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Structure of codimension 1 foliations without holonomy on manifolds with abelian fundamental group

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Introduction

Let M be a connected paracompact smooth manifold and $\mathcal{F}$ a codimension one foliation on M without holonomy. If M is compact and $\mathcal{F}$ is class C^2 then $\mathcal{F}$ is topologically equivalent to a condimension one foliation $\mathcal{F}_\omega$ defined by closed non-singular one form ω (Sacksteder [14]) and the structure of $\mathcal{F}_\omega$ is very well understood ([5], [11], [12], [15]). Of course the compactness of M is essential in these results, but if $\pi_1(M)$ is abelian we can obtain the Sacksteder's theorem for non-compact M.

Theorem I. *Let $\mathcal{F}$ be a transversally orientable codimension one foliation of class C^2 on M. Suppose that $\mathcal{F}$ is without holonomy and $\pi_1(M)$ is a finitely generated abelian group, then we have the following two cases.*

(1) *If there exists no closed transversal curve, then there exists a topological submersion f of M onto* R *or* S^1 which defines $\mathcal{F}$.

(2) *If there exists a closed transversal curve C, let $V_0=Q(C)$ be the saturation of C, then*

(i) *For a suitable choice of differential structure on V_0, there exists a non-singular closed one form ω and $\mathcal{F}|V_0$ is defined by $\omega=0$.*

(ii) *Let V_i be a connected component of $M-V_0$, then there exists a topological submersion f_i of V_i to* R *which defines $\mathcal{F}|V_i$ and any leaf of $\mathcal{F}|V_i$ separates M into two components.*

Remarks I. (1) In (I. 1) and (I. 2. ii), the submersions f and f_i are not necessarily differentiable (see [3]). (2) In (I. 2. i) and (I. 2. ii), ω is not necessarily extendable to $V_0 \cup V_i$ (see [8]). For a curve $c(t)$, where $c(t) \in V_0$ for $0 \leqq t < 1$ and $c(1) \in V_i$, if $\lim_{t\to 1} \int_0^t c^*\omega$ is finite then ω is extendable to $V_0 \cup V_i$. (3) Let X_i be the leaf space of $\mathcal{F}|V_i$, then (I. 2. ii) is equivalent to say that X_i is a simply connected (non-Hausdorff) 1 dimensional manifold with the boundary consisting of one point.

As for the structure of codimension one foliation $\mathcal{F}$ defined by closed one form ω, we showed in [6] that any non-closed leaf is locally dense when ω is a closed one form on a compact manifold M with isolated singularities Σ and $\mathcal{F}$ is defined on $M-\Sigma$. If $\pi_1(M)$ is abelian we can obtain more imformations on $\mathcal{F}$ even if M is not compact.

Let ω be a closed one form on a manifold M, a homomorphism $i(\omega)$ from $\pi_1(M)$ to R is defined by $i(\omega)(\alpha) = \int_c \omega$ where c is a representative of $\alpha \in \pi_1(M)$. We denote $\mathrm{Per}(\omega)$ the set of periods of ω, i.e., the image of $i(\omega)$. The *rank* of ω is defined by the rank (as Z-module) of $\mathrm{Per}(\omega)$. If $\mathrm{Per}(\omega)$ is not finitely generated we define rank ω to be ∞.

Theorem II. *Let ω be a non-singular closed one form of class C^1 on M and $\mathcal{F}$ a codimension one foliation defined by $\omega=0$. If $\pi_1(M)$ is abelian, we have the following two cases.*
(1) *If the rank of ω is* 0 *or* 1, *there is a submersion f of M onto* R *or* S^1 *such that $\mathcal{F}$ is defined by f.*
(2) *If the rank of ω is greater than one, there exists a saturated open set V_0 of M and $\mathcal{F}$ has the following properties.*
(i) *All leaves in V_0 are dense in V_0.*
(ii) *Let α be an element of $\pi_1(M, x)$, $x \in V_0$, α is represented by a closed transversal curve if $i(\omega)(\alpha) \neq 0$ and is represented by a curve in L_x, where L_x is the leaf passing through x, if $i(\omega)(\alpha)=0$.*
(iii) *Let l be a curve in M from x to y, $x, y \in V_0$, then $L_x = L_y$ if and only if $\int_l \omega$ belongs to $\mathrm{Per}(\omega)$.*
(iv) *Same as* (I. 2. ii) *but in this time f_i is differentiable.*

Remarks II. (1) If rank $\omega=1$ and there exists a closed transversal curve C then, putting $V_0=Q(C)$, the properties (II. 2. ii, iii, iv) hold. (2) since $\pi_1(V_0)$ is not necessarily abelian, we cannot apply directly Theorem II to the structure of $\mathcal{F}|V_0$ in (I. 2). But the properties of (II. 2) hold in the case of (I. 2) if we replace $i(\omega)$ by the linear characteristic homomorphism (see § 3). (3) If ∂M is not empty, suppose that, on each component of ∂M, $\mathcal{F}$ is transverse or tangent to ∂M then theorems I and II hold under trivial modifications of (I. 2. ii) and (II. 2. iv). (4) When $\pi_1(M)$ is not abelian, if any commutator of $\pi_1(M)$ is represented by curves in leaves of $\mathcal{F}$ (more precisely, if any commutator of $\pi_1(M)$ is represented by a curve which is simply bordant to a family of curves in leaves of $\mathcal{F}$ (see [9])), then Theorems I and II hold.

As an application we show that "almost all" free R^n actions on $(n+1)$-

manifolds with abelian fundamental group are "essentially" linear actions on $T^k \times R^{n+1-k}$. (For the precise statement see § 5).

(I. 1) and (II. 1) are proved in section 1 by using the results of Haefliger-Reeb [3] and a lemma of Koike (Lemma 1. 2). (I. 2. i) is proved by the same method as [5].

In [5] the proof of the theorem of Saksteder was devided in three steps. At the first step we took a closed transversal curve C and showed that the domain of an element of the holonomy pseudogroup acting on C is extendable to the whole of C and the pseudogroup is really a group acting on C without fixed point. In the second step we related this group to the fundamental group of M and showed that the group is finitely generated. At the third step, by using the theorem of Denjoy, we showed that there exists a metric on C which is invariant under the holonomy pseudogroup. The theorem of Sacksteder follows from this invariant metric.

In the present case we prove the first step by using singular foliations on T^2 astisfying some conditions (see § 2). In § 3 we prove the second step by considering a subspace $M(\mathcal{F}, c)$ of the space of maps of T^2 into M and by introducing the notion of the characteristic homomorphism $\chi(\mathcal{F})$. The third step is the same as in [5] and we prove (I. 2. i) in § 3.

The rest of Theorems I and II are proved in § 4. Here the essential tool is Lemma 4. 4. which is a consequence of the Van Kampen's theorem. In § 6 we show examples which show that the statements of theorems I and II fail if we omit the abelian assumption on $\pi_1(M)$.

§ 1. Proof of (I. 1) and (II. 1)

The following lemma is trivial.

Lemma 1. 1. *Let ω be a closed one from on a manifold M. If the rank of ω is one then there exist the smallest positive number s in Per (ω). If the rank of ω is greater than one then Per(ω) is dense in* R.

Proof of (II. 1). If the rank of ω is zero, then the result is trivial. If the rank of ω is one, choose a point x_0 in M and define $f: M \to S^1 = R/sZ$ by $f(x) = \int_l \omega$ where l is a curve from x_0 to x. Then f defines the foliation $\mathcal{F}$.

The following, easy but usefull, lemma is due to T. Koike [7].

Lemma 1. 2. *Let M be a connected manifold with abelian fundametal group and L_1, L_2 connected codimension one closed submanifolds of M. If $L_1 \cap L_2$ is vide then $M-(L_1 \cup L_2)$ is disconnected.*

Proof. If $M-(L_1 \cup L_2)$ is connected then we can take a closed curve l_i $(i=1, 2)$ in $M-L_j$ $(j \neq i)$ with base point $x \in M-(L_1 \cup L_2)$ which intersects transversally with L_i at only one point. Since $\pi_1(M)$ is abelian, $\ell_1*\ell_2*\ell_1^{-1}*\ell_2^{-1}$ is homotopic to zero. This implies a contradiction by the method of mod 2 degree theory as in [10].

Proof of (I. 1). Let $\pi: M\to M/\mathcal{F}=X$ be the projection to the space of leaves of $\mathcal{F}$. Since there exists no closed transversal curve and $\mathcal{F}$ is transversally orientable, X is an orientable one dimensional manifold. If all leaves of $\mathcal{F}$ separate M into two connected components then X is simply connected and there exits a topological submersion h of X onto R (see [3]). So $f=h\cdot\pi$ is a topological submersion which defines $\mathcal{F}$. If there exists a leaf L such that $M'=M-L$ is connected then by (1.2) any leaf of $\mathcal{F}'=\mathcal{F}|M'$ separates M'. Hence there exists a topological submersion f' of M' onto R which defines $\mathcal{F}'$. Let $c(t)$ be a transverse segument in M passing through $c(0)\in L$. by the proof of Proposition 1 of [3], it is possible to choose f' so that $a_{\pm}=\lim_{t\to\pm 0} f'(c(t))$ exists. If $a_+=a_-$ we can extend f to the submersion $f: M\to R$. If $a_+\neq a_-$ then we define $f: M\to S^1=R/|a_+-a_-|Z$ by $f(x)=f'(x)$ for $x\in M'$ and $f(x)=a_+$ for $x\in L$. Then f defines $\mathcal{F}$.

Remark. In the above proof the case $a_+\neq a_-$ actually occurs. In fact consider the foliation on $R^2-(0,0)$ defined by the family of curves $xy=$ constant for $y\leqq 0$ and $\theta=$ constant for $y\geqq 0$ where θ is the angle from the positive semi x-axis. In [8], this possibility is missed.

Proposition 1. 3. *Let $\mathcal{F}$ be a codimension one foliation defined by closed one form ω. Suppose that $\pi_1(M)$ is abelian and the rank of ω is greater than one then $\mathcal{F}$ has a closed transversal curve.*

Proof. We use the notations of the above proof. Suppose $\mathcal{F}$ has no closed transversal curve, then X is a one dimensional manifold. If X is simply connected then we can take the submersion f of M onto R by using the integral ω (see the proof of Proposition 1 of [3]). So the rank of ω is zero. If X is not simply connected we can take a leaf L so that $X'=M-L/(\mathcal{F}|M-L)$ is simply connected and the rank of $\omega|M-L$ is zero. But if the rank of ω is greater than k then the rank of $\omega|M-L$ is greater than $k-1$. So, under the assumption, the rank of ω is one or zero.

§ 2. Singular foliations on 2-torus

Let ω be a one form on a manifold M with singularity Σ. For $x\in\Sigma$, if there exist a neighborhood V of x, a non-zero function g on V and a Morse function f on V satisfying $g\omega|V=df$, then we call x a *Morse type singular point.*

Let T^2 be the two dimensional torus, ω a one form on T^2 with Morse type singularity Σ, $\mathcal{F}$ a codimension one foliation on $T^2-\Sigma$ defined by $\omega=0$ and X a vector field on T^2 satisfying $i_X\Omega=\omega$, where Ω is a volume form on T^2 and i is the inner product operation. Let φ_τ be the flow generated by X then a leaf of $\mathcal{F}$ is an orbit of φ_τ. We say that a leaf of F is *singular* if the ω- (or α-) limit set of the corresponding orbit is a singular point. A singular leaf L is called a *sepratrix* if the ω- and α-limit sets of the corresponding orbit are singular points.

In this section we make the following assumptions.

A. 1. *X has neither limit cycle nor limit polygon (i.e. a ω- (or α-) limit set of X composed of separatrice and singular points).*
A. 2. *There exists a simple closed curve $c(t)$ of period 1 in $T^2-\Sigma$ which is transverse to $\mathcal{F}$.*

If (A. 2) is satisfied we take the coordinate of T^2 so that $c(t)=(t,0)$ $\in R^2/Z\times Z=T^2$.

Under the assumptions A. 1 and A. 2, we define the *holonomy map* H: $S^1\to S^1$ as follows. If there exists a positive number τ such that $\varphi_\tau((t,0))$ $\in S^1\times 1\subset T^2$, we define $\tau(t)$ the smallest of such τ and define $H(t)\in R/Z$ $=S^1$ by $\varphi_{\tau(t)}(t,0)=(H(t),1)$. We define the *t-map* ℓ_t: $[0,1]\to T^2$ by $\ell_t(\tau)$ $=\varphi_s((t,0))$, $s=\tau(t)\times\tau$. H and ℓ_t are defined on a subset $D(H)$ of S^1 but we have

Proposition 2. 1. *Let $D(H)$ be the domain of H, then $S^1-D(H)$ is a finite set and H is uniquely extended to a homeomorphism of S^1. We use the same notation H for the extended map then H has no fixed point unless H is the identity of S^1.*

Proof. By (A. 1) and the Poincaré-Bendixon theorem, if H is not defined at t then the ω-limit set of $\varphi_\tau((t,0))$ is a singular point of saddle type. Hence $S^1-D(H)$ is a finite set. Suppose that H is not defined at t_0 then it is easy to see that the limits $t_\pm=\lim_{t\to t_0\pm 0} H(t)$ exist. If $t_+\neq t_-$, consider $X=-X$ and the corresponding holonomy map H'. Then X' also satisfies the assumption (A. 1) and the domain of H' is open and dense. But on an interval $I=\widehat{t_-t_+}$ $\subset S^1$, H' is not defined. Thus $t_+=t_-$ and we can extend H by defining $H(t_0)$ $=t_+$. This H is a homeomorphism. If H is not the identity then the existence of a fixed point of H implies the existence of a limit cycle (or limit polygon) of X. So H has no fixed point.

Proposition 2. 2. *Let L be a non-singular leaf of $\mathcal{F}$ and C be the image of c of (A. 2), then $L\cap C$ is vide if and only if L is a compact leaf which is homologous to zero.*

Proof. Clearly $L\cap C=\phi$ if L is compact and homologous to zero. Conversely if L is not compact then, by (A. 1), L is not contained in a plannar domain of T^2 and $L\cap C\neq\phi$. If L is compact and $L\cap C=\phi$ then by considering the intersection number we see that L is homologous to $n\cdot C$ and, since L is a simple curve, $n=0$ or 1. If $n=1$ then C and L bound a domain diffeomorphic to $S^1\times[0,1]$. This contradicts to (A. 1).

Suppose that ω is a closed one form with Morse type singularity and satisfying (A. 2). In this case we parametrize $c(t)$ so that $\omega(\dot{c}(t))\equiv 1$ and put $a=\int_c\omega$. Then we consider T^2 as $R^2/aZ\times Z$ and S^1 as R/aZ.

Proposition 2. 3. *Under the above conditions, H is the rotation of an angle $2\pi b$ and $Per(\omega)$ is generated by a and b.*

Proof. For $t_0, t_1\in D(H)$, put $c_1=c|[H(t_0), H(t_1)]$ and $c_2=c|[t_0, t_1]$. Consider the curve $\ell=\ell_{t_0}*c_1*\ell_{t_1}^{-1}*c_2^{-1}$ then ℓ is homologous to zero. Hence we have

$$0=\int_{\ell}\omega=\int_{c_1}\omega-\int_{c_2}\omega=(H(t_1)-H(t_0))-(t_1-t_0).$$

So $H(t_1)=t_1+b, b=H(t_0)-t_0$. Put $c_3=c|[t_0, H(t_0)]$, then $\ell'=c_3*l_{t_0}^{-1}$ and c generate $\pi_1(T^2)$ and $\mathrm{Per}(\omega)$ is generated by $\int_c\omega=a$ and $\int_{\ell}\omega=b$.

§ 3. Characteristic homomorphism

In this section we suppose that $\pi_1(M)$ is abelian, $\mathcal{F}$ a codimension one foliation without holonomy of class C^2 on M defined by one form ω and $c(t)$ a closed transversal curve of period one. We denote by C the image of c.

Let $M(\mathcal{F}, c)$ be the set of maps f of T^2 to M which satisfy the following conditions.

C. 1. *There exists $\varepsilon>0$ such that, for $-\varepsilon<\tau<\varepsilon$, $f(t,\tau)\equiv c(t)$ where $(t,\tau)\in R^2/Z\times Z=T^2$.*
C. 2. *Singular points of $f^*\omega$ are Morse type.*

For an element f of $M(\mathcal{F}, c)$, let $\mathcal{F}(f)$ be the foliation on T^2 defined by $f^*\omega$ then $\mathcal{F}(f)$ satisfies (A. 1) and (A. 2). We denote by $H(f)$ the holomomy map of $\mathcal{F}(f)$. For $f_1, f_2\in M(\mathcal{F}, c)$ we define $f_1*f_2=f\in M(\mathcal{F}, c)$ by $f(t,\tau)=f_1(t, 2\tau)$, $0\leq\tau\leq\frac{1}{2}$ and $f(t,\tau)=f_2(t, 2\tau-1)$, $\frac{1}{2}\leq\tau\leq 1$. Similarly we define f^{-1} for $f\in M(\mathcal{F}, c)$. Then clearly $H(f_1*f_2)=H(f_2)\circ H(f_1)$ and $H(f^{-1})=H(f)^{-1}$. Therefore the image of the map $H: M(\mathcal{F}, c)\to\mathcal{H}(S^1)$ is a subgroup $G(\mathcal{F})$ where $\mathcal{H}(S^1)$ is the group of orientation preserving homeomorphisms of S^1.

Lemma 3.1. *Let* $\ell:[0,1]\to M$ *be a curve satisfying* $\ell(\tau)=\ell(0)$ *for* $\tau<\varepsilon$ *and* $\ell(\tau)=\ell(1)$ *for* $\tau>1-\varepsilon$ *then*
(i) *if* $\ell(0)=\ell(1)=c(t_0)$ *then there exists an element* f *of* $M(\mathcal{F},c)$ *such that* $f(t_0,\tau)=\ell(\tau)$, *and*
(ii) *if* ℓ *is a curve in a leaf of* $\mathcal{F}$ *from* $c(t_0)$ *to* $c(t_1)$ *then there exists* $f\in M(\mathcal{F},c)$ *such that* $\ell(f)_{t_0}=\ell$ *where* $\ell(f)_{t_0}$ *is defined by* $f\circ\ell_{t_0}$ *for the* t_0*-curve of* $\mathcal{F}(f)$.

This lemma follows from the assumption that $\pi_1(M)$ is abelian. The following lemma is easy.

Lemma 3.2. *For an element* f *of* $M(\mathcal{F},c)$, *let* $L_t(f)$ *be the leaf of* $\mathcal{F}(f)$ *passing through* $(t,0)$. *If* $L_t(f)$ *is singular then, for any neighborhood* V *of the singularity of* $f^*\omega$, *there exists* $f'\in M(\mathcal{F},c)$ *such that* $f=f'$ *on* T^2-V *and* $L_t(f')$ *is non-singular.*

Lemma 3.3. *For two elements* f_1 *and* f_2 *of* $M(\mathcal{F}.c)$, *if* $H(f_1)(t_0)=H(f_2)(t_0)$ *for some* $t_0\in S^1$ *then* $H(f_1)=H(f_2)$.

Proof. Consider $f=f_1*f_2^{-1}$ then $H(f)(t_0)=t_0$. Hence, by (2.1), $H(f)$ is the identity of S^1 and we have $H(f_1)=H(f_2)$.

Lemma 3.4. *For an element* f *of* $M(\mathcal{F},c)$, $H(f)$ *is a diffeomorphism of class* C^2.

Proof. Let $D(H(f))$ be the domain of $H(f)$ then clearly $H(f)$ is class C^2 on $D(H(f))$. For $t_0\in S^1-D(H(f))$ choose $t_1\in D(H(f))$ and U in (3.2) so that $U\cap l(f)_{t_1}([0,1])=\phi$. Then t_0 is contained in $D(H(f'))$ where f' satisfies (3.2) and $H(f')$ is class C^2 near t_0. But, since $H(f)(t_1)=H(f')(t_1)$, $H(f)=H(f')$.

We denote by $\mathrm{Diff}^r(S^1)$ the group of orientation preserving C^r-diffeomorphisms of S^1. Then by (2.1), (3.1) and (3.4) we have the following

Proposition 3.5. $G(\mathcal{F})$ *is a subgroup of* $\mathit{Diff}^2(S^1)$ *whose action on* S^1 *is free. Moreover for any leaf* L *in* $Q(C)$, $C\cap L$ *is an orbit of* $G(\mathcal{F})$ *under the identification of* C *and* S^1.

Let $\pi: R\to S^1=R/Z$ be the natural projection, $\mathrm{Diff}^r(R)_p$ the group of periodic C^r-diffeomorphisms of R where "periodic" means $\gamma(t+1)=\gamma(t)+1$ and $\pi\times 1$ the covering map of $R\times S^1$ on $S^1\times S^1=T^2$. For an element f of

$M(\mathcal{F}, c)$ let $\tilde{f}=f\circ(\pi\times 1)$ be the map of $R\times S^1$ in M and consider the foliation $\tilde{\mathcal{F}}(f)$ on $R\times S^1$ defined by $\tilde{f}^*\omega$. Then we can define an element $\tilde{H}(f)$ of $\mathrm{Diff}^2(R)_p$ analogously to $H(f)$. It is clear that $\pi\circ\tilde{H}(f)=H(f)\circ\pi$ and $\tilde{H}(f_1 * f_2)=\tilde{H}(f_2)\circ\tilde{H}(f_1)$.

We define a map p from $M(\mathcal{F}, c)$ to $\pi_1(M, x)$, $x=c(0)$, by $p(f)=f|(0\times S^1)$. Then clearly $p(f_1 * f_2)=p(f_1)+p(f_2)$, $p(f^{-1})=-p(f)$ and p is surjective by (3.1.i).

Proposition 3.6. *There exists a homomorphism χ of $\pi_1(M, x)$ into $\mathit{Diff}^2(R)_p$ such that $\tilde{H}=\chi\circ p$. We call χ the characteristic homomorphism of $\mathcal{F}$.*

Proof. It is sufficient to show that $p(f)=0$ implies $\tilde{H}(f)(t)=t$ for any $t\in R$. Suppose that $p(f)=0$ and $\tilde{H}(f)$ is not identity then there exists t such that $\ell(f)_t$ is well defined and $\tilde{H}(f)(t)\neq t$. Consider the map $\ell=\ell(f)_t * (c|[t, \tilde{H}(f)(t)])^{-1}$ then ℓ is homotopic to $f|0\times S^1$ and by assumption ℓ is homotopic to zero. Hence, by modifying ℓ, we have a closed transversal curve which is homotopic to zero. This implies the existence of a leaf with non-trivial holonomy group.

Let $\tilde{G}(\mathcal{F})$ be the image of $\tilde{H}$ and we define the order in $\tilde{G}(\mathcal{F})$ by $\tilde{H}(f)\geq\tilde{H}(g)$ if $\tilde{H}(f)(t)\geq\tilde{H}(g)(t)$ for one (and all) $t\in R$. then it is easy to see that $\tilde{G}(\mathcal{F})$ is an Archimedean ordered group and there exists an injective homomorphism i of $\tilde{G}(\mathcal{F})$ into R by the theorem of Hölder where i is unique up to multiplication by positive numbers. We define the *linear characteristic homomorphism* $\chi'=\chi'(\mathcal{F}, c)$ of $\pi_1(M, x)$ to R by $\chi'(\alpha)=i\circ\chi(\alpha)$ where i is chosen so that $\chi'(c)=1$. The following lemma follows from the proof of the theorem of Hölder and the definition of rotation number.

Lemma 3.7. *Let $\pi_*\colon \mathit{Diff}^2(R)_p\to\mathit{Diff}^2(S^1)$ be the map induced by $\pi\colon R\to S^1$ and $\gamma\colon\mathit{Diff}^2(S^1)\to S^1$ the rotation number then we have $\pi\circ\chi'=\gamma\circ\pi_*\circ\chi$. If $\mathcal{F}$ is defined by closed one form ω then $\chi'=i(\omega)$ where ω is normalized so that $\int_c\omega=1$.*

Lemma 3.8. *If the image of χ' contains a irrational number then there exist homeomorphisms h and $\tilde{h}$ of S^1 and R respectively such that $hG(\mathcal{F})h^{-1}\subset SO(2)$ and $\chi'(\alpha)=(\tilde{h}\circ\chi(\alpha)\circ\tilde{h}^{-1})(0)$ respectively. We call h and $\tilde{h}$ the linearization maps.*

This follows immediately from the theorem of Denjoy (for more detail see [5]).

Proof of (I. 2. i). Since $\pi_1(M)$ is finitely generated, the image of χ' is isomorphic to Z or contains a irrational number. In any case there exists a linearization map h. If we define the metric on C by h we obtain a metric which is invariant under the holonomy pseudogroup. Therefore as in [5] we have the result.

We remark that the closed one form ω is differentiable (without changing the differential structure on $Q(C)$) if and only if the linearization map h is differentiable. Therefore by the result of Herman announced in [4], we have the following

Proposition 3. 9. *Suppose $\mathcal{F}$ is class C^∞ then there exists a subset A of measure zero of R such that $\mathcal{F}|Q(C)$ is defined by a closed one form of class C^∞ if the image of χ' is not contained in A.*

§ 4. Proof of the rest of Theorems I and II

In this section $\mathcal{F}$ is a transversally orientable codimension one foliation without holonomy on M with abelian fundamental group and we suppose that $\mathcal{F}$ has a closed transversal curve C with parametrization $c(t)$.

Lemma 4. 1. *Let C_1 and C_2 be closed transversal curves. If $Q(C_1) \cap Q(C_2)$ is not vide then $Q(C_1) = Q(c_2)$.*

Proof. Let c_i: $[0, 1] \to M$ be a parametrization of $C_i (i=1, 2)$. We can suppose $c_1(0) = c_2(0)$. Then there exists a map f of T^2 to M such that $f(t, 0) = c_1(t)$, $f(0, \tau) = c_2(\tau)$ and the singular points of $f^*\mathcal{F}$ are Morse type. Then $S^1 \times 0$ and $0 \times S^1$ are transverse to $f^*\mathcal{F}$. Therefore by (2. 2) all leaves of $f^*\mathcal{F}$ which intersect with $S^1 \times 0$ also intersect with $0 \times S^1$ and we have $Q(C_1) = Q(C_2)$.

Lemma 4. 2. *$\partial Q(C)$ is a union of closed leaves.*

Proof. If $L \subset \partial Q(C)$ is not closed there exists a closed transversal curve C' passing through L. Then by (4. 1) we have $Q(C) = Q(C')$. This is a contradiction.

Lemma 4. 3. *Let C' be a one cycle in $M - Q(C)$ then C and C' are not homologous.*

Proof. Assume that C and C' are homologous then there exists a compact orientable two manifold S in M such that $\partial S = C \cup C'$ and S is transverse to $\partial Q(C)$. Moreover since $\pi_1(M)$ is abelian we can suppose that S is genus zero. Consider the connected component S' of $S \cap \overline{Q(C)}$ which contains C then $\partial S' = C \cup C''$. $\mathcal{F}$ is transverse to C and tangent to C''. Then, since S' is a planner domain, by the Poincaré-Bendixon theorem there exists a limit cycle (or polygon) in $\mathcal{F}|S'$. This is a contradiction.

Lemma 4. 4. *Let A_1, A_2 be connected codimension zero submanifolds of M such that $M = A_1 \cup A_2$ and $A_0 = A_1 \cap A_2 = \partial A_1 = \partial A_2$ is connected then at least one of $H_1(A_i, A_0)$ $(i=1, 2)$ is zero.*

The proof is immediate from the Van Kampen's theorem (see for example [2]).

Proposition 4. 5. *Let L be a closed leaf in $M - Q(C)$ then $M - L$ is not connected. In particular let V_i be a connected component of $M - Q(C)$ then we have Int $V_i \neq \phi$ and ∂V_i is a leaf L_i.*

Proof. We assume that the dimension of M is greater than two (two dimensional case is easy). If $M - L$ is connected then there exists a simple closed curve ℓ such that $\ell \cap C = \phi$ and ℓ intersects transversally with L only one time. Let A_1 be a neighborhood of $L \cup \ell$ such that $L \cup \ell$ is a deformation retract of A_1 and $A_1 \cap C = \phi$. Put $A_2 = M -$ Int A_1 then A_1 and A_2 satisfy the condition of (4. 4). It is easy to see that a cycle in $A_0 = \partial A_1$ is homologous to a cycle in L and ℓ represent a non zero element of $H_1(A_1, A_0)$. Therefore by (4. 4) C must be homologous to a cycle in A_0 and to a cycle in L. This contradicts to (4. 3). The second statement follows from (4. 2).

Proposition 4. 6. *There exists no closed transversal curve in V_i. In particular all leaves in V_i are closed.*

Proof. If there exists a closed transversal curve C' in V_i then, by (4. 3), C' represent a non zero element of $H_1(V_i, L_i)$. Similarly C represents a non zero element of $H_1(M - \text{Int } V_i, L_i)$. This contradicts to (4. 4).

Proposition 4. 7. *Let α be an element of $\pi_1(M, x)$, $x = c(0)$, then α is represented by a curve in L_x if $\chi'(\alpha) = 0$ and by a closed transversal curve if $\chi'(\alpha) \neq 0$.*

Proof. Take an element f of $M(\mathcal{F}, c)$ such that $p(f) = \alpha$ and $\ell(f)_0$ is well defined. Then α is represented by $\ell = \ell(f)_0 * (c|[0, \tilde{H}(f)(0)])^{-1}$. By the definition of χ', $\tilde{H}(f)(0) = 0$ or $\neq 0$ according to $\chi'(\alpha) = 0$ or $\neq 0$. Therefore if $\chi'(\alpha) = 0$ ℓ is a curve in L_x and if $\chi'(\alpha) \neq 0$, by modification of ℓ, we obtain a closed transversal curve homotopic to ℓ.

We suppose that $\mathcal{F}$ is defined by closed one form ω of rank greater than one. Then by (1. 3) there exists a closed transversal curve $c(t)$. We parametrise $c(t)$ so that $\omega(\dot{c}(t))=1$ and let a be the period of $c(t)$.

Lemma 4. 8. *$c(t_1)$ and $c(t_2)$ belong to the same leaf of $\mathcal{F}$ if and only if t_2-t_1 is a peroid of ω.*

Proof. "Only if" part is clear. Suppose that t_2-t_1 is a period of ω then there exists a closed curve $\mathcal{F}$ with base point x such that $\int_l \omega=t$. Consider $f\in M(\mathcal{F},c)$ of (3. 1. i) then $\mathrm{Per}(f^*\omega)$ is generated by a and t. therefore $c(t_1)$ and $c(t_2)$ belong to the same leaf of $\mathcal{F}$ by (2. 3).

Proposition 4. 9. *If the rank of ω is greater than one then any leaf of $\mathcal{F}|Q(C)$ is dense in $Q(C)$*

This is immediate from (1. 1) and (4. 8).

Proposition 4. 10. *For $x_1, x_2\in Q(C)$, let l be a curve from x_1 to x_2 then x_1 and x_2 belong to the same leaf if and only if $\int_l \omega$ is a period of ω.*

Proof. Let l_i be a curve in a leaf of $\mathcal{F}$ from x_i to $c(t_i)$ $(i=1,2)$. Define a closed curve l' by $l'=(l*l_2)*(l_1*c|[t_1,t_2])^{-1}$ then $\int_{l'}\omega=\int_l\omega+(t_1-t_2)$ is a period of ω and t_1-t_2 is a period of ω if and only if $\int_l\omega\in\mathrm{Per}(\omega)$. (4. 10) follows from (4. 9).

Thus Theorems I and II are proved by the propositions of this section together with the arguments in the section one. We remark that, for the proof of (4. 8) (4. 9) and (4. 10), it is sufficient that ω is defined only on $Q(C)$.

§ 5. Free R^n-actions on $(n+1)$-manifolds

Theorem 5. 1. *Let $\mathcal{F}$ be a transversally orientable codimension one foliation of class C^2 on M. Suppose that all leaves of $\mathcal{F}$ are diffeomorphic to R^n and $\pi_1(M)$ is abelian.*

(1) *If $\mathcal{F}$ has no closed transversal curve then there exists a codimension one foliation $\mathcal{F}'$ on R^2 or $S^1\times R$ such that $\mathcal{F}$ is the product of $\mathcal{F}'$ with R^{n-1}*

(2) *If $\mathcal{F}$ has a closed transversal curve C, let U_i be a connected component of $M-Q(C)$. Then there exists a codimension one foliation $\mathcal{F}_i$ on the half plane $H=\{(x,y)\,|y\geq 0\}$ which is tangent to the x-axis such that*

$\mathcal{F}|U_i$ is the product of $\mathcal{F}_i$ with R^{n-1}.

Proof. This theorem is a consequence of the results of Palmaire [13]. If $\mathcal{F}$ has no closed transversal curve then $\pi: M \to X = M/\mathcal{F}$ is a fiberation and $\mathcal{F}$ is the product of a foliation $\mathcal{F}'$ on a two dimensional manifold S with R^{n-1}([13]). By the proof of (I, 1), $\pi_1(X)=0$ or Z and we must have $S=R^2$ or $R\times S^1$. If $\mathcal{F}$ has a closed transversal curve then $X_i = U_i/(\mathcal{F}|U_i)$ is a simply connected manifold whose boundary is one point. Therefore by [13] we obtain the results.

Let $\varphi: R^n \times M \to M$ be a free action of R^n of class C^∞ on an orientable $(n+1)$-manifold M. Then the orbits of φ defines a foliation $\mathcal{F}_\varphi$ on M whose leaves are R^n. If $\pi_1(M)$ is abelian and $\mathcal{F}_\varphi$ has a closed transversal curve C it is easy to see that $\pi_1(Q(C))$ is isomorphic to $\pi_1(M)$ and we say the restrion of φ to $Q(C)$ the *essential part* of φ.

We say that φ is a *linear action* if there exists a diffeomorphism h of M onto $R^{n-k}\times T^{k+1}$ and a $(n+1, n)$-matrix of real numbers A such that $h\circ\varphi(y, x) = h(x) + A\cdot y$ where $x\in M$, $y\in R^n$ and $R^{n-k}\times T^{k+1} = R^{n+1}/Z^{k+1}$, $Z^{k+1} = \{{}^t(0, \cdots, 0, n_1, \cdots, n_{k+1}), n_i \in Z\}$.

Theorem 5. 2. *Let φ be a smooth free R^n-action on an orientable $(n+1)$-manifold M. Suppose $\pi_1(M)$ is abelian and $\chi': \pi_1(M) \to R$ be the linear characteristic homomorphism of $\mathcal{F}_\varphi$. There exists a subset A' of measure zero of R and if the image of χ' is not contained in A' then the essential part of φ is a linear action.*

Proof. The proof of this theorem is exactly the same as the proof of D. Ticshler and R. Tischler [16]. If the rank of $\mathrm{Im}\,\chi'$ is one then $\pi_1(M)=Z$ and we can choose C so that $C\cap L$ is one point for any leaf L in $Q(C)$. Using such C we can extend the action φ to an R^{n+1}-action on $Q(C)$ and we see that $Q(C) = R^n\times S^1$. If the rank of $\mathrm{Im}\,\chi'$ is greater than one then, by (3. 9) if the image of χ' is not contained in $A, \mathcal{F}_\varphi|Q(C)$ is defined by a smooth closed one form. Therefore the return functions of [16] are smooth. (In [16] the differentiability of return functions is used without proof. This gap is saved by the result of Herman [4]) Moreover if there exists an element of $\mathrm{Im}\,\chi'$ which satisfies the Liouville inequality (see [16]) we can extend φ to an R^{n+1}-action on $Q(C)$ and we obtain the result (for more detail see [16]).

§ 6. Counter examples

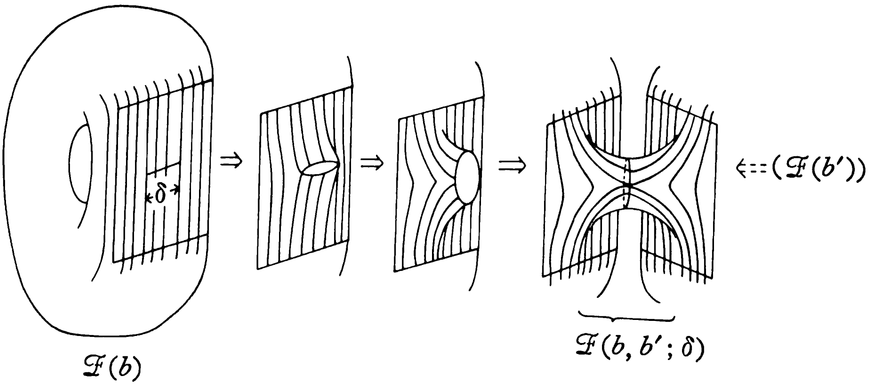
Theorem 6. 1. *There exists a codimension one foliation $\mathcal{F}$ on M of class C^∞ with exceptional leaves under the following assumptions on $\mathcal{F}$ and M.*

(i) *$\mathcal{F}$ is without holonomy and $\pi_1(M)$ is abelian but not finitely generated.*
(ii) *$\mathcal{F}$ is without holonomy and $\pi_1(M)$ is finitely generated but not abelian.*
(iii) *$\mathcal{F}$ is defined by a closed one form and $\pi_1(M)$ is neither abelian nor finitely generated.*

Proof. (i) In [5] (p. 614) we constructed a countable abelian subgroup G of $\mathrm{Diff}^\infty(S^1)$ whose action on S^1 is free and has exceptional orbits. Let N be a manifold such that $\pi_1(N)$ is isomorphic to G and consider $M=\tilde{N}\underset{G}{\times}S^1$ where $\tilde{N}$ is the universal covering of N and G acts on $\tilde{N}\times S^1$ by diagonal action. We consider the codimension one foliation $\mathcal{F}$ on M induced by the product foliation on $\tilde{N}\times S^1$. Then $(M, \mathcal{F})$ has the desired properties. (ii) Cherry [1] constructed a vector field on T^2 with one source and one saddle point as singularities which has exceptional orbits and has no periodic orbit. Thus we obtain the desired $\mathcal{F}$ on T^2-{two points}. (iii) We define f_n: $S^1\to S^1$ by $f_n(t)=t+\frac{2}{3^n}$, $n=0, 1, 2, \cdots$, where $S^1=R/4Z$ and consider open sets I_n and I'_n defined as follows. Put $I_0=(0, 1)$, $I'_0=(2, 3)$ and we define I_n, I'_n inductively. Put $I_{n-1}\cup I'_{n-1}=\bigcup_{i=1}^{2n}(a_i, b_i)$ then define $I_n=\bigcup_{i=1}^{2n}\left(a_i, a_i+\frac{1}{3^n}\right)$ and $I'_n=\bigcup_{i=1}^{\infty}\left(b_i-\frac{1}{3^n}, b_i\right)$. Put $H=\{(x, y)\,|y>0\}$ and consider $M'=H\times S^1\cup\left[\bigcup_{n=0}^{n}\left(\left\{(x, 0)\,|\, n<x<n+\frac{1}{4}\right\}\times I_n\right)\cup\left\{(x, 0)\,|n+\frac{1}{2}<x<n+\frac{3}{4}\right\}\times I'_n\right)\right]$ and we identify $(x, 0)\times t$ with $\left(x+\frac{1}{2}, 0\right)\times f_n(t)$ for $n<x<n+\frac{1}{4}$. Then the obtained manifold M has the foliation $\mathcal{F}$ induced by the product structure of M'. Clearly $\mathcal{F}$ is defined by closed one form and has exceptional leaves.

Remarks. (1) By using the example of Cherry [1] it is easy to construct a codimension one foliation $\mathcal{F}$ without holonomy of class C^∞ which has a closed transversal curve homologous to zero. This gives a negative answer to a conjecture of Lamourex [9].
(2) The authore does not know whether there exists a codimension one foliation defined by closed one form of finite rank which has exceptional leaves.

To construct counter examples to Theorem II we define codimension one foliations $\mathcal{F}(b, b'\colon\delta)$, $b, b', \delta\in R$ and $\delta>0$, on the orientable closed surface V^2 of genus two defined by closed one form with two saddle singular points. Let $\mathcal{F}(b)$ be the codimension one foliation on $T^2=R^2/Z^2$ defined by bdx^1-dx^2. We define $\mathcal{F}(b, b'; \delta)$ as a "connected sum" of $\mathcal{F}(b)$ and $\mathcal{F}(b')$ along transversal segments of length δ as is shown in the figure.



By the construction $\mathcal{F}(b, b'; \delta)$ is defined by closed one form ω and the cohomology class of ω is determined (up to multiplication by non-zero numbers) by b and b'. But the qualitative structure of $\mathcal{F}(b, b': \delta)$ heavily depend on the length δ and properties of Theorem II does not hold. In particular we have

Theorem 6.2. (i) *If b is irrational then $\mathcal{F}(b, -b; \delta)$ is defined by closed one form of rank two. But all leaves are closed in V^2-{singular points}.*
(ii) *Suppose b is rational and b' is irrational. If $0<\delta<b$ then $\mathcal{F}(b, b'; \delta)$ has a closed transversal curve which meets both closed leaves and locally dense leaves. If $\delta>b$ then all leaves are dense.*

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