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Normal coverings of hyperelliptic Riemann surfaces

By

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§ 1. Introduction

As to the hyperellipticity of unramified normal coverings of hyperelliptic Riemann surfaces, H. M. Farkas [2] firstly showed that any unramified two-sheeted covering of a compact Riemann surface of genus 2 is necessarily hyperelliptic. R. D. M. Accola [1] also obtained this result and proved that an unramified two-sheeted covering of a hyperelliptic surface of genus 3 is either hyperelliptic or a ramified covering of a torus.

Later, C. Machlachlan [7] showed using Fuchsian groups that if an n-sheeted unramified normal covering of a hyperelliptic surface of genus $g\geq 2$ is hyperelliptic, then $n=2$ or 4. This was also obtained by Farkas [3] simply using properties of Weierstraß points on hyperelliptic surfaces.

Machlachlan [7] also proved that an unramified two-sheeted covering of a hyperelliptic surface of genus $g\geq 2$ is not necessarily hyperelliptic. In this connection, Farkas [3] proved using Jacobian varieties that there are exactly $\binom{2g+2}{2}$ different unramified two-sheeted coverings which are also hyper elliptic.

Following this, T. Kato [5] showed using Hurwitz representations of automorphisms on hyperelliptic surfaces that there are exactly $\binom{2g+2}{3}$ different unramified four-sheeted coverings of a hyperelliptic surface of genus g which are also hyperelliptic and proved that if an abelian covering of a compact Riemann surface of genus $g\geq 2$ is hyperelliptic, then the basic surface is also hyperelliptic. Recently, H. H. Martens [8] pointed out that the last statement is true of any coverings.

In this paper, we shall investigate normal coverings of hyperelliptic surfaces which are also hyperelliptic (including the ramified case) and determine all such coverings by means of Hurwitz representations of automorphisms on hyperelliptic surfaces following Kato [5].

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§ 2. Hyperelliptic Riemann surfaces

We recall that a hyperelliptic surface S of genus $g\geq 2$ is defined by an algebraic equation $w^2=f(z)$, where f is a polynomial of z and has only simple zeros. S can be considered as a two-sheeted covering of the sphere and $2g+2$ Weierstraß points of S are exactly the branch points of the covering. The projection from S to the sphere is z. We denote the set of all Weierstraß points of S by W. The degree of f is $2g+1$ or $2g+2$ according as the point at infinity is a branch point or not. S admits the hyperelliptic involution J represented as $J(z, w)=(z, -w)$ and as it is central in the automorphism group of S, every conformal automorphism T of S induces an elliptic linear transformation T^* of the sphere.

Hurwitz [4] showed that a hyperelliptic surface admitting an automorphism T of order n can be defined by

$$w^2=g(z^n) \quad \text{or} \quad w^2=zg(z^n)$$

where $g(z)$ is a polynomial of z. The automorphism T is represented as

$$T(z, w)=(\varepsilon z, \pm w)$$

in the first case and

$$T(z, w)=(\varepsilon z, \pm\varepsilon^{1/2}w)$$

in the second case, where ε is a primitive n-th root of unity.

Our purpose is to determine all hyperelliptic surfaces which are normal coverings of a hyperelliptic surface. We assume that $\tilde{S}$ is a hyperelliptic surface of genus $\tilde{g}$ which is an n-sheeted normal covering of a compact Riemann surface S of genus $g\geq 2$. Let G be the covering transformation group of $\tilde{S}$, then the order of G is n and $\tilde{S}/G$ can be identified with S. By the remark of Martens [8], S is hyperelliptic. Thus G cannot contain J. For, if G contains J, the genus of S must be 0, which contradicts our hypothesis. Using the same remark, it is seen that the covering map from $\tilde{S}$ to S can be represented as

$$Z=h(z), \quad W=k(z)w$$

where $\tilde{S}$ and S are represented as $w^2=f(z)$ and $W^2=g(Z)$ respectively and h and k are rational functions of z.

The homomorphism $T\rightarrow T^*$ from G to the linear transformation group is in fact an isomorphism from G to the subgroup G^* of the linear transformation group which contains only elliptic transformations of finite order, since G does not contain J. It is well known that finite subgroups of the linear transformation group are cyclic groups, dihedral groups, tetrahedral group, octahedral group and icosahedral group. Thus we only must look the cases in which G is one of these five groups.

§ 3. The cyclic groups

We assume G is a cyclic group of order n generated by T. By normalizing the fixed points of T^* to be 0 and ∞ and exchanging generator, we may assume

$$T^*(z) = \varepsilon z$$

where $\varepsilon = e^{2\pi i/n}$. From now on, we always assume that this normalization has been done.

According as 0 and ∞ belong to $z(W)$ or not, we may consider that the defining equations of $\tilde{S}$ have following forms.

(1) In case 0 and ∞ do not belong to $z(W)$,

$$w^2 = \prod_{i=1}^{m} (z^n - a_i), \qquad \text{where } mn = 2\tilde{g} + 2 .$$

(2) In case 0 belongs to $z(W)$ and ∞ does not,

$$w^2 = z \prod_{i=1}^{m} (z^n - a_i), \qquad \text{where } mn + 1 = 2\tilde{g} + 2 .$$

(3) In case 0 and ∞ belong to $z(W)$,

$$w^2 = z \prod_{i=1}^{m} (z^n - a_i), \qquad \text{where } mn + 2 = 2\tilde{g} + 2 .$$

Note that in each case, $a_i \neq 0$ and if $i \neq j$, then $a_i \neq a_j$.

First we consider case (1). T is represented as

$$T(z, w) = (\varepsilon z, \pm w),$$

so that we define T_1 and T_2 to be

$$T_1(z, w) = (\varepsilon z, w), \; T_2(z, w) = (\varepsilon z, -w)$$

and we shall see that the number t of the fixed points of T is 0, 2 or 4. The number m plays an important rôle to decide the number of the fixed points of T.

Lemma 1. *Suppose m is odd, then T_1 has two fixed points lying over 0 and T_2 has two fixed points lying over ∞. Suppose m is even, then T_1 has four fixed points lying over 0 and ∞ and T_2 has no fixed point.*

Proof. It is obvious that T_1 fixes two points lying over 0. Suppose m is odd. To see that two points lying over ∞ are interchanged by T_1, we choose a curve L on $\tilde{S}$ which is the lifting of a line $z(L)$ connecting 0 with ∞ and so that joins a point lying over 0 with a point lying over ∞ and does not pass any Weierstraß points. If two points lying over ∞ are fixed by T_1, then $L \cdot T_1(L)^{-1}$ is a closed curve on $\tilde{S}$ and m is the number of points of

$z(W)$ which $z(L\cdot T_1(L)^{-1})$ encircles. This is a contradiction, for m is an odd integer. The rest of the lemma will be proved similarly.

If $t=0$, then $T=T_2$ and m is even by Lemma 1 and as

$$(z, w) = T^n(z, w) = (\varepsilon^n z, (-1)^n w),$$

$n=2n'$, where n' is a positive integer. The degrees of ramification of four points which lie over 0 and ∞ are fixed by T^2 are all $n'-1$. Thus the Riemann-Hurwitz formula gives

$$2\tilde{g}-2=2n'(2g-2)+4(n'-1)$$

and hence

$$2mn'=2\tilde{g}+2=2n'(2g-2)+4n',$$

$$m=2g\,.$$

This agrees with our condition and S is defined by

$$W^2=Z\prod_{i=1}^{2g}(Z-a_i)\ ,$$

where the projection π is $\pi(z, w) = (Z, W) = (z^n, z^{n/2}w)$. Note that the two points on S which lie over 0 and ∞ are Weierstraß points.

If $t=2$, then $T(z, w) = (\varepsilon z, \pm w)$ and m is odd by Lemma 1. We first examine the case where $T=T_1$. By Lemma 1, the two points lying over 0 are fixed and the two points lying over ∞ are interchanged by T_1. From this fact, $n=2n'$ and the degrees of ramification of two points lying over 0 are both $n-1$ and those of two points lying over ∞ are both $n'-1$. The similar computation as in the case $t=0$ gives

$$m=2g+1$$

and S is defined by

$$W^2=\prod_{i=1}^{2g+1}(Z-a_i)$$

where the projection π is $\pi(z, w) = (Z, W) = (z^n, w)$. Note that the two points on S which lie over 0 are non Weierstraß points and the point which lies over ∞ is a Weierstraß point. The case $T=T_2$ can be treated in a similar manner. This time S is defined by

$$W^2=Z\prod_{i=1}^{2g+1}(Z-a_i)\ ,$$

where the projection π is $\pi(z, w) = (Z, W) = (z^n, z^{n/2}w)$. The point on S which lies over 0 is a Weierstraß point and the two points which lie over ∞ are non Weierstraß points.

If $t=4$, then $T(z, w)=(\varepsilon z, w)$ and m is even. The four points lying over 0 and ∞ are fixed and their degrees of ramification are all $n-1$. Hence $m=2g+2$ and S is defined by

$$W^2=\prod_{i=1}^{2g+2}(Z-a_i),$$

where the projection π is $\pi(z, w)=(Z, W)=(z^n, w)$. The four points on S which lie over 0 and ∞ are non Weierstraß points and n is an arbitrary integer greater than 1.

Next we consider case (2) and (3). In these cases T is represented as

$$T(z, w)=(\varepsilon z, \pm\varepsilon^{1/2}w), \quad \text{where } \varepsilon^{1/2}=e^{\pi i/n}$$

so that we define T_1 and T_2 to be

$$T_1(z, w)=(\varepsilon z, \varepsilon^{1/2}w), \; T_2(z, w)=(\varepsilon z, -\varepsilon^{1/2}w).$$

If $T=T_1$, then

$$(z, w)=T^n(z, w)=(\varepsilon^n z \; \varepsilon^{n/2}w)$$

and hence

$$1=\varepsilon^{n/2}=-1.$$

This is a contradiction and so $T=T_2$.

We must as well count up the fixed points of T.

In case (2), we have $mn=2\tilde{g}+1$. Thus m and n are both odd. T fixed the Weierstraß point lying over 0 and the two points lying over ∞. For, as n is odd, the two points lying over ∞ are not interchanged by T. The degrees of ramification of these three points are all $n-1$. Hence

$$m=2g+1$$

and S is defined by

$$W^2=Z\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is $\pi(z, w)=(Z, W)=(z^n, z^{(n-1)/2}w)$. The point on S which lies over 0 is a Weierstraß point and the two points which lie over ∞ are non Weierstraß points.

In case (3), we have $mn+2=2\tilde{g}+2$. Hence $mn=2\tilde{g}$. If n is even,

$$(z, w)=T^n(z, w)=(\varepsilon^n z, (-\varepsilon^{1/2})^n w)=(z, -w),$$

which gives a contradiction. Thus n is odd and m is even. Then T fixes the two Weierstraß points lying over 0 and ∞ and their degrees of ramification are both $n-1$. Hence

$$m=2g$$

and S is defined by

$$W^2=Z\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is $\pi(z, w)=(Z, W)=(z^n, z^{(n-1)/2}w)$. The two points on S which lie over 0 and ∞ are Weierstraß points.

Most part of the above study is due to Kato [5].

We summarize the obtained results in the next two theorems.

Theorem 1. *An n-sheeted ($n\geq 2$)* **cyclic** *covering of a hyperelliptic surface which is also hyperelliptic can be represented as follows. ($a_i\neq 0$ and $a_i\neq a_j$ if $i\neq j$).*

(1) *n: arbitrary.*

$$\pi(z, w)=(Z, W)=(z^n, w), \qquad T(z, w)=(\varepsilon z, w).$$

$$\tilde{S}:\ w^2=\prod_{i=1}^{2g+2}(z^n-a_i), \qquad S:\ W^2=\prod_{i=1}^{2g+2}(Z-a_i).$$

(2) *n: even.*

$$\pi(z, w)=(Z, W)=(z^n, z^{n/2}w), \qquad T(z, w)=(\varepsilon z, -w).$$

$$\tilde{S}_1:\ w^2=\prod_{i=1}^{2g}(z^n-a_i), \qquad S_1:\ W^2=Z\prod_{i=1}^{2g}(Z-a_i)$$

$$\tilde{S}_2:\ w^2=\prod_{i=1}^{2g+1}(z^n-a_i), \qquad S_2:\ W^2=Z\prod_{i=1}^{2g+1}(Z-a_i).$$

(3) *n: odd.*

$$\pi(z, w)=(Z, W)=(z^n, z^{(n-1)/2}w), \qquad T(z, w)=(\varepsilon z, -\varepsilon^{1/2}w)$$

$$\tilde{S}_1:\ w^2=z\prod_{i=1}^{2g}(z^n-a_i), \qquad S_1:\ W^2=Z\prod_{i=1}^{2g}(Z-a_i)$$

$$\tilde{S}_2:\ w^2=z\prod_{i=1}^{2g+1}(z^n-a_i), \qquad S_2:\ W^2=Z\prod_{i=1}^{2g+1}(Z-a_i).$$

Theorem 1′. *The types of the coverings in Theorem 1 are as follows*

(a) *Choose two non Weierstraß points P_1 and P_2, then we have a covering with four branch points of degree $n-1$ lying over P_1, P_2, $J(P_1)$ and $J(P_2)$.*

(b) *Choose a non Weierstraß point P and a Weierstraß point Q. Suppose n is odd, we have a covering with three branch points of degree $n-1$ over P, $J(P)$ and Q. Suppose n is even, we have a covering with two branch points of degree $n-1$ lying over P and $J(P)$ and two branch points of degree $n/2-1$ lying over Q.*

(c) *Choose two Weierstraß points Q_1 and Q_2. Suppose n is odd, we* *have a covering with two branch points of degree $n-1$ lying over Q_1 and Q_2. Suppose n is even, we have a covering with four branch points of degree $n/2-1$ lying over Q_1 and Q_2.*

Remark. If we set $n=2$ in (c), then we have $\binom{2g+2}{2}$ unramified two-sheeted coverings, which were discussed by Farkas in [3]. By considering two-valued functions $\sqrt{Z}$ and $\sqrt{Z-a_i}$ $(i=1, \cdots, 2g)$. it follows that different pairs of Q_1 and Q_2 on S give different unramified two-sheeted coverings of S [5].

As a corollary from our considerations in case (2) and (3), each automorphism of even order not generating J cannot fix any Weierstraß points on hyperelliptic surfaces. This can be generalized: If an automorphism T of prime order p on a compact Riemann surface of genus $g\geq 2$ fixes the Weierstraß point whose first non gap is p, then T can be represented as a covering transformation of the covering over the sphere associated with the Weierstraß point and its first nongap p.

§4. The dihedral groups

We assume G is generated by T of order n and U of order 2.

In case $z(W)$ does not contain 0 and ∞, since W is invariant under any automorphisms of S, we may suppose that T and U are represented as

$$T(z, w) = (\varepsilon z, w), \ U(z, w) = (1/z, w/z^{nm})$$

and $\tilde{S}$ is defined by

$$w^2 = \prod_{i=1}^{m} (z^{2n} - (4a_i - 2) z^n + 1) \qquad a_i \neq 0, 1 .$$

We shall examine all possible cases of n, T and U and determine the numbers m, the defining equations of S and the projections π from $\tilde{S}$ to S. Here we define T_1, T_2, U_1 and U_2 to be

$$T_1(z, w) = (\varepsilon z, w), \ T_2(z, w) = (\varepsilon z, -w),$$

$$U_1(z, w) = (1/z, w/z^{nm}), \ U_2(z, w) = (1/z, -w/z^{nm}).$$

(A) In case n is even and

(A_1) $T=T_1$ and $U=U_1$, then $TU(z, w) = (\varepsilon/z, w/z^{nm})$. T fixes four points lying over 0 and ∞, U fixes two points lying over 1.

If m is even, TU fixes two points lying over $\varepsilon^{1/2}$ and the statements for U and TU are true of congruent points of 1 and $\varepsilon^{1/2}$ so that the Riemann-Hurwitz formula gives

$$2\tilde{g}-2 = 2n(2g-2) + 4(n-1) + 2n(2-1) + 2n(2-1).$$

Now we have

$$2\tilde{g}+2=2mn .$$

Hence

$$m=2g+2$$

and S is defined by

$$W^2=\prod_{i=1}^{2g+2}(Z-a_i),$$

where the projection π is

$$\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, w/(4z^n)^{g+1}).$$

If m is odd, TU has no fixed point. Then

$$m=2g+1$$

and S is defined by

$$W^2=Z\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, (z^n+1)w/(4z^n)^{g+1}).$$

Note that π maps $z=0$ and ∞, congruent points of 1 and congruent points of $\varepsilon^{1/2}$ to $Z=\infty$, 1 and 0 respectively.

The corresponding results for the remaining cases are listed belows.

(A_2) $T=T_1$ and $U=U_2$, then $TU(z, w) = (\varepsilon/z, -w/z^{nm})$.

m: even.

$$m=2g$$

$$W^2=Z(Z-1)\prod_{i=1}^{2g}(Z-a_i)$$

$$\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, (z^{2n}-1)w/(4z^n)^{g+1}).$$

m: odd.

$$m=2g+1$$

$$W^2= (Z-1)\prod_{i=1}^{2g+1}(Z-a_i)$$

$$\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, (z^n-1)/(4z^n)^{g+1}).$$

(A_3) $T=T_2$ and $U=U_1$, then $TU(z, w) = (\varepsilon/z, -w/z^{nm})$.

m: even.

$m=2g$

$W^2=Z\prod_{i=1}^{2g}(Z-a_i)$

$\pi(z, w)=(Z, W)=((z^n+1)^2/4z^n,\ (z^n+1)w/(2z^{n/2})^{2g+1})$.

m: odd.

$m=2g+1$

$W^2=\prod_{i=1}^{2g+1}(Z-a_i)$

$\pi(z, w)=(Z, W)=((z^n+1)^2/4z^n,\ w/(2z^{n/2})^{2g+1})$.

(A_4) $T=T_2$ and $U=U_2$, then $TU(z, w)=(\varepsilon/z, w/z^{nm})$

m: even.

$m=2g$

$W^2=(Z-1)\prod_{i=1}^{2g}(Z-a_i)$

$\pi(z, w)=(Z, W)=((z^n+1)^2/4z^n,\ (z^n-1)w/(2z^{n/2})^{2g+1})$.

m: odd.

$m=2g-1$

$W^2=Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i)$

$\pi(z, w)=(Z, W)=((z^n+1)^2/4z^n,\ (z^{2n}-1)w/(2z^{n/2})^{2g+1})$.

Remark. Since the order of T is even, the points lying over 0 and ∞ cannot be Weierstraß points by the remark in § 3.

(B) In case n is odd, T must be T_1 and the same arguments as (A_1) and (A_2) are available.

In case $z(W)$ contains 0 and ∞, n must be odd by the above remark. $\tilde{S}$ is then defined by

$$w^2=4z^n\prod_{i=1}^{m}(z^{2n}-(4a_i-2)z^n+1) \qquad a_i\neq 0, 1\,.$$

Since n is odd, T and U are represented as

$$T(z, w)=(\varepsilon z, w)\,,\ U(z, w)=(1/z, \pm w/z^{n(m+1)})\,.$$

(C_1) If $U(z, w)=(1/z, w/z^{n(m+1)})$, $TU(z, w)=(\varepsilon/z, w/z^{n(m+1)})$.

T fixes two points lying over 0 and ∞ and U fixes two points lying over 1.

In case m is even, TU has no fixed point and the Riemann-Hurwitz formula gives

$$2\tilde{g}-2=2n(2g-2)+2(n-1)+2n(2-1).$$

In this case, we have

$$2\tilde{g}+2=2mn+2$$

so that

$$m=2g.$$

S is defined by

$$W^2=Z\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=((z^n+1)^2/4z^n,\ (z^n+1)w/(4z^n)^{g+1}).$$

In case m is odd, TU fixes $\varepsilon^{1/2}$ and then

$$m=2g+1.$$

S is define by

$$W^2=\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=((z^n+1)^2/4z^n,\ w/(4z^n)^{g+1}).$$

(C_2) If $U(z, w)=(1/z, -w/z^{n(m+1)})$, $TU(z, w)=(\varepsilon/z, -w/z^{n(m+1)})$. U has no fixed point and TU fixes $\varepsilon^{1/2}$ in case m is even and has no fixed point in case m is odd. Thus we have the next results.

m: even.

$$m=2g$$

$$W^2=(Z-1)\prod_{i=1}^{2g}(Z-a_i)$$

$$\pi(z, w)=(Z, W)=((z^n+1)^2/4z^n,\ (z^n-1)/(4z^n)^{g+1}).$$

m: odd.

$$m=2g-1$$

$$W^2=Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i).$$

We summarize the obtained results in the next theorem.

Theorem 2. *An n-sheeted* ($n\geq 2$) ***dihedral*** *covering of a hyperelliptic surface which is also hyperelliptic can be represented as follows.* ($a_i\neq 0, 1$ *and* $a_i\neq a_j$ *if* $i\neq j$)

(1) $\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, w/(4z^n)^{g+1})$.

$T(z, w) = (\varepsilon z, w)$, $U(z, w) = (1/z, w/z^{(2g+2)n})$.

n: *arbitrary.*

$$\tilde{S}:\ w^2=\prod_{i=1}^{2g+2}(z^{2n}-(4a_i-2)z^n+1), \qquad S:\ W^2=\prod_{i=1}^{2g+2}(Z-a_i).$$

n: *odd.*

$$\tilde{S}:\ w^2=4z^n\prod_{i=1}^{2g+1}(z^{2n}-(4a_i-2)+1), \qquad S:\ W^2=\prod_{i=1}^{2g+1}(Z-a_i).$$

(2) $\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, (z^n+1)w/(4z^n)^{g+1})$.

$T(z, w) = (\varepsilon z, w)$, $U(z, w) = (1/z, w/z^{(2g+1)n})$.

n: *arbitrary.*

$$\tilde{S}:\ w^2=\prod_{i=1}^{2g+1}(z^{2n}-(4a_i-2)z^n+1), \qquad S:\ W^2=Z\prod_{i=1}^{2g+1}(Z-a_i).$$

n: *odd.*

$$\tilde{S}:\ w^2=4z^n\prod_{i=1}^{2g}(z^{2n}-(4a_i-2)z^n+1), \qquad S:\ W^2=Z\prod_{i=1}^{2g}(Z-a_i).$$

(3) $\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, (z^{2n}-1)w/(4z^n)^{g+1})$.

$T(z, w) = (\varepsilon z, w)$, $U(z, w) = (1/z, -w/z^{2gn})$.

n: *arbitrary,*

$$\tilde{S}:\ w^2=\prod_{i=1}^{2g}(z^{2n}-(4a_i-2)z^n+1), \qquad S:\ W^2=Z(Z-1)\prod_{i=1}^{2g}(Z-a_i).$$

n: *odd.*

$$\tilde{S}:\ w^2=4z^n\prod_{i=1}^{2g-1}(z^{2n}-(4a_i-2)z^n+1), \qquad S:\ W^2=Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i).$$

(4) n: *even.*

$\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n, w/(2z^{n/2})^{2g+1})$.

$T(z, w) = (\varepsilon z, -w)$, $U(z, w) = (1/z, w/z^{(2g+1)n})$.

$$\tilde{S}:\ w^2=\prod_{i=1}^{2g+1}(z^{2n}-(4a_i-2)z^n+1), \qquad S:\ W^2=\prod_{i=1}^{2g+1}(Z-a_i).$$

$$\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n,\ (z^n+1)w/(2z^{n/2})^{2g+1}).$$

$$T(z, w) = (\varepsilon z, -w),\ U(z, e) = (1/z, w/z^{2gn}).$$

$$\tilde{S}\colon\ w^2 = \prod_{i=1}^{2g}(z^{2n}-(4a_i-2)z^n+1),\qquad S\colon\ S\colon\ W^2 = Z\prod_{i=1}^{2g}(Z-a_i).$$

$$\pi(z, w) = (Z, W) = ((z^n+1)^2/4z^n,\ (z^{2n}-1)w/(2z^{n/2})^{2g+1}).$$

$$T(z, w) = (\varepsilon z, -w),\ U(z, w) = (1/z, -w/z^{(2g-1)n}).$$

$$\tilde{S}\colon\ w^2 = \prod_{i=1}^{2g-1}(z^{2n}-(4a_i-2)z^n+1),\qquad S\colon\ W^2 = Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i).$$

Here we set $n=2$ and we get the four group. Since abelian groups in our groups are only cyclic group and the four group, we have the next theorem.

Theorem 2′. *All types of the* ***non-cyclic abelian*** *coverings of a hyperelliptic surface which are also hyperelliptic are as follows.*

(a) *Corresponding to the choices of three Weierstraß points, we have* $\binom{2g+2}{3}$ *unramified four-sheeted coverings.*

(b) *Corresponding to the choices of two Weierstraß points, we have* $\binom{2g+2}{2}$ *four-sheeted coverings with four branch points of degree 1 lying over P and $J(P)$, where P is a non Weierstraß point.*

(c) *Corresponding to the choices of one Weiestraß point, we have* $2g+2$ *four-sheeted coverings with eight branch points of degree* 1 *lying over P_1, P_2, $J(P_1)$ and $J(P_2)$, where P_1 and P_2 are non Weierstraß points.*

(d) *We have a four-sheeted covering with twelve branch points of degree* 1 *lying over P_1, P_2, P_3, $J(P_1)$, $J(P_2)$ and $J(P_3)$, where P_1, P_2 and P_3 are non Weierstraß points.*

Remark. Case (a) was shown by Kato [5]. The number of coverings in each case is justified by the same reasoning as the remark in § 3.

§ 5. The tetrahedral group

It is known [6] that the tetrahedral group G^* can be generated by $T^*(z) = (i-z)/(i+z)$ of order 3 and $U^*(z) = i(z-1)/(z+1)$ of order 3. Invariant functions for G^* are

$$f(z) = \frac{(z^4-2\sqrt{3}iz^2+1)^3}{-12\sqrt{3}iz^2(z^4-1)^2}$$

and

$$f(z)-1 = \frac{(z^4+2\sqrt{3}iz^2+1)^3}{-12\sqrt{3}iz^2(z^4-1)^2}$$

which we shall verify.

Define

$$g(z)=z(z^4-1),$$

$$h(z)=(z^4\pm 2\sqrt{3}iz^2+1),$$

then

$$g(T^*(z))=\frac{8i}{(i+z)^6}g(z)$$

$$h(T^*(z))=\frac{2(1\pm\sqrt{3}i)}{(i+z)^4}h(z)$$

$$g(U^*(z))=-\frac{8i}{(z+1)^6}g(z)$$

$$h(U^*(z))=\frac{2(1\mp\sqrt{3}i)}{(z+1)^4}h(z)$$

Substituting these into $f(T^*(z))$ and $f(U^*(z))$ gives the invariance of $f(z)$ for G^*.

Since the order of $(UT)^*(z)=-z$ is 2, it cannot fix any points of $z(W)$. Thus we only have to examine the following cases.

(A) In case T^*, U^* and their conjugates do not fix any points of $z(W)$, S is defined by

$$w^2=(-12\sqrt{3}iz^2(z^4-1)^2)^m\prod_{i=1}^{m}(f(z)-a_i)$$

and as $T^3(z, w)=U^3(z, w)=(z, w)$, T and U are represented as

$$T(z, w)=\left(\frac{i-z}{i+z}, \left(\frac{8i}{(i+z)^6}\right)^m w\right)$$

$$U(z, w)=\left(i\frac{z-1}{z+1}, (-1)^m\left(\frac{8i}{(1+z)^6}\right)^m w\right)$$

and then $UT(z, w)=(-z, (-1)^m w)$.

T fixes two points lying over z which T^* fixes and U fixes two points lying over z which U^* fixes. These are true of congruent points.

If m is even, UT fixes two points over 0 and the Riemann-Hurwitz formula gives

$$2\tilde{g}-2=12(2g-2)+8(3-1)+8(3-1)+12(2-1).$$

Now we have $2\tilde{g}+2=12m$. Then $m=2g+2$ and S is defined by

$$W^2=\prod_{i=1}^{2g+2}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{(-12\sqrt{3}\, iz^2(z^4-1)^2)^{g+1}}\right).$$

If m is odd, UT has no fixed point and then $m=2g+1$. S is defined by

$$W^2=\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{((-12\sqrt{3}\, i)^{1/2} z(z^4-1))^{2g+1}}\right).$$

(B) In case T^* and its conjugates fix points of $z(W)$, $\tilde{S}$ is defined by

$$w^2=(z^4+2\sqrt{3}\, iz^2+1)^3(-12\sqrt{3}\, iz^2(z^4-1)^2)^m \prod_{i=1}^{m}(f(z)-a_i) \qquad a_i\neq 0, 1$$

and T and U are represented as

$$T(z, w)=\left(\frac{i-z}{i+z}, \left(\frac{8i}{(i+z)^6}\right)^{m+1} w\right)$$

$$U(z, w)=\left(i\frac{z-1}{z+1}, (-1)^{m+1}\left(\frac{8i}{(i+z)^6}\right)^{m+1} w\right).$$

Then $UT(z, w)=(-z, (-1)^{m+1}w)$.

If m is even, $m=2g$ and S is defined by

$$W^2=(Z-1)\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{((-12\sqrt{3}\, i)^{1/2} z(z^4-1))^{2g+1}}\right).$$

If m is odd, $m=2g+1$ and S is defined by

$$W^2=(Z-1)\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{(-12\sqrt{3}\, iz^2(z^4-1))^{2g+1}}\right).$$

(C) In case U^* and its conjugates fix points of $z(W)$, $\tilde{S}$ is defined by

$$w^2=(z^4-2\sqrt{3}\, iz^2+1)^3(-12\sqrt{3}\, iz^2(z^4-1)^2)^m \prod_{i=1}^{m}(f(z)-a_i) \qquad a_i\neq 0, 1$$

and T and U are represented as

$$T(z, w) = \left(\frac{i-z}{i+z}, \left(\frac{8i}{(i+z)^6}\right)^{m+1} w\right)$$

$$U(z, w) = \left(i\frac{z-1}{z+1}, (-1)^{m+1}\left(\frac{8i}{(i+z)^6}\right)^{m+1} w\right).$$

Then $UT(z, w) = (-z, (-1)^{m+1}w)$.

If m is even, $m=2g$ and S is defined by

$$W^2 = Z\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z, w) = (Z, W) = \left(f(z), \frac{w}{((-12\sqrt{3}i)^{1/2}z(z^4-1))^{2g+1}}\right).$$

If m is odd, $m=2g+1$ and S is defined by

$$W^2 = Z\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w) = (Z, W) = \left(f(z), \frac{w}{(-12\sqrt{3}iz^2(z^4-1)^2)^{g+1}}\right).$$

(D) In case T^*, U^* and their conjugates fix points of $z(W)$, $\tilde{S}$ is defined by

$$w^2 = (z^8+14z^4+1)^3(-12\sqrt{3}iz^2(z^4-1)^2)^m\prod_{i=1}^{m}(f(z)-a_i) \qquad a_i \neq 0, 1$$

and T and U are represented as

$$T(z, w) = \left(\frac{i-z}{i+z}, \left(\frac{8i}{(i+z)^6}\right)^{m+2} w\right)$$

$$U(z, w) = \left(i\frac{z-1}{z+1}, (-1)^{m+2}\left(\frac{8i}{(i+z)^6}\right)^{m+2} w\right).$$

Then $UT(z, w) = (-z, (-1)^{m+2}w)$.

If m is even, $m=2g$ and S is defined by

$$W^2 = Z(Z-1)\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z, w) = (Z, W) = \left(f(z), \frac{w}{(-12\sqrt{3}iz^2(z^4-1)^2)^{g+1}}\right).$$

If m is odd, $m=2g-1$ and S is defined by

$$W^2=Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i),$$

where the projection π is

$$\pi(z,w)=(Z,W)=\left(f(z),\ \frac{w}{((-12\sqrt{3}i)^{1/2}z(z^4-1))^{2g+1}}\right).$$

Theorem 3. *A* 12-*sheeted* **tetrahedral** *covering of a hyperel-liptic surface which is also hyperelliptic can be represented as follows.* ($a_i\neq 0, 1$ *and* $a_i\neq a_j$ *if* $i\neq j$)

(1) $$\pi(z,w)=\left(f(z),\frac{w}{(-12\sqrt{3}iz^2(z^4-1)^2)^{g+1}}\right)$$

$$T(z,w)=\left(\frac{i-z}{i+z},\left(\frac{8i}{(i+z)^6}\right)^{2g+2}w\right)$$

$$U(z,w)=\left(i\frac{z-1}{z+1},\left(\frac{8i}{(i+z)^6}\right)^{2g+2}w\right)$$

$$\tilde{S}_1:\ w^2=(-12\sqrt{3}iz^2(z^4-1)^2)^{2g+2}\prod_{i=1}^{2g+2}(f(z)-a_i)$$

$$S_1:\ W^2=\prod_{i=1}^{2g+2}(Z-a_i)$$

$$\tilde{S}_2:\ w^2=(z^4-2\sqrt{3}iz^2+1)^3(-12\sqrt{3}iz^2(z^4-1)^2)^{2g+1}\prod_{i=1}^{2g+1}(f(z)-a_i)$$

$$S_2:\ W^2=Z\prod_{i=1}^{2g+1}(Z-a_i)$$

$$\tilde{S}_3:\ w^2=(z^8+14z^4+1)^3(-12\sqrt{3}iz^2(z^4-1)^2)^{2g}\prod_{i=1}^{2g}(f(z)-a_i)$$

$$S_3:\ W^2=Z(Z-1)\prod_{i=1}^{2g}(Z-a_i)$$

(2) $$\pi(z,w)=\left(f(z)\frac{w}{((-12\sqrt{3}i)^{1/2}z(z^4-1))^{2g+1}}\right)$$

$$T(z,w)=\left(\frac{i-z}{i+z},\left(\frac{8i}{(1+z)^6}\right)^{2g+1}w\right)$$

$$U(z,w)=\left(i\frac{z-1}{z+1},-\left(\frac{8i}{(i+z)^6}\right)^{2g+1}w\right)$$

$$\tilde{S}_1:\ w^2=(-12\sqrt{3}iz^2(z^4-1)^2)^{2g+1}\prod_{i=1}^{2g+1}(f(z)-a_i)$$

$$S_1:\ W^2=\prod_{i=1}^{2g+1}(Z-a_i)$$

$$S_2:\ w^2=(z^4-2\sqrt{3}iz^2+1)^3(-12\sqrt{3}iz^2(z^4-1)^2)^{2g}\prod_{i=1}^{2g}(f(z)-a_i)$$

$$S_2:\ W^2 = Z\prod_{i=1}^{2g}(Z-a_i)$$

$$S_3:\ w^2 = (z^8+14z^4+1)^3(-12\sqrt{3}\,iz^2(z^4-1)^2)^{2g-1}\prod_{i=1}^{2g-1}(f(z)-a_i)$$

$$S_3:\ W^2 = Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i)$$

§ 6. The octahedral group

It is known [6] that the octahedral group G^* can be generated by $T^*(z) = (i-z)/(i+z)$ of order 3 and $U^*(z) = iz$ of order 4. Invariant function for G^* are

$$f(z) = \frac{(z^8+14z^4+1)^3}{108z^4(z^4-1)^4}$$

and

$$f(z)-1 = \frac{(z^{12}-33z^8-33z^4+1)^2}{108z^4(z^4-1)^4},$$

which can be verified using the transformation formula of $g(z)$ and $h(z)$ in § 5.

Since the orders of U^* and $(TU)^*(z) = (1-z)/(1+z)$ are even, they cannot fix any points of $z(W)$. Thus we have only to examine the following cases.

(A) In case T^* and its conjugates do not any points of $z(W)$, S is defined by

$$w^2 = (108z^4(z^4-1)^4)^m\prod_{i=1}^{m}(f(z)-z_i) \qquad a_i\neq 0, 1$$

and T and U are represented as

$$T(z, w) = \left(\frac{i-z}{i+z}, (-1)^m\left(\frac{2^6}{(i+z)^{12}}\right)^m w\right)$$

$$U(z, w) = (iz, \pm w).$$

(a) If $U(z, w) = (iz, w)$, then $TU(z, w) = \left(\dfrac{1-z}{1+z}, (-1)^m\left(\dfrac{2^6}{(1+z)^{12}}\right)^m w\right)$.

T fixes two points lying over z which T^* fixes and U fixes four points lying over 0 and ∞. These are true of congruent points.

If m is even, TU fixes two points lying over z which $(TU)^*$ fixes. Now we have $2g+2=24m$ and the Riemann-Hurwitz formula gives

$$2\tilde{g}-2 = 24(2g-2)+16(3-1)+12(4-1)+24(2-1).$$

Then $m=2\tilde{g}+2$ and S is defined by

$$W^2=\prod_{i=1}^{2g+2}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, w)=\left(f(z), \frac{(z^{12}-33z^8-33z^4+1)}{(108z^4(z^4-1)^4)^{g+1}}w\right).$$

(b) If $U(z, w)=(iz, -w)$, then

$$TU(z, w)=\left(\frac{1-z}{1+z}, (-1)^{m+1}\left(\frac{2^6}{(1+z)^{12}}\right)^m w\right).$$

U has no fixed point and the degrees of ramification of the points which lie over 0 are 1.

If m is even, TU has no fixed point and then $m=2g$. S is defined by

$$W^2=(Z-1)\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{(z^{12}-33z^8-33z^4+1)}{(6\sqrt{3}z^2(z^4-1)^2)^{g+1}}w\right).$$

If m is odd, TU fixes two points lying over z which $(TU)^*$ fixes. Then $m=2g+1$ and S is defined by

$$W^2=\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{(6\sqrt{3}z^2(z^4-1)^2)^{2g+1}}\right).$$

(B) In case T^* and its conjugates fix points of $z(W)$, S is defined by

$$w^2=(z^8+14z^4+1)^3(108z^4(z^4-1)^4)^m\prod_{i=1}^{m}(f(z)-a_i) \qquad a\neq 0, 1$$

and T and U are represented as

$$T(z, w)=\left(\frac{i-z}{i+z}, (-1)^{m+1}\left(\frac{2^6}{(i+z)^{12}}\right)^{m+1}w\right)$$

$$U(z, w)=(iz, \pm w).$$

(c) If $U(z, w)=(iz, w)$, then

$$TU(z, w)=\left(\frac{1-z}{1+z}, (-1)^{m+1}\left(\frac{2^6}{(1+z)^{12}}\right)^{m+1}w\right).$$

T fixes two points lying over z which T^* fixes and U fixes two points lying over 0.

If m is even, TU has no fixed point and then $m=2g$. S is defined by

$$W^2=Z(Z-1)\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{(6\sqrt{3}z^2(z^4-1)^2)^{2g+1}}\right).$$

If m is odd, TU fixes two points lying over z which $(TU)^*$ fixes. Then $m=2g+1$ and S is defined by

$$W^2=Z\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{(108z^4(z^4-1)^4)^{g+1}}\right).$$

(d) If $U(z, w)=(iz, -w)$, then

$$TU(z, w)=\left(\frac{1-z}{1+z}, (-1)^{m+2}\left(\frac{2^6}{(1+z)^{12}}\right)^{m+1}w\right).$$

U has no fixed point and the degrees of ramification of the points which lie over 0 are 1.

If m is even, TU fixes two points lying over z which $(TU)^*$ fixes. Then $m=2g$ and S is defined by

$$W^2=Z\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{w}{(6\sqrt{3}z^2(z^4-1)^2)^{2g+1}}\right).$$

If m is odd, TU has no fixed point and then $m=2g-1$. S is defined by

$$W^2=Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=(Z, W)=\left(f(z), \frac{(z^{12}-33z^8-33z^4+1)w}{(6\sqrt{3}z^2(z^4-1)^2)^{2g+1}}\right).$$

Theorem 4. *A 24-sheeted* **octahedral** *covering of a hyperelliptic-surface which is also hyperelliptic can be represented as follows.* ($a_i\neq 0, 1$ *and* $a_i\neq a_j$ *if* $i\neq j$)

(1) $$\pi(z, w) = \left(f(z), \frac{w}{(108z^4(z^4-1)^4)^{g+1}}\right)$$

$$T(z, w) = \left(\frac{i-z}{i+z}, \left(\frac{2^6}{(i+z)^{12}}\right)^{2g+2} w\right)$$

$$U(z, w) = (iz, w)$$

$$\tilde{S}_1: \ w^2 = (108z^4(z^4-1)^4)^{2g+2} \prod_{i=1}^{2g+2} (f(z) - a_i)$$

$$S_1: \ W^2 = \prod_{i=1}^{2g+2} (Z - a_i)$$

$$\tilde{S}_2: \ w^2 = (z^8 + 14z^4 + 1)^3 (108z^4(z^4-1)^4)^{2g+1} \prod_{i=1}^{2g+1} (f(z) - a_i)$$

$$S_2: \ W^2 = Z \prod_{i=1}^{2g+1} (Z - a_i)$$

(2) $$\pi(z, w) = \left(f(z), \frac{(z^{12} - 33z^8 - 33z^4 + 1)w}{(108z^4(z^4-1)^4)^{g+1}}\right)$$

$$T(z, w) = \left(\frac{i-z}{i+z}, -\left(\frac{2^6}{(i+z)^{12}}\right)^{2g+1} w\right)$$

$$U(z, w) = (iz, w)$$

$$\tilde{S}_1: \ w^2 = (108z^4(z^4-1)^4)^{2g+1} \prod_{i=1}^{2g+1} (f(z) - a_i)$$

$$S_1: \ W^2 = (Z-1) \prod_{i=1}^{2g+1} (Z - a_i)$$

$$\tilde{S}_2: \ w^2 = (z^8 + 14z^4 + 1)^3 (108z^4(z^4-1)^4)^{2g} \prod_{i=1}^{2g+1} (f(z) - a_i)$$

$$S_2: \ W^2 = Z \prod_{i=1}^{2g} (Z - a_i)$$

(3) $$\pi(z, w) = \left(f(z), \frac{w}{(6\sqrt{3}\,z^2(z^4-1)^2)^{2g+1}}\right)$$

$$T(z, w) = \left(\frac{i-z}{i+z}, -\left(\frac{2^6}{(i+z)^{12}}\right)^{2g+1} w\right)$$

$$U(z, w) = (iz, -w)$$

$$\tilde{S}_1: \ w^2 = (108z^4(z^4-1)^4)^{2g+1} \prod_{i=1}^{2g+1} (f(z) - a_i)$$

$$S_1: \ W^2 = \prod_{i=1}^{2g+1} (Z - a_i)$$

$$\tilde{S}_2: \ w^2 = (z^8 + 14z^4 + 1)^3 (108z^4(z^4-1)^4)^{2g} \prod_{i=1}^{2g} (f(z) - a_i)$$

$$S_2: \ W^2 = Z \prod_{i=1}^{2g} (Z - a_i)$$

$$(4) \qquad \pi(z, w) = \left(f(z), \frac{(z^{12}-33z^8-33z^4+1)\,w}{(6\sqrt{3}\,z^2(z^4-1)^2)^{2g+1}}\right)$$

$$T(z, w) = \left(\frac{i-z}{i+z}, \left(\frac{2^6}{(i+z)^{12}}\right)^{2g} w\right)$$

$$U(z, w) = (iz, -w)$$

$$\tilde{S}_1:\ w^2 = (108z^4(z^4-1)^4)^{2g} \prod_{i=1}^{2g} (f(z) - a_i)$$

$$S_1:\ W^2 = (Z-1) \prod_{i=1}^{2g} (Z-a_i)$$

$$\tilde{S}_2:\ w^2 = (z^8+14z^4+1)^3 (108z^4(z^4-1)^4)^{2g-1} \prod_{i=1}^{2g-1} (f(z) - a_i)$$

$$S_2:\ W^2 = Z(Z-1) \prod_{i=1}^{2g-1} (Z-a_i).$$

§ 7. The icosahedral group

It is known [6] that the icosahedral group G^* can be generated by

$$T^*(z) = \frac{(1-\varepsilon^4)z + (\varepsilon^4-\varepsilon)}{(\varepsilon-\varepsilon^3)z + (\varepsilon^2-\varepsilon^3)} \qquad \text{where } \varepsilon = e^{2\pi i/5},$$

of order 3 and

$$U^*(z) = \varepsilon z$$

of order 5. Invariant functions for G^* are

$$f(z) = \frac{(-(z^{20}+1) + 228(z^{15}-z^5) - 494z^{10})^3}{1728z^5(z^{10}+11z^5-1)^5}$$

and

$$f(z) - 1 = \frac{-((z^{30}+1) + 522(z^{25}-z^5) - 10005(z^{20}+z^{10}))^2}{1728z^5(z^{10}+11z^5-1)^5}.$$

If we define

$$V = TU^2TU^4T,$$

the order of V is 2 and thus V^* cannot fix any points of $z(W)$. We therefore have only to examine the next cases.

(A) In case T^*, U^* and their conjugates do not fix any points of $z(W)$, $\tilde{S}$ is defined by

$$w^2 = (1728z^5(z^{10}+11z^5-1)^5)^m \prod_{i=1}^{m} (f(z) - a_i) \qquad a_i \neq 0, 1$$

and T and U are represented as

$$T(z, w)=\left(\frac{(1-\varepsilon^4)z+(\varepsilon^4-\varepsilon)}{(\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3)}, \left(\frac{5^{15}}{((\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3))^{30}}\right)^m w\right)$$

$$U(z, w)=(\varepsilon z, w).$$

Then

$$V(z, w)=\left(-\frac{\varepsilon^2}{z}, \frac{w}{z^{30m}}\right)$$

and T fixes two points lying over z which T^* fixes and U fixes two points lying over 0. Of course, these are true of conguruent points.

If m is even, V fixes two points lying over $i\varepsilon$ and the Riemann-Hurwitz formula gives

$$2\tilde{g}-2=60(2g-2)+40(3-1)+24(5-1)+60(2-1).$$

Now we have $2\tilde{g}+2=60m$ and then $m=2g+2$. S is defined by

$$W^2=\prod_{i=1}^{2g+2}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=\left(f(z), \frac{w}{(1728z^5(z^{10}+11z^5-1)^5)^{g+1}}\right).$$

If m is odd, V has no fixed point and then $m=2g+1$. S is defined by

$$W^2=(Z-1)\prod_{i=1}^{2g+1}(Z-a_i),$$

where the projection π is

$$\pi(z, w)=\left(f(z), \frac{i((z^{30}+1)+228(z^{15}-z^5)-494z^{10})}{(1728z^5(z^{10}+11z^5-1)^5)^{g+1}}w\right).$$

(B) In case T^* and its conjugates fix points of $z(W)$, S is defined by

$$w^2=(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3$$

$$\times(1728z^5(z^{10}+11z^5-1)^5)^m\prod_{i=1}^{m}(f(z)-a_i) \qquad a_i\neq 0, 1$$

and T and U are represented as

$$T(z, w)=\left(\frac{(1-\varepsilon^4)z+(\varepsilon^4-\varepsilon)}{(\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3)}, \left(\frac{5^{15}}{((\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3))^{30}}\right)^{m+1} w\right)$$

$$U(z, w)=(\varepsilon z, w).$$

Then

$$V(z, w)=\left(-\frac{\varepsilon^{2}}{z}, \frac{w}{z^{30(m+1)}}\right)$$

and T fixes the point lying over z which T^* fixes and U fixes two points lying over 0.

If m is even, V has no fixed point and then $m=2g$. S is defined by

$$W^{2}=Z(Z-1)\prod_{i=1}^{2g}(Z-a_{i}),$$

where the projection π is

$$\pi(z, w)=\left(f(z), \frac{i((z^{30}+1)+228(z^{15}-z^{5})-494z^{10})}{(1728z^{5}(z^{10}+11z^{5}-1)^{5})^{g+1}}w\right).$$

If m is odd, V fixes two points lying over $i\varepsilon$ and then $m=2g+1$. S is defined by

$$W^{2}=Z\prod_{i=1}^{2g+1}(Z-a_{i}),$$

where the projection π is

$$\pi(z, w)=\left(f(z), \frac{w}{(1728z^{5}(z^{10}+11z^{5}-1)^{5})^{g+1}}\right).$$

(C) In case U^* and its conjugates fix points of $z(W)$, $\tilde{S}$ is defined by

$$w^{2}=(1728z^{5}(z^{10}+11z^{5}-1)^{5})^{m+1}\prod_{i=1}^{m}(f(z)-a_{i}) \qquad a_{i}\neq 0, 1$$

and T and U are represented just same as in (B). Consequently similar arguments as in (B) are applicable and we only list the results.

m: even

$$m=2g$$

$$W^{2}=(Z-1)\prod_{i=1}^{2g}(Z-a_{i})$$

$$\pi(z, w)=\left(f(z), \frac{i((z^{30}+1)+228(z^{15}-z^{5})-494z^{10})}{(1728z^{5}(z^{10}+11z^{5}-1)^{5})^{g+1}}w\right)$$

m: odd

$$m=2g+1$$

$$W^{2}=\prod_{i=1}^{2g+1}(Z-a_{i})$$

$$\pi(z, w)=\left(f(z), \frac{w}{(1728z^{5}(z^{10}+11z^{5}-1)^{g+1}}\right).$$

(D) In case T^*, U^* and their conjugates fix points of $z(W)$, $\tilde{S}$ is defined by

$$w^2=(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3$$
$$\times((1728z^5(z^{10}+11z^5-1)^5)^{m+1}\prod_{i=1}^{m}(f(z)-a_i) \qquad a_i\neq 0,1$$

and T and U are represented as

$$T(z,w)=\left(\frac{(1-\varepsilon^4)z+(\varepsilon^4-\varepsilon)}{(\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3)},\left(\frac{5^{15}}{((\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3))^{30}}\right)^{m+2}w\right)$$
$$U(z,w)=(\varepsilon z,w).$$

Then

$$V(z,)=\left(-\frac{\varepsilon^2}{z},\frac{w}{z^{30(m+2)}}\right)$$

and T fixes the point lying over z which T^* fixes and U fixes the point lying over 0.

If m is even, V fixes two points lying over $i\varepsilon$ and then $m=2g$. S is defined by

$$W^2=Z\prod_{i=1}^{2g}(Z-a_i),$$

where the projection π is

$$\pi(z,w)=\left(f(z),\frac{w}{(1728z^5(z^{10}+11z^5-1)^5)^{g+1}}\right).$$

If m is odd, V has no fixed point and then $m=2g-1$. S is defined by

$$W^2=Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i),$$

where the projection π is

$$\pi(z,w)=\left(f(z),\frac{i((z^{30}+1)+228(z^{15}-z^5)-494z^{10})}{(1728z^5(z^{10}+11z^5-1)^5)^{g+1}}w\right).$$

Theorem 5. *A 60-sheeted* ***icosahedral*** *covering of a hyperelliptic surface which is also hyperelliptic can be represented as follows.* ($a_i\neq 0,1$ *and* $a_i\neq a_j$ *if* $i\neq j$)

(1) $$\pi(z,w)=\left(f(z),\frac{w}{(1728z^5(z^{10}+11z^5-1)^5)^{g+1}}\right)$$

$$T(z,w)=\left(\frac{(1-\varepsilon^4)z+(\varepsilon^4-\varepsilon)}{(\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3)},\left(\frac{5^{15}}{((\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3))^{30}}\right)^{2g+2}w\right)$$

$$U(z,w)=(\varepsilon z,w)$$

$$\tilde{S}_1:\ w^2=(1728z^5(z^{10}+11z^5-1)^5)^{2g+2}\prod_{i=1}^{2g+2}(f(z)-a_i)$$

$$S_1:\ W^2=\prod_{i=1}^{2g+2}(Z-a_i)$$

$$\widetilde{S}_2:\ w^2=(1728z^5(z^{10}+11z^5-1)^5)^{2g+2}\prod_{i=1}^{2g+1}(f(z)-a_i)$$

$$S_2:\ W^2=\prod_{i=1}^{2g+1}(Z-a_i)$$

$$\widetilde{S}_3:\ w^2=(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3$$
$$\times(1728z^5(z^{10}+11z^5-1)^5)^{2g+1}\prod_{i=1}^{2g+1}(f(z)-a_i)$$

$$S_3:\ W^2=Z\prod_{i=1}^{2g+1}(Z-a_i)$$

$$\widetilde{S}_4:\ w^2=(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3$$
$$\times(1728z^5(z^{10}+11z^5-1)^5)^{2g+1}\prod_{i=1}^{2g}(f(z)-a_i)$$

$$S_4:\ W^2=Z\prod_{i=1}^{2g}(Z-a_i)$$

$$\pi(z,w)=\left(f(z),\ \frac{i((z^{30}+1)+228(z^{15}-z^5)-494z^{10})}{(1728z^5(z^{10}+11z^5-1)^5)^{g+1}}w\right) \tag{2}$$

$$T(z,w)=\left(\frac{(1-\varepsilon^4)z+(\varepsilon^4-\varepsilon)}{(\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3)},\ \left(\frac{5^{15}}{((\varepsilon-\varepsilon^3)z+(\varepsilon^2-\varepsilon^3))^{30}}\right)^{2g+1}w\right)$$

$$U(z,w)=(\varepsilon z,w)$$

$$\widetilde{S}_1:\ w^2=(1728z^5(z^{10}+11z^5-1)^5)^{2g+1}\prod_{i=1}^{2g+1}(f(z)-a_i)$$

$$S_1:\ W^2=(Z-1)\prod_{i=1}^{2g+1}(Z-a_i)$$

$$\widetilde{S}_2:\ w^2=(1728z^5(z^{10}+11z^5-1)^5)^{2g+1}\prod_{i=1}^{2g}(f(z)-a_i)$$

$$S_2:\ W^2=(Z-1)\prod_{i=1}^{2g}(Z-a_i)$$

$$\widetilde{S}_3:\ w^2=(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3\prod_{i=1}^{2g}(f(z)-a_i)$$

$$S_3:\ W^2=Z(Z-1)\prod_{i=1}^{2g}(Z-a_i)$$

$$\widetilde{S}_4:\ w^2=(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3$$
$$\times(1728z^5(z^{10}+11z^5-1)^5)^{2g}\prod_{i=1}^{2g-1}(f(z)-a_i)$$

$$S_4:\ W^2=Z(Z-1)\prod_{i=1}^{2g-1}(Z-a_i)$$

§ **8.** We list all the types of dihedral, tetrahedral, octhafedral and icosahedral coverings of a hyperelliptic surface which are also hyperelliptic in the next Table. As a corollary from this Table, it can be seen that if an n-sheeted unramified covering of a hyperelliptic surface is hyperelliptic, then $n=2$ or 4. (Machlachlan [7])

Table

P: non Weierstraß point, Q: Weierstraß point.

	Dihedral group ($2n$)			Tetrahedral group (12)		
n:	even, odd	even	odd			
$P_1, J(P_1)$:	$2(n-1)$			$4(3-1)$		
$P_2, J(P_2)$:	$n(2-1)$			$4(3-1)$		
$P_3, J(P_3)$:	$n(2-1)$			$6(2-1)$		
$P_1, J(P_1)$:	$2(n-1)$	$n(2-1)$	$n(2-1)$	$4(3-1)$	$4(3-1)$	
$P_2, J(P_2)$:	$n(2-1)$	$n(2-1)$	$n(2-1)$	$4(3-1)$	$6(2-1)$	
Q :	0	$4\left(\frac{n}{2}-1\right)$	$2(n-1)$	0	$4(3-1)$	
$P, J(P)$:	$2(n-1)$	$n(2-1)$	$n(2-1)$	$4(3-1)$	$6(2-1)$	
Q_1 :	0	$4\left(\frac{n}{2}-1\right)$	$2(n-1)$	$4(3-1)$	$4(3-1)$	
Q_2 :	0	0	0	0	$4(3-1)$	
Q_1 :		$4\left(\frac{n}{2}-1\right)$	$2(n-1)$	$4(3-1)$		
Q_2 :		0	0	$4(3-1)$		
Q_3 :		0	0	0		
	Octahedral group (24)			Icosahedral group (60)		
$P_1, J(P_1)$:	$6(4-1)$			$12(5-1)$		
$P_2, J(P_2)$:	$8(3-1)$			$20(3-1)$		
$P_3, J(P_3)$:	$12(2-1)$			$30(2-1)$		
$P_1, J(P_1)$:	$6(4-1)$	$8(3-1)$	$6(4-1)$	$12(5-1)$	$20(3-1)$	$12(5-1)$
$P_2, J(P_2)$:	$8(3-1)$	$12(2-1)$	$12(2-1)$	$30(3-1)$	$30(2-1)$	$30(2-1)$
Q :	0	$12(2-1)$	$8(3-1)$	0	$12(5-1)$	$20(3-1)$
$P, J(P)$:	$8(3-1)$	$6(4-1)$	$12(2-1)$	$20(3-1)$	$12(5-1)$	$30(2-1)$
Q_1 :	$12(2-1)$	$8(3-1)$	$8(3-1)$	$12(5-1)$	$20(3-1)$	$12(5-1)$
Q_2 :	0	0	$12(2-1)$	0	0	$20(3-1)$
Q_1 :	$8(3-1)$			$12(5-1)$		
Q_2 :	$12(2-1)$			$20(3-1)$		
Q_3 :	0			0		

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