J. Math. Kyoto Univ. (JMKYAZ)
19-3 (1979) 553-581

On the cohomology mod p of the classifying spaces of the exceptional Lie groups, I

By

Mamoru MIMURA and Yuriko SAMBE

Dedicated to Professor A. Komatu on his 70-th birthday

(Received July 25, 1978)

§ 1. Introduction

Let p be a prime and G a compact, 1-connected simple Lie group. In general, when $H_*(G; \boldsymbol{Z})$ has no p-torsion, the cohomology mod p $H^*(BG; \boldsymbol{Z}_p)$ of the classifying space BG of G is a polynomial algebra. When $H_*(G; \boldsymbol{Z})$ has p-torsion, however, $H^*(BG; \boldsymbol{Z}_p)$ is of complicated form.

Let E_i be a compact, 1-connected exceptional Lie group of rank i ($i=6, 7, 8$). Then $H_*(E_i; \boldsymbol{Z})$ has p-torsion for $(i=6; p\leq 3)$, $(i=7; p\leq 3)$ and $(i=8; p\leq 5)$. Of these the module structures of $H^*(BE_i; \boldsymbol{Z}_2)$ for $i=6, 7$ have already been determined in [5] and [9] respectively.

The purpose of this series of papers is to investigate the structures of $H^*(BE_i; \boldsymbol{Z}_3)$ for $i=6, 7, 8$ and also of $H^*(BE_8; \boldsymbol{Z}_5)$.

Let $\{G: p\}$ be the set of all compact, associative H-spaces X such that $H^*(X; \boldsymbol{Z}_p)\cong H^*(G; \boldsymbol{Z}_p)$ as Hopf algebras over the Steenrod algebra $\mathcal{A}_p$. (We do not necessarily assume the existence of a map between spaces inducing an isomorphism.) For every space X of $\{G: p\}$ we have the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ such that

(1. 1) $$E_2\cong \mathrm{Cotor}_A(\boldsymbol{Z}_p, \boldsymbol{Z}_p) \text{ with } A=H^*(X; \boldsymbol{Z}_p),$$

(1. 2) $$E_\infty\cong \mathcal{G}_r H^*(BX; \boldsymbol{Z}_p).$$

(Refer, for example, to [12] and [13] for the construction and the properties of the Eilenberg-Moore spectral sequence.)

In the present paper, Part I of the series, we determine the E_2-term of the Eilenberg-Moore spectral sequence for X_6 of $\{E_6: 3\}$ and for X_7 of $\{E_7: 3\}$. The main results are Theorems 4. 10 and 5. 20.

The paper is organized as follows. In § 2 we construct an injective resolution of $\boldsymbol{Z}_3$ over $H^*(X_7; \boldsymbol{Z}_3)$. In sections 3 and 4 we determine $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ for $A=H^*(X_7; \boldsymbol{Z}_3)$. In the last section, § 5, we construct an injective resolution of $\boldsymbol{Z}_3$ over $H^*(X_6; \boldsymbol{Z}_3)$ and determine $\mathrm{Cotor}_B(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ for $B=H^*(X_6; \boldsymbol{Z}_3)$. The calculation in §5 is quite similar to but much simpler than that in §§ 3 and 4.

§ 2. An injective resolution of Z_3 over $H^*(X_7; Z_3)$

First we recall the Hopf algebra structure of $H^*(X_7; \boldsymbol{Z}_3)$ (over $\mathcal{A}_3$) from [7]:

(2.1) *As an algebra*

$$H^*(X_7; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}),$$

where $\deg x_i = i$;

(2.2) *The coalgebra structure is given by*

$$\bar{\phi}(x_i) = 0 \qquad \text{for } i = 3, 7, 8, 19,$$

$$\bar{\phi}(x_j) = x_8 \otimes x_{j-8} \qquad \text{for } j = 11, 15, 27,$$

$$\bar{\phi}(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19},$$

where $\bar{\phi}$ *is the reduced diagonal map induced from the multiplication on* X_7.

Notation. $A = H^*(X_7; \boldsymbol{Z}_3)$ and $\bar{A} = \tilde{H}^*(X_7; \boldsymbol{Z}_3)$.

We shall construct an injective resolution of $\boldsymbol{Z}_3$ over A using the same construction and the same notation as those in § 3 of [8].

Take L to be a graded $\boldsymbol{Z}_3$-submodule of $\bar{A}$ generated by

$$\{x_3, x_7, x_8, x_{19}, x_{11}, x_{15}, x_{27}, x_8^2, x_{35}\}.$$

Let θ: $A \to L$ be the projection and ι: $L \to A$ the injection such that $\iota \circ \theta = 1_A$. We name the set of corresponding elements under the suspension s as

(2.3) $$sL = \{a_4, a_8, a_9, a_{20}, b_{12}, b_{16}, b_{28}, c_{17}, e_{36}\}.$$

Define $\bar{\theta}$: $A \to sL$ by $\bar{\theta} = s \circ \theta$ and $\bar{\iota}$: $sL \to A$ by $\bar{\iota} = \iota \circ s^{-1}$. Let $T(sL)$ be the free tensor algebra over sL with the (natural) product ψ. Consider the two sided ideal I of $T(sL)$ generated by $\mathrm{Im}(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi)(\mathrm{Ker}\, \bar{\theta})$, where ϕ is the diagonal map of A. Then I is generated by

(2.4) $[\alpha, \beta]$ *for all pairs* (α, β) *of generators of* $T(sL)$ *except* (a_9, b_j) $(j = 12, 16, 28)$, (a_9, e_{36}) *and* (a_9, c_{17}),

$[a_9, b_j] + c_{17} a_{j-8}$ *for* $j = 12, 16, 28$,

$[a_9, e_{36}] + c_{17} b_{28}$,

where $[\alpha, \beta] = \alpha\beta - (-1)^* \beta\alpha$ *with* $* = \deg \alpha \cdot \deg \beta$.

Put $\overline{W}=T(sL)/I$, that is, $\overline{W}=\boldsymbol{Z}_3\{a_4, a_8, a_9, a_{20}, b_{12}, b_{16}, b_{28}, c_{17}, e_{36}\}$. Note that $\overline{W}$ contains the polynomial algebra

$$R=\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}].$$

We define a map

$$d=-\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi\circ\bar{\iota}\colon\ sL\to T(sL)$$

and extend it naturally over $T(sL)$ as derivation. Since $d(I)\subset I$ holds, d induces a map $\overline{W}\to\overline{W}$, which is again denoted by $d\colon\ \overline{W}\to\overline{W}$ by abuse of notation. It is easy to check that $d\circ d=0$ and so $\overline{W}$ is a differential algebra over $\boldsymbol{Z}_3$. Using the relation

$$d\circ\bar{\theta}+\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi=0\,,$$

we can construct the twisted tensor product $W=A\otimes\overline{W}$ with respect to $\bar{\theta}$ [14]. Namely, W is an A-comodule with the differential operator

$$\bar{d}=1\otimes d+(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)\circ(\phi\otimes 1).$$

More explicitly, the differential operators $\bar{d}$ and d are given by

(2.5)
$$\begin{aligned}
&\bar{d}(x_i\otimes 1)=1\otimes a_{i+1} && \textit{for } i=3, 7, 8, 19\,,\\
&\bar{d}(x_8^2\otimes 1)=1\otimes c_{17}-x_8\otimes a_9\,,\\
&\bar{d}(x_j\otimes 1)=1\otimes b_{j+1}+x_8\otimes a_{j-7} && \textit{for } j=11, 15, 27\,,\\
&\bar{d}(x_{35}\otimes 1)=1\otimes e_{36}+x_8\otimes b_{28}-x_8^2\otimes a_{20};
\end{aligned}$$

(2.6)
$$\begin{aligned}
&da_i=0 && \textit{for } i=4, 8, 9, 20\,,\\
&dc_{17}=a_9^2\,,\\
&db_j=-a_9a_{j-8} && \textit{for } j=12, 16, 28\,,\\
&de_{36}=-a_9b_{28}+c_{17}a_{20}\,.
\end{aligned}$$

Now we define weight in $W=A\otimes\overline{W}$ as follows:

(2.7)

A:	x_3,	x_7,	x_{19},	x_8,	x_8^2,	x_{11},	x_{15},	x_{27},	x_{35}
$\overline{W}$:	a_4,	a_8,	a_{20},	a_9,	c_{17},	b_{12},	b_{16},	b_{28},	e_{36}
weight:	0	0	0	1	2	2	2	2	6

(The weight of a monomial is the sum of the weights of each element.) Define a filtration

(2.8)
$$F_r=\{x\mid \text{weight } x\leq r\}.$$

Put $E_0W=\sum F_i/F_{i-1}$. Then it is easy to see that

$$E_0W\cong\Lambda(x_3, x_7, x_{19}, x_{11}, x_{15}, x_{27}, x_{35})\otimes\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}]\otimes C(Q(x_8))\,,$$

where $C(Q(x_8))$ is the cobar construction of $\boldsymbol{Z}_3[x_8]/(x_8^3)$. The differential formulae (2. 5) and (2. 6) imply that E_0W is acylic, and hence W is acyclic.

Theorem 2. 9. *W is an injective resolution of $\boldsymbol{Z}_3$ over $A=H^*(X_7;\boldsymbol{Z}_3)$.*

By the definition of Cotor we have

Corollary 2. 10. $H(\overline{W}: d)=\operatorname{Ker} d/\operatorname{Im} d\cong \operatorname{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$.

§ 3. Elements with neither a_9 nor c_{17} in $\operatorname{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$

We define an operator ∂ by

(3. 1) $$\partial a_i=0 \qquad \textit{for } i=4, 8, 20,$$
$$\partial b_j=-a_{j-8} \qquad \textit{for } j=12, 16, 28,$$
$$\partial e_{36}=-b_{28},$$

and extend it over $R=\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}]$ so that it satisfies

(3. 2) $$\partial(P+Q)=\partial P+\partial Q \quad \textit{and} \quad \partial(PQ)=\partial P\cdot Q+P\cdot\partial Q$$
for any polynomials P and Q.

Then we have

Lemma 3. 3. *For a polynomial $P\in R$ we have*

$$\partial^3P=0,$$
$$[a_9, P]=c_{17}\partial P,$$
$$dP=a_9\partial P+c_{17}\partial^2P.$$

Proof. (By induction.) Suppose that $\partial^3P=0$ holds for any polynomial P of degree up to l. Then

$$\partial^3(xP)=\partial^3x\cdot P+x\cdot\partial^3P=0.$$

Thus $\partial^3P=0$ holds for a polynomial of degree $l+1$.

Suppose that $[a_9, P]=c_{17}\partial P$ holds for any polynomial P of degree up to l. Then

$$[a_9, xP]=[a_9, x]P+x[a_9, P]=c_{17}\partial x\cdot P+xc_{17}\partial P=c_{17}\partial(xP).$$

Thus the relation holds for a polynomial of degree $l+1$.

Suppose that $dP=a_9\partial P+c_{17}\partial^2 P$ holds for any polynomial of degree up to l. Then

$$\begin{aligned} d(xP) &= dx\cdot P+x\cdot dP \\ &= (a_9\partial x+c_{17}\partial^2 x)P+x(a_9\partial P+c_{17}\partial^2 P) \\ &= a_9\partial x\cdot P+c_{17}\partial^2 x\cdot P+(a_9 x-c_{17}\partial x)\partial P+c_{17}x\partial^2 P \\ &= a_9\partial(xP)+c_{17}\partial^2(xP). \end{aligned}$$

Thus the differential formula holds for a polynomial of degree $l+1$. q.e.d.

Lemma 3.4. *Let P be non-trivial in R. Then P is a non-trivial cocycle if and only if $\partial P=0$.*

Proof. If P is a cocycle, $dP=0$. Then by the differential formula, we have $\partial P=0$.

Conversely, if $\partial P=0$, so does $\partial^2 P$, whence we have $dP=0$ by the differential formula. Since P contains no a_9, it is not in the d-image, hence it is a non-trivial cocycle. q.e.d.

We shall find cocycles in the following steps:

(i) cocycles in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$,

(ii) those in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$,

(iii) those in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}]$,

(iv) those in $\boldsymbol{Z}_3\{a_9, c_{17}\}$,

(v) other cocycles.

(The last two steps will be done in § 4.)

(i) Cocycles in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$

Clearly, a_4, a_8 and a_{20} are cocycles.

A cocycle of degree 1 with respect to b_{12} and b_{16} is of the form $P=Ab_{12}+Bb_{16}$ with $A, B\in\boldsymbol{Z}_3[a_4, a_8, a_{20}]$. The relation $\partial P=-Aa_4-Ba_8=0$ yields an indecomposable cocycle

$$y_{20}=a_8b_{12}-a_4b_{16}.$$

A cocycle of degree 2 with respect to b_{12} and b_{16} is of the form

$$P=Ab_{12}^2+Bb_{16}^2+Cb_{12}b_{16} \quad \text{with } A, B, C\in\boldsymbol{Z}_3[a_4, a_8, a_{20}].$$

Then $\partial P=(Aa_4-Ca_8)b_{12}+(Ba_8-Ca_4)b_{16}=0$ gives rise to

$$Aa_4 - Ca_8 = 0 \quad \text{and} \quad Ba_8 - Ca_4 = 0\,,$$

from which we obtain a decomposable cocycle

$$P = a_8^2 b_{12}^2 + a_4^2 b_{16}^2 + a_4 a_8 b_{12} b_{16} = y_{20}^2.$$

A cocycle of degree 3 with respect to b_{12} and b_{16} is of the form

$$P = Ab_{12}^3 + Bb_{16}^3 + Cb_{12}^2 b_{16} + Db_{12} b_{16}^2.$$

Then $\partial P = -Ca_8 b_{12}^2 + (Ca_4 + Da_8) b_{12} b_{16} - Da_4 b_{16}^2 = 0$ gives rise to $C = D = 0$. Thus we have two new cocycles

$$x_{36} = b_{12}^3 \text{ and } x_{48} = b_{16}^3.$$

A cocycle of degree 4 is of the form

$$P = b_{12}^3 (Ab_{12} + Bb_{16}) + Cb_{12}^2 b_{16}^2 + b_{16}^3 (Db_{12} + Eb_{16})\,.$$

Then $\partial P = 0$ implies that $C = 0$, since no term with $b_{12} b_{16}^2$ appears except for $Ca_4 b_{12} b_{16}^2$. Further $\partial P = 0$ gives rise to $\partial (Ab_{12} + Bb_{16}) = \partial (Db_{12} + Eb_{16}) = 0$, that is, P is decomposed in cocycles b_{12}^3, b_{16}^3, $Ab_{12} + Bb_{16}$ and $Db_{12} + Eb_{16}$. So no new cocycles are obtained.

Similarly, any cocycles of degree higher than 4 is decomposable.

We have obtained

Result (i). *The following are all the indecomposable cocycles in* $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$:

$$a_4,\quad a_8,\quad a_{20},\quad y_{20}(= a_8 b_{12} - a_4 b_{16})\,,\quad x_{36}(= b_{12}^3)\,,\quad x_{48}(= b_{16}^3)\,.$$

(ii) Cocycles in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$

A cocycle of degree 1 with respect to b_{28} is of the form $P = Ab_{28} + B$ with $A, B \in \boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$. Then $\partial P = \partial A \cdot b_{28} - a_{20} A + \partial B = 0$ gives rise to

$$\partial A = 0 \quad \text{and} \quad \partial B = a_{20} A\,.$$

So by Lemma 3.4, A is a cocycle and thus an element in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, y_{20}, x_{36}, x_{48}]$, for which we have to find, if any, a B such that $\partial B = a_{20} A$. Note that it is sufficient to choose one such B, since the difference of two cocyles $P = Ab_{28} + B$ and $P' = Ab_{28} + B'$ is a cocycle without b_{28}:

$$P - P' = (Ab_{28} + B) - (Ab_{28} + B') = B - B'.$$

Note also that, if there is a cocyle P_i corresponding to A_i: $P_i = A_i b_{28} + \cdots$ $(i = 1, 2)$, then cocycles corresponding to the sum $A_1 + A_2$ and to the product $A_1 A_2$ exist and are decomposable:

$$P_1+P_2=(A_1+A_2)b_{28}+\cdots,$$

$$A_1P_2=A_1A_2b_{28}+\cdots.$$

Now A is an element in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, y_{20}, x_{36}, x_{48}]$. In particular, for $A=a_4, a_8$ and $y_{20}^2(=\partial^2(-b_{12}^2b_{16}^2))$, we can choose $B=-a_{20}b_{12}, -a_{20}b_{16}$ and $a_{20}\partial(-b_{12}^2b_{16}^2)$ respectively, and we have corresponding cocycles

$$y_{32}=a_4b_{28}-a_{20}b_{12},$$

$$y_{36}=a_8b_{28}-a_{20}b_{16},$$

$$-y_{68}=\partial^2(-b_{12}^2b_{16}^2)b_{28}+a_{20}\partial(-b_{12}^2b_{16}^2)=-\partial^2(b_{12}^2b_{16}^2b_{28}).$$

Thus for $A=a_4A'+a_8A''+y_{20}^2A'''$, we have a decomposable cocycle $P=y_{32}A'+y_{36}A''-y_{68}A'''$.

A monomial in cocycles for A that has no a_4, a_8 nor y_{20}^2 is of the form $a_{20}^ix_{36}^jx_{48}^k$ or $y_{20}a_{20}^ix_{36}^jx_{48}^k$ (where i, j, k are non-negative integers), for which there is no B satisfying the conditions. Neither is there B for $A=a_{20}^ix_{36}^jx_{48}^k+A'$ and $y_{20}a_{20}^ix_{36}^jx_{48}^k+A'$ whatever a cocycle in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, y_{20}, x_{36}, x_{48}]$ A' is.

We have thus

(3. 5) *The indecomposable cocycles of degree* 1 *with respect to* b_{28} *are* y_{32}, y_{36} *and* y_{68}.

A cocycle of degree 2 with respect to b_{28} is of the form $P=Ab_{28}^2+Bb_{28}+C$ with $A, B, C\in\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$. Then the relation

$$\partial P=\partial A\cdot b_{28}^2+(a_{20}A+\partial B)b_{28}+(-a_{20}B+\partial C)=0$$

gives rise to

$$\partial A=0,\ \partial B=-a_{20}A \quad\text{and}\quad \partial C=a_{20}B.$$

Again by Lemma 3. 4, A is a cocycle, that is, an element in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, y_{20}, x_{36}, x_{48}]$. The difference of two cocycles $P=Ab_{28}^2+Bb_{28}+C$ and $P'=Ab_{28}^2+B'b_{28}+C'$ is a cocycle $(B-B')b_{28}+(C-C')$, that is, a cocycle of lower degree with respect to b_{28}. So it is sufficient to choose, if any, one corresponding cocycle for a cocycle A.

Once again, if there is a cocycle P_i corresponding to A_i $(i=1, 2)$, then there exist cocycles corresponding to A_1+A_2 and to A_1A_2, which are decomposable.

For a cocycle A of the form ∂^2D with $D\in\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$, we have actually a corresponding cocycle $P=\partial^2(Db_{28}^2)$. So we have a cocycle for each of the following:

(3. 6) $\quad a_4^2=\partial^2(-b_{12}^2),\ a_8^2=\partial^2(-b_{16}^2),\ a_4a_8=\partial^2(-b_{12}b_{16}),$

$$a_4y_{20}=\partial^2(b_{12}^2b_{16}),\ a_8y_{20}=\partial^2(-b_{12}b_{16}^2),\ y_{20}^2=\partial^2(-b_{12}^2b_{16}^2).$$

Cocycles $P=\partial^2(-b_{12}^2b_{28}^2)$, $\partial^2(-b_{16}^2b_{28}^2)$ and $\partial^2(-b_{12}b_{16}b_{28}^2)$ corresponding respectively to a_4^2, a_8^2 and a_4a_8 are y_{32}^2, y_{36}^2 and $y_{32}y_{36}$ respectively, and hence they are decomposable.

Now we put

$$y_{80}=\partial^2(b_{12}^2b_{16}b_{28}^2)\ =a_4y_{20}b_{28}^2+\cdots,$$
$$y_{84}=\partial^2(b_{12}b_{16}^2b_{28}^2)\ =-a_8y_{20}b_{28}^2+\cdots,$$
$$y_{96}=\partial^2(b_{12}^2b_{16}^2b_{28}^2)\ =-y_{20}^2b_{28}^2+\cdots.$$

Lemma 3.7. *The cocycles y_{80}, y_{84} and y_{96} are indecomposable and there is no other indecomposable cocycle of degree 2 with respect to b_{28}.*

Proof. First we study A in $Z_3[a_4, a_8, y_{20}]$. For $A=a_4$, we have $B=a_{20}b_{12}+(\partial\text{-kernel})$, but no C in $Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$ such that $\partial C=a_{20}^2b_{12}+(\text{other terms})$, thus there is no cocycle beginning with $a_4b_{28}^2$. Similarly, there is none beginning with $a_8b_{28}^2$. For $A=y_{20}$, there is no B in $Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}]$ such that $\partial B=-a_{20}y_{20}$, and so there is no cocycle that begins with $y_{20}b_{28}^2$. Recall that there is also no cocycle that begins with $y_{20}b_{28}$. And we also see that there is no cocycle beginning with Ab_{28}^2 whatever a sum of a_4, a_8 and y_{20} A is. We conclude that the cocycles y_{80}, y_{84} and y_{96} are indecomposable.

We have seen that each monomial of degree 2 in a_4, a_8 and y_{20} has a corresponding cocycle (decomposable or indecomposable). Therefore any polynomial A in a_4, a_8 and y_{20} of degree higher or equal to 2 has a corresponding cocycle, which is decomposable except for y_{80}, y_{84} and y_{96}. Thus there is no other indecomposable cocycle for $A\in Z_3[a_4, a_8, y_{20}]$.

Now we consider cocycles in $Z_3[a_4, a_8, a_{20}, y_{20}, x_{36}, x_{48}]\cong Z_3[a_4, a_8, y_{20}]\otimes Z_3[a_{20}, x_{36}, x_{48}]$.

As we have noted, there exists a cocycle corresponding to $Aa_{20}^ix_{36}^jx_{48}^k$ provided there is a cocycle corresponsing to A (here in particular, to $A\in Z_3[a_4, a_8, y_{20}]$), although it is decomposable.

Since a_{20}, b_{12} and b_{16} are not in the image $\partial(Z_3[a_4, a_8, b_{12}, b_{16}])$ and since $\partial a_{20}=\partial x_{36}=\partial x_{48}=0$, the elements a_{20}, x_{36} and x_{48} are '*immobile*' under ∂ when seeking B or C. It follows that there is no cocycle corresponding to $Aa_{20}^ix_{36}^jx_{48}^k+(\text{other cocycles})$ if there is none corresponding to A. Therefore there is no other indecomposable cocycle of degree 2 with respect to b_{28}. q.e.d.

Finally, $x_{84}=b_{28}^3$ is the only indecomposable cocycle of degree 3 with respect to b_{28}. It is easy to see that there is no new cocycle of degree higher than 3.

We have

Result (ii). *The following are all the indecomposable cocycles in* $Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$:

$$a_4, \quad a_8, \quad a_{20} = \partial^2 e_{36},$$

$$x_{36} = b_{12}^3, \quad x_{48} = b_{16}^3, \quad x_{84} = b_{28}^3 = \partial^2(-b_{28}e_{36}^2),$$

$$y_{20} = a_8 b_{12} - a_4 b_{16},$$

$$y_{32} = a_4 b_{28} - a_{20} b_{12} = \partial^2(-b_{12}e_{36}),$$

$$y_{36} = a_8 b_{28} - a_{20} b_{16} = \partial^2(-b_{16}e_{36}),$$

$$y_{68} = \partial^2(b_{12}^2 b_{16}^2 b_{28}) = y_{20}^2 b_{28} + \cdots,$$

$$y_{80} = \partial^2(b_{12}^2 b_{16} b_{28}^2) = a_4 y_{20} b_{28}^2 + \cdots,$$

$$y_{84} = \partial^2(b_{12} b_{16}^2 b_{28}^2) = -a_8 y_{20} b_{28}^2 + \cdots,$$

$$y_{96} = \partial^2(b_{12}^2 b_{16}^2 b_{28}^2) = -y_{20}^2 b_{28}^2 + \cdots.$$

(Result (i) is included in Result (ii).)

The following will be needed in the calculation in step (iii).

Lemma 3.8. *The elements* $y_{68}, y_{80}, y_{84}, y_{96}$ *and the following elements appear in the image* $\partial^2(Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}])$:

$$a_4^2 = \partial^2(-b_{12}^2), \qquad a_8^2 = \partial^2(-b_{16}^2), \qquad a_{20}^2 = \partial^2(-b_{28}^2),$$

$$a_4 a_8 = \partial^2(-b_{12}b_{16}), \quad a_4 a_{20} = \partial^2(-b_{12}b_{28}), \quad a_8 a_{20} = \partial^2(-b_{16}b_{28}),$$

$$a_4 y_{20} = \partial^2(b_{12}^2 b_{16}), \qquad a_8 y_{20} = \partial^2(-b_{12}b_{16}^2),$$

$$a_4 y_{32} = \partial^2(-b_{12}^2 b_{28}), \quad a_{20} y_{32} = \partial^2(b_{12}b_{28}^2),$$

$$a_8 y_{36} = \partial^2(-b_{16}^2 b_{28}), \quad a_{20} y_{36} = \partial^2(b_{16}b_{28}^2),$$

$$a_{20}y_{20} - a_8 y_{32} = -a_{20}y_{20} - a_4 y_{36} = a_4 y_{36} + a_8 y_{32} = \partial^2(b_{12}b_{16}b_{28}),$$

$$y_{20}^2 = \partial^2(-b_{12}^2 b_{16}^2), \quad y_{32}^2 = \partial^2(-b_{12}^2 b_{28}^2), \quad y_{36}^2 = \partial^2(-b_{16}^2 b_{28}^2),$$

$$y_{20}y_{32} = \partial^2(b_{12}^2 b_{16} b_{28}), \quad y_{20}y_{36} = \partial^2(-b_{12}b_{16}^2 b_{28}),$$

$$y_{32}y_{36} = \partial^2(-b_{12}b_{16}b_{28}^2),$$

and $P \cdot \partial^2 Q = \partial^2(P \cdot Q)$ *for any cocycle* P *and any polynomial* Q.

Proof is by direct calculation.

(iii) Cocycles in $Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}]$

A cocycle of degree 1 with respect to e_{36} is of the form $P = Ae_{36} + B$ with $A, B \in Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$. Then $\partial P = \partial A \cdot e_{36} - Ab_{28} + \partial B = 0$ gives rise to

$$\partial A = 0 \quad \text{and} \quad Ab_{28} = \partial B.$$

Thus A is a cocycle, for which it is sufficient to find, if any, one corresponding cocycle just as in (ii). For a cocycle A of the form $\partial^2 C$ with $C \in \mathbf{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$, we have actually a corresponding cocycle $P = \partial^2(Ce_{36})$ and for A of the form $\partial^2(Ce_{36})$ with C as above, we have a corresponding cocycle $P = \partial^2(-Ce_{36}^2)$.

Result (iii-1). *We have the following indecomposable cocycles of degree* 1 *with respect to* e_{36}:

$$z_{56} = \partial^2(e_{36}^2) = -a_{20}e_{36} + \cdots,$$
$$z_{44} = \partial^2(b_{12}^2 e_{36}) = -a_4^2 e_{36} + \cdots,$$
$$z_{48} = \partial^2(b_{12}b_{16}e_{36}) = -a_4 a_8 e_{36} + \cdots,$$
$$z_{52} = \partial^2(b_{16}^2 e_{36}) = -a_8^2 e_{36} + \cdots,$$
$$z_{68} = \partial^2(b_{12}e_{36}^2) = y_{32}e_{36} + \cdots,$$
$$z_{72} = \partial^2(b_{16}e_{36}^2) = y_{36}e_{36} + \cdots,$$
$$z_{60} = \partial^2(b_{12}^2 b_{16}e_{36}) = a_4 y_{20}e_{36} + \cdots,$$
$$z_{64} = \partial^2(b_{12}b_{16}^2 e_{36}) = -a_8 y_{20}e_{36} + \cdots,$$
$$z_{76} = \partial^2(b_{12}^2 b_{16}^2 e_{36}) = -y_{20}^2 e_{36} + \cdots,$$
$$z_{104} = \partial^2(b_{12}^2 b_{16}^2 b_{28}e_{36}) = y_{68}e_{36} + \cdots,$$
$$z_{116} = \partial^2(b_{12}^2 b_{16}b_{28}^2 e_{36}) = y_{80}e_{36} + \cdots,$$
$$z_{120} = \partial^2(b_{12}b_{16}^2 b_{28}^2 e_{36}) = y_{84}e_{36} + \cdots,$$
$$z_{132} = \partial^2(b_{12}^2 b_{16}^2 b_{28}^2 e_{36}) = y_{96}e_{36} + \cdots.$$

Proof. We have indecomposable cocycles $-z_{56}$, z_{68}, z_{72}, z_{104}, z_{116}, z_{120} and z_{132} corresponding respectively to a_{20}, y_{32}, y_{36}, y_{68}, y_{80}, y_{84} and y_{96}. Therefore, if each term of a cocycle A contains one of a_{20}, y_{32}, y_{36}, y_{68}, y_{80}, y_{84} or y_{96}, a cocycle beginning with Ae_{36} is decomposable.

Cocycles that we have to consider next as A are polynomials in a_4, a_8, y_{20}, x_{36}, x_{48} and x_{84}. Recall that x_{36} and x_{48} as well as $x_{84} = b_{28}^3$ are '*immobile*' under ∂ as before and so there is no cocycle corresponding to $Ax_{36}^i x_{48}^j x_{84}^k$ + (other cocycles) if there is none corresponding to A. In particular, we have none corresponding to $x_{36}^i x_{48}^j x_{84}^k$.

We have now only to consider those A in $\mathbf{Z}_3[a_4, a_8, y_{20}]$.

For A a sum of a_4, a_8 and y_{20}, there is no B satisfying $Ab_{28} = \partial B$, whence there is no cocycle that begins with $a_4 e_{36}$, $a_8 e_{36}$ or $y_{20}e_{36}$. So the cocycles corresponding to a_4^2, $a_4 a_8$, a_8^2, $a_4 y_{20}$, $a_8 y_{20}$ and y_{20}^2 are all indecomposable. Any monomial A in a_4, a_8 and y_{20} of degree higher than 2 has one of a_4^2, $a_4 a_8$, a_8^2, $a_4 y_{20}$, $a_8 y_{20}$ or y_{20}^2, whence any cocycle corresponding to such A is decomposable.

We have shown that the cocycles in (iii-1) are all the indecomposable ones of degree 1 with respect to e_{36}. q.e.d.

A cocycle of degree 2 with respect to e_{36} is of the form $P=Ae_{36}^2+Be_{36}+C$ with $A, B, C\in \boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$. Then $\partial P=0$ gives rise to

$$\partial A=0, \quad Ab_{28}=-\partial B \quad \text{and} \quad Bb_{28}=\partial C.$$

Therefore A is a cocycle, for which it is again sufficient to find, if any, one corresponding cocycle. We have actually

(3.9) *There is a cocycle $P=\partial^2(De_{36}^2)$ corresponding to a cocycle A of the form $\partial^2 D$ with $D\in \boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$.*

Result (iii-2). *The following are the indecomposable cocycles of degree 2 with respect to e_{36}:*

$$\begin{aligned}
w_{80}&=\partial^2(b_{12}^2e_{36}^2) = -a_4^2e_{36}^2+\cdots,\\
w_{84}&=\partial^2(b_{12}b_{16}e_{36}^2) = -a_4a_8e_{36}^2+\cdots,\\
w_{88}&=\partial^2(b_{16}^2e_{36}^2) = -a_8^2e_{36}^2+\cdots,\\
w_{96}&=\partial^2(b_{12}b_{28}e_{36}^2) = -a_4a_{20}e_{36}^2+\cdots,\\
w_{100}&=\partial^2(b_{16}b_{28}e_{36}^2) = -a_8a_{20}e_{36}^2+\cdots,\\
v_{96}&=\partial^2(b_{12}^2b_{16}e_{36}^2) = a_4y_{20}e_{36}^2+\cdots,\\
v_{100}&=\partial^2(b_{12}b_{16}^2e_{36}^2) = -a_8y_{20}e_{36}^2+\cdots,\\
v_{108}&=\partial^2(b_{12}^2b_{28}e_{36}^2) = -a_4y_{32}e_{36}^2+\cdots,\\
v_{112}&=\partial^2(b_{12}b_{16}b_{28}e_{36}^2) = (a_{20}y_{20}-a_8y_{32})\,e_{36}^2+\cdots\\
&\qquad = (-a_{20}y_{20}-a_4y_{36})\,e_{36}^2+\cdots\\
&\qquad = (a_4y_{36}+a_8y_{32})\,e_{36}^2+\cdots,\\
v_{116}&=\partial^2(b_{16}^2b_{28}e_{36}^2) = -a_8y_{36}e_{36}^2+\cdots,\\
u_{112}&=\partial^2(b_{12}^2b_{16}^2e_{36}^2) = -y_{20}^2e_{36}^2+\cdots,\\
u_{124}&=\partial^2(b_{12}^2b_{16}b_{28}e_{36}^2) = y_{20}y_{32}e_{36}^2+\cdots,\\
u_{128}&=\partial^2(b_{12}b_{16}^2b_{28}e_{36}^2) = -y_{20}y_{36}e_{36}^2+\cdots,\\
p_{140}&=\partial^2(b_{12}^2b_{16}^2b_{28}e_{36}^2) = y_{68}e_{36}^2+\cdots,\\
p_{152}&=\partial^2(b_{12}^2b_{16}b_{28}^2e_{36}^2) = y_{80}e_{36}^2+\cdots,\\
p_{156}&=\partial^2(b_{12}b_{16}^2b_{28}^2e_{36}^2) = y_{84}e_{36}^2+\cdots,\\
p_{168}&=\partial^2(b_{12}^2b_{16}^2b_{28}^2e_{36}^2) = y_{96}e_{36}^2+\cdots.
\end{aligned}$$

Proof. By virtue of (3.9) we have 23 cocycles corresponding to the 23 elements in Lemma 3.8. Of these cocycles 17 are indecomposable and the 6 corresponding to a_{20}^2, $a_{20}y_{32}$, $a_{20}y_{36}$, y_{32}^2, y_{36}^2 and $y_{32}y_{36}$ are decomposable.

The cocycles p_{140}, p_{152}, p_{156} and p_{168} corresponding respectively to y_{68}, y_{80}, y_{84} and y_{96} are indecomposable, and any cocycle which corresponds to a monomial A having one of y_{68}, y_{80}, y_{84} and y_{96} is decomposable.

Note again that there is no cocycle corresponding to $Ax_{36}^i x_{48}^j x_{84}^k +$ (other cocycles) if there is none corresponding to A. In particular, there is no cocycle corresponding to $x_{36}^i x_{48}^j x_{84}^k$. Thus we have only to consider polynomials in a_4, a_8, a_{20}, y_{20}, y_{32} and y_{36} as A.

For $A=a_4$ and a_8, there is no B satisfying the conditions, that is, there are no cocycles beginning with either $a_4e_{36}^2$ or $a_8e_{36}^2$. Recalling that there is no cocycle beginning with either a_4e_{36} or a_8e_{36}, we conclude that $w_{80}=-a_4^2e_{36}^2+\cdots$, $w_{84}=-a_4a_8e_{36}^2+\cdots$ and $w_{88}=-a_8^2e_{36}^2+\cdots$ are indecomposable.

For $A=a_{20}$ we have $B=-b_{28}^2+$ (∂-kernel) but no C in $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}]$ such that $\partial C=Bb_{28}=-b_{28}^3+$ (other terms). Thus there is no cocycle beginning with $a_{20}e_{36}^2$ and we conclude that $w_{96}=-a_4a_{20}e_{36}^2+\cdots$ and $w_{100}=-a_8a_{20}e_{36}^2+\cdots$ are indecomposable. However, a cocycle beginning with $a_{20}^2e_{36}^2$ is decomposable, since z_{56}^2 begins with $a_{20}^2e_{36}^2$.

There is no cocycle corresponding to y_{20} as there is no B such that $\partial B=-y_{20}b_{28}$, and there is no cocycle beginning with $y_{20}e_{36}$. Therefore, $v_{96}=a_4y_{20}e_{36}^2+\cdots$, $v_{100}=-a_8y_{20}e_{36}^2+\cdots$ and $u_{112}=-y_{20}^2e_{36}^2+\cdots$ are indecomposable.

For $A=a_{20}y_{20}$ we have $B=-y_{20}b_{28}^2+$ (∂-kernel) but no C such that $\partial C=-y_{20}b_{28}^3+$ (other terms). Thus there is no cocycle beginning with $a_{20}y_{20}e_{36}^2$. On the other hand, we have a cocycle $v_{112}=(a_{20}y_{20}-a_8y_{32})e_{36}^2+\cdots=(-a_{20}y_{20}-a_4y_{36})e_{36}^2+\cdots$. Hence we conclude here that there is no cocycle beginning with either $a_8y_{32}e_{36}^2$ or $a_4y_{36}e_{36}^2$, and also that v_{112} is indecomposable.

For $A=y_{32}$ and y_{36}, we have $B=b_{12}b_{28}^2+$ (∂-kernel) and $b_{16}b_{28}^2+$ (∂-kernel) respectively but no C, that is, there is no cocycle corresponding to y_{32} or y_{36}. We conclude that v_{108}, v_{116}, u_{124} and u_{128} corresponding respectively to $-a_4y_{32}$, $-a_8y_{36}$, $y_{20}y_{32}$ and $-y_{20}y_{36}$ are indecomposable.

One can easily see that cocycles corresponding to $a_{20}y_{32}$, $a_{20}y_{36}$, y_{32}^2, y_{36}^2 and $y_{32}y_{36}$ are decomposed in terms of the elements $z_{56}=-a_{20}e_{36}+\cdots$, $z_{68}=-y_{32}e_{36}+\cdots$, $z_{68}=-y_{32}e_{36}+\cdots$ and $z_{72}=-y_{36}e_{36}+\cdots$.

We have proved that the cocycles in (iii-2) are all the indecomposable ones of degree 2 with respect to e_{36}. q.e.d.

Obviously we have

Result (iii-3). *The element $x_{108}=e_{36}^3$ is the only indecomposable cocycle of degree 3 with respect to e_{36}.*

It is easy to see that there are no indecomposable cocycles of degree higher than 3. Thus we have shown

Proposition 3.10. *Cocycles in* Results (ii), (iii-1), (iii-2) *and* (iii-3) *are all the indecomposable ones with neither a_9 nor c_{17}. Any cocycle that has neither a_9 nor c_{17} is trivial if and only if it is 0 as a polynomial in* $\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}]$.

We see

Remark 3.11. (1) *The generators are in the ∂^2-image except a_4, a_8, y_{20}, x_{36}, x_{48} and x_{108};*
(2) *a_4 and a_8 are in the ∂-image, but not in the ∂^2-image;*
(3) *y_{20}, x_{36}, x_{48}, x_{108} are not in the ∂-image.*

Using the above and Lemma 3.8 we see that

(**3.12.1**) *A cocycle is in the ∂^2-image if and only if it has no term of the form $a_4x_{36}^ix_{48}^jx_{108}^k$, $a_8x_{36}^ix_{48}^jx_{108}^k$, $x_{36}^ix_{48}^jx_{108}^k$ or $y_{20}x_{36}^ix_{48}^jx_{108}^k$;*

(**3.12.2**) *A cocycle is in the ∂-image but not in the ∂^2-image if and only if it is a sum of $a_4x_{36}^ix_{48}^jx_{108}^k$, $a_8x_{36}^ix_{48}^jx_{108}^k$ and ∂^2-image;*

(**3.12.3**) *A cocycle is not the ∂-image, if it is a sum of $x_{36}^ix_{48}^jx_{108}^k$, $y_{20}x_{36}^ix_{48}^jx_{108}^k$ and any other terms.*

§ 4. Elements with a_9 and c_{17} in $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$

Now we study cocycles with a_9 and c_{17}.

(iv) Cocycles in $\boldsymbol{Z}_3\{a_9, c_{17}\}$

Clearly a_9 is a cocycle and $a_9^2 = dc_{17}$. It is easy to see that $x_{26} = [a_9, c_{17}]$ is also a cocycle. The following lemma provides a convenient manner of writing elements in $\boldsymbol{Z}_3\{a_9, c_{17}\}$.

Lemma 4.1. *An element α_n in $\boldsymbol{Z}_3\{a_9, c_{17}\}$ of degree n can be written as follows:*

$$\alpha_{2k-1} = d\alpha_{2k-2} + \sum_{i=0}^{k-1} x_{26}^i c_{17}\alpha_{2k-2i-2} + \varepsilon x_{26}^{k-1}a_9\,,$$

$$\alpha_{2k} = d\alpha_{2k-1} + \sum_{i=0}^{k-1} x_{26}^i c_{17}\alpha_{2k-2i-1} + \varepsilon x_{26}^{k},$$

where α_j are elements in $\boldsymbol{Z}_3\{a_9, c_{17}\}$ of degree j and α_0, $\varepsilon \in \boldsymbol{Z}_3$.

Proof. (By induction.) Suppose that the lemma is true for degrees up to $2k$. Then

$$\begin{aligned}\alpha_{2k+1}&=\alpha_{2k}c_{17}+\alpha'_{2k}a_9\\&=(d\alpha_{2k-1}+\sum_{i=0}^{k-1}x_{26}{}^ic_{17}\alpha_{2k-2i-1}+\varepsilon x_{26}{}^k)\,c_{17}\\&\quad+(d\alpha'_{2k-1}+\sum_{i=0}^{k-1}x_{26}{}^ic_{17}\alpha'_{2k-2i-1}+\varepsilon' x_{26}{}^k)\,a_9\\&=d(\alpha_{2k-1}c_{17}+\alpha'_{2k-1}a_9)+\sum_{i=0}^{k-1}x_{26}{}^ic_{17}(\alpha_{2k-2i-1}c_{17}+\alpha'_{2k-2i-1}a_9)\\&\quad+\varepsilon x_{26}{}^kc_{17}+\varepsilon' x_{26}{}^ka_9+\alpha_{2k-1}a_9{}^2.\end{aligned}$$

Now the last term $\alpha_{2k-1}a_9{}^2$ can be rewritten as follows:

$$\begin{aligned}\alpha_{2k-1}a_9{}^2&=(d\alpha_{2k-2}+\sum_{i=0}^{k-1}x_{26}{}^ic_{17}\alpha_{2k-2i-2}+\varepsilon'' x_{26}{}^{k-1}a_9)\,a_9{}^2\\&=d(\alpha_{2k-2}a_9{}^2+\varepsilon'' x_{26}{}^{k-1}c_{17}a_9)+\sum_{i=0}^{k-1}x_{26}{}^ic_{17}\alpha_{2k-2i-2}a_9{}^2.\end{aligned}$$

Thus α_{2k+1} can be written in the required form.

Similarly,

$$\begin{aligned}\alpha_{2k+2}&=\alpha_{2k+1}c_{17}+\alpha'_{2k+1}a_9\\&=(d\alpha_{2k}+\sum_{i=0}^{k}x_{26}{}^ic_{17}\alpha_{2k-2i}+\varepsilon x_{26}{}^ka_9)\,c_{17}\\&\quad+(d\alpha'_{2k}+\sum_{i=0}^{k}x_{26}{}^ic_{17}\alpha'_{2k-2i}+\varepsilon' x_{26}{}^ka_9)\,a_9\\&=d(\alpha_{2k}c_{17}+\alpha'_{2k}a_9+\varepsilon' x_{26}{}^kc_{17})+\varepsilon x_{26}{}^{k+1}-\varepsilon x_{26}{}^kc_{17}a_9\\&\quad+\sum_{i=0}^{k}x_{26}{}^ic_{17}(\alpha_{2k-2i}c_{17}+\alpha'_{2k-2i}a_9)-\alpha_{2k}a_9{}^2,\end{aligned}$$

and the last term $\alpha_{2k}a_9{}^2$ can be rewritten as

$$\begin{aligned}\alpha_{2k}a_9{}^2&=d\alpha_{2k-1}+\sum_{i=0}^{k-1}x_{26}{}^ic_{17}\alpha_{2k-2i-1}+\varepsilon'' x_{26}{}^k)\,a_9{}^2\\&=d(\alpha_{2k-1}a_9{}^2+\varepsilon'' x_{26}{}^kc_{17})+\sum_{i=0}^{k-1}x_{26}{}^ic_{17}\alpha_{2k-2i-1}a_9{}^2.\end{aligned}$$

Hence, α_{2k+2} can also be written in the required form. q.e.d.

Proposition 4.2. *The elements a_9 and x_{26} are the only indecomposable cocycles in $\boldsymbol{Z}_3\{a_9, c_{17}\}$.*

Proof. Writing an element α_{2k-1} in $\boldsymbol{Z}_3\{a_9, c_{17}\}$ of degree $2k-1$ as in Lemma 4.1, we have

$$d\alpha_{2k-1}=\sum_{i=0}^{k-1}x_{26}{}^i(a_9{}^2\alpha_{2k-2i-2}-c_{17}d\alpha_{2k-2i-2}).$$

Thus $d\alpha_{2k-1}=0$ gives rise to $\alpha_{2k-2i-2}=0$ $(0\leq i\leq k-1)$ (and also $d\alpha_{2k-2i-2}=0$). Conversely, if $\alpha_{2k-2i-2}=0$ $(0\leq i\leq k-1)$, then $d\alpha_{2k-1}$ is clearly 0. So α_{2k-1} is a cocycle if and only if it is $\pm x_{26}{}^{k-1}a_9$.

Similarly,

$$d\alpha_{2k}=\sum_{i=0}^{k-1}x_{26}{}^{i}(a_9{}^2\alpha_{2k-2i-1}+c_{17}d\alpha_{2k-2i-1})=0$$

if and only if $\alpha_{2k-2i-1}=0$ $(0\leq i\leq k-1)$ if and only if $\alpha_{2k}=\pm x_{26}{}^{k}$.

Therefore a_9 and x_{26} are the only indecomposable cocycles in $Z_3\{a_9, c_{17}\}$.

q.e.d.

(v) Other cocycles

We shall find other cocycles with a_9 and c_{17}. We shall use the letter f to denote elements in $\overline{W}$.

Lemma 4.3. *An element f_n of degree n with respect to a_9 and c_{17} can be written as*

$$f_{2k}=\sum_{i=0}^{k-1}x_{26}{}^{i}c_{17}f_{2k-2i-1}+x_{26}{}^{k}P+(d\text{-image}),$$

$$f_{2k+1}=\sum_{i=0}^{k-1}x_{26}{}^{i}c_{17}f_{2k-2i}+x_{26}{}^{k}(c_{17}P+a_9Q)+(d\text{-image}),$$

where P and Q are elements of $R=Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}]$.

Proof. We shall use the letter α to denote, as before, elements of $Z_3\{a_9, c_{17}\}$ and the letter P to denote elements with neither a_9 nor c_{17}. Now, the following identities will be needed in the calculation:

$$\begin{aligned}d\alpha_{2k-1}\cdot P&=d(\alpha_{2k-1}P)+\alpha_{2k-1}dP\\&=d(\alpha_{2k-1}P)+(d\alpha_{2k-2}+\sum_{i=0}^{k-1}x_{26}{}^{i}c_{17}\alpha_{2k-2i-2}+\varepsilon x_{26}{}^{k-1}a_9)\,dP\\&=d(\alpha_{2k-1}P+\alpha_{2k-2}dP-\varepsilon x_{26}{}^{k-1}a_9P)\\&\quad+\sum_{i=0}^{k-1}x_{26}{}^{i}c_{17}\alpha_{2k-2i-2}(a_9\partial P+c_{17}\partial^2P)\\&=\sum_{i=0}^{k-1}x_{26}{}^{i}c_{17}f_{2k-2i-1}+(d\text{-image}),\end{aligned}$$

and

$$\begin{aligned}d\alpha_{2k}\cdot P&=d(\alpha_{2k}P)-\alpha_{2k}dP\\&=d(\alpha_{2k}P)-(d\alpha_{2k-1}+\sum_{i=0}^{k-1}x_{26}{}^{i}c_{17}\alpha_{2k-2i-1}+\varepsilon x_{26}{}^{k})\,dP\\&=d(\alpha_{2k}P-\alpha_{2k-1}dP-\varepsilon x_{26}{}^{k}P)-\sum_{i=0}^{k-1}x_{26}{}^{i}c_{17}\alpha_{2k-2i-1}(a_9\partial P+c_{17}\partial^2P)\end{aligned}$$

$$=\sum_{i=0}^{k-1} x_{26}{}^{i} c_{17} f_{2k-2i} + (d\text{-image}).$$

Consider now an element of the form $\alpha_{2k}P$:

$$\alpha_{2k}P = (d\alpha_{2k-1} + \sum_{i=0}^{k-1} x_{26}{}^{i} c_{17}\alpha_{2k-2i-1} + \varepsilon x_{26}{}^{k}) P$$

$$= d\alpha_{2k-1}\cdot P + \sum_{i=0}^{k-1} x_{26}{}^{i} c_{17} (\alpha_{2k-2i-1}P) + x_{26}{}^{k} (\varepsilon P)$$

$$= \sum_{i=0}^{k-1} x_{26}{}^{i} c_{17} (f_{2k-2i-1} + \alpha_{2k-2i-1}P) + x_{26}{}^{k} (\varepsilon P) + (d\text{-image}).$$

Thus an element f_{2k} of degree $2k$ can be written in the required form.

Similarly,

$$\alpha_{2k+1}P = (d\alpha_{2k} + \sum_{i=0}^{k} x_{26}{}^{i} c_{17}\alpha_{2k-2i} + \varepsilon x_{26}{}^{k} a_9) P$$

$$= \sum_{i=0}^{k-1} x_{26}{}^{i} c_{17} (f_{2k-2i} + \alpha_{2k-2i}P) + x_{26}{}^{k} (c_{17}\alpha_0 P + \varepsilon a_9 P)$$

and an element f_{2k+1} can also be written in the required form. q.e.d.

Writing an element of $\overline{W}$ as in the previous lemma, we have

(4. 3. 1) $$df_{2k} = \sum_{i=0}^{k-1} x_{26}{}^{i} a_9^2 f_{2k-2i-1} - \sum_{i=0}^{k-1} x_{26}{}^{i} c_{17} df_{2k-2i-1} + x_{26}{}^{k} (a_9\partial P + c_{17}\partial^2 P).$$

Thus $df_{2k}=0$ gives rise to $f_{2k-2i-1}=0$ $(0\leq i\leq k-1)$ (and $df_{2k-2i-1}=0$) and $\partial P=0$ (that is, P is a cocycle), and the converse is clear. Therefore we have

(4. 4. 1) *$df_{2k}=0$ if and only if f_{2k} is of the form $f_{2k}=x_{26}{}^{k}A$, where A is a cocycle with neither a_9 nor c_{17}.*

Similarly,

(4. 3. 2) $$df_{2k+1} = \sum_{i=0}^{k-1} x_{26}{}^{i} a_9^2 f_{2k-2i} - \sum_{i=0}^{k-1} x_{26}{}^{i} c_{17} df_{2k-2i} + x_{26}{}^{k} a_9^2 (P-\partial Q)$$
$$- x_{26}{}^{k} c_{17} (a_9\partial (P-\partial Q) + c_{17}\partial^2 P) - x_{26}{}^{k+1}\partial^2 Q.$$

Thus $df_{2k+1}=0$ if and only if $f_{2k-2i}=0$ $(0\leq i\leq k-1)$, $P=\partial Q$ and $\partial P=\partial^2 Q=0$. That is,

(4. 4. 2) *$df_{2k+1}=0$ if and only if f_{2k+1} is of the form*

$$f_{2k+1} = x_{26}{}^{k} (c_{17}\partial Q + a_9 Q) \quad \textit{with} \quad \partial^2 Q = 0.$$

Thus we have only to determine cocycles of the form

$$f_1 = c_{17}\partial Q + a_9 Q \quad \text{with} \quad \partial^2 Q = 0\,.$$

In case $\partial Q = 0$, f_1 is a product $a_9 Q$ with Q a cocycle. We obtain no new cocycle.

In case $\partial Q \neq 0$, ∂Q is a cocycle as $\partial^2 Q = 0$. If there is another Q' such that $\partial Q' = \partial Q$, then the difference of $f_1 = a_9 Q + c_{17}\partial Q$ and $f_1' = a_9 Q' + c_{17}\partial Q$ is a decomposable cocycle $a_9(Q - Q')$. Thus, it is sufficient to choose one Q for a cocycle ∂Q.

Now, if ∂Q is in the ∂^2-image, say $\partial Q = \partial^2 R$, we can choose $Q = \partial R$, but then

$$f_1 = a_9 \partial R + c_{17}\partial^2 R = dR\,.$$

By (3. 12. 2), the only cocycle of the form ∂Q but not of the form $\partial^2 R$ is a sum of

$$a_4 x_{36}{}^i x_{48}{}^j x_{108}{}^k, \quad a_8 x_{36}{}^i x_{48}{}^j x_{108}{}^k \quad \text{and} \quad \partial^2\text{-image } \partial^2 R'.$$

In particular, for $\partial Q = a_4$ and a_8, taking $Q = -b_{12}$ and $-b_{16}$ respectively, we have

$$-y_{21} = -a_9 b_{12} + c_{17} a_4 \quad \text{and} \quad -y_{25} = -a_9 b_{16} + c_{17} a_8\,.$$

For $Q = \sum \alpha(i, j, k) a_4 x_{36}{}^i x_{48}{}^j x_{108}{}^k + \sum \beta(i, j, k) a_8 x_{36}{}^i x_{48}{}^j x_{108}{}^k + \partial^2 R'$ $(\alpha(i, j, k),$ $\beta(i, j, k) \in \boldsymbol{Z}_3$ and $i, j, k = 0, 1, 2, \cdots)$ we have a decomposable cocycle

$$f_1 = -\sum \alpha(i, j, k) y_{21} x_{36}{}^i x_{48}{}^j x_{108}{}^k - \sum \beta(i, j, k) y_{25} x_{36}{}^i x_{48}{}^j x_{108}{}^k + dR'.$$

Thus we have

Result (v). *We obtain in* (v) *two new cocycles*

$$y_{21} = a_9 b_{12} - c_{17} a_4 \quad \textit{and} \quad y_{25} = a_9 b_{16} - c_{17} a_8\,.$$

Now we can express (4. 4. 2) more concretely as follows (incidentally we repeat (4. 4. 1) just for convenience).

Lemma 4. 4. (1) *For an element f_{2k} of degree $2k$, $df_{2k} = 0$ if and only if f_{2k} is of the form $x_{26}{}^k A$ with A a cocycle with neither a_9 nor c_{17}.*
(2) *For an element f_{2k+1} of degree $2k+1$, $df_{2k+1} = 0$ if and only if f_{2k+1} is a sum of*

$$x_{26}{}^k a_9 A, \quad x_{26}{}^k y_{21} x_{36}{}^i x_{48}{}^j x_{108}{}^h \quad \textit{and} \quad x_{26}{}^k y_{25} x_{36}{}^i x_{48}{}^j x_{108}{}^h,$$

where A is a cocycle with neither a_9 nor c_{17}.

Now that we have found all generators of $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$, we shall check the commutativity among them and seek relations between generators.

Proposition 4. 5. $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ *is commutative.*

Proof. To begin with we have the following d-images:

(4. 6. 1)
$$a_9^2 = dc_{17}, \quad y_{21}^2 = d(c_{17}b_{12}^2), \quad y_{25}^2 = d(c_{17}b_{16}^2),$$
$$a_9y_{21} + x_{26}a_4 = y_{21}a_9 - x_{26}a_4 = -[a_9, y_{21}] = d(c_{17}b_{12}),$$
$$a_9y_{25} + x_{26}a_8 = y_{25}a_9 - x_{26}a_8 = -[a_9, y_{25}] = d(c_{17}b_{16}),$$
$$y_{21}y_{25} + x_{26}y_{20} = y_{25}y_{21} - x_{26}y_{20} = -[y_{21}, y_{25}] = d(c_{17}b_{12}b_{16}),$$
$$[a_9, x_{26}] = d(c_{17}^2),$$
$$[y_{21}, x_{26}] = d(c_{17}^2b_{12}), \quad [y_{25}, x_{26}] = d(c_{17}^2b_{16}).$$

In $\overline{W}$, $[a_9, P] = c_{17}\partial P$ holds. If A is a cocycle with neither a_9 nor c_{17}, we have

(4. 6. 2) $[a_9, A] = 0.$

Since A commutes with a_9, c_{17}, a_4, a_8, b_{12} and b_{16}, we also have

(4. 6. 3) $[y_{21}, A] = 0$, $[y_{25}, A] = 0$ *and* $[x_{26}, A] = 0$.

Therefore commutativity holds in $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$. q.e.d.

Recall that the differential operator d augments the degree with respect to a_9 and c_{17} by 1. Therefore, $\sum f_l \in d$-image occurs with different degrees l only when each $f_l \in d$-image.

Lemma 4. 7. *The following elements are non-trivial:*
$$x_{26}^k x_{36}^i x_{48}^j x_{108}^h, \quad x_{26}^k y_{20} x_{36}^i x_{48}^j x_{108}^h,$$
$$x_{26}^k a_4 x_{36}^i x_{48}^j x_{108}^h, \quad x_{26}^k a_8 x_{36}^i x_{48}^j x_{108}^h,$$
$$x_{26}^k a_9 x_{36}^i x_{48}^j x_{108}^h, \quad x_{26}^k a_9 y_{20} x_{36}^i x_{48}^j x_{108}^h,$$
$$x_{26}^k y_{21} x_{36}^i x_{48}^j x_{108}^h, \quad x_{26}^k y_{25} x_{36}^i x_{48}^j x_{108}^h$$
and they are linearly independent, where k, i, j, h are non-negative integers.

Proof. By Lemma 4. 4 a cocycle f_{2k+1} of degree $2k+1$ is of the form
$$f_{2k+1} = x_{26}^k a_9 A + x_{26}^k \sum\alpha(i, j, h)\, y_{21} x_{36}^i x_{48}^j x_{108}^h + x_{26}^k \sum\beta(i, j, h)\, y_{25} x_{36}^i x_{48}^j x_{108}^h$$
$$= x_{26}^k (a_9 (A + \sum\alpha(i, j, h)\, b_{12} x_{36}^i x_{48}^j x_{108}^h + \sum\beta(i, j, h)\, b_{16} x_{36}^i x_{48}^j x_{108}^h)$$

$$+c_{17}(-\sum\alpha(i,j,h)a_4x_{36}{}^ix_{48}{}^jx_{108}{}^h-\sum\beta(i,j,h)a_8x_{36}{}^ix_{48}{}^jx_{108}{}^h)),$$

where $\alpha(i,j,h)$, $\beta(i,j,h)\in \boldsymbol{Z}_3$, A is a cocycle with neither a_9 nor c_{17} and $\sum\alpha(i,j,h)a_4x_{36}{}^ix_{48}{}^jx_{108}{}^h+\sum\beta(i,j,h)a_8x_{36}{}^ix_{48}{}^jx_{108}{}^h$ is not in the ∂^2-image by (3. 12. 2). On the other hand any element f_{2k} of degree $2k$ is written as in Lemma 4. 3:

$$f_{2k}=\sum_{i=0}^{k-1}x_{26}{}^ic_{17}f_{2k-2i-1}+x_{26}{}^kP+(d\text{-image}),$$

and its d-image is calculated as in (4. 3. 1):

$$df_{2k}=\sum_{i=0}^{k-1}x_{26}{}^ia_9{}^2f_{2k-2i-1}-\sum_{i=0}^{k-1}x_{26}{}^ic_{17}df_{2k-2i-1}+x_{26}{}^k(a_9\partial P+c_{17}\partial^2P).$$

Comparing our cocycle f_{2k+1} with df_{2k}, we see that f_{2k+1} is not in the d-image so long as it has a term $x_{26}{}^ky_{21}x_{36}{}^ix_{48}{}^jx_{108}{}^h$ or $x_{26}{}^ky_{25}x_{36}{}^ix_{48}{}^jx_{108}{}^h$. That is to say, $x_{26}{}^ky_{21}x_{36}{}^ix_{48}{}^jx_{108}{}^h$ and $x_{26}{}^ky_{25}x_{36}{}^ix_{48}{}^jx_{108}{}^h$ are non-trivial, and they and $x_{26}{}^ka_9A$ (if the latter is non-trivial) are linearly independent.

Comparing $x_{26}{}^ia_9A$ again with df_{2k}, we see that $x_{26}{}^ka_9A$ is in the d-image only when $A=\partial P$ and $x_{26}{}^ka_9\partial P=d(x_{26}{}^kP)$. Referring to (3. 12. 3), we see that

$$x_{26}{}^ka_9x_{36}{}^ix_{48}{}^jx_{108}{}^h,\qquad x_{26}{}^ka_9y_{20}x_{36}{}^ix_{48}{}^jx_{108}{}^h$$

and their sum are non-trivial.

A cocycle of degree $2k+2$ is, by Lemma 4. 4, of the form $x_{26}{}^{k+1}A$ where A is a cocycle with neither a_9 nor c_{17}. And any element f_{2k+1} of degree $2k+1$ is written as in Lemma 4. 3:

$$f_{2k+1}=\sum_{i=0}^{k-1}x_{26}{}^ic_{17}f_{2k-2i}+x_{26}{}^k(c_{17}P+a_9Q)+(d\text{-image}),$$

and its d-image is calculated as (4. 3. 2):

$$df_{2k+1}=\sum_{i=0}^{k-1}x_{26}{}^ia_9{}^2f_{2k-2i}-\sum_{i=0}^{k-1}x_{26}{}^ic_{17}df_{2k-2i}+x_{26}{}^ka_9{}^2(P-\partial Q)$$

$$-x_{26}{}^kc_{17}(a_9\partial(P-\partial Q)+c_{17}\partial^2P)-x_{26}{}^{k+1}\partial^2Q.$$

Comparing $x_{26}{}^{k+1}A$ with df_{2k+1}, we see that $x_{26}{}^{k+1}A$ is in the d-image only when $A=\partial^2(-Q)$, and then

$$x_{26}{}^{k+1}\partial^2(-Q)=d(x_{26}{}^k(a_9Q+c_{17}\partial Q)).$$

So by (3. 12. 2) and (3. 12. 3) we see that

$$x_{26}{}^kx_{36}{}^ix_{48}{}^jx_{108}{}^h,\qquad x_{26}{}^ky_{20}x_{36}{}^ix_{48}{}^jx_{108}{}^h,$$

$$x_{26}{}^{k}a_4x_{36}{}^{i}x_{48}{}^{j}x_{108}{}^{h},\quad x_{26}{}^{k}a_8x_{36}{}^{i}x_{48}{}^{j}x_{108}{}^{h}$$

and their sum are non-trivial. q.e.d.

We prove

Lemma 4.8. *A cocycle with a_9 and c_{17} is either trivial or a linear combination of cocyles in* Lemma 4.7.

Proof. Recall from (4.6.1) the relations:

$$a_9{}^2=dc_{17},\quad y_{21}{}^2=d(c_{17}b_{12}{}^2),\quad y_{25}{}^2=d(c_{17}b_{16}{}^2),$$
$$a_9y_{21}=-x_{26}a_4+d(c_{17}b_{12}),\quad a_9y_{25}=-x_{26}a_8+d(c_{17}b_{16}),$$
$$y_{21}y_{25}=-x_{26}y_{20}+d(c_{17}b_{12}b_{16}).$$

Thus any cocycle is reduced to a cocycle, each term of which has at most one of a_9, y_{21} or y_{25}.

We have shown that

(**4.9.1**) $x_{26}\partial^2Q=d(-a_9Q-c_{17}\partial Q)$,

and $a_9\partial P=dP$, in pariticular,

(**4.9.2**) $a_9\partial^2Q=d(\partial Q)$, $a_9a_4=d(-b_{12})$, $a_9a_8=d(-b_{16})$.

Finally, we have the following d-images:

(**4.9.3**)
$$y_{21}\partial^2Q=d(a_4Q+b_{12}\partial Q),\quad y_{25}\partial^2Q=d(a_8Q+b_{16}\partial Q),$$
$$y_{21}a_4=d(b_{12}{}^2),\quad y_{25}a_8=d(b_{16}{}^2),$$
$$y_{21}a_8+a_9y_{20}=y_{25}a_4-a_9y_{20}=d(b_{12}b_{16}),$$
$$y_{21}y_{20}=d(-b_{12}{}^2b_{16}),\quad y_{25}y_{20}=d(b_{12}b_{16}{}^2).$$

Using these relations we see that any monomial in cocycles is either trivial or equivalent to one of cocycles in Lemma 4.7. q.e.d.

Theorem 4.10. *For $A=H^*(X_7;\boldsymbol{Z}_3)$, we have as algebra*

$$\mathrm{Cotor}_A(\boldsymbol{Z}_3,\boldsymbol{Z}_3)\cong\boldsymbol{Z}_3[a_9,y_{21},y_{25},x_{26},a_4,a_8,a_{20},x_{36},x_{48},x_{84},x_{108},$$
$$y_{20},y_{32},y_{36},y_{68},y_{80},y_{84},y_{96},z_{56},z_{44},z_{48},z_{52},z_{68},z_{72},z_{60},$$
$$z_{64},z_{76},z_{104},z_{116},z_{120},z_{132},w_{80},w_{84},w_{88},w_{96},w_{100},$$
$$v_{96},v_{100},v_{108},v_{112},v_{116},u_{112},u_{124},u_{128},p_{140},p_{152},p_{156},p_{168}]/\rho,$$

where ρ is the ideal generated by

i) *elements which are* 0 *as polynomial in* $\boldsymbol{Z}_3[a_4,a_8,a_{20},b_{12},b_{16},b_{28},e_{36}]$,

ii) $a_9{}^2$, $y_{21}{}^2$, $y_{25}{}^2$,

$a_9y_{21}+x_{26}a_4$, $a_9y_{25}+x_{26}a_8$, $y_{21}y_{25}+x_{26}y_{20}$,

iii) $a_9\partial^2Q$, $y_{21}\partial^2Q$, $y_{25}\partial^2Q$, $x_{26}\partial^2Q$,

a_9a_4, a_9a_8, $y_{21}a_4$, $y_{25}a_8$, $y_{21}y_{20}$, $y_{25}y_{20}$,

$y_{21}a_8+a_9y_{20}=y_{25}a_4-a_9y_{20}=y_{21}a_8+y_{25}a_4$.

(See Results (ii), (iii-1), (iii-2), (iii-3) and (v) for the expression of the generators in terms of a_i's, b_j's, c_{17} and e_{36}. See also Remark 3.11 for practical use of the ∂^2-image.)

§ 5. $\mathrm{Cotor}_B(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ with $B=H^*(X_6; \boldsymbol{Z}_3)$

The Hopf algebra structure of $H^*(X_6; \boldsymbol{Z}_3)$ is very alike that of $H^*(X_7; \boldsymbol{Z}_3)$ ([6] and [7]):

(5.1) $H^*(X_6; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17})$,

where $\deg x_i = i$;

(5.2) $\bar{\phi}(x_i)=0$ *for* $i=3, 7, 8, 9$,

$\bar{\phi}(x_j)=x_8 \otimes x_{j-8}$ *for* $j=11, 15, 17$,

where $\bar{\phi}$ *is the reduced diagonal map induced from the multiplication on* X_6.

Notation. $B=H^*(X_6; \boldsymbol{Z}_3)$ and $\bar{B}=\tilde{H}^*(X_6; \boldsymbol{Z}_3)$.

We construct an injective resolution of $\boldsymbol{Z}_3$ over B quite similarly to that in § 2, taking M to be a graded $\boldsymbol{Z}_3$-submodule of $\bar{B}$ generated by

$$\{x_3, x_7, x_8, x_9, x_{11}, x_{15}, x_{17}, x_8^2\}$$

and naming the set of the corresponding elements under the suspension s as

(5.3) $sM=\{a_4, a_8, a_9, a_{10}, b_{12}, b_{16}, b_{18}, c_{17}\}$.

Now, we put

$$\bar{V}=T(sM)/J$$
$$=\boldsymbol{Z}_3\{a_4, a_8, a_9, a_{10}, b_{12}, b_{16}, b_{18}, c_{17}\}/J,$$

where J is the ideal generated by

(5.4) $[\alpha, \beta]$ *for all pairs* (α, β) *of generators of* $\bar{V}$ *except* (a_9, b_j) $(j=12, 16, 18)$ *and* (a_9, c_{17}),

$[a_9, b_j]+c_{17}a_{j-8}$ *for* $j=12, 16, 18$,

where $[\alpha, \beta]=\alpha\beta-(-1)^*\beta\alpha$ *with* $*=\deg\alpha\cdot\deg\beta$.

We construct the twisted tensor product $V=B\otimes\bar{V}$ as in § 2. Then the differential operators $\bar{d}$ in V and d in $\bar{V}$ are given by

(5.5) $\bar{d}(x_i\otimes 1)=1\otimes a_{i+1}$ for $i=3, 7, 8, 9$,

$\bar{d}(x_8^2\otimes 1)=1\otimes c_{17}-x_8\otimes a_9$

$\bar{d}(x_j\otimes 1)=1\otimes b_{j+1}+x_8\otimes a_{j-7}$ for $j=11, 15, 17$;

(5.6) $da_i=0$ for $i=3, 8, 9, 10$,

$dc_{17}=a_9^2$,

$db_j=-a_9a_{j-8}$ for $j=12, 16, 18$.

Note that $\bar{V}$ contains the polynomial algebra $S=\boldsymbol{Z}_3[a_4, a_8, a_{10}, b_{12}, b_{16}, b_{18}]$. Quite similarly as before one can show

Theorem 5.7. *V is an injective resolution of $\boldsymbol{Z}_3$ over $B=H^*(X_6; \boldsymbol{Z}_3)$.*

Corollary 5.8. $H(\bar{V}: d)=\mathrm{Ker}\, d/\mathrm{Im}\, d\cong \mathrm{Cotor}_B(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$.

Before we calculate $H(\bar{V}: d)$ we observe that the Hopf algebra $H^*(X_6; \boldsymbol{Z}_3)$ is obtained by replacing x_{19} and x_{27} in $H^*(X_7; \boldsymbol{Z}_3)$ with x_9 and x_{17} respectively and by omitting x_{35} in $H^*(X_7; \boldsymbol{Z}_3)$. This corresponds to the fact that $\bar{V}$ is obtained by replacing a_{20} and b_{28} in $\bar{W}$ with a_{10} and b_{18} respectively and by omitting e_{36} in $\bar{W}$. Thus the calculation of cocycles is done almost similarly to but more simply than the case of X_7.

Parallel to the case of X_7 we shall find cocycles in the following steps:

(i)′ cocycles in $\boldsymbol{Z}_3[a_4, a_8, a_{10}, b_{12}, b_{16}]$,

(ii)′ those in $\boldsymbol{Z}_3[a_4, a_8, a_{10}, b_{12}, b_{16}, b_{18}]$,

(iii)′ (this is not necessary, since there is no e_{36}),

(iv)′ those in $\boldsymbol{Z}_3\{a_9, c_{17}\}$,

(v)′ other cocycles.

In order to calculate $H(\bar{V}: d)$ we define an operator which we denote also by ∂:

(5.9) $\partial a_4=0,\quad \partial a_8=0,\quad \partial a_{10}=0,$

$\partial b_{12}=-a_4,\ \partial b_{16}=-a_8,\ \partial b_{18}=-a_{10},$

and extend it over $S=\boldsymbol{Z}_3[a_4, a_8, a_{10}, b_{12}, b_{16}, b_{18}]$ by (3.2). Lemmas 3.3 and 3.4 again hold for P in S.

(i)′, (ii)′, (iii)′ Cocycles in $\boldsymbol{Z}_3[a_4, a_8, a_{10}, b_{12}, b_{16}, b_{18}]$

The calculation of cocycles with neither a_9 nor c_{17} is as above except that we have no step (iii):

Proposition 5.10. *We have the following indecomposable cocycles with neither a_9 nor c_{17}:*

$$a_4,\ a_8,\ a_{10},$$

$$x_{36}=b_{12}^3,\ x_{48}=b_{16}^3,\ x_{54}=b_{18}^3,$$

$$y_{20}=a_8b_{12}-a_4b_{16},\ y_{22}=a_4b_{18}-a_{10}b_{12},\ y_{26}=a_8b_{18}-a_{10}b_{16},$$

$$y_{58}=\partial^2(b_{12}^2b_{16}^2b_{18}),\ y_{60}=\partial^2(b_{12}^2b_{16}b_{18}^2),$$

$$y_{64}=\partial^2(b_{12}b_{16}^2b_{18}^2),\ y_{76}=\partial^2(b_{12}^2b_{16}^2b_{18}^2).$$

Lemma 5.11. *The elements y_{58}, y_{60}, y_{64}, y_{76} and the following products appear in the ∂^2-image:*

$$a_4^2=\partial^2(-b_{12}^2),\quad a_8^2=\partial^2(-b_{16}^2),\quad a_{10}^2=\partial^2(-b_{18}^2),$$

$$a_4a_8=\partial^2(-b_{12}b_{16}),\quad a_4a_{10}=\partial^2(-b_{12}b_{18}),\quad a_8a_{10}=\partial^2(-b_{16}b_{18}),$$

$$a_4y_{20}=\partial^2(b_{12}^2b_{16}),\quad a_8y_{20}=\partial^2(-b_{12}b_{16}^2),$$

$$a_4y_{22}=\partial^2(-b_{12}^2b_{18}),\quad a_{10}y_{22}=\partial^2(b_{12}b_{18}^2),$$

$$a_8y_{26}=\partial^2(-b_{16}^2b_{18}),\quad a_{10}y_{26}=\partial^2(b_{16}b_{18}^2),$$

$$a_4y_{26}+a_8y_{22}=-a_8y_{22}+a_{10}y_{20}=-a_{10}y_{20}-a_4y_{26}=\partial^2(b_{12}b_{16}b_{18}),$$

$$y_{20}^2=\partial^2(-b_{12}^2b_{16}^2),\quad y_{22}^2=\partial^2(-b_{12}^2b_{18}^2),\quad y_{26}^2=\partial^2(-b_{16}^2b_{18}^2),$$

$$y_{20}y_{22}=\partial^2(b_{12}^2b_{16}b_{18}),\ y_{20}y_{26}=\partial^2(-b_{12}b_{16}^2b_{18}),\ y_{22}y_{26}=\partial^2(-b_{12}b_{16}b_{18}^2),$$

and $P\cdot\partial^2Q=\partial^2(PQ)$ for any cocycle P with neither a_9 nor c_{17}.

Lemma 5.11 is just the interpretation of the corresponding Lemma 3.8. Note the following:

(5.11.1) *The generators a_{10}, y_{22}, y_{26} and x_{54} correspond to $a_{20}=\partial^2e_{36}$, $y_{32}=\partial^2(-b_{12}e_{36})$, $y_{36}=\partial^2(-b_{16}e_{36})$ and $x_{84}=\partial^2(-b_{28}e_{36}^2)$ respectively in the case of X_7, but they are not in the ∂^2-image;*

(5.11.2) *In* Lemma 5.11 *we have a ∂^2-image*

$$a_{10}y_{20}-a_8y_{22}=-a_{10}y_{20}-a_4y_{26}=a_4y_{26}+a_8y_{22}=\partial^2(b_{12}b_{16}b_{18}),$$

though $a_{10}y_{20}$, a_8y_{22} and a_4y_{26} are not in the ∂^2-image, which did not occur in the final résumé of ∂^2-image in X_7 (Remark (3.5)).

Remark 5.12. *Of the generators,*

(1) *y_{58}, y_{60}, y_{64} and y_{76} are in the ∂^2-image;*

(2) $a_4=\partial(-b_{12})$, $a_8=\partial(-b_{16})$ *and* $a_{10}=\partial(-b_{18})$ *are in the* ∂*-image, but not in the* ∂^2*-image*;

(3) y_{20}, y_{22}, y_{26}, x_{36}, x_{48} *and* x_{54} *are not in the* ∂*-image.*

Using the above and Lemma 5. 11 we see that

(5. 12. 1) *Monomials in cocycles with neither* a_9 *nor* c_{17} *except the ones in* (5. 12. 2) *and* (5. 12. 3) *are in the* ∂^2*-image*;

(5. 12. 2)
$$a_4x_{36}^ix_{48}^jx_{54}^k,\ a_8x_{36}^ix_{48}^jx_{54}^k,\ a_{10}x_{36}^ix_{48}^jx_{54}^k,$$
$$a_4y_{26}x_{36}^ix_{48}^jx_{54}^k,\ a_8y_{22}x_{36}^ix_{48}^jx_{54}^k \text{ and } a_{10}y_{20}x_{36}^ix_{48}^jx_{54}^k$$

are in the ∂*-image, but not in the* ∂^2*-image*;

(5. 12. 3) $x_{36}^ix_{48}^jx_{54}^k$, $y_{20}x_{36}^ix_{48}^jx_{54}^k$, $y_{22}x_{36}^ix_{48}^jx_{54}^k$ *and* $y_{26}x_{36}^ix_{48}^jx_{54}^k$ *are not in the* ∂*-image.*

Note (cf. (5. 11. 2)) that

$$a_4y_{26}x_{36}^ix_{48}^jx_{54}^k=-a_{10}y_{20}x_{36}^ix_{48}^jx_{54}^k+(\partial^2\text{-image}),$$
$$a_8y_{22}x_{36}^ix_{48}^jx_{54}^k=a_{10}y_{20}x_{36}^ix_{48}^jx_{54}^k+(\partial^2\text{-image}).$$

From now on until the end of the calculation of $\mathrm{Cotor}_B(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$, we shall always replace $a_4y_{26}x_{36}^ix_{48}^jx_{54}^k$ and $a_8y_{22}x_{36}^ix_{48}^jx_{54}^k$ by the right hand sides of the above relations.

Such replacement done, we have the following:

(5. 13. 1) *A cocycle is in the* ∂^2*-image if and only if each term of the cocycle is in the* ∂^2*-image*;

(5. 13. 2) *A cocylce is in the* ∂*-image but not in the* ∂^2*-image if and only if it is of the form*

$$\begin{pmatrix}\text{a sum of } a_4x_{36}^ix_{48}^jx_{54}^k,\ a_8x_{36}^ix_{48}^jx_{54}^k, \\ a_{10}x_{36}^ix_{48}^jx_{54}^k \text{ and } a_{10}y_{20}x_{36}^ix_{48}^jx_{54}^k\end{pmatrix}+(\partial^2\text{-image});$$

(5. 13. 3) *A cocycle is not in the* ∂*-image if and only if it is of the form* (*a sum of monomials in* (5. 12. 3)) + (*any cocycle*).

(iv)′ Cocycles in $\boldsymbol{Z}_3\{a_9, c_{17}\}$

No change is needed in step (iv)′ and we obtain

Proposition 5. 13. *The elements* a_9 *and* $x_{26}=[a_9, c_{17}]$ *are the only indecomposable cocycles in* $\boldsymbol{Z}_3\{a_9, c_{17}\}$.

(v)$'$ **Other cocycles**

The argument is almost the same as in (v) and we have

(5. 14. 1) *For an element f_{2k} of degree $2k$ with respect to a_9 and c_{17}, $df_{2k}=0$ if and only if f_{2k} is of the form*

$$f_{2k}=x_{26}{}^k A \quad \textit{with } A \textit{ a cocycle with neither } a_9 \textit{ nor } c_{17};$$

(5. 14. 2) *For an element f_{2k+1} of degree $2k+1$ with respect to a_9 and c_{17}, $df_{2k+1}=0$ if and only if f_{2k+1} is of the form*

$$f_{2k+1}=x_{26}{}^k(a_9Q+c_{17}\partial Q) \quad \textit{with } \partial^2Q=0\,.$$

Using these formulae we see that we have only to determine cocycles of the form

$$f_1=a_9Q+c_{17}\partial Q \quad \text{with } \partial^2Q=0\,.$$

In case $\partial Q=0$, f_1 is a decomposable cocycle a_9Q with Q a cocycle.

In case $\partial Q\neq 0$, ∂Q is a cocycle as $\partial^2Q=0$. If $\partial Q=\partial^2R$ for some R, then choosing Q to be ∂R, we have $f_1=a_9\partial R+c_{17}\partial^2R=dR$, which is a trivial cocycle. By (5. 13. 2) a cocycle of the form ∂Q but not of the form ∂^2R is a sum of

$$a_4x_{36}{}^ix_{48}{}^jx_{54}{}^k,\ a_8x_{36}{}^ix_{48}{}^jx_{54}{}^k,\ a_{10}x_{36}{}^ix_{48}{}^jx_{54}{}^k,\ a_{10}y_{20}x_{36}{}^ix_{48}{}^jx_{54}{}^k \text{ and } (\partial^2\text{-image}).$$

In particular, for $a_4=\partial(-b_{12})$, $a_8=\partial(-b_{16})$ and $a_{10}=\partial(-b_{18})$, we have

$$y_{21}=a_9b_{12}-c_{17}a_4,\ y_{25}=a_9b_{16}-c_{17}a_8 \text{ and } y_{27}=a_9b_{18}-c_{17}a_{10}\,.$$

And for ∂Q of the form of a sum above, we have a sum of

$$-y_{21}x_{36}{}^ix_{48}{}^jx_{54}{}^k,\quad -y_{25}x_{36}{}^ix_{48}{}^jx_{54}{}^k,$$
$$-y_{27}a_{10}x_{36}{}^ix_{48}{}^jx_{54}{}^k,\quad -y_{27}y_{20}x_{36}{}^ix_{48}{}^jx_{54}{}^k \text{ and } (d\text{-image}).$$

Thus we have three new indecomposable cocycles with a_9 and c_{17}, namely, y_{21}, y_{25} and y_{27}.

Remark. *The cocycles y_{21} and y_{25} are the same as in* (v), *and the cocycle in* (v) *that corresponds to y_{27} is a trivial one $a_9b_{28}-c_{17}a_{20}=d(-e_{36})$.*

Looking at (5. 12. 1) $\sim$ (5. 12. 3), we see that we have found all cocycles to be found in (v)$'$.

Result (v)$'$. *We have three cocycles in step* (v)$'$:

$$y_{21}=a_9b_{12}-c_{17}a_4,\ y_{25}=a_9b_{16}-c_{17}a_8 \quad \textit{and} \quad y_{27}=a_9b_{18}-c_{17}a_{10}\,.$$

We have seen also

Lemma 5.14. (1) *For an element f_{2k} of degree $2k$, $df_{2k}=0$ if and only if f_{2k} is of the form $x_{26}^k A$ where A is a cocycle with neither a_9 nor c_{17};*
(2) *For an element f_{2k+1} of degree $2k+1$, $df_{2k+1}=0$ if and only if f_{2k+1} is of the form*

$$x_{26}^k\left\{a_9A+\begin{pmatrix}\text{a sum of } y_{21}x_{36}^i x_{48}^j x_{54}^h,\ y_{25}x_{36}^i x_{48}^j x_{54}^h \\ y_{27}x_{36}^i x_{48}^j x_{54}^h \text{ and } y_{27}y_{20}x_{36}^i x_{48}^j x_{54}^h,\end{pmatrix}\right\}$$

where A is a cocycle with neither a_9 nor c_{17}.

Here we have the following d-images:

(5.15.1) *In addition to the d-images in* (4.6.1) *we have*

$$y_{27}^2=d(c_{17}b_{18}^2),\ [y_{27}, x_{26}]=d(c_{17}^2b_{18}),$$

$$a_9y_{27}+x_{26}a_{10}=y_{27}a_9-x_{26}a_{10}=-[a_9, y_{27}]=d(c_{17}b_{18}),$$

$$y_{21}y_{27}-x_{26}y_{22}=y_{27}y_{21}+x_{26}y_{22}=-[y_{21}, y_{27}]=d(c_{17}b_{12}b_{18}),$$

$$y_{25}y_{27}-x_{26}y_{26}=y_{27}y_{25}+x_{26}y_{26}=-[y_{25}, y_{27}]=d(c_{17}b_{12}b_{18}).$$

In addition to the relations (4.6.2) and (4.6.3) we have similarly

(5.15.2) $[y_{27}, P]=0$ *for P with neither a_9 nor c_{17}.*

Thus we have

Proposition 5.16. $\mathrm{Cotor}_B(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ *is commutative.*

Lemma 5.17. *The following elements are non-trivial and they are linearly independent.*

$$x_{26}^k x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k a_4x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k a_8x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k a_{10}x_{36}^i x_{48}^j x_{54}^h,$$

$$x_{26}^k y_{20}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k y_{22}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k y_{26}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k a_{10}y_{20}x_{36}^i x_{48}^j x_{54}^h,$$

$$x_{26}^k a_9x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k a_9y_{20}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k a_9y_{22}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k a_9y_{26}x_{36}^i x_{48}^j x_{54}^h,$$

$$x_{26}^k y_{21}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k y_{25}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k y_{27}x_{36}^i x_{48}^j x_{54}^h,\quad x_{26}^k y_{27}y_{20}x_{36}^i x_{48}^j x_{54}^h.$$

Proof. The argument is the same as in the proof of Lemma 4.7. A cocycle f_{2k+1} of degree $2k+1$ is, by Lemma 5.14, of the form

$$x_{26}^k a_9A+\begin{pmatrix}\text{a sum of } x_{26}^k y_{21}x_{36}^i x_{48}^j x_{54}^h,\ x_{26}^k y_{25}x_{36}^i x_{48}^j x_{54}^h, \\ x_{26}^k y_{27}x_{36}^i x_{48}^j x_{54}^h \text{ and } x_{26}^k y_{27}y_{20}x_{36}^i x_{48}^j x_{54}^h\end{pmatrix}$$

$$=x_{26}^k(a_9Q+c_{17}\partial Q),$$

where the ∂Q was taken not to be in the ∂^2-image and A is a cocycle with neither a_9 nor c_{17}. Comparing f_{2k+1} with df_{2k} as in the proof of Lemma 4. 7, we see that such an f_{2k+1} is not in the d-image so long as it has a term $x_{26}{}^k y_{21} x_{36}{}^i x_{48}{}^j x_{54}{}^h$, $x_{26}{}^k y_{25} x_{36}{}^i x_{48}{}^j x_{54}{}^h$, $x_{26}{}^k y_{27} x_{36}{}^i x_{48}{}^j x_{54}{}^h$ or $x_{26}{}^k y_{27} y_{20} x_{36}{}^i x_{48}{}^j x_{54}{}^h$. In other words, $x_{26}{}^k y_{21} x_{36}{}^i x_{48}{}^j x_{54}{}^h$, $x_{26}{}^k y_{25} x_{36}{}^i x_{48}{}^j x_{54}{}^h$, $x_{26}{}^k y_{27} x_{36}{}^i x_{48}{}^j x_{54}{}^h$ and $x_{26}{}^k y_{27} y_{20} x_{36}{}^i x_{48}{}^j x_{54}{}^h$ are non-trivial and they and $x_{26}{}^k a_9 A$ (if it is non-trivial) are linearly independent.

Comparing $x_{26}{}^k a_9 A$ again with df_{2k}, we see that $x_{26}{}^k a_9 A$ is in the d-image only when $A=\partial P$, and then $x_{26}{}^k a_9 \partial P = d(x_{26}{}^k P)$. Referring to (5. 12. 3), we see that $x_{26}{}^k a_9 x_{36}{}^i x_{48}{}^j x_{54}{}^h$, $x_{26}{}^k a_9 y_{21} x_{36}{}^i x_{48}{}^j x_{54}{}^h$, $x_{26}{}^k a_9 y_{22} x_{36}{}^i x_{48}{}^j x_{54}{}^h$, $x_{26}{}^k a_9 y_{26} x_{36}{}^i x_{48}{}^j x_{54}{}^h$ and their sum are non-trivial.

Again by Lemma 5. 14, a cocycle of degree $2k+2$ is of the form $x_{26}{}^{k+1}A$, where A is a cocycle with neither a_9 nor c_{17}. Comparing $x_{26}{}^{k+1}A$ with df_{2k+1} as in the proof of Lemma 4. 7, we see that $x_{26}{}^{k+1}A$ is in the d-image only when $A=\partial^2(-Q)$, and then

$$x_{26}{}^{k+1}\partial^2(-Q) = d(x_{26}{}^k(a_9 Q + c_{17}\partial Q)).$$

By (5. 12. 1) $\sim$ (5. 12. 3) we see that

$$x_{26}{}^k x_{36}{}^i x_{48}{}^j x_{54}{}^h, \quad x_{26}{}^k a_4 x_{36}{}^i x_{48}{}^j x_{54}{}^h, \ x_{26}{}^k a_8 x_{36}{}^i x_{48}{}^j x_{54}{}^h, \ x_{26}{}^k a_{10} x_{36}{}^i x_{48}{}^j x_{54}{}^h,$$
$$x_{26}{}^k y_{20} x_{36}{}^i x_{48}{}^j x_{54}{}^h, \ x_{26}{}^k y_{22} x_{36}{}^i x_{48}{}^j x_{54}{}^h, \ x_{26}{}^k y_{26} x_{36}{}^i x_{48}{}^j x_{54}{}^h, \ x_{26}{}^k a_{10} y_{20} x_{36}{}^i x_{48}{}^j x_{54}{}^h$$

and their sum are non-trivial.

We have shown that the elements in the lemma are non-trivial and that they are linearly independent. q.e.d.

Finally,

Lemma 5. 18. *A cocycle with a_9 and c_{17} is either trivial or a linear combination of the cocycles in Lemma* 5. 17.

Proof. Recall from (4. 6. 1) and (5. 15. 1) the relations

$$a_9{}^2 = dc_{17}, \quad y_{21}{}^2 = d(c_{17}b_{12}{}^2), \quad y_{25}{}^2 = d(c_{17}b_{16}{}^2), \quad y_{27}{}^2 = d(c_{17}b_{18}{}^2),$$

$$a_9 y_{21} = -x_{26}a_4 + d(c_{17}b_{12}), \quad a_9 y_{25} = -x_{26}a_8 + d(c_{17}b_{16}), \ a_9 y_{27} = -x_{26}a_{10} + d(c_{17}b_{18}),$$

$$y_{21}y_{25} = -x_{26}y_{20} + d(c_{17}b_{12}b_{16}), \ y_{21}y_{27} = x_{26}y_{22} + d(c_{17}b_{12}b_{18}), \ y_{25}y_{27} = x_{26}y_{26} + d(c_{17}b_{16}b_{18}).$$

Therefore, any monomial in cocycles is reduced to a monomial that has at most one of a_9, y_{21}, y_{25} or y_{27}.

We have shown that

(5. 19. 1) $x_{26}\partial^2 Q = d(-a_9 Q - c_{17}\partial Q)$,

and $a_9 \partial Q = dQ$, in particular,

(5. 19. 2) $a_9\partial^2 Q=d(\partial Q)$, $a_9a_4=d(-b_{12})$, $a_9a_8=d(-b_{16})$, $a_9a_{10}=d(-b_{18})$.

Finally we have the following d-images:

(5. 19. 3)
$$y_{21}\partial^2 Q=d(a_4Q+b_{12}\partial Q),\quad y_{25}\partial^2 Q=d(a_8Q+b_{16}\partial Q),$$
$$y_{27}\partial^2 Q=d(a_{10}Q+b_{18}\partial Q),$$
$$y_{21}a_4=d(b_{12}^2),\qquad y_{25}a_8=d(b_{16}^2),\qquad y_{27}a_{10}=d(b_{18}^2),$$
$$y_{21}a_8+a_9y_{20}=y_{25}a_4-a_9y_{20}=d(b_{12}b_{16}),$$
$$y_{21}a_{10}-a_9y_{22}=y_{27}a_4+a_9y_{22}=d(b_{12}b_{18}),$$
$$y_{25}a_{10}-a_9y_{26}=y_{27}a_8+a_9y_{26}=d(b_{16}b_{28}),$$
$$y_{21}y_{20}=d(-b_{12}^2b_{16}),\qquad y_{25}y_{20}=d(b_{12}b_{16}^2),$$
$$y_{21}y_{22}=d(b_{12}^2b_{18}),\qquad y_{27}y_{22}=d(-b_{12}b_{18}^2),$$
$$y_{25}y_{26}=d(b_{16}^2b_{18}),\qquad y_{27}y_{26}=d(-b_{16}b_{18}^2),$$
$$y_{27}y_{20}-y_{25}y_{22}=y_{25}y_{22}+y_{21}y_{26}=-y_{27}y_{20}-y_{21}y_{26}=d(-b_{12}b_{16}b_{18}).$$

Using these relations we see that any monomial in cocycles with a_9 and c_{17} is either trivial or equivalent to one of the elements in Lemma 5. 17.

q.e.d.

Thus we have

Theorem 5. 20. *For $B=H^*(X_6; \boldsymbol{Z}_3)$, we have as algebra*

$$\mathrm{Cotor}_B(\boldsymbol{Z}_3, \boldsymbol{Z}_3)\cong \boldsymbol{Z}_3[a_9, x_{26}, y_{21}, y_{25}, y_{27}, a_4, a_8, a_{10}, x_{36}, x_{48}, x_{54},$$
$$y_{20}, y_{22}, y_{26}, y_{58}, y_{60}, y_{64}, y_{76}]/\eta,$$

where η is the ideal generated by

i) *elements which are 0 as polynomial in $\boldsymbol{Z}_3[a_4, a_8, a_{10}, b_{12}, b_{16}, b_{18}]$;*

ii) a_9^2, y_{21}^2, y_{25}^2, y_{27}^2,
$a_9y_{21}+x_{26}a_4$, $a_9y_{25}+x_{26}a_8$, $a_9y_{27}+x_{26}a_{10}$,
$y_{21}y_{25}+x_{26}y_{20}$, $y_{21}y_{27}-x_{26}y_{22}$, $y_{25}y_{27}-x_{26}y_{26}$,

iii) $a_9\partial^2 Q$, $y_{21}\partial^2 Q$, $y_{25}\partial^2 Q$, $y_{27}\partial^2 Q$, $x_{26}\partial^2 Q$,
a_9a_4, a_9a_8, a_9a_{10},
$y_{21}a_4$, $y_{25}a_8$, $y_{27}a_{10}$,
$y_{21}a_8+a_9y_{20}=y_{25}a_4-a_9y_{20}$,
$y_{21}a_{10}-a_9y_{22}=y_{27}a_4+a_9y_{22}$,
$y_{25}a_{10}-a_9y_{26}=y_{27}a_8+a_9y_{26}$,
$y_{21}y_{20}$, $y_{25}y_{20}$, $y_{21}y_{22}$, $y_{27}y_{22}$, $y_{25}y_{26}$, $y_{27}y_{26}$,
$y_{27}y_{20}-y_{25}y_{22}=y_{25}y_{22}+y_{21}y_{26}=-y_{27}y_{20}-y_{21}y_{26}$.

(See Result (v)′ and Propositions 5.10 and 5.13 for the expression of the generators in terms of a_i's, b_j's and c_{17}. See also Remark 5.12 for practical use of the ∂^2-image.)

INSTITUTE OF MATHEMATICS
YOSHIDA COLLEGE
KYOTO UNIVERSITY;
FACULTY OF MANAGEMENT
SENSHU UNIVERSITY

References

[1] S. Araki: On the non-commutativity of Pontrjagin rings mod 3 of some compact exceptional groups, Nagoya Math. J., 17 (1960), 225-260.
[2] E. H. Brown Jr.: Twisted tensor product I, Ann. Math., 69 (1959), 223-246.
[3] A. Iwai - A. Shimada: A remark on resolutions for Hopf algebras, Publ. RIMS of Kyoto Univ., 1 (1966), 187-198.
[4] A Kono: Hopf algebra structure of simple Lie groups, J. Math. Kyoto Univ., 17 (1977), 259-298.
[5] A. Kono - M. Mimura: Cohomology mod 2 of the classifying space of the compact connected Lie group of type E_6, J. Pure and Applied Algebra, 6 (1975), 61-81.
[6] A. Kono - M. Mimura: Cohomology mod 3 of the classifying space of the compact, 1-connected Lie group of type E_6, Preprint Series 1974/75 No. 30., Matematisk Institut, Aarhus Universitet.
[7] A. Kono - M. Mimura: Cohomology operations and the Hopf algebra structure of the compact exceptional Lie groups E_7 and E_8, Proc. London Math. Soc., (3) 35 (1977), 345-358.
[8] A. Kono - M. Mimura - N. Shimada: Cohomology of classifying space of certain associative H-spaces, J. Math. Kyoto Univ., 15 (1975), 607-617.
[9] A. Kono - M. Mimura - N. Shimada: On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E_7, J. Pure and Applied Algebra, 8 (1976), 267-283.
[10] M. Mori: A note on Steenrod operations in the Eilenberg-Moore spectral sequence, Proc. Japan Acad., 53 (1977), 112-114.
[11] M. Mori: The Steenrod operations in the Eilenberg-Moore spectral sequence, Hiroshima Math. J., 9 (1979), 17-34.
[12] M. Rothenberg - N. E. Steenrod: The cohomology of classifying spaces of H-spaces, Bull. AMS, 71 (1965), 872-875.
[13] M. Rothenberg - N. E. Steenrod: The cohomology of classifying spaces of H-spaces, (mimeographed notes).
[14] N. Shimada - A. Iwai: On the cohomology of some Hopf algebra, Nagoya Math. J., 30 (1971), 103-111.