\{\int_{\gamma_i} f(p)h(p,\ t)dp\right\}dt.$$

We may assume that γ_i is a plane curve. By definition of the integral we may write as

$$dG_i(t)=\{\lim_{m\to\infty}\sum_{\Delta} f(z'_j)h(z_j,\ t)(z_j-z_{j-1})\}dt,$$

where Δ is a division of γ_i by points z_j, $1\leqq j\leqq m$, z'_j being a point on γ_i between z_j and z_{j-1}. Let $H_U(z_j,\ z_k)$ be the inner product of $h(z_j,\ t)dt$ and $h(z_k,\ t)dt$ on U, then by the Schwarz's inequality we obtain

$$|H_U(z_j,\ z_k)|\leqq\sqrt{H_U(z_j,\ z_j)}\sqrt{H_U(z_k,\ z_k)}.$$

We put $H_U(z_j,\ z_j)=H_{W_n}(z_j)=H_n(z_j)$ for simplicity. Now we put

$$dG_{i\Delta}(t)=\sum_{i=1}^{m} f(z'_j)h(z'_j,\ t)(z_j-z_{j-1})dt$$

and $t=\xi+i\eta$. Then

$$\int_U|G'_{i\Delta}(t)|^2d\xi d\eta=\sum_{j=1}^{m}\sum_{k=1}^{m} f(z'_j)\overline{f(z'_k)}H_U(z'_j,\ z'_k)(z_j-z_{j-1})\overline{(z_k-z_{k-1})}$$

$$\leqq\sum_{j=1}^{m}\sum_{k=1}^{m}|f(z'_j)|\,|f(z'_k)|\sqrt{H_n(z'_j)}\sqrt{H_n(z'_k)}|z_j-z_{j-1}|\,|z_k-z_{k-1}|$$

$$=(\sum_{j=1}^{m}|f(z'_j)|\sqrt{H_n(z'_j)}\,|z_j-z_{j-1}|)^2.$$

Thus by the Fatou's lemma

$$\int_U|G'_i(t)|^2d\xi d\eta\leqq\varliminf_{\Delta}\int_U|G'_{i\Delta}(t)|^2d\xi d\eta$$

$$\leqq\left(\int_{\gamma_i}|f(p)|\sqrt{H_n(p)}\,|dp|\right)^2.$$

Since $\sqrt{H_n(p)}\,|dp|$ is a continuous invariant form by lemma 2.4

$\int_{\gamma_i} |f(p)| \sqrt{H_n(p)}\,|dp|$ is finite. Hence the norm of $dG_i(t)$ on U is finite and so is the norm of $dF_i(t)$. q.e.d.

2.2. *A decomposition of meromorphic differentials.* In this section we define a decomposition of meromorphic differentials and this decomposition will play important roles in §§ 3 and 4.

Proposition 2.1. *Let dv be an element of $\mathscr{D}'_P$ with a representative $d\tilde{v}$ in $A_0(P, U)$. Let fix q in W_1, then*

$$i(d\tilde{v}) = -\frac{1}{2\pi i}\lim_{n\to\infty} d_p \int_{\partial W_n} \Pi^{s,t}_{p,q}\, d\tilde{v}(s),$$

is a meromorphic differential on W and is also a representative of dv, that is, $\eta_P(i(d\tilde{v})) = dv$. Here $\Pi^{s,t}_{p,q}$ is a single valued branch of the integral of $d\Pi^{s,t}_{p,q}$ on $W_0 - C$, where C is a path from q to p in W_0.

Proof. First we remark that for n such that $\partial W_n \subset U$ $\int_{\partial W_n} \Pi^{s,t}_{p,q}\, d\tilde{v}(s)$ does not depend on the point t. For let t' be another point of $W_0 - \cup D_n$. Then the difference $d\Pi^{s,t}_{p,q} - d\Pi^{s,t'}_{p,q}$ is a holomorphic square integrable differential on W such that all of its A-periods are zero. By lemma 1.1 it vanishes identically and therefore $\Pi^{s,t}_{p,q} - \Pi^{s,t'}_{p,q}$ does not depend on s. Hence

$$\int_{\partial W_n} \Pi^{s,t}_{p,q}\, d\tilde{v}(s) = \int_{\partial W_n} \Pi^{s,t'}_{p,q}\, d\tilde{v}(s) + \int_{\partial W_n} (\Pi^{s,t}_{p,q} - \Pi^{s,t'}_{p,q})\, d\tilde{v}(s)$$

$$= \int_{\partial W_n} \Pi^{s,t'}_{p,q}\, d\tilde{v}(s).$$

Choose a sufficiently large n_0 so that ∂W_n is contained in U for $n \geqq n_0$. Then it is easy to see that

$$-\frac{1}{2\pi i}\lim_{n\to\infty} d_p \int_{\partial W_n} \Pi^{s,t}_{p,q}\, d\tilde{v}(s) =$$

$$-\frac{1}{2\pi i} d_p \int_{\partial W_{n_0}} \Pi^{s,t}_{p,q}\, d\tilde{v}(s) \qquad\qquad p \in W_{n_0}$$

$$-\frac{1}{2\pi i}d_p\int_{\partial W_{n_0}}\Pi^{s,t}_{p,q}d\tilde{v}(s)+d\tilde{v}(p)\qquad p\notin W_{n_0}$$

Since $\Pi^{s,t}_{p,q}$ has no non-zero periods along each dividing cycle in $W-C'$, where C' is a path from s to t in W, and also has no non-zero A-periods, we see

$$\int_{\gamma_{nl}}\frac{1}{2\pi i}d_p\int_{\partial W_{n_0}}\Pi^{s,t}_{p,q}d\tilde{v}(s)=0,\qquad \int_{A_l}\frac{1}{2\pi i}d_p\int_{\partial W_{n_0}}\Pi^{s,t}_{p,q}d\tilde{v}(s)=0$$

for $n>n_0$ and for all l. Therefore

$$\int_{\beta_{nj}}i(d\tilde{v})=0\quad\text{and}\quad\int_{A_l}i(d\tilde{v})=0$$

for $n>n_0$ and for all l. This means that $i(d\tilde{v})\in A_0(P, W-\overline{W}_{n_0})$. The differential $\frac{1}{2\pi i}d_p\int_{\partial W_{n_0}}\Pi^{s,t}_{p,q}d\tilde{v}(s)$ is of finite norm on a neighborhood of ∂W by lemma 2.5. Hence the above differential belongs to $A_d(Q, U')$ for some canonical end $U'\subset W-W_{n_0}$. Since

$$i(d\tilde{v})=d\tilde{v}+\Big(-\frac{1}{2\pi i}d_p\int_{\partial W_{n_0}}\Pi^{s,t}_{q,p}d\tilde{v}(s)\Big)$$

outside of W_n, $i(d\tilde{v})\equiv d\tilde{v}\bmod A_d$. Hence $\eta_P(i(d\tilde{v}))=dv$. q.e.d.

Proposition 2.2. *Let η be a meromorphic differential on W and assume that η is represented as*

$$\eta=\sigma+\phi+\lambda$$

on a canonical end U, where $\sigma\in m(Q, U)$, $\phi\in A_0(P, U)$ and $\lambda\in A_d(Q, U)$. We put $(\eta)=\delta_1-\delta_2$, where δ_1 and δ_2 are positive inner divisors, and put $\delta_2(n)=\sum_{j=1}^{k_2(n)}n_jq_j=\delta_2|W_n$. If the singular part of η is $\sum_{k=1}^{n_j}\frac{b_{jk}}{z^k}dz$ at q_j then,

$$H_n(p)=\sum_{i=2}^{k_2(n)}b_{i1}\Pi^{p,q}_{q_i,1}-\sum_{i=1}^{k_2(n)}\sum_{k=2}^{n_i}\frac{b_{ik}}{k-1}Y^{p,q}_{q_i,k-1}+\sum_{j=1}^{g(n)}\Big(\int_{A_j}\eta\Big)\int_q^p dw_j$$

$$-\frac{1}{2\pi i}\int_{\partial W_n}\Pi^{s,t}_{p,q}\sigma(s)$$

converges to a (multi-valued) meromorphic function H uniformly on compact subsets of $W_0-|\delta_2|$ and

$$\eta(p)=d_p\Big(\lim_{n\to\infty}H_n(p)-\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial W_n}\Pi^{s,t}_{p,q}\phi(s)\Big) \tag{2.2}$$

$$=\lim_{n\to\infty}dH_n(p)-\lim_{n\to\infty}\frac{1}{2\pi i}d_p\int_{\partial W_n}\Pi^{s,t}_{p,q}\phi(s).$$

Proof. We remark at first that

$$\sum_{i=1}^{k_2(n)}b_{i1}=0 \tag{2.3}$$

for sufficiently large n. Let p_0 be an arbitrary point of W_0 and and K be a relatively compact simply connected neighborhood of p_0 in W_0 such that $K\cap|\delta_2|=\emptyset$. It is sufficient to show the uniform convergence of $H_n(p)$ on such K. We choose n_1 such that $K\subset W_{n_1}$. Let Ξ_n denote the restriction of Ξ to W_n. For a fixed q in $W_1-\Xi_1-|\delta_2(1)|$ we take a path ρ from q to p_0 and a narrow strip $K'\supset\rho$ such that $K\cup K'$ is simply connected domain in $W_{n_1}-\Xi_{n_1}-|\delta_2(n_1)|$. We denote by U_j a simply connected neighborhood of $\{q_j\}\cup\{q_1\}$ such that $U_j\cap(K\cup K')=\emptyset$ and $U_j\subset W_n$ for $q_j\in W_n$, where we have assumed $q_1\in W_1-\Xi_1$. Then by the Stokes' theorem

$$0=(d\Pi_{p,q},\overline{^*\eta})_{W_n-(K\cup K')-\bigcup_{i=1}^{k_2(n)}U_j}$$

$$=-\int_{\partial W_n}\Pi^{s,t}_{p,q}\eta+\int_{\partial(\bigcup_{i=2}^{k_2(n)}U_i)}\Pi^{s,t}_{p,q}\eta+\int_{\partial(K\cup K')}\Pi^{s,t}_{p,q}\eta+$$

$$+\sum_{j=1}^{g(n)}\Big(\int_{A_j}d\Pi^{s,t}_{p,q}\int_{B_j}\eta-\int_{A_j}\eta\int_{B_j}d\Pi^{s,t}_{p,q}\Big).$$

If we use (2.3) and proposition 1.1, then we see

$$\int_{\partial(\bigcup_{i=2}^{k_2(n)}U_i)}\Pi^{s,t}_{p,q}\eta(s)=2\pi i\sum_{i=2}^{k_2(n)}b_{i1}\Pi^{p,q}_{q_i,q_1}-$$

$$-2\pi i\sum_{i=1}^{k_2(n)}\sum_{k=2}^{n_i}\frac{b_{ik}}{k-1}Y^{p,q}_{q_i,k-1}$$

On the other hand

$$\int_{\partial(K\cup K')} \Pi^{s,t}_{p,q}\eta(s) = -2\pi i\int_q^p \eta(s), \quad \int_{B_j} d\Pi^{s,t}_{p,q} = -2\pi i\int_q^p dw_j.$$

Therefore we have

$$\int_q^p \eta(s) = H_n(p) - \frac{1}{2\pi i}\int_{\partial W_n} \Pi^{s,t}_{p,q}\phi(s) - \frac{1}{2\pi i}\int_{\partial W_n} \Pi^{s,t}_{p,q}\lambda(s)$$

for each sufficiently large n. By lemma 2.2

$$\left|\int_{\partial W_n} \Pi^{s,t}_{p,q}\lambda(s)\right| \leqq \frac{2\pi}{\min\limits_{1\leqq i\leqq 1(n)} v^i_n}\|d\Pi_{p,q}\|_{D_n}\|\lambda\|_{D_n}.$$

Since $p\in K\subset W_{n_1}$, there is an n_0 such that

$$\left|\int_{\partial W_n} \Pi^{s,t}_{p,q}\lambda(s)\right| \leqq \frac{2\pi}{\min\limits_{1\leqq i\leqq 1(n)} v^i_n}\|d\Pi_{p,q}\|_{W-W_{n-1}}\|\lambda\|_{\cup D_n}$$

for each integer $n\geqq n_0$. Since $\|d\Pi_{p,q}\|_{W-W_{n-1}}$ is continuous with respect to p on K by lemma 2.4 and $\lim\limits_{n\to\infty}\|d\Pi_{p,q}\|_{W-W_{n-1}}=0$, for a given $\varepsilon'>0$ there is an integer n'_0 such that

$$\|d\Pi_{p,q}\|_{D_n} < \|d\Pi_{p,q}\|_{W-W_{n-1}} < \varepsilon'$$

for each $n\geqq n'_0$ and for each $p\in K$. Since $d\Pi_{p,q}$ and ϕ have no non-zero A-periods on their respective domains, it is easy to see that there is an integer n''_0 such that

$$\int_{\partial W_n} \Pi^{s,t}_{p,q}\phi(s) = \int_{\partial W_m} \Pi^{s,t}_{p,q}\phi(s)$$

for integers $n>m\geqq n''_0$. Therefore for a given $\varepsilon>0$,

$$\left|\int_q^p \eta(s) + \frac{1}{2\pi i}\int_{\partial W_n} \Pi^{s,t}_{p,q}\phi(s) - H_n(p)\right| < \varepsilon$$

for each $n>\max(n_0, n'_0, n''_0)$ and $p\in K$. This completes a proof.

Corollary. *Let $\Delta=(\delta, \partial)$ be a P_Y-divisor on W and η be a mero-*

morphic differential such that $\Delta|\eta$. *If* η *has two representations*

$$\eta=\tau_1+\phi_1+\lambda_1=\tau_2+\phi_2+\lambda_2$$

on a canonical end U, *where* $\tau_i\in m(Q, U)$ *with* $\|\tau_i\|_{U\cap(\cup D_n)}<\infty$, $\phi_i\in A_0(P, U)$ *and* $\lambda_i\in A_d(Q, U)$, *then*

$$H_{n_i}(p)=\sum_{i=2}^{k_2(n)} b_{i1}\Pi^{p,q}_{q_i,q_1}-\sum_{i=1}^{k_2(n)}\sum_{k=2}^{n_i}\frac{b_{ik}}{k-1}Y^{p,q}_{q_i,k-1}+\sum_{j=1}^{g(n)}\left(\int_{A_j}\eta\right)\int_q^p dw_j$$

$$-\frac{1}{2\pi i}\int_{\partial W_n}\Pi^{s,t}_{p,q}\tau_i(s),\qquad i=1,2,$$

tend to the same limit.

Proof. Since $\|\tau_i\|_{U\cap(\cup D_n)}<\infty$ and $\int_{\gamma^i_n}\tau_i=0$,

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial W_n}\Pi^{s,t}_{p,q}\tau_i(s)=0$$

by lemma 2.2. q.e.d.

Let $\Delta=(\delta, \partial)$ be a P_V-divisor on W, η be a meromorphic differential such that $\Delta|\eta$. Then η has a representation

$$\eta=\tau+\phi+\lambda \tag{2.3}$$

on a canonical end U, where $\phi\in A_0(P, U)$, $\lambda\in A_d(Q, U)$ and $\tau\in m(Q, U)$ such that $\|\tau\|_{U\cap(\cup D_n)}<\infty$. If we put

$$h_V(\eta)(p)=\lim_{n\to\infty} dH_n(p)=d\lim_{n\to\infty} H_n(p),$$

$$t_V(\eta)(p)=-\frac{1}{2\pi i}\lim_{n\to\infty} d_p\int_{\partial W_n}\Pi^{s,t}_{p,q}\phi(s),$$

then $h_V(\eta)$ and $t_V(\eta)$ do not depend on a representation (2.3) by the above corollary.

Definition 2.1. The decomposition

$$\eta=h_V(\eta)+t_V(\eta)$$

is called V-decomposition of η.

Proposition 2.3. *$t_V(\eta)$ has no non-zero A-periods. $h_V(\eta)$ is semi-exact if η has no residues. $\int_{\gamma_n^i} h_V(\eta)=0$ for every sufficiently large n. Furthermore $h_V(\eta)$ has a finite norm on $\bigcup_{n=1}^{\infty} D_n$.*

Proof. From the constructions of $h_V(\eta)$, all the A-periods of $h_V(\eta)$ and those of η are equal, and $h_V(\eta)$ is semi-exact if η has no residues. Therefore

$$\int_{A_j} t_V(\eta)=\int_{A_j}\eta-\int_{A_j} h_V(\eta)=0.$$

To show that $\|h_V(\eta)\|_{\cup D_n}<\infty$ it is sufficient to prove that $h_V(\eta)$ has a finite norm on $U_0=(\bigcup_{n=1}^{\infty} D_n)\cap U$, where U is a canonical end on which η is represented as $\eta=\tau+\phi+\lambda$.

$$\|h_V(\eta)\|_{U_0}=\|\eta-t_V(\eta)\|_{U_0}\leqq\|\tau\|_{U_0}+\|\phi-t_V(\eta)\|_{U_0}+\|\lambda\|_{U_0}.$$

From proposition 2.1 $\phi\equiv t_V(\eta) \bmod A_d$ and thus $\|\phi-t_V(\eta)\|_{U_0}<\infty$. Therefore $\|h_V(\eta)\|_{U_0}<\infty$. On the other hand if $\gamma_n^i\subset U$, then

$$\int_{\gamma_n^i} h_V(\eta)=\int_{\gamma_n^i}(\tau+\lambda+\phi-t_V(\eta))$$

$$=\int_{\gamma_n^i}\tau+\int_{\gamma_n^i}\lambda+\int_{\gamma_n^i}(\phi-t_V(\eta))=0. \qquad \text{q.e.d.}$$

Now let δ be a positive inner divisor and ∂ be an arbitrary subspace of $\mathscr{S}(P_V)$, where $\mathscr{S}(P_V)$ is the space of P_V-singularities. If η is a semi-exact meromorphic differential which is square integrable outside of a compact subset, $(\eta)\geqq-\delta$ and furthermore has no non-zero A-periods on a canonical end, then clearly $\Delta=(-\delta, \partial)|\eta$. This means that such a differential as above has V-decomposition.

Proposition 2.4. $dw_j=h_V(dw_j)$, $dY_{p,n}=h_V(dY_{p,n})$ and $d\Pi_{p,q}=h_V(d\Pi_{p,q})$.

§3. Duality theorems

As in §2, we fix an open Riemann surface W satisfying the condition (A).

3.1. *The main duality theorem.* Let $\partial W = \alpha \cup \beta \cup \gamma$ be a regular partition of ∂W such that $\beta \cup \gamma \neq \emptyset$ and α may be empty. We denote this partition by P_0. Since P_0 is regular, α, β and γ are closed subsets of $W^* = W \cup \partial W$. W^* is a compact Hausdorff space and therefore W^* is a normal space. Hence α, β and γ are separated by open subsets of W^*. Let $U(\alpha)$, $U(\beta)$ and $U(\gamma)$ be open neighborhoods of α, β and γ respectively such that they are mutually disjoint. $U(\alpha) \cup U(\beta) \cup U(\gamma)$ is an open neighborhood of ∂W and if we put $(\partial W_n) \cap U(\alpha) = \alpha_n$, $(\partial W_n) \cap U(\beta) = \beta_n$ and $(\partial W_n) \cap U(\gamma) = \gamma_n$, then $\partial W_n = \alpha_n \cup \beta_n \cup \gamma_n$ is a partition of ∂W_n induced by P_0. Let $\delta_1 = \sum_{j=1}^{k_1} m_j p_j$ and $\delta_2 = \sum_{j=1}^{k_2} n_j q_j$, $1 \leqq k_1, k_2 \leqq \infty$, be positive inner divisors in $W_0 - \bigcup_{n=1}^{\infty} D_n$ such that $|\delta_1| \cap |\delta_2| = \emptyset$. Q is the canonical partition of ∂W. Let V_1 be a (Q)-divisor at boundary such that $\mathrm{Supp.}\, V_1 \subset \beta$. $\Delta_1 = (-\delta_1, \partial_1)$ is a Q_{V_1}-divisor. Let P be a regular partition of ∂W which is a refinement of P_0, that is to say, each part of P is contained in a part of P_0. Let V_2 be a (P)-divisor at boundary such that $\mathrm{Supp.}\, V_2 \subset \gamma$ and let $\Delta_2 = (-\delta_2, \partial_2)$ be a P_{V_2}-divisor. We consider the following two complex vector spaces.

$N(\Delta_1) = \{f | f$: an additive meromorphic function such that

$$(1)\ \Delta_1 | f\ (2) \int_{\gamma_n^i} df = 0 \text{ for all } \gamma_n^i \text{ and } (3) \int_{A_j} df = 0$$

for all $j\}$,

$W(\Delta_2) = \{dv | dv$: a meromorphic differential such that

$$(1)\ \Delta_2 | dv \text{ and } (2) \int_{\beta_{nj}} dv = 0 \text{ for all } \beta_{nj}\},$$

where $\partial W_n = \bigcup_{j=1}^{k(n)} \beta_{nj}$ is the partition of ∂W_n induced by P which is used to define $(P)m$ and $(P)A_0$, etc.. We assume that the partition of

∂W_n induced by P is a refinement of the partition of ∂W_n induced by P_0.

Let f be an element of $N(\Delta_1)$. Since $<df>_{V_1} \in \partial_1$, df has the V_1-decomposition

$$df = h_{V_1}(df) + t_{V_1}(df).$$

If dv is an element of $W(\Delta_2)$, then dv has the V_2-decomposition

$$dv = h_{V_2}(dv) + t_{V_2}(dv).$$

Since df has no residues, $h_{V_1}(df)$ is semi-exact and thus

$$\int_{\gamma_n^i} t_{V_1}(df) = \int_{\gamma_n^i} df - \int_{\gamma_n^i} h_{V_1}(df) = 0.$$

From this we see that df, $h_{V_1}(df)$ and $t_{V_1}(df)$ have single valued integrals on $W_0 - |\delta_1|$. Now choose single valued integrals f, $\int h_{V_1}(df)$ and $\int t_{V_1}(df)$ on $W_0 - |\delta_1|$ so that $f = \int h_{V_1}(df) + \int t_{V_1}(df)$. We put $\delta_1(n) = \delta_1 | W_n$ and $\delta_2(n) = \delta_2 | W_n$. Let U_j and $\tilde{U}_j$ be simply connected neighborhoods of $p_j \in \delta_1(n)$ and $q_j \in \delta_2(n)$ in $W_n \cap W_0$ respectively. We assume $U_j \cap U_i = \emptyset$, $\tilde{U}_j \cap \tilde{U}_i = \emptyset$ for $j \neq i$ and $U_i \cap \tilde{U}_j = \emptyset$ for all i and j. Since df is semi-exact, by the Stokes' theorem,

$$\begin{aligned}
0 &= (df, \overline{{}^*h_{V_2}(dv)})_{W_n - \cup U_j - \cup \tilde{U}_j} \\
&= -\int_{\partial W_n} f h_{V_2}(dv) + 2\pi i \sum_{p_j \in \delta_1(n)} \operatorname*{Res}_{p_j}(f h_{V_2}(dv)) \\
&\quad + 2\pi i \sum_{q_j \in \delta_2(n)} \operatorname*{Res}_{q_j}(f h_{V_2}(dv)) + \\
&\quad + \sum_{j=1}^{g(n)} \left\{ \int_{A_j} df \int_{B_j} h_{V_2}(dv) - \int_{A_j} h_{V_2}(dv) \int_{B_j} df \right\} \\
&= -\int_{\partial W_n} \left\{ \int h_{V_1}(df) + \int t_{V_1}(df) \right\} h_{V_2}(dv) + 2\pi i \sum_{p_j \in \delta_1(n)} \operatorname*{Res}_{p_j}(f h_{V_2}(dv)) \\
&\quad + 2\pi i \sum_{q_j \in \delta_2(n)} \operatorname*{Res}_{q_j}(f h_{V_2}(dv)) - \sum_{j=1}^{g(n)} \int_{A_j} h_{V_2}(dv) \int_{B_j} df.
\end{aligned}$$

Since $\|h_{V_1}(df)\|_{\cup D_n}<\infty$ and $\|h_{V_2}(dv)\|_{\cup D_n}<\infty$, we have

$$\lim_{n\to\infty}\int_{\partial W_n}\left(\int h_{V_1}(df)\right)h_{V_2}(dv)=0$$

by lemma 2.2. Here we have used the fact that $\int_{\gamma_n^i}h_{V_2}(dv)=0$ for sufficiently large n. Therefore

$$\lim_{n\to\infty}\Big\{-\int_{\partial W_n}\left(\int t_{V_1}(df)\right)h_{V_2}(dv)+2\pi i\sum_{p_j\in\delta_1(n)}\operatorname*{Res}_{p_j}(fh_{V_2}(dv))+$$

$$+2\pi i\sum_{q_j\in\delta_2(n)}\operatorname*{Res}_{q_j}(fh_{V_2}(dv))-\sum_{j=1}^{g(n)}\int_{A_j}h_{V_2}(dv)\int_{B_j}df\Big\}=0.$$

From this identity we see the following lemma.

Lemma 3.1. *Let* $f\in N(\Delta_1)$ *and* $dv\in W(\Delta_2)$. *Let* $\int h_{V_1}(df)$ *and* $\int t_{V_1}(df)$ *be single valued integrals on* $W_0-|\delta_1|$ *respectively. Then in order that*

$$\mathscr{L}_1(f, dv)=$$

$$\lim_{n\to\infty}\Big\{-\int_{\partial W_n}\left(\int t_{V_1}(df)\right)h_{V_2}(dv)+2\pi i\sum_{p_j\in\delta_1(n)}\operatorname*{Res}_{p_j}(fh_{V_2}(dv))\Big\}$$

converges, it is necessary and sufficient that

$$\mathscr{L}_1^*(f, dv)=$$

$$\lim_{n\to\infty}\Big\{\sum_{j=1}^{g(n)}\int_{A_j}h_{V_2}(dv)\int_{B_j}df-2\pi i\sum_{q_j\in\delta_2(n)}\operatorname*{Res}_{q_j}(fh_{V_2}(dv))\Big\}$$

converges.

Proof. We have only to show that $\mathscr{L}_1(f, dv)$ and $\mathscr{L}_1^*(f, dv)$ are independent of the choice of the branches $\int t_{V_1}(df)$ and f. Since $\int_{\gamma_n^i}h_{V_2}(dv)=0$ for sufficiently large n, we see

$$\int_{\partial W_n}h_{V_2}(dv)=0\quad\text{and}\quad\sum_{q_j\in\delta_2(n)}\operatorname*{Res}_{q_j}(h_{V_2}(dv))=0.$$

Hence if $\int t_{V_1}(df)$ and f are replaced by constants in $\mathscr{L}_1(f, dv)$, it is easy to see $\mathscr{L}_1(f, dv)=0$. This means $\mathscr{L}_1(f, dv)$ does not depend on the special branches of $\int t_{V_1}(df)$ and f. By the same reason $\mathscr{L}_1^*(f, dv)$ does not depend on the choices of the branches of $\int t_{V_1}(df)$ and f.

q.e.d.

Lemma 3.2. *Let $f \in N(\Delta_1)$ and $dv \in W(\Delta_2)$. Then*

$$\mathscr{L}_2(f, dv) = -\lim_{n\to\infty}\int_{\beta_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv)$$

and

$$\mathscr{L}_2^*(f, dv) = \lim_{n}\int_{\gamma_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv)$$

exist and $\mathscr{L}_2(f, dv)=\mathscr{L}_2^*(f, dv)$.

Proof. If we put $\partial W_n = \alpha_n \cup \beta_n \cup \gamma_n$ for large n, then β_n and β_m are homologous to each other. By the Stokes' theorem

$$\int_{\beta_m}\left(\int t_{V_1}(df)\right)t_{V_2}(dv) - \int_{\beta_n}\left(t_{V_1}(df)\right)t_{V_2}(dv)$$

$$= \sum_{W_m - W_n}{}' \left\{\int_{A_j} t_{V_1}(df)\int_{B_j} t_{V_2}(dv) - \int_{A_j} t_{V_2}(dv)\int_{B_j} t_{V_1}(df)\right\},$$

where Σ' means that the sum is taken with respect to A_j, B_j contained in the part of $W_m - W_n$ surrounded by β_m and β_n. This sum vanishes for sufficiently large n and m by proposition 2.3. Thus

$$\int_{\beta_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv) = \int_{\beta_m}\left(\int t_{V_1}(df)\right)t_{V_2}(dv)$$

and $\mathscr{L}_2(f, dv)$ exists. By the same reason we see that $\mathscr{L}_2^*(f, dv)$ exists. Again by the Stokes' theorem we have easily

$$0 = (t_{V_1}(df), \overline{{}^*t_{V_2}(dv)})_{W_n}$$

$$= -\int_{\partial W_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv)$$

$$= -\int_{\alpha_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv) - \int_{\beta_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv)$$

$$-\int_{\gamma_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv).$$

Since we have assumed that Supp. $V_1 \subset \beta$ and Supp. $V_2 \subset \gamma$, we see by proposition 1.2 that $t_{V_1}(df)$ and $t_{V_2}(dv)$ are semi-exact and of finite norm in a neighborhood of α.

$$\lim_{n\to\infty}\int_{\alpha_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv) = 0$$

by lemma 2.2. On the other hand we see easily

$$\int_{\alpha_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv) = \int_{\alpha_m}\left(\int t_{V_1}(df)\right)t_{V_2}(dv).$$

Thus $\int_{\alpha_n}\left(\int t_{V_1}(df)\right)t_{V_2}(dv) = 0$ and from this fact we have $\mathscr{L}_2(f, dv) = \mathscr{L}_2^*(f, dv)$. q.e.d.

We introduce the following three spaces.

$N_0(\Delta_1) = \{f \in N(\Delta_1) | \mathscr{L}_1(f, dv)$ exists for all dv in $W(\Delta_2)\}$

$N_0(\Delta_1 \| \Delta_2) = \{f \in N_0(\Delta_1) | f$: single valued, $(f) \geqq \delta_2 - \delta_1$ and

$\mathscr{L}_2(f, dv) = 0$ for all dv in $W(\Delta_2)\}$

$W(\Delta_2 \| \Delta_1) = \{dv \in W(\Delta_2) | (h_{V_2}(dv)) \geqq \delta_1 - \delta_2$ and

$\lim_{n\to\infty}\int_{\beta_n}\left(\int t_{V_1}(df)\right)dv = 0$ for all f in $N_0(\Delta_1)\}$.

Theorem I. $\dim_C N_0(\Delta_1)/\{N_0(\Delta_1 \| \Delta_2) + C\} = \dim_C W(\Delta_2)/W(\Delta_2 \| \Delta_1)$, *where* $\dim_C$ *stands for the complex dimension and the above formula permits of infinite dimensions.*

Proof. We define a bilinear mapping of $N_0(\Delta_1) \times W(\Delta_2)$ onto C by $T(f, dv) = \mathscr{L}_1(f, dv) + \mathscr{L}_2(f, dv)$. We denote by K_r the right kernel of T and by K_l the left kernel of T. We have only to show that

$$K_l = N_0(\Delta_1 \| \Delta_2) + C \quad \text{and} \quad K_r = W(\Delta_2 \| \Delta_1).$$

Let f be an element of K_l. For simplicity we assume that $\delta_2 \neq 0$. All the normalized Abelian differentials of the first kind are in $W(\Delta_2)$. Since $t_{V_2}(dw_j)=0$, we have $\mathscr{L}_2(f, dw_j)=0$ and hence $T(f, dw_j)=\mathscr{L}_1(f, dw_j)=0$. Since $\mathscr{L}_1 = \mathscr{L}_1^*$, $\mathscr{L}_1^*(f, dw_j)=0$. This means that all the B-periods of f are zero. Since f is semi-exact, f is single valued. Next we substitute $dY_{q_j,n}$ for dv, where $1 \leqq n \leqq n_j - 1$ and $1 \leqq j \leqq k_2$. Since $t_{V_2}(dY_{q_j,n})=0$, we have $\mathscr{L}_1(f, dY_{q_j,n})=0$ and this means $f^{(n)}(q_j)=0$ for $1 \leqq n \leqq n_j$. If $k_2 \geqq 2$, then we put $dv = d\Pi_{q_j,q_1}$. Since $t_{V_2}(d\Pi_{q_j,q_1})=0$, $\mathscr{L}_1(f, d\Pi_{q_j,q_1})=0$. Therefore we have $f(q_j)=f(q_1)$. In both cases of $k_2 \geqq 2$ and $k_2 = 1$ we obtain $(f - f(q_1)) \geqq \delta_2 - \delta_1$. But it is easy to see that $\mathscr{L}_2(f(q_1), dv)=0$ and $\mathscr{L}_1(f(q_1), dv)=0$ for all dv in $W(\Delta_2)$. Therefore

$$\begin{aligned} \mathscr{L}_2(f - f(q_1), dv) &= \mathscr{L}_2(f, dv) = T(f, dv) - \mathscr{L}_1(f, dv) \\ &= -\mathscr{L}_1(f, dv) = -\mathscr{L}_1(f - f(q_1), dv) = 0. \end{aligned}$$

This means that $f = f - f(q_1) + f(q_1) \in N_0(\Delta_1 \| \Delta_2) + C$. Therefore $K_l \subset N_0(\Delta_1 \| \Delta_2) + C$.

It is easy to see that $K_l \supset C$. Now assume that f is an element of $N_0(\Delta_1 \| \Delta_2)$. Since f is single valued, all the B-periods of f are zero. Since $(f) \geqq \delta_2 - \delta_1$, $\sum_{q_j \in \delta_2(n)} \operatorname{Res}_{q_j}(f h_{V_2}(dv))=0$ for all n. Thus $\mathscr{L}_1^*(f, dv)=0$ and hence $\mathscr{L}_1(f, dv)=0$ for all dv in $W(\Delta_2)$. Of course $\mathscr{L}_2(f, dv)=0$. Therefore $T(f, dv)=0$ for all dv in $W(\Delta_2)$ and this means $K_l \supset N_0(\Delta_1 \| \Delta_2)$. We have proved $K_l = N_0(\Delta_1 \| \Delta_2) + C$.

Now we shall prove $K_r = W(\Delta_2 \| \Delta_1)$. Let $d\tilde{v}$ be an element of K_r. It is easy to see that $Y_{p_j,m}$, $1 \leqq m \leqq m_j - 1$ belong to $N_0(\Delta_1)$. In fact since $t_{V_1}(dY_{p_j,m})=0$, $\mathscr{L}_1(Y_{p_j,m}, dv) = 2\pi i \operatorname{Res}_{p_j}(Y_{p_j,m} h_{V_2}(dv))$ exists for all dv in $W(\Delta_2)$. Furthermore $\mathscr{L}_2(Y_{p_j,m}, dv)=0$ for all dv in $W(\Delta_2)$ and therefore $\mathscr{L}_1(Y_{p_j,m}, d\tilde{v})=0$. Therefore $\operatorname{Res}_{p_j}(Y_{p_j,m} h_{V_2}(d\tilde{v}))=0$. We conclude that $(h_{V_2}(d\tilde{v})) \geqq \delta_1 - \delta_2$ and

$$0 = T(f, d\tilde{v})$$

$$= \lim_{n\to\infty} \left\{ -\int_{\partial W_n} \left(\int t_{V_1}(df) \right) h_{V_2}(d\tilde{v}) \right\} + \lim_{n\to\infty} \left\{ -\int_{\beta_n} \left(\int t_{V_1}(df) \right) t_{V_2}(d\tilde{v}) \right\}$$

for all f in $N_0(\Delta_1)$. Since Supp. $V_1 \subset \beta$,

$$\lim_{n\to\infty} \int_{\partial W_n} \left(\int t_{V_1}(df) \right) h_{V_2}(d\tilde{v}) = \lim_{n\to\infty} \int_{\beta_n} \left(\int t_{V_1}(df) \right) h_{V_2}(d\tilde{v})$$

by lemma 2.2 and proposition 1.2. Hence we obtain

$$\lim_{n\to\infty} \int_{\beta_n} \left(\int t_{V_1}(df) \right) d\tilde{v} = 0$$

for all f in $N_0(\Delta_1)$. We have proved $d\tilde{v} \in W(\Delta_2 \| \Delta_1)$, i.e. $K_r \subset W(\Delta_2 \| \Delta_1)$. Conversely let $d\tilde{v}$ be an element of $W(\Delta_2 \| \Delta_1)$. Then

$$T(f, d\tilde{v}) = \lim_{n\to\infty} \left\{ -\int_{\partial W_n} \left(\int t_{V_1}(df) \right) h_{V_2}(d\tilde{v}) - \int_{\beta_n} \left(\int t_{V_1}(df) \right) t_{V_2}(d\tilde{v}) \right\}$$

$$= \lim_{n\to\infty} \left\{ -\int_{\beta_n} \left(\int t_{V_1}(df) \right) h_{V_2}(d\tilde{v}) - \int_{\beta_n} \left(t_{V_1}(df) \right) t_{V_2}(d\tilde{v}) \right\}$$

$$= -\lim_{n\to\infty} \int_{\beta_n} \left(\int t_{V_1}(df) \right) d\tilde{v} = 0.$$

Therefore $d\tilde{v} \in K_r$ and this means $K_r = W(\Delta_2 \| \Delta_1)$. This completes a proof.

3.2. *Duality theorems for divisors of restricted types.* In this section we shall prove two duality theorems by restricting the types of divisors. Let $\delta_1 = \sum_{j=1}^{k_1} m_j p_j$ and $\delta_2 = \sum_{j=1}^{k_2} n_j q_j$ be two positive inner divisors in $W_0 - \bigcup_{n=1}^{\infty} D_n$ such that $|\delta_1| \cap |\delta_2| = \emptyset$. We introduce the following two complex vector spaces.

$\hat{M}(-\delta_1) = \{f | f$: an additive meromorphic function such that

(1) $(f) \geqq -\delta_1$, (2) $\int_{A_j} df = 0$, for all j, (3) $\int_{\gamma_n^i} df = 0$ for all γ_n^i and (4) $\|df\|_{\cup D_n} < \infty\}$

$W(-\delta_2) = \{dv | dv$: a meromorphic differential such that

(1) $(dv)\geqq -\delta_2$, (2) $\int_{\gamma_n^i} dv=0$ for all γ_n^i and

(3) $\|dv\|_{\cup D_n}<\infty\}$.

Then $d\hat{M}(-\delta_1)=\{df\,|\,f\in\hat{M}(-\delta_1)\}$ defines a subspace of $(Q)m$. If we denote by O the zero dimensional subspace of $\mathscr{D}'_Q$ and if we put $\partial_1=\eta_Q^O\circ\eta_Q(d\hat{M}(-\delta_1))$, then ∂_1 is a subspace of $\mathscr{S}(Q_O)$. Similarly $W(-\delta_2)$ determines a subspace of $(Q)m$. If we put $\partial_2=\eta_Q^O\circ\eta_Q(W(-\delta_2))$, then ∂_2 is a subspace of $\mathscr{S}(Q_O)$. Since Supp. $O=\emptyset$ and Q is the finest partition of ∂W, the conditions of theorem I are satisfied for an arbitrary regular partition P_0 of ∂W into three parts α, β and γ.

Lemma 3.3. *If we put* $\Delta_1=(-\delta_1, \partial_1)$ *and* $\Delta_2=(-\delta_2, \partial_2)$, *then*

$$N(\Delta_1)=\hat{M}(-\delta_1), \qquad W(\Delta_2)=W(-\delta_2).$$

Proof. First we show that $N(\Delta_1)=\hat{M}(-\delta_1)$. Let $f\in N(\Delta_1)$, then, by definition of $N(\Delta_1)$, we see $(f)\geqq -\delta_1$ and $<df>_O=\eta_Q^O\circ\eta_Q(df)\in\partial_1$. Therefore, as is easily seen, $df=h_O(df)$ and thus $\|df\|_{\cup D_n}=\|h_O(df)\|_{\cup D_n}<\infty$. It is evident that f has no non-zero A-periods. Furthermore df has no residues. This shows that $\int_{\gamma_n^i} df=0$. Hence we conclude $f\in\hat{M}(-\delta_1)$, i.e. $N(\Delta_1)\subset\hat{M}(-\delta_1)$. Conversely if we take any element g of $\hat{M}(-\delta_1)$, $<dg>_O\in\partial_1$ and $(g)\geqq -\delta_1$. Hence $\Delta_1|g$. Furthermore $\int_{A_j} dg=0$ and $\int_{\gamma_n^i} dg=0$ by definition of $\hat{M}(-\delta_1)$. Hence $N(\Delta_1)\supset\hat{M}(-\delta_1)$, i.e. $N(\Delta_1)=\hat{M}(-\delta_1)$.

Next we show that $W(\Delta_2)=W(-\delta_2)$. It is clear that $W(-\delta_2)\subset W(\Delta_2)$. Let dv be an element of $W(\Delta_2)$. By definition of $W(\Delta_2)$, we see $(dv)\geqq -\delta_2$ and $<dv>_O\in\eta_Q^O\circ\eta_Q(W(-\delta_2))=\partial_2$. If we choose a canonical end U, then

$$dv=dw+\lambda$$

on U, where $dw\in W(-\delta_2)$ and $\lambda\in A_d(Q, U)$. Therefore

$$\|dv\|_{U\cap(\cup D_n)}\leqq\|dw\|_{U\cap(\cup D_n)}+\|\lambda\|_U<\infty.$$

This means that $\|dv\|_{\cup D_n}<\infty$. Furthermore $\int_{\gamma_n^i} dv=0$. Hence dv belongs to $W(-\delta_2)$, i.e. $W(\varDelta_2)=W(-\delta_2)$. q.e.d.

In the present case of $\varDelta_1$ and $\varDelta_2$ the bilinear mapping T: $N_0(\varDelta_1)\times W(\varDelta_2)\to C$ degenerates to $\mathscr{L}_1$ since Supp. $O=\emptyset$.

$$T(f, dv)=\mathscr{L}_1(f, dv)$$

$$=\lim_{n\to\infty}\left\{-\int_{\partial W_n}\left(\int t_O(df)\right)h_O(dv)+2\pi i\sum_{p_j\in\delta_1(n)}\operatorname*{Res}_{p_j}(fh_O(dv))\right\}$$

$$=\lim_{n\to\infty}2\pi i\sum_{p_j\in\delta_1(n)}\operatorname*{Res}_{p_j}(fh_O(dv))$$

since $t_O(df)=0$ for all f in $N(\varDelta_1)$. We put $\delta=\delta_1-\delta_2$. If we put

$$\hat{M}_0(-\delta_1)=\{f\in M(-\delta_1)|\mathscr{L}_1(f, dv) \text{ exists for all } dv \text{ in } W(-\delta_2)\}$$

$$M_0(-\delta)=\{f\in M_0(-\delta_1)|f \text{ is a single valued meromorphic function such that } (f)\geqq-\delta\}$$

then the space $M_0(-\delta)$ is nothing but $N_o(\varDelta_1\|\varDelta_2)$. Similarly $W(\delta)=\{dv\in W(-\delta_2)|(dv)\geqq\delta\}$ is $W(\varDelta_2\|\varDelta_1)$. With these notations we have the following duality theorem.

Theorem II. (Sainouchi [4], theorem 5)

$$\dim_C \hat{M}_0(-\delta_1)/\{M_0(-\delta)+C\}=\dim_C W(-\delta_2)/W(\delta).$$

Next we shall be concerned with another divisor. Let P_0 be a regular partition of ∂W into three parts α, β and γ. Further let P be a regular partition of ∂W which is a refinement of P_0. Let V_Q be a (Q)-divisor at boundary and V_P be a (P)-divisor at boundary such that Supp. $V_Q\subset\beta$ and Supp. $V_P\subset\gamma$. We put

$$\mathscr{M}(V_Q)=\{f|f \text{ is a semi-exact holomorphic additive function such that } \eta_Q(df)\in V_Q \text{ and } \int_{A_j}df=0 \text{ for all } j\}$$

$$\mathscr{D}(V_P)=\{dv|dv \text{ is a holomorphic differential such that } \int_{\beta_{nj}}dv=0$$

and $\eta_P(dv)\in V_P\}$.

Further we put

$$\mathscr{M}(V_Q\| V_P)=\{f\in\mathscr{M}(V_Q)|f \text{ is single valued and } \lim_{n\to\infty}\int_{\gamma_n} ft_{V_P}(dv)=0 \text{ for all } dv \text{ in } \mathscr{D}(V_P)\}$$

$$\mathscr{D}(V_P\| V_Q)=\{dv\in\mathscr{D}(V_P)|\lim_{n\to\infty}\int_{\beta_n}\left(\int t_{V_Q}(df)\right)dv=0 \text{ for all } f \text{ in } \mathscr{M}(V_Q)\}.$$

Then we can prove the following theorem.

Theorem III. $\dim_C\mathscr{M}(V_Q)/\{\mathscr{M}(V_Q\| V_P)+C\}=\dim_C\mathscr{D}(V_P)/\mathscr{D}(V_P\| V_Q)$.

Proof. Let $\{O\}_Q$ and $\{O\}_P$ be the zero dimensional subspaces of $\mathscr{D}_Q(V_Q)$ and $\mathscr{D}_P(V_P)$ respectively, and consider the Q_{V_Q}-divisor $\varDelta_1=(0, \{O\}_Q)$ and P_{V_P}-divisor $\varDelta_2=(0, \{O\}_P)$. It is easy to see that $N(\varDelta_1)=\mathscr{M}(V_Q)$ and $W(\varDelta_2)=\mathscr{D}(V_P)$. For these $\varDelta_1$ and $\varDelta_2$, the bilinear mapping T: $N(\varDelta_1)\times W(\varDelta_2)\to C$ is written as

$$T(f, dv)=\lim_{n\to\infty}\left\{-\int_{\partial W_n}\left(\int t_{V_Q}(df)\right)h_{V_P}(dv)-\int_{\beta_n}\left(\int t_{V_Q}(df)\right)t_{V_P}(dv)\right\}$$

and this exists for all (f, dv) in $N(\varDelta_1)\times W(\varDelta_2)$. In fact if n and m are sufficiently large, then

$$\int_{\partial W_n}\left(t_{V_Q}(df)\right)h_{V_P}(dv)=\int_{\partial W_m}\left(\int t_{V_Q}(df)\right)h_{V_P}(dv)$$

since $h_{V_P}(dv)$ has no non-zero A-periods outside of a large W_{n_0} and $t_{V_Q}(df)$ has no non-zero A-periods. Thus $N_0(\varDelta_1)=N(\varDelta_1)=\mathscr{M}(V_Q)$. Therefore

$$N_0(\varDelta_1\|\varDelta_2)=\{f\in\mathscr{M}(V_0)|f \text{ is single valued and } \mathscr{L}_2(f, dv)=0 \text{ for all } dv \text{ in } \mathscr{D}(V_P)\}$$

and

$$W(\varDelta_2\|\varDelta_1)=\{dv\in\mathscr{D}(V_P)|\lim_{n\to\infty}\int_{\beta_n}\left(\int t_{V_Q}(df)\right)dv=0 \text{ for all } f \text{ in } \mathscr{M}(V_Q)\}$$

But if $f \in N_0(\Delta_1 \| \Delta_2)$, then the V_Q-decomposition of df is

$$df = h_{V_Q}(df) + t_{V_Q}(df) = t_{V_Q}(df)$$

since $\|h_{V_Q}(df)\|_{\cup D_n} < \infty$ and $\int_{A_j} h_{V_Q}(df) = 0$ for all j. From this fact we see $\mathscr{L}_2(f, dv) = \lim_{n\to\infty} \int_{\gamma_n} f t_{V_P}(dv)$. Therefore we have proved $N_0(\Delta_1 \| \Delta_2) = \mathscr{M}(V_Q \| V_P)$ and $W(\Delta_1 \| \Delta_2) = \mathscr{D}(V_P \| V_Q)$. Theorem I completes a proof.

§4. Interpolation theorems

As in the preceeding sections we fix an open Riemann surface W satisfying the condition (A).

4.1. *An interpolation theorem for multiplicative meromorphic functions.* Let P be a regular partition of ∂W and V be a (P)-divisor at boundary. Let σ be a P_V-singularity and δ be an inner divisor in $W_0 - \bigcup_{n=1}^{\infty} D_n$ such that the degree of $\delta(n) = \delta | W_n$ is zero for each n. We may put $\delta(n) = \sum_{j=1}^{k(n)} p_j - \sum_{j=1}^{k(n)} q_j$. Let γ_j be a singular 1-chain in $W_n - \Xi$ such that $\partial\gamma_j = q_j - p_j$ and put $C(n) = \sum_{j=1}^{k(n)} \gamma_j$.

Theorem IV. *Let* $\{\chi_{A_j}, \chi_{B_j}\}_{j=1}^{g}$ *be a sequence of complex numbers. Then there exists a multiplicative meromorphic function* f *on* W *such that*

$$(f) = \delta, \qquad \langle d \log f \rangle_V = \sigma \tag{4.1}$$

and

$$\int_{A_j} d \log f = \chi_{A_j}, \qquad \int_{B_j} d \log f = \chi_{B_j}, \tag{4.2}$$

if and only if the following conditions (4.3), (4.4) *and* (4.5) *hold*:

$$M_n(p) = \sum_{j=1}^{k(n)} \Pi_{p_j, q_j}^{p, q} + \sum_{j=1}^{g(n)} \chi_{A_j} \int_q^p dw_j \tag{4.3}$$

converges uniformly on compact subsets in $W_0 - \sum_{i=1}^{\infty} C(i)$, (4.4) $\|dM\|_{\cup D_n} < \infty$ *and* $<dM>_V = \sigma$, *where* $M(p) = \lim_{n\to\infty} M_n(p)$, (4.5) *there is a holomorphic differential* ϕ *such that*

(i) $\int_{A_j} \phi = 0$ for all j, $\int_{\beta_{nj}} \phi = 0$ for all β_{nj} and $\eta_P(\phi) \in V$,

(ii) $\lim_{n\to\infty} \{2\pi i \int_{C(n)} dw_j + \sum_{i=1}^{g(n)} \tau_{ij}\chi_{A_i}\} = \chi_{B_j} + \int_{B_j} \phi$,

where $\tau_{ij} = \int_{B_j} dw_i = \int_{B_i} dw_j$.

If the conditions (4.3)–(4.5) *hold, then one of the desired function* f *is given by* $\exp(M + \int \phi)$.

Proof. Let us assume that there exists a function f satisfying the conditions (4.1) and (4.2). Then $d\log f$ has V-decomposition

$$d\log f = h_V(d\log f) + t_V(d\log f).$$

Then $\eta_P(t_V(d\log f)) \in V$ and $<h_V(d\log f)>_V = \sigma$. By definition of V-decomposition

$$M_n(p) = \sum_{j=1}^{k(n)} \Pi_{p_j, q_j}^{p,q} + \sum_{j=1}^{g(n)} \chi_{A_j} \int_q^p dw_j$$

converges uniformly on compact subsets of $W_0 - |\delta_2|$ and $h_V(d\log f) = d\lim_{n\to\infty} M_n(p)$. Therefore $M_n(p)$ converges uniformly on compact subsets of $W_0 - \sum_{i=1}^{\infty} C(i)$ and $\|dM\|_{\cup D_n} = \|h_V(d\log f)\|_{\cup D_n} < \infty$. Furthermore $<dM>_V = \sigma$. Let U_j be a simply connected neighborhood of $\{p_j\} \cup \{q_j\}$ such that if $p_j, q_j \in |\delta(n)|$, then $U_j \subset W_n$. We assume $U_i \cap U_j = \emptyset$ if $i \neq j$. We put $t_V(d\log f) = \phi$. ϕ satisfies the conditions (4.5), (i). By the Stokes' theorem

$$0 = (dw_j, \overline{*d\log f})_{W_n - \bigcup_{j=1}^{k(n)} U_j}$$

$$= -\int_{\partial W_n} \left(\int dw_j\right)(h_V(d\log f) + t_V(d\log f)) + \sum_{i=1}^{k(n)} \int_{\partial U_i} \left(\int dw_j\right) d\log f +$$

$$+\sum_{i=1}^{g(n)}\left\{\int_{A_i} dw_j \int_{B_i} d\log f - \int_{A_i} d\log f \int_{B_i} dw_j\right\}.$$

Since $\|h_V(d\log f)\|_{\cup D_n} < \infty$,

$$\lim_{n\to\infty}\int_{\partial W_n}\left(\int dw_j\right)h_V(d\log f)=0$$

by lemma 2.2. On the other hand

$$\sum_{i=1}^{k(n)}\int_{\partial U_i}\left(\int dw_j\right)h_V(d\log f) = -2\pi i\int_{C(n)} dw_j.$$

Moreover since $\int_{A_i}\phi=0$ for all i, we see

$$\lim_{n\to\infty}\int_{\partial W_n}\left(\int dw_j\right)\phi = -\int_{B_j}\phi.$$

Therefore

$$\lim_{n\to\infty}\left\{2\pi i\int_{C(n)} dw_j + \sum_{i=1}^{g(n)}\tau_{ij}\chi_{A_i}\right\} = \chi_{B_j} + \int_{B_j}\phi.$$

Hence we have shown that (4.3), (4.4) and (4.5) hold.

Conversely assume that (4.3)-(4.5) hold. By making use of $M(p)$ of (4.4) and ϕ of (4.5) we put $dF=dM+\phi$ and $f=\exp\left(\int dF\right)$. Then it is clear that $(f)=\delta$ and $<d\log f>_V = <dM+\phi>_V = <dM>_V = \sigma$, since $<\phi>_V=0$. Since the A_j-period of M is χ_{A_j},

$$\int_{A_j} dF = \int_{A_j} dM + \int_{A_j}\phi = \int_{A_j} dM = \chi_{A_j}.$$

By the Stokes' theorem

$$0=(dw_j, \overline{{}^*dF})_{W_n - \bigcup_{i=1}^{k(n)} U_i}$$

$$= -\int_{\partial W_n}\left(\int dw_j\right)dF - 2\pi i\int_{C(n)} dw_j + \sum_{i=1}^{g(n)}\left\{\int_{A_i} dw_j\int_{B_i} dF - \int_{A_i} dF\int_{B_i} dw_j\right\}.$$

Hence

$$\lim_{n\to\infty}\left\{2\pi i\int_{C(n)} dw_j + \sum_{i=1}^{g(n)}\tau_{ij}\chi_{A_i}\right\} = \int_{B_j} dF + \int_{B_j}\phi.$$

If we compare the above identity with (ii) of (4.5), we see

$$\int_{B_j} dF=\chi_{B_j}, \quad \text{i.e.} \int_{B_j} d\log f=\chi_{B_j}. \qquad \text{q.e.d.}$$

Corollary. *Let V be a (Q)-divisor at boundary and σ be a Q_V-singularity. Let δ be an inner divisor in $W_0-\bigcup_{n=1}^{\infty} D_n$ such that* $\deg.\delta(n)=0$ *for each n. Then there exists a meromorphic function f such that*

$$(4.6) \qquad (f)=\delta, \qquad <d\log f>_V=\sigma$$

if and only if there exist a sequence $\{n_{A_j}, n_{B_j}\}_{j=1}^{g}$ of integers and a semi-exact holomorphic differential ϕ satisfying the following properties:

$$(4.7) \qquad M_n(p)=\sum_{j=1}^{k(n)} \Pi_{p_j,q_j}^{p,q}+\sum_{j=1}^{g(n)} 2\pi i n_{A_j}\int_q^p dw_j$$

converges uniformly on compact subsets of $W_0-\sum_{j=1}^{\infty} C(j)$,

$$(4.8) \qquad \|dM\|_{\cup D_n}<\infty \text{ and } <dM>_V=\sigma, \text{ where } M(p)=\lim_{n\to\infty} M_n(p),$$

$$(4.9) \qquad \text{(i)} \int_{A_j}\phi=0 \text{ for all } j \text{ and } \eta_Q(\phi)\in V,$$

$$\text{(ii)} \quad \lim_{n\to\infty} 2\pi i\left\{\int_{C(n)} dw_j+\sum_{i=1}^{g(n)} \tau_{ij} n_{A_i}\right\}=2\pi i n_{B_j}+\int_{B_j}\phi.$$

4.2. *An interpolation theorem for a singularity of restricted type.* In theorem IV we substitute the zero dimensional subspace of $\mathscr{D}'_Q$ for V. Then we have the following theorem due to Sainouchi [4].

Theorem V. *In order to exist a meromorphic function f such that $(f)=\delta$ and $\|d\log f\|_{\cup D_n}<\infty$, it is necessary and sufficient that there is a sequence $\{n_{A_j}, n_{B_j}\}_{j=1}^{g}$ of integers such that*

$$(4.10) \qquad M_n(p)=\sum_{j=1}^{k(n)} \Pi_{p_j,q_j}^{p,q}+\sum_{j=1}^{g(n)} 2\pi i n_{A_j}\int_q^p dw_j$$

converges uniformly on compact subsets of $W_0-\sum_{i=1}^{\infty} C(i)$,

$$(4.11) \qquad \|dM\|_{UD_n} < \infty, \ \textit{where} \ M(p) = \lim_{n\to\infty} M_n(p)$$

$$(4.12) \qquad \lim_{n\to\infty}\left\{\int_{C(n)} dw_j + \sum_{i=1}^{g(n)} \tau_{ij} n_{A_i}\right\} = n_{B_j}.$$

Proof. Suppose that there exist a desired function f. We define Q_O-singularity σ by $<d\log f>_O$. By corollary to theorem IV, there are a sequence $\{n_{A_j}, n_{B_j}\}_{j=1}^{g}$ of integers and a semi-exact holomorphic differential ϕ satisfying the conditions (4.7)–(4.9). Since $\eta_Q(\phi) \in V = \{0\}$ from (i) of (4.9), ϕ is square integrable, semi-exact and has no non-zero A-periods. Thus $\phi = 0$. Therefore (ii) of (4.9) is of the form (4.12) in the present case. Conversely assume that (4.10)–(4.12) hold. If we put $<dM>_O = \sigma$, then by corollary to theorem IV we have a meromorphic function such that $(f) = \delta$ and $<d\log f>_O = \sigma$. On a canonical end U, $d\log f = dM + \lambda$, where $\lambda \in A_d(Q, U)$. Therefore $\|d\log f\|_{UD_n} < \infty$. This completes a proof.

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