J. Math. Kyoto Univ. (JMKYAZ)
19-3 (1979) 435-470

A simple expression of the characters of certain discrete series representations

By

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(Received May 10, 1978)

Introduction

The purpose of this paper is to show that Hirai's general character formula (in [1]) is reduced to a simpler form for certain discrete series unitary representations of real simple Lie group of type B, $SO_0(p, q)$ ($p+q$ is odd). And we write it down by means of the determinants of matrices.

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. It is known that when G has a compact Cartan subgroup B with Lie algebra $\mathfrak{b}$, it has the discrete series unitary representations. We denote by $\mathfrak{b}_c^*$ the complex dual of $\mathfrak{b}$ and $\mathfrak{b}_B^*$ the lattice in $\mathfrak{b}_c^*$ consisting of such $\Lambda \in \mathfrak{b}_c^*$ that the mapping ξ_Λ: $B \ni \exp X \mapsto e^{\Lambda(X)}$ defines a unitary character of B. Let $W_G(\mathfrak{b})$ be the little Weyl group. Then there exists an invariant analytic function π'_Λ on the set G' of the regular elements corresponding to a tempered invariant eigendistribution π_Λ*) which is expressed as

$$\pi_\Lambda = \Big(\sum_{w \in W_G(\mathfrak{b})} \operatorname{sgn}(w)\, \xi_{w\Lambda}\Big)\, (\Delta^{\mathfrak{b}}(h, P))^{-1}$$

on $B \cap G'$ (see § 1). When Λ is regular, π_Λ is equal to the character of a discrete series unitary representation except the known multiplicative sign ± 1.

In [1], Hirai gave a global formula of π'_Λ on G' valid for any $\Lambda \in \mathfrak{b}_B^*$. When a root system canonically attached to a given connected component of a Cartan subgroup of G is not of class I (for example, of type B or C) (for the definitions, see [1]), the formula on this component is very complicated. In the case $G \cong Sp(n, \boldsymbol{R})$, he showed in [2] that for the special classes of Λ's, the formula of π'_Λ can be more simplified. Especially, the holomorphic discrete series representations and their contragradient ones are in such a case.

In this paper, we consider the group G of type B, $SO_0(p, q)$. Let K be a maximal compact subgroup of G. In this case, there exists no G-invariant

*) Harish-Chandra denotes this invariant eigendistribution by Θ_Λ in his papers.

Hermitian structure on G/K and G has no holomorphic discrete series repre sentation except when p or $q=2$. But we show that when Λ is dominant with respect to a *special* positive system on the root system Σ of $(\mathfrak{g}_c, \mathfrak{h}_c)$, the global formula is reduced to a simpler form (Theorem 1$\sim$6). A positive system on Σ is called *special* when only one simple root is singular imaginary and all the other simple roots are compact.

Let us explain our method. At first, we consider the group $G \cong SO_0(2n, 2n+1)$. Let $H^{(0,0)}$ be a Cartan subgroup corresponding to $\mathfrak{h}^{(0,0)}$ whose toroidal part is $\{0\}$. By operating suitable Cayley transformations on $\mathfrak{h}^{(0,0)}$ repeatedly, we get a complete collection of Cartan subalgebras not conjugate mutually under inner automorphisms of G. On a connected component A of a Cartan subgroup corresponding to a Cartan subalgebra obtained in this manner, we write down the general formula in [1] by matrix elements of $h \in A \cap G'$ concretely. Since $W_G(\mathfrak{b})$ is generated by the reflections with respect to the compact roots of $(\mathfrak{g}_c, \mathfrak{b}_c)$, at first we classify roots in Σ by means of the property of Cayley transformations, and then we characterize the elements in $W_G(\mathfrak{b})$ by certain properties with respect to the permutation and the changes of signs. Finally, using the fundamental lemmas (Lemma 3$\sim$5), we can simplify π'_Λ for the above Λ's. These processes are essential when we consider the simple Lie groups of type B in general.

In § 1, we prepare necessary facts about a Cayley transformation and definitions and notations. In § 2$\sim$§ 5, we consider π'_Λ on various types of Cartan subgroups. In § 6, at first we note that the toroidal parts of Cartan subalgebras (so, Catran subgroups) has no effect on our procedure. Then we show that the same method is applicable to the group $G \cong SO_0(n, n+2m+1)$, after changing the root vectors if necessary and modifying the lemmas slightly.

Very recently, Vargas showed similar results for such cases that includes $SO_0(n, n+1)$ by a different method in [6].

§ 1. Preliminaries

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. For a real root α of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we define a "Cayley transformation" ν_α and repeated Cayley transformations on $\mathfrak{g}_c$ and prepare the fundamental results about them in this section.

Denote by $\Sigma(\mathfrak{h})$ the set of roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$. Here for a Lie algebra $\mathfrak{n}$, $\mathfrak{n}_c$ means the complexification of $\mathfrak{n}$. Let $\boldsymbol{P}$ be an order on $\Sigma(\mathfrak{h})$ and denote the totality of positive roots by $P(\mathfrak{h})$. For $\alpha \in \Sigma(\mathfrak{h})$, we choose a root vector X_α from $\mathfrak{g}_c$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, where $H_\alpha \in \mathfrak{h}_c$ is the element corresponding to α under the Killing form B of $\mathfrak{g}_c$. Put $H'_\alpha = 2B(H_\alpha, H_\alpha)^{-1} H_\alpha$ and $X'_\alpha = 2\sqrt{B(H_\alpha, H_\alpha)}^{-1} X_\alpha$. A root α is called real when $\alpha(\mathfrak{h}) \subseteq \boldsymbol{R}$, and $\Sigma_R(\mathfrak{h})$ denotes the totality of real roots in $\Sigma(\mathfrak{h})$. Let θ be a Cartan involution of $\mathfrak{g}$ such that $\theta\mathfrak{h} = \mathfrak{h}$. For $\alpha \in \Sigma_R(\mathfrak{h})$, we can assume that X_α belong to $\mathfrak{g}$ and $\theta X_\alpha = -X_{-\alpha}$.

We denote the corresponding Cartan decomposition of $\mathfrak{g}$ by $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k}=\{X\in\mathfrak{g};\theta X=X\}$ and $\mathfrak{p}=\{X\in\mathfrak{g};\theta X=-X\}$. An imaginary root β (that is, $\beta(\mathfrak{h})\subseteq\sqrt{-1}\boldsymbol{R}$) is called compact (resp. singular imaginary) when X_β belongs to $\mathfrak{k}_c$ (resp. $\mathfrak{p}_c$).

For $\alpha\in\Sigma_R(\mathfrak{h})$, we define an automorphism of $\mathfrak{g}_c$, ν_α, as follows:

$$\nu_\alpha=\exp\{-\sqrt{-1}\pi\,\mathrm{ad}(X'_\alpha+X'_{-\alpha})/4\}.$$

Put $\mathfrak{h}^\alpha=\nu_\alpha(\mathfrak{h}_c)\cap\mathfrak{g}$, then it is a Cartan subalgebra of $\mathfrak{g}$. In fact, $\nu_\alpha(H'_\alpha)=\sqrt{-1}(X'_\alpha-X'_{-\alpha})$, so $\mathfrak{h}^\alpha=\sigma_\alpha+\boldsymbol{R}(X'_\alpha-X'_{-\alpha})$, where $\sigma_\alpha=\{H\in\mathfrak{h};\alpha(H)=0\}$. We know that $\Sigma_R(\mathfrak{h}^\alpha)=\{\nu_\alpha\beta|\beta\in\Sigma_R(\mathfrak{h}),\alpha\perp\beta\}$ and $\nu_\alpha\alpha$ is a singular imaginary root of $(\mathfrak{g}_c,\mathfrak{h}_c^\alpha)$. Here $\nu_\alpha\beta=\beta\cdot\nu_\alpha^{-1}|\mathfrak{h}_c^\alpha$.

For two roots α and β, they are called strongly orthogonal if neither $\alpha+\beta$ nor $\alpha-\beta$ is a root. When $\alpha\in\Sigma_R(\mathfrak{h})$ and $\beta\in\Sigma(\mathfrak{h})$ are strongly orthogonal, we get $\nu_\alpha(X_\beta)=X_\beta$ because $[X'_{\pm\alpha},X_\beta]=0$.

Lemma 1. *For $\alpha\in\Sigma_R(\mathfrak{h})$ and $\beta\in\Sigma(\mathfrak{h})$ such that $\alpha\perp\beta$ and $\alpha\pm\beta\in\Sigma(\mathfrak{h})$, we have $\nu_\alpha(X_\beta)=-(\sqrt{-1}/2)\,\mathrm{ad}(X'_\alpha+X'_{-\alpha})X_\beta$.* Furthermore if β is *is a compact (resp. singular imaginary) root of $\mathfrak{h}$, then $\nu_\alpha\beta$ is a singular imaginary (resp. compact).*

Proof. Since $\alpha\perp\beta$ and $\pm2\alpha+\beta\notin\Sigma(\mathfrak{h})$, $\mathrm{ad}(X'_\alpha)\mathrm{ad}(X'_{-\alpha})X_\beta=\mathrm{ad}(X'_{-\alpha})\mathrm{ad}(X'_\alpha)X_\beta=2X_\beta$. Therefore $(\mathrm{ad}(X'_\alpha+X'_{-\alpha}))^2X_\beta=4X_\beta$ and

$$\begin{aligned}\nu_\alpha(X_\beta)&=\sum_{m=0}^{\infty}\frac{1}{(2m)!}\left(-\frac{\sqrt{-1}\pi}{4}\right)^{2m}(\mathrm{ad}(X'_\alpha+X'_{-\alpha}))^{2m}X_\beta\\&\quad+\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(-\frac{\sqrt{-1}\pi}{4}\right)^{2m+1}(\mathrm{ad}(X'_\alpha+X'_{-\alpha}))^{2m+1}X_\beta\\&=\sum_{m=0}^{\infty}\frac{1}{(2m)!}(-1)^m\left(\frac{\pi}{2}\right)^{2m}X_\beta\\&\quad+\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{-\sqrt{-1}}{2}\right)(-1)^m\left(\frac{\pi}{2}\right)^{2m+1}\mathrm{ad}(X'_\alpha+X'_{-\alpha})X_\beta\\&=\frac{-\sqrt{-1}}{2}\,\mathrm{ad}(X'_\alpha+X'_{-\alpha})X_\beta\,.\end{aligned}$$

By the choice of X_α and $X_{-\alpha}$, $\theta(X'_\alpha+X'_{-\alpha})=-X'_\alpha-X'_{-\alpha}$ so it belongs to $\mathfrak{p}$. Therefore if $X_\beta\in\mathfrak{k}_c$, $[X'_\alpha+X'_{-\alpha},X_\beta]\in\mathfrak{p}_c$ and similarly if $X_\beta\in\mathfrak{p}_c$, $[X'_\alpha+X'_{-\alpha},X_\beta]\in\mathfrak{k}_c$. As $\nu_\alpha(X_\beta)$ is a root vector corrresponding to $\nu_\alpha\beta$, the required results are obtained. Q.E.D.

For $\alpha,\beta\in\Sigma_R(\mathfrak{h})$ such that $\alpha\perp\beta$, we can take a root vector $X_{\nu_\alpha\beta}$ for $\nu_\alpha\beta\in\Sigma_R(\mathfrak{h}^\alpha)$ as follows:

1) If α and β are strongly orthogonal, $X_{\nu_\alpha\beta}=X_\beta$.

2) Otherwise, $X_{\nu_\alpha\beta}=-\varepsilon\sqrt{-1}\nu_\alpha(X_\beta)=(-\varepsilon/2)[X'_\alpha+X'_{-\alpha},X_\beta]$, where ε is a sign of β.

Then $X_{\nu_\alpha\beta}$ and $X_{-\nu_\alpha\beta}$ belong to $\mathfrak{g}$ and they satisfy $[X_{\nu_\alpha\beta},X_{-\nu_\alpha\beta}]=H_{\nu_\alpha\beta}=\nu_\alpha(H_\beta)$ $=H_\beta$ and $\theta(X_{\nu_\alpha\beta})=-X_{-\nu_\alpha\beta}$.

Next let $E=(\alpha_1,\alpha_2,\cdots,\alpha_s)$ be an ordered set of real roots of $\mathfrak{h}$ and E^* the underlying set of E.

First put $\beta_s=\alpha_s$, $\mathfrak{h}_s=\mathfrak{h}^{\beta_s}=\nu_{\beta_s}(\mathfrak{h}_c)\cap\mathfrak{g}$. If $\mathfrak{h}_j$ and β_j are defined for $i\leqq j$ $\leqq s$, we put

$$\beta_{i-1}=\nu_{\beta_i}\nu_{\beta_{i+1}}\cdots\nu_{\beta_s}\alpha_{i-1},\ \mathfrak{h}_{i-1}=\nu_{\beta_{i-1}}((\mathfrak{h}_i)_c)\cap\mathfrak{g}\ \text{ and finally}$$

$$\mathfrak{h}^E=\mathfrak{h}_1,\ \nu_E\ (\text{or } \nu(E))=\nu_{\beta_1}\nu_{\beta_2}\cdots\nu_{\beta_s}.$$

When any two roots in E^* are mutually strongly orthogonal, $\nu_E=\nu_{\alpha_1}\nu_{\alpha_2}\cdots\nu_{\alpha_s}$. Furthermore $\nu_{\alpha_i}\nu_{\alpha_j}=\nu_{\alpha_j}\nu_{\alpha_i}$ because $[X_{\pm\alpha_i},X_{\pm\alpha_j}]=0$ for $i\neq j$.

For later use, we prepare some notations and definitions here. For a Cartan subalgebra $\mathfrak{h}$, we denote by $H^{\mathfrak{h}}$ the Cartan subgroup corresponding to it. For $\alpha\in\Sigma(\mathfrak{h})$, we define a character ξ_α on $H^{\mathfrak{h}}$ by

$$\mathrm{Ad}(h)X_\alpha=\xi_\alpha(h)X_\alpha\quad(h\in H^{\mathfrak{h}}).$$

Let A be a connected component of $H^{\mathfrak{h}}$. Put

$$\Sigma_R(A)=\{\alpha\in\Sigma_R(\mathfrak{h});\xi_\alpha(A)>0\}.$$

We introduce an order on $\Sigma_R(A)$ by restricting the order $\boldsymbol{P}$ on $\Sigma(\mathfrak{h})$ to $\Sigma_R(A)$ and we denote this order on $\Sigma_R(A)$ by $P_R(A)$. Furthermore put $A(P)$ $=\{h\in A;\xi_\alpha(h)>1$ for any positive α in $\Sigma_R(A)\}$.

In general, for a given root system Σ and an order $\boldsymbol{P}$ on it, we define $M^{or}(\boldsymbol{P})$, $P(E)$ and $\varepsilon(E)$ as follows:

Case I) The system Σ is simple and the lengths of all roots in Σ are uniform (e.g. D type). Let E be an ordered set of positive roots satisfying the following conditions:

1) For $E=(\alpha_1,\alpha_2,\cdots,\alpha_s)$, the underlying set E^* is a matximal orthogonal subset in P, where P is the set of all positive roots in Σ.

2) $\alpha_1>\alpha_2>\cdots>\alpha_s$.

In this case $M^{or}(\boldsymbol{P})$ is defined as the totality of E as above. Put $P(E)$ $=E^*$ and $\varepsilon(E)=(-1)^s$.

Case II) The system Σ is simple and the lengths of roots are not uniform (e.g. B type). Let $E=(\alpha_1,\alpha_2,\cdots,\alpha_s)$ be an ordered set of positive roots satisfying above 1) and the next two conditions:

3) The long roots are placed before the short ones. Put l be the number of long roots in E, and $m=[l/2]$. Then $\alpha_1>\alpha_3>\cdots>\alpha_{2m-1}$, $\alpha_{2i-1}>\alpha_{2i}$ $(i=1,2,\cdots,m)$ and $\alpha_{l+1}>\alpha_{l+2}>\cdots>\alpha_s$.

4) For $i=1,2,\cdots,m$, both $(\alpha_{2i-1}+\alpha_{2i})/2$ and $(\alpha_{2i-1}-\alpha_{2i})/2$ belong to P.

In this case $M^{or}(\boldsymbol{P})$ is defined as the totality of E as above. Put $P(E)$

$=E^{*}\cup\{(\alpha_1+\alpha_2)/2, (\alpha_1-\alpha_2)/2, \cdots, (\alpha_{2m-1}+\alpha_{2m})/2, (\alpha_{2m-1}-\alpha_{2m})/2\}$ and $\varepsilon(E)=(-1)^{s}(-1)^{m}$.

Case III) The system Σ is not simple. Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_p$ be the simple components of Σ and $\boldsymbol{P}_i$ be the order on Σ_i induced by $\boldsymbol{P}$. In this case $M^{or}(\boldsymbol{P})$ is defined as the totality of ordered set $E=(E_1, E_2, \cdots, E_p)$ with $E_i \in M^{or}(\boldsymbol{P}_i)$ for $i=1, \cdots, p$. Put $P(E)=P(E_1)\cup P(E_2)\cup\cdots\cup P(E_p)$ and $\varepsilon(E)=\varepsilon(E_1)\varepsilon(E_2)\cdots\varepsilon(E_p)$, where $P(E_i)$ and $\varepsilon(E_i)$ are already defined in case I or II.

In each case, for $E=(\alpha_1, \alpha_2, \cdots, \alpha_s)\in M^{or}(\boldsymbol{P})$, we define $W(E, \boldsymbol{P})$ by $W(E,\boldsymbol{P})=\{w\in W(\Sigma); wE=(w\alpha_1, w\alpha_2, \cdots, w\alpha_s)\in M^{or}(\boldsymbol{P})\}$, where $W(\Sigma)$ denotes the Weyl group of Σ.

For case I, an element E in $M^{or}(\boldsymbol{P})$ is called standard when α_j is the highest root in $\Sigma_{(j)}=\{\alpha\in\Sigma; \alpha_i\perp\alpha, i=1, \cdots, j-1\}$. For case II, an element $E\in M^{or}(\boldsymbol{P})$ is called standard when α_{2i-1} is the highest root in $\Sigma_{(2i-1)}$ and α_{2i} is the higherst root in $\Sigma_{(2i)}$ such that $2^{-1}(\alpha_{2i-1}\pm\alpha_{2i})\in\Sigma$ for $i=1, \cdots, m$ and α_l is the highest long root in $\Sigma_{(l)}$ if l is odd, and α_i is the highest short root in $\Sigma_{(i)}$ for $i=l+1, \cdots, s$. In general, an element $E=(E_1, \cdots, E_p)\in M^{or}(\boldsymbol{P})$ for case III is called standard when each E_i is standard.

Finally, we describe Hirai's theorem in [1] in such cases as we consider later. Let $\mathfrak{h}$ be one of Cartan subslgebras and $\boldsymbol{P}$ be the order on $\Sigma(\mathfrak{h})$ taken in the succeeding sections. Let $\{E_0, E_1, \cdots, E_r\}$ be the set of all standard elements in $M^{or}(P_B(\Lambda))$ and E_0 be strongly orthogonal. Take a system of root vectors $X_{\pm\gamma}(\gamma\in E_0^{*})$ which satisfies Condition 5.1 in [1], and put $\mathfrak{b}=\mathfrak{h}^{E_0}$. Set $W_G(\mathfrak{b})=N_G(\mathfrak{b})/Z_G(\mathfrak{b})$, where $N_G(\mathfrak{b})=\{g\in G; \mathrm{Ad}(g)(\mathfrak{b})\subseteq\mathfrak{b}\}$, and $Z_G(\mathfrak{b})=\{g\in G; \mathrm{Ad}(g)X=X \text{ for any } X\in\mathfrak{b}\}$.

For $E\in M^{or}(P_B(\Lambda))$ and a regular element $\Lambda\in(\mathfrak{b}_B^{*})$, we can define $\mathrm{sgn}_{P(E)}(\Lambda)$ by $\mathrm{sgn}\{\prod_{\gamma\in P(E)}(\Lambda, \nu_{E_0}\gamma)\}$ because $\nu_{E_0}\gamma$ is an imaginary root of $\mathfrak{b}$. Here B is the Cartan subgroup corresponding to $\mathfrak{b}$.

Theorem ([1]). *Let $\Lambda\in\mathfrak{b}_B^{*}$ be regular and π_Λ the tempered invariant eigendistribution on G described in Introduction, and put $\tilde{\kappa}^{\mathfrak{h}}(h, P(\mathfrak{h}))=\Delta^{\mathfrak{h}}(h, P(\mathfrak{h}))\pi'_\Lambda(h)$ for $h\in H^{\mathfrak{h}}\cap G'$. then $\tilde{\kappa}^{\mathfrak{h}}$ on a connected componenet A of $H^{\mathfrak{h}}$ is given as follows: For $h\in A(P)$,*

$$\tilde{\kappa}^{\mathfrak{h}}(h, P(\mathfrak{h}))=\sum_{i=0}^{r}\varepsilon(E_i)Z(h, E_i, \Lambda, P_B(\Lambda)),$$

where

$$\begin{aligned}Z(h, E_i, \Lambda, P_B(\Lambda)) &= \sum_{s\in W_G(\mathfrak{b})}\ \sum_{u\in W(E_i, P_B(\Lambda))}\mathrm{sgn}(s^{-1})\\ &\times \mathrm{sgn}_{P(E_i)}(s^{-1}\Lambda)\,\xi_{s^{-1}\Lambda}(h_K)\\ &\times \prod_{\alpha\in E_i^{*}}\exp\{-u\alpha(X)\,|(\nu_{E_0}\alpha, s^{-1}\Lambda)|/|\alpha|^2\}, \qquad \cdots\cdots(1)\end{aligned}$$

where $h=h_K\exp(X)$ $(h_K\in H^{\mathfrak{h}}\cap K$ and $X\in\mathfrak{h}\cap\mathfrak{p})$, and

$$\Delta^{\mathfrak{h}}(h, P(\mathfrak{h})) = \xi_\rho(h) \prod_{\alpha\in P(\mathfrak{h})} (1-\xi_\alpha(h)^{-1}) \quad \left(\rho = \frac{1}{2}\sum_{\alpha\in P(\mathfrak{h})} \alpha\right).$$

Note. (1) Since $\tilde{\kappa}^{\mathfrak{h}}(uh, P(\mathfrak{h})) = \mathrm{sgn}(u)\tilde{\kappa}^{\mathfrak{h}}(h, P(\mathfrak{h}))$ for $u\in W(\Sigma_R(A))$, $\tilde{\kappa}^{\mathfrak{h}}(h, P(\mathfrak{h}))$ is given on $A' = \{h\in A; \xi_\alpha(h) \neq 1\ (\alpha\in\Sigma_R(A))\}$.

(2) Put $q = (\dim G - \dim K)/2$, $\varepsilon(\Lambda) = \mathrm{sgn} \prod_{\alpha\in P(\mathfrak{h})} (\Lambda, \gamma)$. Then $(-1)^q \varepsilon(\Lambda)\pi_\Lambda$ is the character of the discrete series representation.

In the following sections, we shall reduce the above formula into simpler form for certain cases.

§ 2. The calculation on $H^{(0,0)}$ (I)

Put $J = \begin{pmatrix} 0_{2n} & 1_{2n} & 0 \\ 1_{2n} & 0_{2n} & 0 \\ 0 & 0 & -1 \end{pmatrix}$, where 1_{2n} and 0_{2n} denote the identity matrix and the zero matrix of order $2n$ respectively.

Put $\tilde{G} = \{g\in GL(4n+1, \boldsymbol{R}); {}^t gJg = J\}$ and we consider its connected component G which contains the identity. We denote its Lie algebra by $\mathfrak{g}$, then

$$\mathfrak{g} = \{X\in \mathfrak{gl}(4n+1, \boldsymbol{R}); {}^t XJ + JX = 0\}.$$

The map θ defined by the rule $\theta X = -{}^t X (X\in\mathfrak{g})$ is a Cartan involution of $\mathfrak{g}$ and we denote the corresponding Cartan decomposition by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \{X\in\mathfrak{g}; \theta X = X\}$ and $\mathfrak{p} = \{X\in\mathfrak{g}; \theta X = -X\}$. Obviously, $G = P\cdot SO_0(2n, 2n+1)\cdot P^{-1}$, where

$$P = \begin{pmatrix} (1/\sqrt{2})1_{2n} & (1/\sqrt{2})1_{2n} & 0 \\ (1/\sqrt{2})1_{2n} & (-1/\sqrt{2})1_{2n} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Put $\mathfrak{h}^{(0,0)}$ as follows:

$$\mathfrak{h}^{(0,0)} = \{X = \mathrm{diag}(h_1, h_2, \cdots, h_{2n}, -h_1, -h_2, \cdots, -h_{2n}, 0);\ h_i\in\boldsymbol{R}\}.$$

Then $\mathfrak{h}^{(0,0)}$ is a Cartan subalgebra of $\mathfrak{g}$ whose toroidal part is trivial, i.e. $\mathfrak{h}^{(0,0)} \subseteq \mathfrak{p}$.

For X as above, we define the linear functional e_i by $e_i(X) = h_i$ $(i=1, 2, \cdots, 2n)$. Then

$$\Sigma_R(\mathfrak{h}^{(0,0)}) = \Sigma(\mathfrak{h}^{(0,0)}) = \begin{Bmatrix} \pm e_i \pm e_j & (1\leqq i<j\leqq 2n), \\ \pm e_i & (1\leqq i\leqq 2n) \end{Bmatrix}.$$

Then we can take $X_\alpha(\alpha\in\Sigma(\mathfrak{h}^{(0,0)}))$ as following forms:

$$X_{e_i} = E_{i,4n+1} + E_{4n+1,2n+1}, \quad X_{-e_i} = E_{4n+1,i} + E_{2n+i,4n+1}$$

$$X_{e_i+e_j}=E_{i,2n+j}-E_{j,2n+i} \quad (i<j),$$

$$X_{e_i-e_j}=E_{i,j}-E_{2n+j,2n+i} \quad (i<j),$$

where $E_{p,q}$ denotes the matrix unit of (p, q)-element. Note that $[X_{e_i}, X_{\pm e_j}]=X_{e_i\pm e_j}$ for $i<j$ and that this choice of root vectors satisfies Condition 5.1 in [1]. We introduce a lexicographic order in $\Sigma(\mathfrak{h}^{(0,0)})$ with respect to $(e_1, e_2, \cdots, e_{2n})$.

In this case,

$$H^{\mathfrak{h}^{(0,0)}}=\left\{\begin{matrix} h=\mathrm{diag}(\rho_1 e^{h_1}, \rho_2 e^{h_2}, \cdots, \rho_1 e^{-h_1}, \rho_2 e^{-h_2}, \cdots, 1); \\ h_i\in \boldsymbol{R}, \quad \rho_i=\pm 1, \quad \prod_{i=1}^{2n}\rho_i=1 \end{matrix}\right\}.$$

Put $\rho=(\rho_1, \rho_2, \cdots, \cdots, \rho_{2n})$, then it determines a connected component. In this section we will treat the connected component $A^{(0)}$ which corresponds to $(1, 1, \cdots, 1)$. Therefore $\Sigma_R(A^{(0)})=\Sigma_R(\mathfrak{h}^{(0,0)})$ and it is of type B_{2n}.

In this case, the standard maximal orthogonal systems with respect to the order are as follows:

$$E_0=(e_1+e_2, e_1-e_2, \cdots, e_{2n-1}+e_{2n}, e_{2n-1}-e_{2n}),$$

$$E_1=(e_1+e_2, e_1-e_2, \cdots, e_{2n-3}-e_{2n-2}, e_{2n-1}, e_{2n}),$$

$$\cdots\cdots$$

$$E_n=(e_1, e_2, \cdots, e_{2n}).$$

For each E_i, $\mathfrak{h}^{E_i}$ is a compact Cartan subalgebra. In fact, by calculating successively we get that for $X=\mathrm{diag}(h_1, \cdots, h_{2n}, -h_1, \cdots, -h_{2n}, 0)$, $-\sqrt{-1}\nu_{E_i}(X)=P\cdot\mathrm{diag}(A_1, A_3, \cdots, A_{2n-1}, {}^tA_2, \cdots, {}^tA_{2n-2i}, 0, A_{2n-2i+2}, \cdots, A_{2n})\cdot P^{-1}$, where $\mathrm{diag}(\cdots)$ denotes the blockwise diagonal matrix with diagonal entries indicated and $A_j=\begin{pmatrix} 0 & h_j \\ -h_j & 0 \end{pmatrix}$. So there exists $k_i\in K$ such that $\nu_{E_0}|\mathfrak{h}_c^{(0,0)}=Ad(k_i^{-1})\nu_{E_i}|\mathfrak{h}_c^{(0,0)}$, where K is the maximal compact subgroup of G corresponding to $\mathfrak{k}$.

Put $\mathfrak{b}=(\mathfrak{h}^{(0,0)})^{E_0}$ and $B=\exp\mathfrak{b}=H^{\mathfrak{b}}$. Note that $\nu_{E_0}(e_i)\in\Sigma(\mathfrak{b})$ is compact if and only if so is $\nu_{E_n}(e_i)\in\Sigma(\mathfrak{h}^{E_n})$, because $k_n\in K$. Hence using Lemma 1 repeatedly, we see that $\nu_{E_0}(e_i)$ is compact if and only if i is even and the other short roots are singular imaginary. For long roots, $\nu_{E_0}(e_i\pm e_j)$ is compact if and only if i and j have the same parity, because $[\mathfrak{k}, \mathfrak{k}]\subseteqq\mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}]\subseteq\mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}]\subseteq\mathfrak{p}$.

Denote by $\mathfrak{b}_c^*$ the dual space of $\mathfrak{b}$ over $\boldsymbol{C}$ and $\mathfrak{b}_B^*$ its additive subgroup consisting of such $\Lambda\in\mathfrak{b}_c^*$ that $\xi_\Lambda(\exp X)=e^{\Lambda(X)}$ $(X\in\mathfrak{b})$ defines a unitary character of B. Since $\{-\sqrt{-1}\nu_{E_0}(H_{e_i})\}$ $(1\leqq i\leqq 2n)$ forms a basis of $\mathfrak{b}$, each element Λ in $\mathfrak{b}_B^*$ is parametrized by the sequence $(l_1, l_2, \cdots, l_{2n})$, where $l_i=\Lambda(\nu_{E_0}(H_{e_i}))$. Then they are all integers or half-integers.

Since $W_G(\mathfrak{b})$ is generated by the reflections corresponding to compact roots of $\mathfrak{b}$, every element w in $W_G(\mathfrak{b})$ can be expressed as $w(\nu_{E_0}e_j)=$

$\nu_{E_0}(\varepsilon_j e_{s(j)})$ $(1 \leqq j \leqq 2n)$, where $\varepsilon_j = \pm 1$, $\prod_{j=1}^{n} \varepsilon_{2j-1} = 1$ and s is an element of the $2n$-th symmetric group S_{2n} such that $s(\{1, 3, \cdots, 2n-1\}) \subseteq \{1, 3, \cdots, 2n-1\}$. Put $S = \{s \in S_{2n}; s(\{1, 3, \cdots, 2n-1\}) \subseteq \{1, 3, \cdots, 2n-1\}\}$ and $\hat{S} = \{\hat{s} = (s, \varepsilon); s \in S$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2n})$, $\varepsilon_i = \pm 1$, $\prod_{i=1}^{n} \varepsilon_{2i-1} = 1\}$. Then $W_G(\mathfrak{b})$ is isomorphic to $\hat{S}$ and $\operatorname{sgn}(\hat{s}) = \operatorname{sgn}(s) \times \varepsilon_2 \varepsilon_4 \cdots \varepsilon_{2n}$. Here $\operatorname{sgn}(\hat{s})$ means the signature of $\hat{s}$ in $W(\Sigma(\mathfrak{b}))$.

For $\Lambda = (l_1, l_2, \cdots, l_{2n}) \in \mathfrak{b}_B^*$ such that $l_1 > l_3 > \cdots > l_{2n-1} > l_2 > l_4 > \cdots > l_{2n} > 0$ and all l_i's are integers, we have the following lemma.

Lemma 2. *For* $\hat{s} \in W_G(\mathfrak{b})$, $\operatorname{sgn}(\hat{s}^{-1}) \operatorname{sgn}_{P(E_i)}(\hat{s}^{-1}\Lambda) = \operatorname{sgn}(s)$.

Proof. Apparently, $\hat{s}^{-1}\Lambda$ is parametrized by $(\varepsilon_1 l_{s(1)}, \varepsilon_2 l_{s(2)}, \cdots, \varepsilon_{2n} l_{s(2n)})$. Since $s \in S$ does not change the parity, $l_{s(2p-1)} > l_{s(2q)}$ for any p and q $(1 \leqq p, q \leqq n)$. Therefore

$$\operatorname{sgn}_{P(E_i)}(\hat{s}^{-1}\Lambda) = \operatorname{sgn} \prod_{p=1}^{n-i} (\varepsilon_{2p-1}^2 l_{s(2p-1)}^2 - \varepsilon_{2p}^2 l_{s(2p)}^2) \operatorname{sgn} \prod_{i=1}^{2n} \varepsilon_i l_{s(i)}$$
$$= \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2n} = \varepsilon_2 \varepsilon_4 \cdots \varepsilon_{2n} \quad (\because \varepsilon_1 \varepsilon_3 \cdots \varepsilon_{2n-1} = 1).$$

On the other hand, $\operatorname{sgn}(\hat{s}^{-1}) = \operatorname{sgn}(\hat{s}) = \operatorname{sgn}(s) \varepsilon_2 \varepsilon_4 \cdots \varepsilon_{2n}$. Thus we get the lemma. Q.E.D.

Note. It follows from the result in [5] that there exists a *special* order on $\Sigma(\mathfrak{b})$. An order on it is called *special* when the only one simple root is singular imaginary and the other simple roots are compact. In our case the set $\{\nu_{E_0}(e_1 - e_3), \nu_{E_0}(e_3 - e_5), \cdots, \nu_{E_0}(e_{2n-3} - e_{2n-1}), \nu_{E_0}(e_{2n-1} - e_2), \nu_{E_0}(e_2 - e_4), \cdots, \nu_{E_0}(e_{2n-2} - e_{2n}), \nu_{E_0}(e_{2n})\}$ is a fundamental system of $\Sigma(\mathfrak{b})$ and it gives the *special* order on it. In fact $\nu_{E_0}(e_{2n-1} - e_2)$ is singular imaginary and the others are compact. With respect to this order Λ as above is dominant integral.

From now on, we show that $\tilde{\kappa}^{\mathfrak{h}}(h, P(\mathfrak{h}))$ is reduced to simpler form for Λ as above. We do not deal with the orders except the lexicographic order defined by $(e_1, e_2, \cdots, e_{2n})$, so we will omit $P_R(\Lambda)$ in $Z(\cdots)$ and $W(E, \cdots)$ *for* simplicity.

At first, we study the term $Z(h, E_0, \Lambda)$ in (1) for $h = \exp X \in A^{(0)}(P)$. For $\hat{s} \in W(\mathfrak{h}^{E_0})$ and $u \in W(E_0)$, we have

$$\prod_{\alpha \in E_0^*} \exp\{-u\alpha(X) \,|\, (\hat{s}^{-1}\Lambda, \nu_{E_0}\alpha) \,|/|\alpha|^2\}$$
$$= \delta_{u(1)}^{-l_{s(1)}} \delta_{u(2)}^{-\varepsilon_1 \varepsilon_2 l_{s(2)}} \times \cdots \times \delta_{u(2n-1)}^{-l_{s(2n-1)}} \delta_{u(2n)}^{-\varepsilon_{2n-1}\varepsilon_{2n} l_{s(2n)}}, \qquad \cdots\cdots(2)$$

where $\delta_i = \exp e_i(X)$. It follows from Lemma 2, $\varepsilon_1 \varepsilon_3 \cdots \varepsilon_{2n-1} = 1$ and (2) that

$$Z(h, E_0, \Lambda) = \sum_{u \in W(E_0)} \sum_{\hat{s} \in \hat{S}} \operatorname{sgn} s \prod_{i=1}^{n} \delta_{u(2i-1)}^{-l_{s(2i-1)}} \delta_{u(2i)}^{-\varepsilon_{2i-1}\varepsilon_{2i} l_{s(2i)}}$$

$$=2^{n-1}\sum_{s\in S}\sum_{u\in w(E_0)}\sum_{\eta}\operatorname{sgn} s\prod_{i=1}^{n}\delta_{u(2i-1)}^{-l_{s(2i-1)}}\delta_{u(2i)}^{-\eta_i l_{s(2i)}}$$

where $\eta=(\eta_1, \eta_2, \cdots, \eta_n)$ and $\eta_i=\pm 1$.

In this case we get that

$$W(E_0)=\left\{\begin{array}{ll} u\in S_{2n};\ u(1)<u(3)<\cdots<u(2n-1), & \\ \qquad\qquad u(2i-1)<u(2i) & \text{for } i=1, 2, \cdots, n \end{array}\right\}.$$

Concerning to this set, we have the following lemma.

Lemma 3. *We can divide $W(E_0)\setminus\{e\}$ into two subsets $W_1=\{u_1, u_2, \cdots, u_m\}$ and $W_2=\{u_{m+1}, u_{m+2}, \cdots, u_{2m}\}$ such that $u_i\neq u_j$ for $i\neq j$ and they satisfy the following condition:*

Put $t_i=u_i^{-1}u_{i+m}$, then it is a transposition of two even numbers.

Proof. We prove this lemma by induction on n.

I) For $n=2$, $W(E_0)\setminus\{e\}$ consists of two permutations, that is

$$u=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 2 & 4\end{pmatrix} \quad\text{and}\quad u'=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix}.$$

Then $u'^{-1}u=(2, 4)$, where (p, q) means the transposition between p and q.

II) For $n\geqq 3$, put $W'(E_0)$ and $W''(E_0)$ as follows:

$$W'(E_0)=\{u\in W(E_0)\setminus\{e\}\,;\, u(3)=2\},$$

$$W''(E_0)=\{u\in W(E_0)\setminus\{e\}\,;\, u(3)=3\}.$$

For $u\in W(E_0)$, $u(3)$ must be equal to either 2 or 3, so we have

$$W(E_0)\setminus\{e\}=W'(E_0)\cup W''(E_0) \quad\text{and}\quad W''(E_0)\cap W''(E_0)=\phi.$$

As to $W'(E_0)$, we can divide it into two subsets W_1' and W_2', where $W_1'=\{u\in W'(E_0)\,;\, u(2)<u(4)\}$ and $W_2'=\{u\in W'(E_0)\,;\, u(2)>u(4)\}$. Put $s=(2, 4)$, then $W_1'=W_2'\cdot s$.

For $u\in W''(E_0)$, $u(1)=1$ and $u(2)=2$. So by the hypothesis of induction, we can divide $W''(E_0)$ into two subsets W_1'' and W_2'' such that they satisfy the required conditions. Put $W_1=W_1'\cup W_1''$ and $W_2=W_2'\cup W_2''$. Then the pair W_1 and W_2 is a required one. Q.E.D.

Then

$$Z(h, E_0, \Lambda)=2^{n-1}\sum_{s\in S}\sum_{j=1}^{m}\sum_{\eta}\operatorname{sgn}(s)\prod_{i=1}^{n}\delta_{u_j(2i-1)}^{-l_{s(2i-1)}}\delta_{u_j(2i)}^{-\eta_i l_{s(2i)}}$$

$$+2^{n-1}\sum_{s\in S}\sum_{j=1}^{m}\sum_{\eta}\operatorname{sgn}(s)\prod_{i=1}^{n}\delta_{u_j(t_j(2i-1))}^{-l_{s(2i-1)}}\delta_{u_j(t_j(2i))}^{-\eta_i l_{s(2i)}}$$

$$+2^{n-1}\sum_{s\in S}\sum_{\eta}\operatorname{sgn}(s)\prod_{i=1}^{n}\delta_{2i-1}^{-l_s(2i-1)}\delta_{2i}^{-\eta_i l_s(2i)}$$

$$=2^{n-1}\sum_{j=1}^{m}\sum_{s\in S}\{(\operatorname{sgn}(s)+\operatorname{sgn}(st_j))\sum_{\eta}\prod_{i=1}^{n}\delta_{u_j(2i-1)}^{-l_s(2i-1)}\delta_{u_j(2i)}^{-\eta_i l_s(2i)}\}$$

$$+2^{n-1}\sum_{s\in S}\sum_{\eta}\operatorname{sgn}(s)\prod_{i=1}^{n}\delta_{2i-1}^{-l_s(2i-1)}\delta_{2i}^{-\eta_i l_s(2i)}$$

$$=2^{n-1}\sum_{s\in S}\sum_{\eta}\operatorname{sgn}(s)\prod_{i=1}^{n}\delta_{2i-1}^{-l_s(2i-1)}\delta_{2i}^{-\eta_i l_s(2i)}\,. \qquad \cdots\cdots(4)$$

Next we consider $Z(h, E_1, \Lambda)$. In this case

$$W(E_1)=\left\{\begin{array}{l} u\in S_{2n};\, u(1)<u(3)<\cdots<u(2n-3),\\ \qquad u(2i-1)<u(2i)\ (1\leqq i\leqq n)\end{array}\right\}.$$

Put $W^1=\left\{\begin{array}{l} u\in W(E_1);\, u(1)<u(2)<u(3)<\cdots<u(2n-2),\\ \qquad u(2n-1) \text{ is odd and } u(2n)=u(2n-1)+1\end{array}\right\}$. Then the following lemma holds.

Lemma 4. *We can divide $W(E_1)\backslash W^1$ into two subsets $W_1^1=\{u_1, u_2, \cdots, u_{m_1}\}$ and $W_2^1=\{u_{m_1+1}, u_{m_1+2}, \cdots, u_{2m_1}\}$ such that $u_i\neq u_j$ for $i\neq j$ and they satisfy the next condition*:

Put $t_i=u_i^{-1}u_{m_1+i}$, then $t_i=(2p_i, 2q_i)$ $(p_i<q_i\leqq n-1)$ or $t_i=(2p_i-1, 2q_i-1)$ $(1\leqq p_i<q_i\leqq n)$, for $i=1, 2, \cdots, m_1$.

Proof. Put $\widetilde{W}^1=\{u\in S_{2n};\, u(1)<u(2)<\cdots<u(2n-2),\, u(2n-1)<u(2n)\}$. By Lemma 3, $W(E_1)\backslash\widetilde{W}^1=W_{01}^1\cup W_{02}^1$, where W_{01}^1 and W_{02}^1 have the above properties. Put

$$W_{i1}^1=\{u\in\widetilde{W}^1\backslash W^1;\, u(2n-1)=2i-1\}$$

and

$$W_{i2}^1=\{u\in\widetilde{W}^1\backslash W^1;\, u(2n-1)=2i\},\quad \text{for } i=1, 2, \cdots, n-1\,.$$

Then $\widetilde{W}^1\backslash W^1=\bigcup_{i=1}^{n-1}W_{i1}^1\bigcup_{i=1}^{n-1}W_{i2}^1$ and they are mutually disjoint. Put $t_i=(2i-1, 2n-1)$, then $W_{i1}^1\cdot t_i=W_{i2}^1$ for $i=1, 2, \cdots, n-1$. So put $W_1^1=\bigcup_{i=0}^{n-1}W_{i1}^1$ and $W=\bigcup_{i=0}^{n-1}W_{i2}^1$, then they have the required properties. Q.E.D.

Then it follows from Lemma 4 and $\varepsilon_1\varepsilon_3\cdots\varepsilon_{2n-1}=1$ that

$Z(h, E_1, \Lambda)$

$$=\sum_{\hat{s}\in\hat{S}}\sum_{u\in W(E_1)}\operatorname{sgn}(s)\prod_{i=1}^{n-1}\delta_{u(2i-1)}^{-l_s(2i-1)}\delta_{u(2i)}^{-\varepsilon_{2i-1}\varepsilon_{2i}l_s(2i)}\times\delta_{u(2n-1)}^{-l_s(2n-1)}\delta_{u(2n)}^{-l_s(2n)}$$

$$=2^n\sum_{s\in S}\sum_{j=1}^{m_1}\sum_{\eta^{(n)}}\operatorname{sgn}(s)\prod_{i=1}^{n-1}\delta_{u_j(2i-1)}^{-l_s(2i-1)}\delta_{u_j(2i)}^{-\eta_i l_s(2i)}\times\delta_{u_j(2n-1)}^{-l_s(2n-1)}\delta_{u_j(2n)}^{-l_s(2n)}$$

$$+2^n \sum_{s\in S} \sum_{j=1}^{m_1} \sum_{\eta^{(n)}} \mathrm{sgn}(s) \prod_{i=1}^{n-1} \delta_{u_j(t_j(2i-1))}^{-l_{s(2i-1)}} \delta_{u_j(t_j(2i))}^{-\eta_i l_{s(2i)}} \times \delta_{u_j(t_j(2n-1))}^{-l_{s(2n-1)}} \delta_{u_j(t_j(2n))}^{-l_{s(2n)}}$$

$$+2^n \sum_{s\in S} \sum_{u\in W^1} \sum_{\eta^{(n)}} \mathrm{sgn}(s) \prod_{i=1}^{n-1} \delta_{u(2i-1)}^{-l_{s(2i-1)}} \delta_{u(2i)}^{-\eta_i l_{s(2i)}} \times \delta_{u(2n-1)}^{-l_{s(2n-1)}} \delta_{u(2n)}^{-l_{s(2n)}}. \qquad \cdots\cdots(5)$$

Here $\eta^{(n)} = (\eta_1, \eta_2, \cdots, \eta_{n-1})$ and $\eta_i = \pm 1$. The sum of the first two summations in (5) turns out to be zero through the same argument as for $Z(h, E_0, \Lambda)$. Furthermore we note that for any $w \in W^1$, it belongs to S and $\mathrm{sgn}(w) = 1$.

The subset W^1 consists of n elements and for each even number $2p\,(\leqq 2n)$, there exists a unique element u_p in W^1 such that $u_p(2n) = 2p$. Then, for each p

$$\sum_{s\in S} \sum_{\eta^{(n)}} \mathrm{sgn}(s) \prod_{i=1}^{n-1} \delta_{u_p(2i-1)}^{-l_{s(2i-1)}} \delta_{u_p(2i)}^{-\eta_i l_{s(2i)}} \times \delta_{u_p(2n-1)}^{-l_{s(2n-1)}} \delta_{u_p(2n)}^{-l_{s(2n)}}$$

$$= \sum_{s\in S} \sum_{\eta^{(p)}} \mathrm{sgn}(s) \prod_{i=1}^{n} \delta_{2i-1}^{-l_{sw(2i-1)}} \times \prod_{i=1,\, i\neq p}^{n} \delta_{2i}^{-\eta_i l_{sw(2i)}} \times \delta_{2p}^{-l_{sw(2p)}}$$

$$= \sum_{s\in S} \sum_{\eta^{(p)}} \mathrm{sgn}(s) \prod_{i=1}^{n} \delta_{2i-1}^{-l_{s(2i-1)}} \prod_{i=1,\, i\neq p}^{n} \delta_{2i}^{-\eta_i l_{s(2i)}} \times \delta_{2p}^{-l_{s(2p)}}, \qquad \cdots\cdots(6)$$

where $w = u_p^{-1}$ and $\eta^{(p)} = (\eta_1, \cdots, \check{\eta}_p, \cdots, \eta_n)$ and $\check{\eta}_p$ means that η_p is absent.

In general, for a subset I of $I_n = \{1, 2, \cdots, n\}$, set

$$\Delta(h, n, \Lambda, I) = \sum_{s\in S} \sum_{\eta^{(I)}} \mathrm{sgn}(s) \prod_{i=1}^{n} \delta_{2i-1}^{-l_{s(2i-1)}} \prod_{i\in I_n\setminus I} \delta_{2i}^{-\eta_i l_{s(2i)}} \prod_{i\in I} \delta_{2i}^{-l_{s(2i)}},$$

where $\eta^{(I)} = (\eta_i;\, i \in I_n \setminus I)$, $\eta_i = \pm 1$. Then,

$$Z(h, E_1, \Lambda) = 2^n \sum_{p=1}^{n} \Delta(h, n, \Lambda, \{p\}).$$

For general E_j, by definition

$$W(E_j) = \left\{ \begin{array}{l} u \in S_{2n};\ u(1) < u(3) < \cdots < u(2n-2j-1) \\ \qquad u(2i-1) < u(2i) \quad \text{for } i = 1, 2, \cdots, n-j \\ \qquad u(2n-2j+1) < u(2n-2j+2) < \cdots < u(2n) \end{array} \right\}.$$

Put

$$W^j = \left\{ \begin{array}{l} u \in W(E_j);\ u(1) < u(2) < \cdots < u(2n-2j), \\ \qquad u(2i-1) \text{ is odd and } u(2i) = u(2i-1) + 1 \\ \qquad\qquad \text{for } n-j+1 \leqq i \leqq n \end{array} \right\}.$$

Then the following lemma holds.

Lemma 5. *We can divide $W(E_j) \setminus W^j$ into two subsets $W_1^j = \{u_1, u_2, \cdots, u_{m_j}\}$ and $W_2^j = \{u_{m_j+1}, u_{m_j+2}, \cdots, u_{2m_j}\}$ such that $u_i \neq u_j$ for $i \neq j$ and they satisfy the following condition:*

Put $t_i = u_i^{-1} u_{m_j+1}$, then $t_i = (2p_i, 2q_i)$ $(1 \leqq p_i < q_i \leqq n-j)$ or $t_i = (2p_i - 1,$

$2q_i-1)$ *for* $1\leqq i\leqq n-j$.

Proof. Using Lemma 3 and Lemma 4 repeatedly, we can prove this lemma eassily.

Then following the same way for $Z(h, E_1, \Lambda)$, we get that

$$Z(h, E_j, \Lambda)=2^{n+j-1}\sum_{I\subseteq I_n, |I|=j}\Delta(h, n, \Lambda, I).$$

Here $|I|$ denotes the number of elements of I.

Now we calculate the alternating sum of $Z(h, E_j, \Lambda)$'s for $j=0, 1, \cdots, n$. We determine the coefficient of the term

$$\operatorname{sgn}(s)\,(\prod_{i=1}^{n}\delta_{2i-1}^{-l_{s(2i-1)}}\delta_{2i}^{-\eta_i l_{s(2i)}})$$

in it. Let p be the number of δ_{2i}'s which have the positive exponent and denote by $a_{j,p}$ the coefficient of the above term in $Z(h, E_j, \Lambda)$. Then

$$a_{j,p}=\begin{cases}\binom{n-p}{j}2^{n+j-1} & \text{for } 0\leqq j\leqq n-p\\ 0 & \text{for } n-p+1\leqq j\leqq n.\end{cases}$$

Therefore we get that

$$\sum_{j=0}^{n}(-1)^j a_{j,p}=\sum_{j=0}^{n-p}(-1)^j 2^{n+j-1}\binom{n-p}{j}$$
$$=2^{n-1}\sum_{j=0}^{n-p}(-1)^j 2^j\binom{n-p}{j}=2^{n-1}(-1)^{n-p}.$$

Since $\varepsilon(E_0)=(-1)^n$, the following theorem holds.

Theorem 1. *Let* $\Lambda=(l_1, l_2, \cdots, l_{2n})$ *be an element of* $\mathfrak{h}_B^*$ *such that* $l_1>l_3>\cdots>l_{2n-1}>l_2>l_4>\cdots>l_{2n}>0$ and *all* l_i*'s are integers. For* $h=\operatorname{diag}(\delta_1, \delta_2, \cdots, \delta_{2n}, \delta_1^{-1}, \delta_2^{-1}, \cdots, \delta_{2n}^{-1}, 1)\in A^{(0)}(P)$

$$\tilde{\kappa}(h, P)=2^{p-1}\times\begin{vmatrix}\delta_1^{-l_1}\delta_1^{-l_3}\cdots\delta_1^{-\kappa_{2n-1}}\\ \delta_3^{-l_1}\cdots\cdots\delta_3^{-l_{2n-1}}\\ \vdots\\ \delta_{2n-1}^{-l_1}\cdots\cdots\delta_{2n-1}^{-l_{2n-1}}\end{vmatrix}\begin{vmatrix}\delta_2^{-l_2}-\delta_2^{l_2}, \delta_2^{-l_4}-\delta_2^{l_4}\cdots\delta_2^{-l_{2n}}-\delta_2^{l_{2n}}\\ \delta_4^{-l_2}-\delta_4^{l_2}\cdots\cdots\cdots\cdots\delta_4^{-l_{2n}}-\delta_4^{l_{2n}}\\ \vdots\\ \delta_{2n}^{-l_1}-\delta_{2n}^{l_2}\cdots\cdots\cdots\cdots\delta_{2n}^{-2n}-\delta_{2n}^{l_{2n}}\end{vmatrix}.$$

(Here and in the following, we denote $\tilde{\kappa}(h, P)$ *for* $\tilde{\kappa}^{\mathfrak{h}}(h, P(\mathfrak{h}))$ *since it cannot be misunderstood.)*

§ 3. The calculation on $H^{(0,0)}$ (II)

In § 2, we considered the character formula for Λ only on $A^{(0)}$. In this section, we study it on the other connected components of $H^{(0,0)}$. We denote by $A^{(k)}$ the connected component of $H^{(0,0)}\,(=H^{\mathfrak{h}^{(0,0)}})$ which corresponds to

$\rho^k = (\overbrace{-1, -1, \cdots, -1}^{2k}, 1, 1, \cdots, 1)$. Then we get that

$$\Sigma_R(A^{(k)}) = \left\{ \begin{array}{ll} \pm e_i \pm e_j & (1 \leqq i < j \leqq 2k \text{ or } 2k+1 \leqq i < j \leqq 2n), \\ \pm e_i & (2k+1 \leqq i \leqq 2n) \end{array} \right\}.$$

That is, $\Sigma_R(\mathrm{A}^{(k)})$ is a root system of type $D_{2k} \times B_{2n-2k}$. We introduce an order on it by restricting that on $\Sigma_R(\mathfrak{h}^{(0,0)})$ in § 2. Then the standard maximal orthogonal systems are as follows:

$$E_0 = (e_1 + e_2, e_1 - e_2, \cdots, e_{2k-1} + e_{2k}, e_{2k-1} - e_{2k}, \cdots, e_{2n-1} + e_{2n}, e_{2n-1} - e_{2n})$$

$$E_1 = (e_1 + e_2, e_1 - e_2, \cdots, e_{2n-3} - e_{2n-2}\ e_{2n-1}, e_{2n})$$

$$\cdots\cdots$$

$$E_{n-k} = (e_1 + e_2, e_1 - e_2, \cdots, e_{2k-1} - e_{2k}, e_{2k+1}, \cdots, e_{2n})$$

Lemma 6. *Let* $\hat{s} \in W_G(\mathfrak{b})$, *then* $\operatorname{sgn}(\hat{s}^{-1}) \operatorname{sgn}_{P(E_i)}(\hat{s}^{-1}\Lambda) = \operatorname{sgn}(s) \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2k}$.

Proof. Since $\Sigma_R(A^{(k)})$ is of type $D_{2k} \times B_{2n-2k}$, $P(E_i) = E_i^* \cup \{e_{2k+1}, e_{2k+2}, \cdots, e_{2n}\}$. On the other hand, $l_{2i-1} > l_{2j}$ for any i and j. Therefore

$$\begin{aligned} \operatorname{sgn}(\hat{s}^{-1}) \operatorname{sgn}_{P(E_i)}(\hat{s}^{-1}\Lambda) &= \operatorname{sgn}(s) \varepsilon_2 \varepsilon_4 \cdots \varepsilon_{2n} \varepsilon_{2k+1} \varepsilon_{2k+2} \cdots \varepsilon_{2n} \\ &= \operatorname{sgn}(s) \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2k} \quad (\because \varepsilon_1 \varepsilon_3 \cdots \varepsilon_{2n-1} = 1). \end{aligned}$$

Q.E.D.

Put $A_K^{(k)} = A^{(k)} \cap K$, then an element in $A^{(k)}$ can be expressed uniquely as $h = h_K \exp X$, where $h_K = \operatorname{diag}(-1_{2k}, 1_{2n-2k}, -1_{2k}, 1_{2n-2k+1}) \in A_K^{(k)}$ and $X \in \mathfrak{h}^{(0,0)}$. Furthermore, $h_K = \exp[\pi\{(X'_{e_1-e_2} - X'_{e_2-e_1}) + \cdots + (X'_{e_{2k-1}-e_{2k}} - X'_{e_{2k}-e_{2k-1}})\}] \in \exp \mathfrak{b}$. So we get that

$$\begin{aligned} \xi_{\hat{s}^{-1}\Lambda}(h_K) &= \exp(-\pi\sqrt{-1}(l_{s(1)} - l_{s(2)} + \cdots + l_{s(2k-2)} - l_{s(2k)})) \\ &= (-1)^{l_{s(1)} + l_{s(2)} + \cdots + l_{s(2k)}}. \end{aligned}$$

Here we denote $W(E_i)$ by $W(E_i)_{(k)}$. Since $\Sigma_R(A^{(k)})$ is of type $D_{2k} \times B_{2n-2k}$, we get that

$$W(E_0)_{(k)} = \left\{ \begin{array}{l} u \in S_{2n};\ u(1) < u(3) < \cdots < u(2n-1), \\ \qquad u(2i-1) < u(2i)\ (1 \leqq i \leqq n),\ 1 \leqq u(j) \leqq 2k\ (1 \leqq j \leqq 2k) \end{array} \right\}.$$

Put $\widehat{W}(E_0) = \{u \in W(E_0)_{(k)};\ u(i) = i \text{ for } i = 1, 2, \cdots, 2k\}$. Applying Lemma 3 for $W(E_0)_{(k)} \backslash \widehat{W}(E_0)$ and $\widehat{W}(E_0) \backslash \{e\}$, we can divide $W(E_0)_{(k)} \backslash \{e\}$ into two subsets $W_{1,(k)}$ and $W_{2,(k)}$ where $W_{1,(k)} = \{u_1, \cdots, u_{m_0}\}$ and $W_{2,(k)} = \{u_{m_0+1}, \cdots, u_{2m_0}\}$ such that $u_i \neq u_j$ for $i \neq j$ and the following condition holds:

Put $t_i = u_i^{-1} u_{m_0+i}$, then $t_i = (2p_i, 2q_i)$, where $1 \leqq p_i < q_i \leqq k$ or $k+1 \leqq p_i < q_i \leqq n$. Therefore for $h = h_K \exp X \in A^{(k)}(P)$, we have

$$
\begin{aligned}
Z(h, E_0, \Lambda) =& \sum_{\hat{s}\in\hat{S}} \sum_{u\in W(E_0)_{(k)}} \operatorname{sgn}(\hat{s}^{-1}) \operatorname{sgn}_{P(E_0)}(\hat{s}^{-1}\Lambda) \xi_{\hat{s}^{-1}\Lambda}(h_K) \\
& \times \left(\prod_{i=1}^{n} \delta_{u(2i-1)}^{-l_{s(2i-1)}} \delta_{u(2i)}^{-\varepsilon_{2i-1}\varepsilon_{2i} l_{s(2i)}}\right) \\
=& 2^{n-1} \sum_{j=1}^{m_0} \Big\{ \sum_{s\in S} \sum_{\eta} \operatorname{sgn}(s) \eta_1\cdots\eta_k (-1)^{l_{s(1)}+\cdots+l_{s(2k)}} \\
& \times \left(\prod_{i=1}^{n} \delta_{u_j(2i-1)}^{-l_{s(2i-1)}} \delta_{u_j(2i)}^{-\eta_i l_{s(2i)}} + \prod_{i=1}^{n} \delta_{u_j(t_j(2i-1))}^{-l_{s(2i-1)}} \delta_{u_j(t_j(2i))}^{-\eta_i l_{s(2i)}}\right) \Big\} \\
& + 2^{n-1} \sum_{s\in S} \sum_{\eta} \operatorname{sgn}(s) \eta_1\cdots\eta_k (-1)^{l_{s(1)}+\cdots+l_{s(2k)}} \\
& \times \left(\prod_{i=1}^{n} \delta_{2i-1}^{-l_{s(2i-1)}} \delta_{2i}^{-\eta_i l_{s(2i)}}\right), \qquad \cdots\cdots(9)
\end{aligned}
$$

where $\delta_i = \exp(e_i(X))$, $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$, $\eta_i = \pm 1$. Since each t_i transform $\{2, 4, \cdots, 2k\}$ and $\{2k+2, 2k+4, \cdots, 2n\}$ into themselves respectively, $\eta_{t_i(2)/2}\cdots\eta_{t_i(2k)/2} = \eta_1\cdots\eta_k$ and $\sum_{j=1}^{2k} l_{st_i(j)} = \sum_{j=1}^{2k} l_{s(j)}$ for any $s\in S$. For each $t=t_i$ and $u=u_i$,

$$
\begin{aligned}
& \sum_{s\in S} \sum_{\eta} \operatorname{sgn}(s) \eta_1\cdots\eta_k (-1)^{l_{s(1)}+\cdots+l_{s(2k)}} \prod_{i=1}^{n} \delta_{u(t(2-i))}^{-l_{s(2i-1)}} \delta_{u(t(2i))}^{-\eta_i l_{s(2i)}} \\
& = \sum_{s\in S} \sum_{\eta} \operatorname{sgn}(s) \eta_1\cdots\eta_k (-1)^{l_{st(1)}+\cdots+l_{st(2k)}} \prod_{1=1}^{n} \delta_{u(2i-1)}^{-l_{st(2i-1)}} \delta_{u(2i)}^{-\eta_i l_{st(2i)}} \\
& = -\sum_{s\in S} \sum_{\eta} \operatorname{sgn}(s) \eta_1\cdots\eta_k (-1)^{l_{s(1)}+\cdots+l_{s(2k)}} \prod_{i=1}^{n} \delta_{u(2i-1)}^{-l_{s(2i-1)}} \delta_{u(2i)}^{-\eta_i l_{s(2i)}} .
\end{aligned}
$$

Therefore the summstion $\sum_{j=1}^{m_0}$ in (9) turns out to be zero. Hence

$$
\begin{aligned}
Z(h, E_0, \Lambda) &= 2^{n-1} \sum_{s\in S} \sum_{\eta} \operatorname{sgn}(s) \eta_1\cdots\eta_k (-1)^{l_{s(1)}+\cdots+l_{s(2k)}} \prod_{i=1}^{n} \delta_{2i-1}^{-l_{s(2i-1)}} \delta_{2i}^{-\eta_i l_{s(2i)}} \\
&= 2^{n-1} \sum_{s\in S} \sum_{\eta} \operatorname{sgn}(s) \eta_1\cdots\eta_k \prod_{i=1}^{k} (-\delta_{2i-1})^{-l_{s(2i-1)}} (-\delta_{2i})^{-\eta_i l_{s(2i)}} \\
& \quad \times \prod_{i=k+1}^{n} \delta_{2i-1}^{-l_{s(2i-1)}} \delta_{2i}^{-\eta_i l_{s(2i)}} . \qquad \cdots\cdots(10)
\end{aligned}
$$

Secondly, we consider $Z(h, E_j, \Lambda)$ for $1\leqq j\leqq n-k$. In this case,

$$
W(E_j)_{(k)} = \left\{ \begin{aligned} & u\in S_{2n};\, u(1) < u(3) < \cdots < u(2n-2j-1), \\ & \qquad u(2i-1) < u(2i)\ (1\leqq i\leqq n-j),\ 1\leqq u(i) \leqq 2k\,(1\leqq i\leqq 2k), \\ & \qquad u(2n-2j+1) < u(2n-2j+2) < \cdots < u(2n) \end{aligned} \right\}.
$$

As in § 2, put

$$
W^j_{(k)} = \left\{ \begin{aligned} & u\in W(E_j)_{(k)};\, u(i) = i \quad (1\leqq i\leqq 2k), \\ & \qquad u(2k+1) < u(2k+2) < \cdots < u(2n-2j),\ u(2n-2j+2r-1) \\ & \qquad \text{is odd and } u(2n-2j+2r-1)+1 = u(2n-2j+2r) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1\leqq r\leqq j \end{aligned} \right\}.
$$

Then using Lemma 3, Lemma 4 and the results for $W(E_0)_{(k)}$, we can divide

$W(E_j)_{(k)}\backslash W^j_{(k)}$ into two subset $W^j_1=\{u_1, u_2, \cdots, u_{m_j}\}$ and $W^j_2=\{u_{m_j+1}, u_{m_j+2}, \cdots, u_{2m_j}\}$ such that $u_i\neq u_j$ for $i\neq j$ and that the following condition holds:

Put $t_i=u_i^{-1}u_{m_j+i}$, then t_i is a transposition either between two even numbers $2p_i$ and $2q_i$ $(1\leqq p_i<q_i\leqq k$ or $k+1\leqq p_i<q_i\leqq n-j)$ or between two odd numbers $2p_i+1$ and $2q_i+1$ $(k\leqq p_i\leqq q_i\leqq n-1)$.

For a positive integer m, put $I_m=\{1, 2, \cdots, m\}$. For $k\leqq n$ and a subset I of $I_n\backslash I_k=\{k+1, k+2, \cdots, n\}$, set

$$\Delta(h, n, \Lambda, I; k)=\sum_{\eta^{(I)}}\sum_{s\in S}\operatorname{sgn}(s)\eta_1\cdots\eta_k(-1)^{l_s(1)+\cdots+l_s(2k)}\prod_{i=1}^{n}\delta_{2i-1}^{-l_s(2i-1)}$$

$$\times\prod_{i\in I_n\backslash I}\delta_{2i}^{-\eta_i l_s(2i)}\prod_{i\in I}\delta_{2i}^{-l_s(2i)}.$$

The cancellation in $Z(h, E_j, \Lambda)$ is caused by the term

$$\prod_{\alpha\in E_j^*}[\exp\{-u\alpha(X)\,|\,(\nu_{E_0}\alpha, \hat{s}^{-1}\Lambda)\,|/|\alpha|^2\}]$$

in it. Therefore we can trace the argument in § 2 step by step and we get that for $h\in A^{(k)}(P)$,

$$Z(h, E_j, \Lambda)=2^{n+j-1}\sum_{I\subset(I_n\backslash I_k),\,|I|=j}\Delta(h, n, \Lambda, I; k),$$

where $h=h_K\cdot\exp X$ and $\delta_i=\exp(e_i(X))$, $\eta^{(I)}=(\eta_i; i\in I_n\backslash I)$, $\eta_i=\pm 1$.

Let $a^{(k)}_{j,p,q}$ be the coefficient of

$$\operatorname{sgn}(s)\prod_{i=1}^{k}(-\delta_{2i-1})^{-l_s(2i-1)}(-\delta_{2i})^{-\eta_i l_s(2i)}\times\prod_{i=k+1}^{n}\delta_{2i-1}^{-l_s(2i-1)}\delta_{2i}^{-\eta_i l_s(2i)}$$

in $Z(h, E_j, \Lambda)$, where p is the number of δ_{2i}'s which have the positive exponent among $\{\delta_2, \cdots, \delta_{2k}\}$ and q is the number of those among $\{\delta_{2k+2}, \cdots, \delta_{2n}\}$. Then

$$a^{(k)}_{j,p,q}=\begin{cases}(-1)^p\binom{n-k-q}{j}2^{n+j-1} & \text{for}\quad 0\leqq j\leqq n-k-q\\ 0 \quad \text{for}\quad n-k-q+1\leqq j\leqq n.\end{cases}$$

Therefore

$$\sum_{j=0}^{n-k}(-1)^j a^{(k)}_{j,p,q}=\sum_{j=0}^{n-k}(-1)^j(-1)^p\binom{n-k-q}{j}2^{n+j-1}$$

$$=(-1)^p 2^{n-1}(-1)^{n-k-q}=(-1)^{p+q}2^{n-1}(-1)^{n-k}.$$

By definition, $\varepsilon(E_0)=(-1)^{n-k}$. Hence we obtained the following theorem.

Theorem 2. *Let* $\Lambda=(l_1, l_2, \cdots, l_{2n})$ *be an element of* $\mathfrak{b}_B^*$ *such that* $l_1>l_3>\cdots>l_{2n-1}>l_2>l_4>\cdots>l_{2n}>0$ *and all* l_i*'s are integers. Fix* k *such that* $1\leqq k\leqq n$. *For* $h=\mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_{2n}, \delta_1^{-1}, \delta_2^{-1}, \cdots, \delta_{2n}^{-1}, 1)\in A^{(k)}(P)$, *the function* $\tilde{\kappa}(h, P)$ *is expressed by the same formula as in Theorem* 1. (*Note that for* $h=h_K\exp X$, *the* δ_j*'s in this theorem are expressed as* $\delta_j=-\exp(e_j(X))$ *for* $j\leqq 2k$ *and* $\delta_j=\exp(e_j(X))$ *for* $j>2k$.)

Note. For a conected component A of $H^{(0,0)}$ except $\{A^{(i)}\}_{i=0,1,\cdots,n}$, let m be the number of negative matrix elements of any g in A. Then there exists an element k in K such that $\mathrm{Ad}(k)A=A^{(m)}$, because for any i and $j\,(i\neq j)$ $w_{i,j}=\exp((\pi/2)(X'_{e_i-e_j}-X'_{e_j-e_i}))\in K$ and $\mathrm{Ad}(w_{i,j})$ induces a replacement between (i,i)-element and (j,j)-element in $H^{(0,0)}$. Thus the function $\tilde{\kappa}(h,P)$ on $A(P)$ can be reduced to that on $A^{(m)}(P)$, for the character is invariant under inner automorphisms.

§4. The calculation on $H^{(m,0)}$ and $H^{(m,0)'}$

In this section, we consider the character formula on another type of Cartan subgroups.

Let $E^{(m)}$ and $E^{(m)'}$ be the following ordered sets of positive roots in $\Sigma_R(\mathfrak{h}^{(0,0)})$:

$$E^{(m)}=(e_1+e_2, e_1-e_2, \cdots, e_{2m-1}+e_{2m}, e_{2m-1}-e_{2m}) \quad \text{for } m\leqq n\,,$$

$$E^{(m)'}=(e_1+e_2, e_1-e_2, \cdots, e_{2m-1}+e_{2m}, e_{2m-1}-e_{2m}, e_{2n}) \quad \text{for } m\leqq n-1\,.$$

Put $\mathfrak{h}^{(m,0)}=\mathfrak{h}^{(0,0)E^{(m)}}$ and $\mathfrak{h}^{(m,0)'}=\mathfrak{h}^{(0,0)E^{(m)'}}$. Since any two roots in $E^{(m)}$ (resp. $E^{(m)'}$) are strongly orthogonal, for $H=\mathrm{diag}(h_1, h_2, \cdots, h_{2n}, -h_1, -h_2, \cdots, -h_{2n}, 0)\in\mathfrak{h}^{(0,0)}$,

$$\begin{aligned}\nu(E^{(m)})(H)-\sqrt{-1}\sum_{i=1}^{m}\{&(h_{2i-1}+h_{2i})(X'_{e_{2i-1}+e_{2i}}-X'_{-e_{2i-1}-e_{2i}})/2\\&+(h_{2i-1}-h_{2i})(X'_{e_{2i-1}-e_{2i}}-X'_{-e_{2i-1}+e_{2i}})/2\}\\&+\mathrm{diag}(0_{2m}, h_{2m+1}, \cdots, h_{2n}, 0_{2m}, -h_{2m+1}, \cdots, -h_{2n}, 0),\end{aligned}$$

$$\begin{aligned}(\text{resp. }\nu(E^{(m)'})(H)=\sqrt{-1}\sum_{i=1}^{m}\{&(h_{2i-1}+h_{2i})(X'_{e_{2i-1}+e_{2i}}-X'_{-e_{2i-1}-e_{2i}})/2\\&+(h_{2i-1}-h_{2i})(X'_{e_{2i-1}-e_{2i}}-X'_{-e_{2i-1}+e_{2i}})/2\}+\sqrt{-1}(X'_{e_{2n}}-X'_{-e_{2n}})\\&+\mathrm{diag}(0_{2m}, h_{2m+1}, \cdots, h_{2n-1}, 0_{2m+1}, -h_{2m+1}, -h_{2n-1}, 0, 0)).\end{aligned}$$

Therefore a general element in $\mathfrak{h}^{(m,0)}$ or $\mathfrak{h}^{(m,0)'}$ has the following form respectively:

$$X=\left(\begin{array}{ccccccccc}
A_1 & & & B_1 & & & 0 \\
& \ddots & & & \ddots & & \vdots \\
& & A_m & & & B_m & 0 \\
& & H & & & 0 & x \\
B_1 & & & A_1 & & & 0 \\
& \ddots & & & \ddots & & \vdots \\
& & B_m & & & A_m & 0 \\
& & 0 & & & -H & -x \\
\multicolumn{2}{c}{\underbrace{0\cdots\cdots 0}_{2n-1}} & -x & \multicolumn{2}{c}{\underbrace{0\cdots\cdots 0}_{2n-1}} & x & 0
\end{array}\right)\begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\\ x\end{array}}\right\}2n-1 \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\\ x\end{array}}\right\}2n-1 \\ \\ \end{array},$$

where

$$A_i=\begin{pmatrix} 0 & (h_{2i-1}-h_{2i})/2 \\ -(h_{2i-1}-h_{2i})/2 & 0 \end{pmatrix}\quad B_i=\begin{pmatrix} 0 & (h_{2i-1}+h_{2i})/2 \\ -(h_{2i-1}+h_{2i})/2 & 0 \end{pmatrix},$$

$H=\operatorname{diag}(h_{2m+1},\cdots,h_{2n})$ and $x=0$ for $\mathfrak{h}^{(m,0)}$, and $H=\operatorname{diag}(h_{2m+1},\cdots,h_{2n-1},0)$ and $x=h_{2n}$ for $\mathfrak{h}^{(m,0)'}$.

The corresponding Cartan subgroups, which we denote by $H^{(m,0)}$ and $H^{(m,0)'}$ respectively, consist of the elements of the following form;

$$H^{(m,0)}:\ h=\left(\begin{array}{ccccccc}
C_1 & & & D_1 & & & 0 \\
& \ddots & & & \ddots & & \vdots \\
& & C_m & & & D_m & 0 \\
& & M & & & 0 & \\
D_1 & & & C_1 & & & 0 \\
& \ddots & & & \ddots & & \vdots \\
& & D_m & & & C_m & \vdots \\
& & 0 & & & M^{-1} & 0 \\
\multicolumn{3}{c}{0\cdots\cdots\cdots 0} & \multicolumn{3}{c}{0\cdots\cdots\cdots 0} & 0
\end{array}\right)$$

$$H^{(m,0)'}:\ h=\left(\begin{array}{cccccccccc}
C_1 & & & & D_1 & & & & 0 \\
& \ddots & & & & \ddots & & & \vdots \\
& & C_m & & & & D_m & & \vdots \\
& & & M' & & & & 0 & 0 \\
& & & (1+c_n)/2 & & & & (1-c_n)/2 & s_n/\sqrt{2} \\
D_1 & & & & C_1 & & & & 0 \\
& \ddots & & & & \ddots & & & \vdots \\
& & D_m & & & & C_m & & \vdots \\
& & & 0 & & & & M'^{-1} & 0 \\
& & & (1-c_n)/2 & & & & (1+c_n)/2 & -s_n/\sqrt{2} \\
\multicolumn{3}{c}{\underbrace{0\cdots\cdots\cdots 0}_{2n-1}} & -s_n/\sqrt{2} & \multicolumn{3}{c}{\underbrace{0\cdots\cdots\cdots 0}_{2n-1}} & s_n/\sqrt{2} & c_n
\end{array}\right)\begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \vdots\\ \vdots\\ 0\\ x\end{array}}\right\}2n-1 \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\ \vdots\\ 0\\ x\end{array}}\right\}2n-1 \\ \\ \end{array},$$

where $s_n=\sin h_{2n}$, $c_n=\cos h_{2n}$, and

$$C_i=\begin{pmatrix} (\cos h_{2i-1}+\cos h_{2i})/2 & (\sin h_{2i-1}-\sin h_{2i})/2 \\ -(\sin h_{2i-1}-\sin h_{2i})/2 & (\cos h_{2i-1}+\cos h_{2i})/2 \end{pmatrix},$$

$$D_i=\begin{pmatrix} (\cos h_{2i-1}-\cos h_{2i})/2 & (\sin h_{2i-1}+\sin h_{2i})/2 \\ -(\sin h_{2i-1}+\sin h_{2i})/2 & (\cos h_{2i-1}-\cos h_{2i})/2 \end{pmatrix},$$

$$M=\operatorname{diag}(\rho_{2m+1}e^{h_{2m+1}}, \cdots, \rho_{2n}e^{h_{2n}})$$

$$M'=\operatorname{diag}(\rho_{2m+1}e^{h_{2m+1}}, \cdots, \rho_{2n-1}e^{h_{2n-1}}).$$

Here $\rho_i=\pm 1$ and $\prod_i \rho_i=1$.

We know that $\nu(E^{(m)})(\alpha)$ (resp. $\nu(E^{(m)\prime})(\alpha)$) is a root of $\mathfrak{h}^{(m,0)}$ (resp. $\mathfrak{h}^{(m,0)\prime}$) for $\alpha\in\Sigma(\mathfrak{h}^{(0,0)})$. In fact

$$(\nu(E^{(m)})(e_i))(X)=\begin{cases} h_i & \text{for } 2m+1\leqq i\leqq 2n, \\ -\sqrt{-1}h_i & \text{for } 1\leqq i\leqq 2m; \end{cases}$$

$$(\nu(E^{(m)\prime})(e_i))(X)=\begin{cases} h_i & \text{for } 2m+1\leqq i\leqq 2n-1, \\ -\sqrt{-1}h_i & \text{for } 1\leqq i\leqq 2m \text{ or } i=2n. \end{cases}$$

For simplicity, we denote it again by α. Then

$$\Sigma(\mathfrak{h}^{(m,0)})=\left\{\begin{matrix} \pm e_i\pm e_j & (1\leqq i<j\leqq 2n), \\ \pm e_i & (1\leqq i\leqq 2n) \end{matrix}\right\}.$$

As in § 2, we introduce a lexicographic order on it. Furthermore

$$\Sigma_R(\mathfrak{h}^{(m,0)})=\left\{\begin{matrix} \pm e_i\pm e_j & (2m+1\leqq i<j\leqq 2n), \\ \pm e_i & (2m+1\leqq i\leqq 2n \end{matrix}\right\},$$

and

$$\Sigma_R(\mathfrak{h}^{(m,0)\prime})=\left\{\begin{matrix} \pm e_i\pm e_j & (2m+1\leqq i<j\leqq 2n-1), \\ \pm e_i & (2m+1\leqq i\leqq 2n-1) \end{matrix}\right\}.$$

At first, we consider the character formula on $H^{(m,0)}$. For convenience we assume that $m<n$.

Let $H_0^{(m,0)}$ be the connected component of $H^{(m,0)}$ containing the identity. Then $\Sigma_R(H_0^{(m,0)})=\Sigma_R(\mathfrak{h}^{(m,0)})$ and the standard maximal orthogonal systems of it are as follows:

$$E_0^{(m)}=(e_{2m+1}+e_{2m+2}, e_{2m+1}-e_{2m+2}, \cdots, e_{2n-1}+e_{2n}, e_{2n-1}-e_{2n})$$

$$E_1^{(m)}=(e_{2m+1}+e_{2m+2}, \cdots, e_{2n-3}-e_{2n-2}, e_{2n-1}, e_{2n})$$

$$\cdots\cdots$$

$$E_{n-m}^{(m)}=(e_{2m+1}, e_{2m+2}, \cdots, e_{2n})$$

Then $P(E_i^{(m)})=(E_i^{(m)})^*\cup\{e_{2m+1}, e_{2m+2}, \cdots, e_{2n}\}$.

Since $\nu(E_0)=\nu(E_0^{(m)})\cdot\nu(E^{(m)})$, we have $\nu(E_0^{(m)})\mathfrak{h}_c^{(m,0)}\cap\mathfrak{g}=\mathfrak{b}$. So the way in this section is quite similar to that in § 2 except the treatement of the toroidal part of $\mathfrak{h}^{(m,0)}$.

For $h\in H_0^{(m,0)}$, put

$$\delta_i=\begin{cases}\exp(-\sqrt{-1}h_i), & \text{for } i=1,2,\cdots,2m,\\ \exp(h_i), & \text{for } i=2m+1,2m+2,\cdots,2n.\end{cases}$$

It follows from $\nu(E_0^{(m)})(Y)=Y$ $(Y\in\mathfrak{h}^{(m,0)}\cap\mathfrak{k})$ and the definition of Λ that

$$\xi_{\hat{s}^{-1}\Lambda}(h_K)=\prod_{i=1}^{2m}\delta_i^{\varepsilon_i l_{s(i)}}$$

Since $P(E_i^{(m)})=(E_i^{(m)})^*\cup\{e_{2m+1},e_{2m+2},\cdots,e_{2n}\}$, we have that for $\hat{s}\in W_G(\mathfrak{b})$,

$$\begin{aligned}\operatorname{sgn}(\hat{s}^{-1})\operatorname{sgn}_{P(E_i^{(m)})}(\hat{s}^{-1}\Lambda)&=\operatorname{sgn}(s)\,\varepsilon_2\varepsilon_4\cdots\varepsilon_{2n}\varepsilon_{2m+1}\varepsilon_{2m+2}\cdots\varepsilon_{2n}\\&=\operatorname{sgn}(s)\,\varepsilon_1\varepsilon_2\cdots\varepsilon_{2m}\quad(\because\ \varepsilon_1\varepsilon_3\cdots\varepsilon_{2n-1}=1).\end{aligned}$$

For the treatment of $W(E_0^{(m)})$, we can use the same method as in § 2. Thus we get for $h\in H_0^{(m,0)}(P)$,

$$\begin{aligned}Z(h,E_0^{(m)},\Lambda)=2^{n-m-1}&\sum_{s\in S}\sum_{\varepsilon}\sum_{\eta}\operatorname{sgn}(s)\,\varepsilon_1\varepsilon_2\cdots\varepsilon_{2m}\\&\times\prod_{j=1}^{2m}\delta_j^{\varepsilon_j l_{s(j)}}\prod_{j=m+1}^{n}(\delta_{2j-1}^{-l_{s(2j-1)}}\delta_{2i}^{-\eta_j l_{s(2j)}}),\end{aligned}\qquad\cdots\cdots(12)$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{2m})$ and $\eta=(\eta_{m+1},\eta_{m+2},\cdots,\eta_{2n})$, $\varepsilon_i=\pm 1$ and $\eta_i=\pm 1$.

Similarly, for $i=1,2,\cdots,n-m$,

$$\begin{aligned}Z(h,E_i^{(m)},\Lambda)=2^{n-m+i-1}&\sum_{s\in S}\sum_{\varepsilon}\sum_{\substack{I\subset I_n\setminus I_m\\|I|=i}}\operatorname{sgn}(s)\,\varepsilon_1\varepsilon_2\cdots\varepsilon_{2m}\\&\times\prod_{j=1}^{2m}\delta_j^{\varepsilon_j l_{s(j)}}\{\sum_{\eta^{(I)}}\prod_{j=m+1}^{n}\delta_{2j-1}^{-l_{s(2j)}}\prod_{j\in I_n\setminus(I_m\cup I)}\delta_{2j}^{-\eta_j l_{s(2j)}}\prod_{j\in I}\delta_{2j}^{-l_{s(2j)}}\},\end{aligned}$$

where $\eta^{(I)}=(\eta_i;\,i\in I_n\setminus(I\cup I_m))$.

As in § 2, we take the alternating sum of $Z(h,E_i^{(m)},\Lambda)$'s for $i=0,1,\cdots,n-m$. Then for each triplet (s,ε,η), the coefficient of $\operatorname{sgn}(s)\prod_{i=1}^{2m}(\varepsilon_i\delta_{2i}^{\varepsilon_i l_{s(i)}})\times\prod_{i=m+1}^{n}\delta_{2i-1}^{-l_{s(2i-1)}}\delta_{2i}^{-\eta_i l_{s(2i)}}$ in it is equal to $2^{n-m-1}(-1)^{n-m+q}$, where q is the number of δ_{2i}'s which have the positive exponents among $\{\delta_{2m+2},\cdots,\delta_{2n}\}$. For $m=n$, we have $H^{(n,0)}=\mathrm{B}$ and the signs (ε_i) must vary under $\varepsilon_1\varepsilon_3\cdots\varepsilon_{2n-1}=1$. Since $\varepsilon(E_0)=(-1)^{n-m}$, we get the following theorem.

Theorem 3. *Let $\Lambda=(l_1,l_2,\cdots,l_{2n})$ be an element of $\mathfrak{b}_B^*$ such that $l_1>l_3>\cdots>l_{2n-1}>l_2>l_4>\cdots>l_{2n}>0$ and all l_i's are integers. Let $m<n$. Then for $h\in H_0^{(m,0)}(P)$,*

$$\tilde{\kappa}(h, P) = 2^{n-m-1} \sum_{\varepsilon} \begin{vmatrix} \varepsilon_1 \delta_1^{\varepsilon_1 l_1} & \varepsilon_1 \delta_1^{\varepsilon_1 l_3} \cdots \varepsilon_1 \delta_1^{\varepsilon_1 l_{2n-1}} \\ \varepsilon_3 \delta_3^{\varepsilon_3 l_1} & \varepsilon_3 \delta_3^{\varepsilon_3 l_3} \cdots \varepsilon_3 \delta_3^{\varepsilon_3 l_{2n-1}} \\ \vdots & \\ \varepsilon_{2m-1} \delta_{2m-1}^{\varepsilon_{2m-1} l_1} & \cdots \varepsilon_{2m-1} \delta_{2m-1}^{\varepsilon_{2m-1} l_{2n-1}} \\ \delta_{2m+1}^{-l_1} & \cdots\cdots\cdots\cdots \delta_{2m+1}^{-l_{2n-1}} \\ \vdots & \vdots \\ \delta_{2n-1}^{-l_1} & \cdots\cdots\cdots\cdots \delta_{2n-1}^{-l_{2n-1}} \end{vmatrix} \times \begin{vmatrix} \varepsilon_2 \delta_2^{\varepsilon_2 l_2} & \varepsilon_2 \delta_2^{\varepsilon_{2m} l_4} \cdots \varepsilon_2 \delta_2^{\varepsilon_2 l_{2n}} \\ \vdots & \vdots \\ \varepsilon_{2m} \delta_{2m}^{\varepsilon_{2m} l_2} & \cdots\cdots\cdots \varepsilon_{2m} \delta_{2m}^{\varepsilon_{2m} l_{2n}} \\ \delta_{2m+2}^{-l_2} - \delta_{2m+2}^{l_2} & \cdots \delta_{2m+2}^{-l_{2n}} - \delta_{2m+2}^{l_{2n}} \\ \vdots & \vdots \\ \delta_{2n}^{-l_2} - \delta_{2n}^{l_2} & \cdots\cdots \delta_{2n}^{-l_{2n}} - \delta_{2n}^{l_{2n}} \end{vmatrix}. \qquad \cdots\cdots(14)$$

Secondly, we consider $\tilde{\kappa}$ on $H^{(m,0)'}$. Then $\Sigma_R(H_0^{(m,0)'}) = \Sigma_R(\mathfrak{h}^{(m,0)'})$ and the standard maximal orthogonal systems of it are as follows:

$$E_0^{(m)'} = (e_{2m+1} + e_{2m+2}, e_{2m+1} - e_{2m+2}, \cdots, e_{2n-3} + e_{2n-2}, e_{2n-3} - e_{2n-2}, e_{2n-1})$$

$$E_1^{(m)'} = (e_{2m+1} + e_{2m+2}, \cdots, e_{2n-5} - e_{2n-4}, e_{2n-3}, e_{2n-2}, e_{2n-1})$$

$$\vdots$$

$$E_{n-m-1}^{(m)'} = (e_{2m+1}, e_{2m+2}, \cdots, e_{2n-1})$$

Clearly, $P(E_i^{(m)'}) = (E_i^{(m)'})^* \cup \{e_{2m+1}, e_{2m+2}, \cdots, e_{2n-1}\}$. Since $\Sigma_R(\mathfrak{h}^{(m,0)'})$ is of type $B_{2n-2m-1}$, we get that

$$W(E_i^{(m)'}) = \left\{ \begin{array}{l} u \in S_{2n-1}^{2m+1};\ u(2m+1) < u(2m+3) < \cdots < u(2n-2i-3) \\ \qquad u(2m+2j-1) < u(2m+2j) \quad j = 1, 2, \cdots, n-m-i-1, \\ \qquad u(2n-2i-1) < u(2n-2i) < \cdots < u(2n-1) \end{array} \right\},$$

where S_{2n-1}^{2m+1} denotes the permutation group of $\{2m+1, 2m+2, \cdots, 2n-1\}$. Now we prepare the following analogous lemmas concerning to $W(E_i^{(m)'})$'s.

Lemma 3′. *We can divide $W(E_0^{(m)'}) \setminus \{e\}$ into two subsets $W_1 = \{u_1, u_2, \cdots, u_{m_0}\}$ and $W_2 = \{u_{m_0+1}, u_{m_0+2}, \cdots, u_{2m_0}\}$ such that $u_i \neq u_j$ for $i \neq j$ and they satisfy the following condition:*

Put $t_i = u_i^{-1} u_{m_0+i}$, then $t_i = (p_i, q_i)$, where $2m+1 \leqq p_i < q_i \leqq 2n-1$ and they have the same parity.

Proof. Put $\widetilde{W} = \{u \in W(E_0^{(m)'});\ u(2m+1) < u(2m+2) < \cdots < u(2n-2)\}$. By Lemma 3, we can divide $W(E_0^{(m)'}) \setminus \widetilde{W}$ into two subsets W_1' and W_2' such that each t chosen as above is a transposition of two even numbers less than or equal to $2n-2$. Put $\widetilde{W}_1 = \{u \in \widetilde{W} \setminus \{e\};\ u(2n-1)$ is even$\}$ and $\widetilde{W}_2 = \{u \in \widetilde{W} \setminus \{e\};\ u(2n-1)$ is odd$\}$. For each $u \in \widetilde{W}_1$, put $s_u = (u(2n-1), 2n-1)$. Then s_u is a non-trivial transposition of two odd numbers and $u \cdot s_u$ belongs to $\widetilde{W}_2$. Obviously, this correspondence is a one-to-one mapping from $\widetilde{W}_1$ to $\widetilde{W}_2$. So we get to the conclusion. Q.E.D.

Let $\widetilde{W}^{(k)}$ be the subset of $W(E_k^{(m)'})$ consisting of u which satisfies that $u(2m+1) < u(2m+2) < \cdots < u(2n-2k-2)$, $u(2j-1)$ is odd and $u(2j) = u(2j-1) + 1$ for $n-k \leqq j \leqq n-1$, and $u(2n-1) = 2n-1$.

Lemma 4′. *We can divide* $W(E_k^{(m)\prime})\backslash\widetilde{W}^{(k)}$ *into two subsets* $W_1^{(k)}=\{u_1, u_2, \cdots, u_{m_k}\}$ *and* $W_2^{(k)}=\{u_{m_k+1}, u_{m_k+2}, \cdots, u_{2m_k}\}$ *such that the* $u_i\neq u_j$ *for* $i\neq j$ *and the following property holds*:

Put $t_i=u_i^{-1}u_{m_k+i}$, *then* $t_i=(2p_i, 2q_i)$, *where* $m+1\leqq p_i<q_i\leqq n-k-1$ *or* t_i *is a transition between two odd numbers.*

Proof. Put

$$\widetilde{W}_0^{(k)}=\{u\in W(E_k^{(m)\prime})\,;\,u(2m+1)<u(2m+2)<\cdots<u(2n-2k-2)\},$$

$$\widetilde{W}_{00}^{(k)}=\{u\in \widetilde{W}_0^{(k)}\,;\,u(2j-1) \text{ is odd and } u(2j)=u(2j-1)+1 \text{ for } n-k\leqq j\leqq n-1\}.$$

Then by Lemma 3 we can divide $W(E_k^{(m)\prime})\backslash\widetilde{W}_0^{(k)}$ into two subsets $\widetilde{W}_3^{(k)}$ and $\widetilde{W}_4^{(k)}$ which hold the required properties. Next by Lemma 4, $\widetilde{W}_0^{(k)}\backslash\widetilde{W}_{00}^{(k)}$ can be divided into two subsets $\widetilde{W}_5^{(k)}$ and $\widetilde{W}_6^{(k)}$ which satisfy the condition. Furthermore $\widetilde{W}_{00}^{(k)}\backslash\widetilde{W}^{(k)}$ can be divided into $\widetilde{W}_7^{(k)}$ and $\widetilde{W}_8^{(k)}$ by the analogous way in Lemma 3′. Hence the two subsets $\widetilde{W}_1^{(k)}=\widetilde{W}_3^{(k)}\cup\widetilde{W}_5^{(k)}\cup\widetilde{W}_7^{(k)}$ and $\widetilde{W}_2^{(k)}=\widetilde{W}_4^{(k)}\cup\widetilde{W}_6^{(k)}\cup\widetilde{W}_8^{(k)}$ satisfy the required properties. Q.E.D.

For $h\in H_0^{(m,0)\prime}(P)$, put

$$\delta_i=\begin{cases}\exp(-\sqrt{-1}h_i) & \text{for } 1\leqq i\leqq 2m \text{ or } i=2n,\\ \exp(h_i) & \text{for } 2m+1\leqq i\leqq 2n-1.\end{cases}$$

Put $\mathfrak{b}'=\nu(E_0^{(m)\prime})\mathfrak{h}_c^{(m,0)\prime}\cap\mathfrak{g}$. Then since $\nu(E_0^{(m)\prime})\cdot\nu(E^{(m)\prime})=\nu(E_1)=\mathrm{Ad}(k_1)\nu(E_0)$, we have $\mathfrak{b}'=\mathrm{Ad}(k_1)\mathfrak{b}$ and $W_G(\mathfrak{b}')$ is isomorphic to $W_G(\mathfrak{b})$. Furthermore, for $u\in\widetilde{W}^{(k)}$, we can regard it as an element in $W_G(\mathfrak{b}')$ which never changes the signs and its signature is equal to $+1$.

It follows from these remarks that $\mathrm{sgn}(\hat{s}^{-1})\,\mathrm{sgn}_{P(E_t{}^{(m)\prime})}(\hat{s}^{-1}\mathrm{Ad}(k_1)\Lambda)=\mathrm{sgn}(s)\,\varepsilon_1\varepsilon_2\cdots\varepsilon_{2m}\varepsilon_{2n}$ and

$$Z(h, E_0^{(m)\prime}, \mathrm{Ad}(k_1)\Lambda)=2^{n-m-1}\sum_{s\in S}\sum_{\varepsilon'}\sum_{\eta}\mathrm{sgn}(s)\,\varepsilon_1\varepsilon_2\cdots\varepsilon_{2m}\varepsilon_{2n}$$
$$\times\prod_{i=1}^{2m}(\delta_i^{\varepsilon_i l_s(i)})\prod_{i=m+1}^{n-1}(\delta_{2i-1}^{-l_s(2i-1)}\delta_{2i}^{-\eta_i l_s(2i)})\times\delta_{2n-1}^{-l_s(2n-1)}\delta_{2n}^{\varepsilon_{2n}l_s(2n)},$$

and

$$Z(h, E_j^{(m)\prime}.\,\mathrm{Ad}(k_1)\Lambda)=2^{n-m-1-j}\sum_{s\in S}\sum_{\varepsilon'}\mathrm{sgn}(s)\,\varepsilon_1\varepsilon_2\cdots\varepsilon_{2m}\varepsilon_{2n}$$
$$\times\prod_{i=1}^{2m}\delta_i^{\varepsilon_i l_s(i)}\delta_{2n-1}^{-l_s(2n-1)}\delta_{2n}^{\varepsilon_{2n}l_s(2n)}\times\Big(\sum_{I\subset I_{n-1}\backslash I_m,\,|I|=j}$$
$$\sum_{(\eta^{(I)})}\prod_{i\in I_{n-1}\backslash(I_m\cup I)}\delta_{2i}^{-\eta_i l_s(2i)}\prod_{i\in I}\delta_{2i}^{-l_s(2i)}\times\prod_{i=1+m}^{n-1}\delta_{2i-1}^{-l_s(2i-1)}\Big), \qquad \cdots\cdots(15)$$

where $\varepsilon'=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2m}, \varepsilon_{2n})$, $\eta=(\eta_{m+1}, \cdots, \eta_{n-1})$ and $\eta^{(I)}=(\eta_i\,;\,i\in I_{n-1}\backslash(I_m\cup I))$ and $\varepsilon_i=\pm1$, $\eta_i=\pm1$.

As in §2, we take the alternating sum of $Z(h, E_j^{(m)\prime}, \mathrm{Ad}(k_1)\Lambda)$'s for $0\leqq j\leqq n-m-1$. Then for each triplet (s, ε, η) the coefficient of

$$\operatorname{sgn}(s) \prod_{i=1}^{2m} (\varepsilon_i \delta_i^{\varepsilon_i l_{s(i)}}) \prod_{i=m+1}^{n-1} (\delta_{2i-1}^{-l_{s(2i-1)}} \delta_{2i}^{-\eta_i l_{s(2i)}}) \delta_{2n-1}^{-l_{s(2n-1)}} \times \varepsilon_{2n} \delta_{2n}^{\varepsilon_{2n} l_{s(2n)}}$$

in it is equal to $2^{n-m-1}(-1)^{n-m-1-q}$ where q is the number of δ_{2i}'s which have the positive exponents among $\{\delta_{2m+2}, \cdots, \delta_{2n-2}\}$. Since $\varepsilon(E_0^{(m)\prime}) = (-1)^{n-m}$, we get the following analogous theorem.

Theorem 3′. *Let $\Lambda = (l_1, l_2, \cdots, l_{2n})$ be an element in $\mathfrak{h}_B^*$ such that $l_1 > l_3 > \cdots > l_{2n-1} > l_2 > l_4 > \cdots > l_{2n} > 0$ and all l_i's are integers. Let $m < n$. Then for $h \in H_0^{(m,0)}(P)$,*

$$\tilde{\kappa}(h, P) = -2^{n-m-1} \sum_{\varepsilon'} \begin{vmatrix} \varepsilon_1 \delta_1^{\varepsilon_i l_1} & \cdots\cdots & \varepsilon_1 \delta_1^{\varepsilon_1 l_{2n-1}} \\ \vdots & & \vdots \\ \varepsilon_{2m-1} \delta_{2m-1}^{\varepsilon_{2m-1} l_1} & \cdots & \varepsilon_{2m-1} \delta_{2m-1}^{\varepsilon_{m-1} l_{2n-1}} \\ \delta_{2m+1}^{-l_1} & \cdots\cdots & \delta_{2m+1}^{-l_{2n-1}} \\ \vdots & & \vdots \\ \delta_{2n-1}^{-l_1} & \cdots\cdots & \delta_{2n-1}^{-l_{2n-1}} \end{vmatrix} \times \begin{vmatrix} \varepsilon_2 \delta_2^{\varepsilon_2 l_2} & \cdots\cdots & \varepsilon_2 \delta_2^{\varepsilon_2 l_{2n}} \\ \vdots & & \vdots \\ \varepsilon_{2m} \delta_{2m}^{\varepsilon_{2m} l_2} & \cdots\cdots & \varepsilon_{2m} \delta_{2m}^{\varepsilon_{2m} l_{2n}} \\ \delta_{2m+2}^{-l_2} - \delta_{2m+2}^{l_2} & \cdots & \delta_{2m+2}^{-l_{2n}} - \delta_{2m+2}^{l_{2n}} \\ \vdots & & \vdots \\ \delta_{2n-2}^{-l_2} - \delta_{2n-2}^{l_2} & \cdots & \delta_{2n-2}^{-l_{2n}} - \delta_{2n-2}^{l_{2n}} \\ \varepsilon_{2n} \delta_{2n}^{\varepsilon_{2n} l_2} & \cdots\cdots & \varepsilon_{2n} \delta_{2n}^{\varepsilon_{2n} l_{2n}} \end{vmatrix}.$$

§ 5. The calculation on $H^{(m-k,k)}$

Lastly, we treat the character formula on the remaining Cartan subgroups. Let $E^{(m-k,k)}$ and $E^{(m-k,k)\prime}$ be the following ordered sets of positive roots in $\Sigma(\mathfrak{h}^{(0,0)})$:

$E^{(m-k,k)} = (e_1 + e_2, e_1 - e_2, \cdots, e_{2m-2k-1} + e_{2m-2k}, e_{2m-2k-1} - e_{2m-2k}, e_{2m-2k+1} + e_{2m-2k+2}, \cdots, e_{2m-1} + e_{2m})$ for $0 < k < m \leqq n$, and $E^{(m-k,k)\prime} = (e_1 + e_2, e_1 - e_2, \cdots, e_{2m-2k-1} + e_{2m-2k}, e_{2m-2k-1} - e_{2m-2k}, e_{2m-2k+1} + e_{2m-2k+2}, \cdots, e_{2m-1} + e_{2m}, e_{2n})$ for $0 < k < m \leqq n-1$.

Put $\mathfrak{h}^{(m-k,k)} = \nu(E^{(m-k,k)}) \mathfrak{h}_c^{(0,0)} \cap \mathfrak{g}$ and $\mathfrak{g}^{(m-k,k)\prime} = \nu(E^{(m-k,k)\prime}) \mathfrak{h}_c^{(0,0)} \cap \mathfrak{g}$. Then their general elements have the following forms respectively:

$$X = \left(\begin{array}{ccccccccccccc} A_1 & & & & & & B_1 & & & & & & 0 \\ & \ddots & & & & & & \ddots & & 0 & & & \vdots \\ & & A_{m-k} & & & & & & \ddots & & & & \vdots \\ & & & F_{m-k+1} & & & & 0 & & \ddots & & & \vdots \\ & 0 & & & \ddots & & & & & & & & \vdots \\ & & & & & F_m & & & & & B_m & & 0 \\ & & & & & & H & & & & & 0 & x \\ B_1 & & & & & & & A_1 & & & & & 0 \\ & \ddots & & & & & & & \ddots & & & & \vdots \\ & & \ddots & & & & & & & A_{m-k} & & & \vdots \\ & & & \ddots & & & & & & & -F_{m-k+1} & & \vdots \\ & 0 & & & \ddots & & & & & & & \ddots & \vdots \\ & & & & & B_m & & & & & & -F_m & 0 \\ & & & & & & 0 & 0 & & & & -H & -x \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -x & 0 & \cdots & \cdots & \cdots & 0 \;\; x & 0 \end{array} \right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} 2n-1 \\ \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} 2n-1 \\ \\ \\ \end{array},$$

$$\underbrace{\qquad\qquad\qquad}_{2n-1} \qquad\qquad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{2n-1}$$

where $F_i = \mathrm{diag}((h_{2i-1}-h_{2i})/2, -(h_{2i-1}-h_{2i})/2)$, and for $\mathfrak{h}^{(m-k,k)}$, $H = \mathrm{diag}(h_{2m+1}, \cdots, h_{2n})$ and $x=0$, for $\mathfrak{h}^{(m-k,k)'}$, $H = \mathrm{diag}(h_{2m+1}, \cdots, h_{2n-1}, 0)$ and $x = h_{2n}$.

We denote the corresponding Cartan subgroups by $H^{(m-k,k)}$ and $H^{(m-k,k)'}$ respectively. Put $r_i = (h_{2i-1}-h_{2i})/2$ and $\theta_i = (h_{2i-1}+h_{2i})/2$ $(i = m-k+1, \cdots, m)$. Here we give the form of a general element in $H^{(m-k,k)}$ only.

$$H^{(m-k,k)}: h = \begin{pmatrix}
C_1 & & & & 0 & & & D_1 & & & & 0 & & & 0\\
& \ddots & & & & & & & \ddots & & & & & & \vdots\\
& & C_{m-k} & & & & & & & D_{m-k} & & & & & \vdots\\
& & & L_{m-k+1} & & & & & & & -{}^tD_{m-k+1} & & & & \vdots\\
& 0 & & & \ddots & & & & 0 & & & \ddots & & & \vdots\\
& & & & & L_m & & & & & & & -{}^tD_m & & \vdots\\
& & & & & & M & & & & & & & 0 & \vdots\\
D_1 & & & & & & & C_1 & & & & & & & \vdots\\
& \ddots & & & 0 & & & & \ddots & & & 0 & & & \vdots\\
& & D_{m-k} & & & & & & & C_{m-k} & & & & & \vdots\\
& & & D'_{m-k+1} & & & & & & & L'_{m-k+1} & & & & \vdots\\
& 0 & & & \ddots & & & & 0 & & & \ddots & & & \vdots\\
& & & & & D'_m & & & & & & & L'_m & & \vdots\\
& & & & & & 0 & & & & & & & M^{-1} & 0\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix},$$

where $D'_i = \begin{pmatrix} 0 & e^{-r_i}\sin\theta_i \\ -e^{r_i}\sin\theta_i & 0 \end{pmatrix}$ $L_i = \mathrm{diag}(e^{r_i}\cos\theta_i, e^{-r_i}\cos\theta_i)$ $L'_i = \mathrm{diag}(e^{-r_i}\cos\theta_i, e^{r_i}\cos\theta_i)$, and C_i, D_i, and M denote the same matrices as in § 4.

Since $\nu(E^{(m-k,k)})(\alpha)$ (resp. $\nu(E^{(m-k,k)'})(\alpha)$) $(\alpha \in \Sigma(\mathfrak{h}^{(0,0)}))$ is a root of $\mathfrak{h}^{(m-k,k)}$ (resp. $\mathfrak{h}^{(m-k,k)'}$), we denote it also by α. Then

$$\Sigma(\mathfrak{h}^{(m-k,k)}) = \{\pm e_i \pm e_j;\ (1 \leqq i < j \leqq 2n),\ \pm e_i\ (1 \leqq i \leqq 2n)\}.$$

As in § 2, we introduce a lexicographic order on it. Furthermore

$$\Sigma_R(\mathfrak{h}^{(m-k,k)}) = \left\{\begin{array}{l} \pm(e_{2i-1}-e_{2i}) \quad (m-k+1 \leqq i \leqq m), \\ \pm e_i \pm e_j \quad (2m+1 \leqq i < j \leqq 2n), \\ \pm e_i \quad (2m+1 \leqq i \leqq 2n) \end{array}\right\},$$

$$\Sigma_R(\mathfrak{h}^{(m-k,k)'}) = \left\{\begin{array}{l} \pm(e_{2i-1}-e_{2i}) \quad (m-k+1 \leqq i \leqq m). \\ \pm e_i \pm e_j \quad (2m+1 \leqq i < j \leqq 2n-1), \\ \pm e_i \quad (2m+1 \leqq i \leqq 2n-1) \end{array}\right\}.$$

Hence $\Sigma_R(\mathfrak{h}^{(m-k,k)})$ (resp. $\Sigma_R(\mathfrak{h}^{(m-k,k)'})$) is a root system of type $\overbrace{A_1 \times \cdots \times A_1}^{k} \times B_{2n-2m}$ (resp. $\overbrace{A_1 \times \cdots \times A_1}^{k} \times B_{2n-2m-1}$). Both $H^{(m-k,k)}$ and $H^{(m-k,k)'}$ can be treated completely likewise, so we consider $H^{(m-k,k)}$ only in the following.

For convenience, we assume that $m<n$.

Let $H_0^{(m-k,k)}$ be the connected component of $H^{(m-k,k)}$ containing the identity. Then $\Sigma_R(H_0^{(m-k,k)})=\Sigma_R(\mathfrak{h}^{(m-k,k)})$, and as to the above order, we have the following standard maximal orthogonal systems:

$$\begin{aligned}E_0^{(m-k,k)} &= (e_{2m-2k+1}-e_{2m-2k+2}, \cdots, e_{2m-1}-e_{2m}, e_{2m+1}+e_{2m+2},\\ &\quad e_{2m+1}-e_{2m+2}, \cdots, e_{2n-1}+e_{2n}, e_{2n-1}-e_{2n}),\\ E_1^{(m-k,k)} &= (e_{2m-2k+1}-e_{2m-2k+2}, \cdots, e_{2m-1}-e_{2m}, e_{2m+1}+e_{2m+2},\\ &\quad \cdots, e_{2n-3}-e_{2n-2}, e_{2n-1}, e_{2n})\\ &\ \vdots\\ E_{n-m}^{(m-k,k)} &= (e_{2m-2k+1}-e_{2m-2k+2}, \cdots, e_{2m-1}-e_{2m}, e_{2m+1}, \cdots, e_{2n}).\end{aligned}$$

Then $P(E_i^{(m-k,k)})=(E_i^{(m-k,k)})^* \cup \{e_{2m+1}, e_{2m+2}, \cdots, e_{2n}\}$.

Since $\nu(E_0)=\nu(E_0^{(m-k,k)})\nu(E^{(m-k,k)})$, we have $\nu(E_0^{(m-k,k)})\mathfrak{h}_c^{(m-k,k)}\cap\mathfrak{g}=\mathfrak{b}$. Now we consider the character formula term by term. For $\hat{s}\in W_G(\mathfrak{b})$, we have

$$\begin{aligned}&\operatorname{sgn}(\hat{s}^{-1})\operatorname{sgn}_{P(E_i{}^{(m-k,k)})}(\hat{s}^{-1}\Lambda)\\ &\quad=\operatorname{sgn}(s)\,\varepsilon_2\varepsilon_4\cdots\varepsilon_{2n}\varepsilon_{2m-2k+1}\varepsilon_{2m-2k+3}\cdots\varepsilon_{2m-1}\varepsilon_{2m+1}\varepsilon_{2m+2}\cdots\varepsilon_{2n}\\ &\quad=\operatorname{sgn}(s)\,(\varepsilon_1\varepsilon_2\cdots\varepsilon_{2m-2k})\,(\varepsilon_{2m-2k+2}\varepsilon_{2m-2k+4}\cdots\varepsilon_{2m}).\end{aligned}$$

For $h\in H_0^{(m-k,k)}(P)$, put

$$\delta_i=\begin{cases}\exp(-\sqrt{-1}h_i) & \text{for } 1\leqq i\leqq 2m-2k,\\ \exp(h_i) & \text{for } 2m+1\leqq i\leqq 2n.\end{cases}$$

Under the decomposition $H^{(m-k,k)}=(H^{(m-k,k)}\cap K)\cdot\exp(\mathfrak{h}^{(m-k,k)}\cap\mathfrak{p})$, we have $h=h_K\exp X$, where

$$h_K=\exp\begin{pmatrix} A_1 & & & B_1 & & & 0\\ & \ddots & & & \ddots & & \vdots\\ & & A_{m-k} & & & B_m & \vdots\\ & & 0_{2(n-m+k)} & & & 0_{2n-2m} & \vdots\\ B_1 & & & A_1 & & & \vdots\\ & \ddots & & & \ddots & & \vdots\\ & & B_m & & & A_{m-k} & \vdots\\ & & 0_{2n-2m} & & & 0_{2(n-m+k)} & \vdots\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\end{pmatrix},$$

$$\begin{aligned}X=\operatorname{diag}(&0_{2m-2k}, F_{m-k+1}, \cdots, F_m, h_{2m+1}, h_{2m+2}, \cdots, h_{2n},\\ &0_{2m-2k}, -F_{m-k+1}, \cdots, -F_m, -h_{2m+1}, -h_{2m+2}, \cdots, -h_{2n}, 0).\end{aligned}$$

Therefore for each $j=0, 1, \cdots, n-m$,

$$\xi_{\hat{s}^{-1}\Lambda}(h_K)=\prod_{i=1}^{2m-2k}\delta_i^{\varepsilon_i l_{s(i)}}\prod_{i=m-k+1}^{m}\exp(-\sqrt{-1}\theta_i(\varepsilon_{2i-1}l_{s(2i-1)}+\varepsilon_{2i}l_{s(2i)}))$$

and

$$\prod_{\alpha\in(E_j{}^{(m-k,k)})^*} \exp\{-u\alpha(X)\,|\,(\nu(E_0^{(m-k,k)})(\alpha),\hat{s}^{-1}\Lambda)/|\alpha|^2\}$$

$$=\prod_{i=m-k+1}^{m} \exp(-r_i(l_{s(2i-1)}-\varepsilon_{2i-1}\varepsilon_{2i}l_{s(2i)}))$$

$$\times\prod_{i=+1-l}^{n-j}\{\delta_{u(2i-1)}^{-l_s(2i-1)}\delta_{u(2i)}^{-\varepsilon_{2i-1}\varepsilon_{2i}l_s(2i)}\}\times\prod_{i=n-j+1}^{n}\delta_{u(2i-1)}^{-l_s(2i-1)}\delta_{u(2i)}^{-l_s(2i)}\,.$$

Put $\delta_{2i-1}=\exp(r_i+\sqrt{-1}\theta_i)$ and $\delta_{2i}=\exp(r_i-\sqrt{-1}\theta_i)$ for $i=m-k+1,\cdots,m$, then

$$\sum_{\substack{\varepsilon_{2i-1}=\pm1\\ \varepsilon_{2i}=\pm1}}\varepsilon_{2i}\exp(-\sqrt{-1}\theta_i(\varepsilon_{2i-1}l_{s(2i-1)}+\varepsilon_{2i}l_{s(2i)}))\cdot\exp(-r_i(l_{s(2i-1)}-\varepsilon_{2i-1}\varepsilon_{2i}l_{s(2i)}))$$

$$=\sum_{\substack{\varepsilon_{2i}=\pm1\\ \tau_i=0,1}}(-1)^{\tau_i}\varepsilon_{2i}\delta_{2i-1+\tau_i}^{-l_s(2i-1)}\delta_{2i-\tau_i}^{\varepsilon_{2i}l_s(2i)}\,.$$

Let $a=\{a(1),a(2),\cdots,a(k)\}$ (resp. $b=\{b(1),b(2),\cdots,b(k)\}$) be a subset consisting of k odd (resp. even) numbers such that $1\leqq a(j)\leqq 2n-1$ (resp. $2\leqq b(j)\leqq 2n$). We define a subset $S_{a,b}$ of S as follows:

$$S_{a,b}=\{s\in S;\ s(2m-2k+2i-1)\in a,\ s(2m-2k+2i)\in b\ (1\leqq i\leqq k)\}.$$

Obviously, $S=\bigcup_{(a,b)}S_{a,b}$ (disjoint union). In each subset, there exists a unique element $\sigma_{(a,b)}$ such that $\sigma_{(a,b)}(i)<\sigma_{(a,b)}(i+2)$ holds for any i. Then by easy computation, $\operatorname{sgn}\sigma_{(a,b)}=(-1)^N$ with $N=\{\sum_i(a(i)+b(i))+k\}/2\}$. In the following, we denote the permutation group of $\{t_1,\cdots,t_p\}$ by $S(t_1,\cdots,t_p)$.

Put

$T_1^k=S(1,3,\cdots,2m-2k-1,2m+1,\cdots,2n-1)$,

$T_2^k=S(2,4,\cdots,2m-2k,2m+2,2m+4,\cdots,2n)$,

$T_3^k=S(2m-2k+1,2m-2k+3,\cdots,2m-1)$,

$T_4^k=S(2m-2k+2,2m-2k+4,\cdots,2m)$, and $T^k=T_1^k\times T_2^k\times T_3^k\times T_4^k$.

Then for any (a,b), $S_{a,b}=\sigma_{(a,b)}\cdot T^k$.

In our case, only the simple component of type B in $\Sigma_R(\mathfrak{h}^{(m-k,k)})$ contributes the simplification of the character formula. The treatement of that part can be carried through the same way as in § 2. Therefore we can caluculate the alternating sum of $Z(h,E_j^{(m-k,k)},\Lambda)$'s for $h\in H_0^{(m-k,k)}(P)$ similarly as we could in § 2 and § 3.

Paying attention to the changes of parities in the subindeces of δ's, we get that for any $(\tau_{m-k+1},\tau_{m-k+2},\cdots,\tau_m)$ ($\tau_i=0$ or 1) and $(\eta_{m+1},\cdots,\eta_n)$ ($\eta_i=\pm1$), the coefficient of

$$\prod_{i=1}^{2m-2k}(\varepsilon_i\delta_i^{\varepsilon_i l_s(i)})(-1)^{\Sigma\tau_i}\prod_{i=m-k+1}^{m}(\varepsilon_{2i}\delta_{2i-1+\tau_i}^{-l_s(2i-1)}\delta_{2i-\tau_i}^{\varepsilon_{2i}l_s(2i)})\prod_{i=m+1}^{n}(\delta_{2i-1}^{-l_s(2i-1)}\delta_{2i}^{-\eta_i l_s(2i)})$$

in $\tilde{\kappa}(h, P)$ is equal to $2^{n-m-1}\operatorname{sgn}(s)(-1)^{n-m+q}$, where q is the number of δ_{2i}'s which have the positive exponents among $\{\delta_{2m+2}, \cdots, \delta_{2n}\}$.

We denote the complementary set of a (resp. b) in $\{1, 3, \cdots, 2n-1\}$ (resp. $\{2, 4, \cdots, 2n\}$) by $a' = \{a'(1), a'(2), \cdots, a'(n-k)\}$ (resp. $b' = \{b'(1), b'(2), \cdots, b'(n-k)\}$). In each subset, we assume, the elements are arranged in order. Since $\varepsilon(E_0^{(m-k,k)}) = (-1)^{n-m+k}$ and $S = \bigcup_{(a,b)} \sigma_{(a,b)} \cdot T^k$, we get the following theorem.

Theorem 4. *Let $\Lambda = (l_1, l_2, \cdots, l_{2n})$ be an element in $\mathfrak{h}_B^*$ such that $l_1 > l_3 > \cdots > l_{2n-1} > l_2 > l_4 > \cdots > l_{2n} > 0$ and all l_i's are integers.*
For $h \in H_0^{(m-k,k)}(P)$,

$$\tilde{\kappa}(h, P) = (-1)^k 2^{n-k-1} \sum_{(a,b)} \sum_{\varepsilon} \sum_{\tau} \det(a_i^{\,j}) \det(b_i^{\,j})$$
$$\times \operatorname{sgn}(a, b) \operatorname{sgn}(\tau) \det(c_i^{\,j}) \det(d_i^{\,j}), \qquad \cdots\cdots (17)$$

where

$$a_i^{\,j} = \begin{cases} \varepsilon_{2i-1}\delta_{2i-1}^{\varepsilon_{2i-1} l_{a'(j)}} & \text{for} \quad i = 1, 2, \cdots, m-k, \\ \delta_{2i-1+2k}^{-l_{a'(j)}} & \text{for} \quad i = m-k+1, \cdots, n-k, \end{cases}$$

$$b_i^{\,j} = \begin{cases} \varepsilon_{2i}\delta_{2i}^{\varepsilon_{2i} l_{b'(j)}} & \text{for} \quad i = 1, 2, \cdots, m-k, \\ \delta_{2i+2k}^{-l_{b'(j)}} - \delta_{2i+2k}^{l_{b'(j)}} & \text{for} \quad i = m-k+1, \cdots, n-k, \end{cases}$$

$$c_i^{\,j} = \delta_{2m-2k+2i-1+\tau_{m-k+1}}^{-l_{a(j)}} \qquad 1 \leqq i, \ j \leqq k,$$

$$d_i^{\,j} = \varepsilon_{2m-2k+2i}\delta_{2m-2k+2i-\tau_{m-k+i}}^{\varepsilon_{2m-2k+2i} l_{b(j)}} \qquad 1 \leqq i, \ j \leqq k,$$

and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{2m-2k}, \varepsilon_{2m-2k+2}, \cdots, \cdots, \varepsilon_{2m-2}, \varepsilon_{2m})$, $\tau = (\tau_{m-k+1}, \cdots, \tau_m)$, $\varepsilon_i = \pm 1$, $\tau_i = 0$ or 1, and $\operatorname{sgn}(a, b) = (-1)^N$, $\operatorname{sgn}(\tau) = (-1)^{N'}$ with $2N = \sum_{i=1}^{k} (a(i) + b(i)) + k$ and $N' = \sum_{i=1}^{k} \tau_{m-k+i}$.

Note: For each $i = 2m-2k+1, 2m-2k+3, \cdots, 2m-1$, if $\varepsilon_{2i-1} = -1$ then $\tau_i = 1$ and vice versa. So when $m = n$, for each $(\varepsilon_1, \varepsilon_2, \varepsilon_{2n-2k})$ the sequences $(\tau_{n-k+1}, \tau_{n-k+2}, \cdots, \tau_n)$ must vary under the condition that $(-1)^{\Sigma\tau_i} = \varepsilon_1\varepsilon_3\cdots\varepsilon_{2n-1}$. So for $h \in H_0^{(n-k,k)}(P)$, we get the expression of $\tilde{\kappa}(h, P)$ by replacing 2^{n-m-1} with 1 and $(\tau_{m-k+1}, \tau_{m-k+2}, \cdots, \tau_m)$ with $(\tau_{n-k+1}, \tau_{n-k+2}, \cdots, \tau_n)$ satisfying $(-1)^{\Sigma\tau_i} = \varepsilon_1\varepsilon_3\cdots\varepsilon_{2n-2k-1}$ in (17).

Remark 1. As to $H^{(m-k,k)'}$, we can get the expression of $\tilde{\kappa}(h, P)$ by replacing $\delta_{2n}^{-l_{2i}} - \delta_{2n}^{l_{2i}}$ with $\varepsilon_{2n}\delta_{2n}^{\varepsilon_{2n} l_{2i}}$ and $\sum_{\varepsilon}$ with $(-1)\sum_{\varepsilon'}$, where $\varepsilon' = (\varepsilon_1, \cdots, \varepsilon_{2m-2k}, \varepsilon_{2n})$.

Remark 2. The function $\tilde{\kappa}(h, P)$ is expressed in the same way on the other connected components of $H^{(m-k,k)}$ and $H^{(m-k,k)'}$ by changing the signs of

δ_i's which correspond to the negative matrix elements among $\{\rho_{2m+1}e^{h_{2m+1}}, \rho_{2m+2}e^{h_{2m+2}}, \cdots, \rho_{2n-1}e^{h_{2n-1}}, (\rho_{2n}e^{h_{2n}})\}$.

Remark 3. It follows from Theorem 5 in [4] that the set $\{H^{(i,j)}$; for $0\leqq i, j\leqq n$ and $0\leqq i+j\leqq n$, $H^{(i,j)\prime}$; for $0\leqq i, j\leqq n-1$ and $0\leqq i+j\leqq n-1\}$ exhausts the totality of Cartan subgroups modulo conjugation by G. Therefore we can express π'_Λ concretely on G', because each element in G' is conjugate to an element in some Cartan subgoup of G.

§ 6. The case of $SO_0(n, n+2m+1)$

Put

$$J_{n,m}=\begin{pmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & -1_{2m+1} \end{pmatrix}, \quad P=\frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & 1_n & 0 \\ 1_n & -1_n & 0 \\ 0 & 0 & \sqrt{2}\,1_{2m+1} \end{pmatrix}$$

and

$$\widetilde{G}=\{g\in GL(2n+2m+1, \boldsymbol{R})\,;\,{}^tgJ_{n,m}g=J_{n,m}\} \quad (m\geqq 0).$$

We consider the connected component G of $\widetilde{G}$ containing the identity. Then $G=P\cdot SO_0(n, n+2m+1)\cdot P^{-1}$. Let $\mathfrak{g}$ be the Lie algebra of G then

$$\mathfrak{g}=\{X\in\mathfrak{gl}(2n+2m+1, \boldsymbol{R})\,;\,{}^tXJ_{n,m}+J_{n,m}X=0\}.$$

Let $\mathfrak{h}^{(0,0)}$ be a Cartan subalgebra of $\mathfrak{g}$ consisting of the elements of the form

$$X=\mathrm{diag}(h_1, \cdots, h_n, -h_1, \cdots, -h_n, 0, L_1, L_2, \cdots, L_m),$$

where diag$(\cdots)$ denotes the blockwise matrix with diagonal entries indicated and $L_i=\begin{pmatrix} 0 & h_{n+i} \\ -h_{n+i} & 0 \end{pmatrix}$ for $1\leqq i\leqq m$, $h_j\in\boldsymbol{R}$. The map θ defined by $\theta X=-{}^tX$ for $X\in\mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$ such that $\theta\mathfrak{h}^{(0,0)}=\mathfrak{h}^{(0,0)}$. The vector part of $\mathfrak{h}^{(0,0)}$ has the maximal dimension.

Let $H^{(0,0)}$ be the Cartan subgroup corresponding to $\mathfrak{h}^{(0,0)}$, then it consists of the elements of the form

$$h=\mathrm{diag}(\rho_1e^{h_1}, \cdots, \rho_ne^{h_n}, \rho_1e^{-h_1}, \cdots, \rho_ne^{-h_n}, 1, K_{n+1}, \cdots, K_{n+m}),$$

where $K_i=\begin{pmatrix} \cos h_i & \sin h_i \\ -\sin h_i & \cos h_i \end{pmatrix}$, $\rho_i=\pm 1$, $\prod\rho_i=1$ and $h_i\in\boldsymbol{R}$. Let $A^{(k)}$ be the connected component of $H^{(0,0)}$ corresponding to the signs $\rho_1=\rho_2=\cdots=\rho_{2k}=-1$ and $\rho_{2k+1}=\cdots=\rho_{2n}=1$.

Let e_i be the complex linear functional on $\mathfrak{h}_c^{(0,0)}$ defined as

$$e_i(X)=h_i \quad \text{for } 1\leqq i\leqq n, \quad \text{or} \quad =-\sqrt{-1}h_i \quad \text{for } n+1\leqq i\leqq n+m.$$

Then

$$\Sigma(\mathfrak{h}^{(0,0)}) = \{\pm e_i \pm e_j (1 \leqq i < j \leqq n+m),\ \pm e_i (1 \leqq i \leqq n+m)\},$$

$$\Sigma_R(\mathfrak{h}^{(0,0)}) = \{\pm e_i \pm e_j (1 \leqq i < j \leqq n),\ \pm e_i (1 \leqq i \leqq n)\}.$$

We introduce a lexicographic order on $\Sigma(\mathfrak{h}^{(0,0)})$ by $e_1, e_2, \cdots, e_{n+m}$. We choose a root vector X_α for $\alpha \in \Sigma_R(\mathfrak{h}^{(0,0)})$ as follows:

$$X_{e_i} = E_{i,2n+1} + E_{2n+1,i+n},\quad X_{-e_i} = E_{2n+1,i} + E_{i+n,2n+1} \quad (1 \leqq i \leqq n),$$

$$X_{e_i+e_j} = (-1)^n (E_{i,n+j} - E_{j,n+i}) \quad (1 \leqq i < j \leqq n),$$

$$X_{e_i-e_j} = E_{i,j} - E_{n+j,n+i} \quad (1 \leqq i < j \leqq n),$$

where $E_{p,q}$ is the matrix unit of (p, q)-element. Note that $[X_{e_i}, X_{e_j}] = (-1)^n X_{e_i+e_j}$, $[X_{e_i}, X_{-e_j}] = X_{e_i-e_j}$ for $i<j$, whence Condition 5.1 in [1] is satisfied.

Put $p = [n/2]$. Then the stndard maximal orthogonal systems in $\Sigma_R(\mathrm{A}^{(0)}) = \Sigma_R(\mathfrak{h}^{(0,0)})$ is as follows:

$$E_0 = (e_1+e_2, e_1-e_2, \cdots, e_{2p-1}+e_{2p}, e_{2p-1}-e_{2p}, [e_n])$$

$$E_1 = (e_1+e_2, e_1-e_2, \cdots, e_{2p-3}-e_{2p-2}, e_{2p-1}, e_{2p}, [e_n])$$

$$\cdots\cdots$$

$$E_p = (e_1, e_2, \cdots, e_{2p}, [e_n]).$$

In each E_i, the last element e_n appears only when n is odd.

Under Cayley transforms, the toroidal part of $\mathfrak{h}^{(0,0)}$ is invariant. So for each E_i, $\mathfrak{h}^{(0,0)E_i}$ is a compact Cartan subalgebra. In fact we get that when $n = 2p+1$,

$$-\sqrt{-1}\nu_{E_i}(X) = P\cdot\mathrm{diag}(-h_2 I_2, \cdots, -h_{2p-2i} I_2, 0, h_{2p-2i+2} I_2, \cdots, h_{2p} I_2, h_1 I_2, h_3 I_2, \cdots, h_n I_2, h_{n+1} I_2, \cdots, h_{n+m} I_2)\cdot P^{-1}$$

and when $n = 2p$,

$$-\sqrt{-1}\nu_{E_i}(X) = P\cdot\mathrm{diag}(h_1 I_2, h_3 I_2, \cdots, h_{2p-1}, -h_2 I_2, \cdots, -h_{2p-2i} I_2, 0, h_{2p-2i+2} I_2, \cdots, h_2 I_2, h_{n+1} I_2, \cdots, h_{n+m} I_2) P^{-1},$$

where $I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore for any i, there exists $k_i \in K$ such that $\mathrm{Ad}(k_i)\nu_{E_i}|\mathfrak{h}_c^{(0,0)} = \nu_{E_0}|\mathfrak{h}_c^{(0,0)}$, where K is the maximal compact subgroup of G corresponding to $\mathfrak{k} = \{X \in \mathfrak{g}; \theta X = X\}$.

Using lemma 1 repeatedly, we get that when n is even, the root $\nu_{E_0}(e_j)$ of $\mathfrak{h}^{(0,0)E_i}$ is compact if and only if j is even or $j \geqq n+1$ and all the other short roots are singular imaginary. When n is odd, the root $\nu_{E_0}(e_j)$ is compact if and only if j is even and $j \leqq n$.

Put $\mathfrak{b} = \mathfrak{h}^{(0,0)E_0}$ and $B = \exp\mathfrak{b} = H^{\mathfrak{b}}$. Then every element Λ in $\mathfrak{b}_B^*$ is parametrized by $(l_1, l_2, \cdots, l_{n+m})$, where $l_i = \Lambda(\nu_{E_0}(H_{e_i}))$. Here H_{e_i} is the element of $\mathfrak{h}_c^{(0,0)}$ corresponding to the root e_i with respect to the Killing form of $\mathfrak{g}_c$, that is, $H_{e_i}=E_{i,i}-E_{n+i,n+i}$ for $1\leqq i\leqq n$ and $H_{c_i}=\sqrt{-1}(E_{2i+1,2i+2}-E_{2i+2,2i+1})$ for $n+1\leqq i\leqq n+m$.

Let S_{n+m} be the $(n+m)$-th symmetric group. According to the parity of n, we define the set $\hat{S}$ as follows:

I) When $n=2p$,

$$\hat{S}=\{\hat{s}=(s,\varepsilon);\, s\in S_{n+m} \text{ and } s(\{1,3,\cdots,2p-1\})\subseteq\{1,3,\cdots,2p-1\},$$
$$\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+m}),\quad \varepsilon_i=\pm 1,\ \prod_{j=1}^{p}\varepsilon_{2j-1}=1\}.$$

II) When $n=2p+1$,

$$\hat{S}=\{\hat{s}=(s,\varepsilon);\, s\in S_{n+m} \text{ and } s(\{2,4,\cdots,2p\})\subseteq\{2,4,\cdots,2p\}$$
$$\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+m}),\quad \varepsilon_i=\pm 1,\ \prod_{j=1}^{p}\varepsilon_{2j-1}\prod_{j=n}^{n+m}\varepsilon_j=1\}.$$

Then $W_G(\mathfrak{b})$ can be identified with $\hat{S}$ under the correspondence $w(\nu_{E_0}(e_i))=\nu_{E_0}(\varepsilon_i e_{s(i)})$ for $1\leqq i\leqq n+m$ $(w\in W_G(\mathfrak{b}))$. Furthermore, $\operatorname{sgn}(w)=\operatorname{sgn}(s)\prod_{i=1}^{n+m}\varepsilon_i$.

We consider the character formula corresponding to $\Lambda=(l_1,l_2,\cdots,l_{n+m})$ such that all l_i's are integers and $l_1>l_3>\cdots>l_{2p-1}>l_2>\cdots>l_{2p}>l_{n+1}\cdots>l_{n+m}>0$ (when n is even) or $l_1>l_3>\cdots>l_{2p-1}>l_n>l_{n+1}>\cdots>l_{n+m}>l_2>l_4>\cdots>l_{2p}>0$ (when n is odd). For these Λ, we get that

$$\operatorname{sgn}(\hat{s}^{-1})\operatorname{sgn}_{P(E_i)}(\hat{s}^{-1}\Lambda)=\operatorname{sgn}(s)\,\varepsilon_{n+1}\varepsilon_{n+2}\cdots\varepsilon_{n+m},$$

where $\hat{s}\in\hat{S}$ and $P(E_i)=E_i^*\cup\{e_1,e_2,\cdots,e_n\}$ as in § 1.

Since $A^{(0)}=A_K^{(0)}\cdot\exp(\mathfrak{b}^{(0,0)}\cap\mathfrak{p})$, each element $h\in A^{(0)}(P)$ can be expressed as $h=h_K\exp X$ $(h_K\in A_K^{(0)}=A^{(0)}\cap K$ and $X\in\mathfrak{h}^{(0,0)}\cap\mathfrak{p})$. Furthermore $A_K^{(0)}\subseteq B$, so there exists an element Y in $\mathfrak{h}^{(0,0)}\cap\mathfrak{k}$ such that $h_K=\exp Y$. Then we have,

$$\xi_{\hat{s}^{-1}\Lambda}(h_K)=\delta_{n+1}^{\varepsilon_{n+1}l_{s(n+1)}}\cdots\delta_{n+m}^{\varepsilon_{n+m}l_{s(n+m)}},$$
$$\prod_{\alpha\in E_i^*}\exp\{-u\alpha(X)\,|(\hat{s}^{-1}\Lambda,\nu_{E_0}\alpha)|/|\alpha|^2\}$$
$$=\prod_{j=1}^{p-i}\delta_{u(2j-1)}^{-l_{s(2j-1)}}\delta_{u(2j)}^{-\varepsilon_{2j-1}\varepsilon_{2j}l_{s(2j)}}\times\prod_{j=2p-2i+1}^{2p}\delta_{u(j)}^{-l_{s(j)}}[\delta_{u(n)}^{-l_{s(n)}}],$$

where $\delta_i=\exp e_i(X)$ for $1\leqq i\leqq n$ and $\delta_i=\exp e_i(Y)$ for $n+1\leqq i\leqq n+m$, and $u\in W(E_i)$. The last term $\delta_{u(n)}^{-l_{s(n)}}$ appears only when n is odd.

Since the signs $\{\varepsilon_i\}$ vary under the condition $\varepsilon_1\varepsilon_3\cdots\varepsilon_{2p-1}=1$ (when n is even) or $\varepsilon_1\varepsilon_3\cdots\varepsilon_{2p-1}\varepsilon_n\varepsilon_{n+1}\cdots\varepsilon_{n+m}=1$ (when n is odd), we get that

$$Z(h,E_i,\Lambda)=2^{p-1+i}\sum_{\eta}\sum_{\varepsilon}\sum_{u\in W(E_i)}\sum_{s\in S}\prod_{j=n+1}^{n+m}(\varepsilon_j\delta_j^{\varepsilon_j l_{s(j)}})$$
$$\times\operatorname{sgn}(s)\prod_{j=1}^{p-i}(\delta_{u(2j-1)}^{-l_{s(2j-1)}}\delta_{u(2j)}^{-\eta_j l_{s(2j)}})\prod_{j=2p-2i+1}^{2p}\delta_{u(j)}^{-l_{s(j)}}\times[2\delta_{u(n)}^{-l_{s(n)}}],$$

where $\eta=(\eta_1,\cdots,\eta_{p-i})$, $\varepsilon=(\varepsilon_{n+1},\cdots,\varepsilon_{n+m})$ and $\eta_i=\pm 1$, $\varepsilon_i=\pm 1$.

When n is even, the division of $W(E_i)\backslash W^i$ into two subsets W_1^i and W_2^i in Lemma 3, 4, and 5 causes the cancellations in $Z(h, E_i, \Lambda)$. When n is odd, put

$$W^i = \left\{ \begin{array}{l} u \in S_n : u(1) < u(2) < \cdots < u(2p-2i),\ u(n) = n,\ u(2r-1) \\ \qquad \text{is odd and } u(2r) = u(2r-1)+1 \text{ for } p-i+1 \leqq r \leqq p \end{array} \right\}.$$

Then using Lemma 3′ and 4′, we have the division of $W(E_i)\backslash W^i$ into two subsets $W_1^i = \{u_1, u_2, \cdots, u_{m_i}\}$ and $W_2^i = \{u_{m_i+1}, u_{m_i+2}, \cdots, u_{2m_i}\}$ such that $u_j^{-1}u_{m_i+j} = (2p_j, 2q_j)$ where $1 \leqq p_j < q_j \leqq p-i$ or $u_j^{-1}u_{m_i+j} = (2p_j-1, 2q_j-1)$ where $1 \leqq p_j < q_j \leqq p+1$ for $j=1, \cdots, m_i$.

Thus for any i, we get that

$$Z(h, E_i, \Lambda) = 2^{p+i-1} \sum_{\eta} \sum_{\varepsilon} \sum_{u \in W^i} \sum_{s \in S} \{ \prod_{j=n+1}^{n+m} (\varepsilon_j \delta_j^{\varepsilon_j l_{s(j)}})$$
$$\times \operatorname{sgn}(s) \prod_{j=1}^{p-i} \delta_{u(2j-1)}^{-l_{s(2j-1)}} \delta_{u(2j)}^{-\eta_j l_{s(2j)}} \prod_{j=2p-2i+1}^{2p} \delta_{u(j)}^{-l_{s(j)}} \times [2\delta_n^{-l_{s(n)}}] \}.$$

Now we take the alternating sum of $Z(h, E_i, \Lambda)$'s for $0 \leqq i \leqq p$ as in § 2. For any $s \in S$ and $\varepsilon_0 = (\varepsilon_{n+1}, \varepsilon_{n+2}, \cdots, \varepsilon_{n+m})$, put

$$Z_{s,\varepsilon_0}(h, E_i, \Lambda) = \operatorname{sgn}(s) \prod_{j=n+1}^{n+m} \varepsilon_j \delta_j^{\varepsilon_j l_{s(j)}} (2^{p+i-1} \sum_{\eta} \sum_{u \in W^i}$$
$$(\prod_{j=1}^{p-i} \delta_{u(2j-1)}^{-l_{s(2j-1)}} \delta_{u(2j)}^{-\eta_j l_{s(2j)}}) (\prod_{j=2p-2i+1}^{2p} \delta_{u(j)}^{-l_{s(j)}})) \quad (\text{when } n=2p), \quad \text{or,}$$
$$= \operatorname{sgn}(s) \prod_{j=n+1}^{n+m} ((\varepsilon_j \delta_j^{\varepsilon_j l_{s(j)}}) \delta_n^{-l_{s(n)}} (2^{p+i} \sum_{\eta} \sum_{u \in W^i}$$
$$(\prod_{j=1}^{p-i} \delta_{u(2j-1)}^{-l_{s(2j-1)}} \delta_{u(2j)}^{-\eta_j l_{s(2j)}}) (\prod_{j=2p-2i+1}^{2p} \delta_{u(j)}^{-l_{s(j)}})) \quad (\text{when } n=2p+1).$$

Then

$$\sum_{i=1}^{p} (-1)^i Z(h, E_i, \Lambda) = \sum_{s \in S} \sum_{\varepsilon_0} (\sum_{i=0}^{p} (-1)^i Z_{s,\varepsilon_0}(h, E_i, \Lambda)). \qquad \cdots\cdots (18)$$

We can apply the results in § 2 to the alternating sum of $Z_{s,\varepsilon_0}(h, E_i, \Lambda)$'s in (18) for any $s \in S$ and ε_0. Therefore for any $s \in S$ and $\varepsilon_0 = (\varepsilon_{n+1}, \cdots, \varepsilon_{n+m})$ and $\eta = (\eta_1, \cdots, \eta_p)$, the coefficient of

$$\operatorname{sgn}(s) \prod_{j=1}^{p} \delta_{2j-1}^{-l_{s(2j-1)}} \delta_{2j}^{-\eta_j l_{s(2j)}} \prod_{j=n+1}^{n+m} \varepsilon_j \delta_j^{\varepsilon_j l_{s(j)}} [\delta_n^{-l_{s(n)}}]$$

in (18) is equal to $2^{p-1}(-1)^{p-q}$ (when $n=2p$) or $2^p(-1)^{p-q}$ (when $n=2p+1$). Here q is the number of δ_{2j}'s which have the positive exponents among $\{\delta_2, \delta_4, \cdots, \delta_{2p}\}$ and the last term $\delta_n^{-l_{s(n)}}$ appears only when n is odd.

Thus the following theorem holds.

Theorem 5. *Let $\Lambda = (l_1, l_2, \cdots, l_{n+m})$ be an element in $\mathfrak{h}_B^*$ such that all*

l_i's are integers. Moreover, when $n=2p$, we assume that it satisfies the inequalities $l_1>l_3>\cdots>l_{2p-1}>p_2>l_4>\cdots>l_{2p}>l_{n+1}>\cdots>l_{n+m}>0$ and when $n=2p+1$, we assume that it satisfies $l_1>l_3>\cdots>l_{2p-1}>l_n>l_{n+1}>\cdots>l_{n+m}>l_2>l_4>\cdots>l_{2p}>0$.

Then when $n=2p$,

$$\tilde{\kappa}(h, P)=\sum_{\varepsilon_0} 2^{p-1} \det (c_j^i)_{1 \leqq i, j \leqq p} \det (d_j^i)_{1 \leqq i, j \leqq p+m},$$

where

$$c_i{}^j=\delta_{2i-1}^{-l_{2j-1}}$$

and

$$d_i{}^j=\begin{cases} \delta_{2i}^{-l_{2j}}-\delta_{2i}^{l_{2j}} & (1 \leqq i,\ j \leqq p), \\ \delta_{2i}^{-l_{j+p}}-\delta_{2i}^{l_{j+p}} & (1 \leqq i \leqq p,\ p+1 \leqq j \leqq p+m), \\ \varepsilon_{i+p} \delta_{i+p}^{\varepsilon_{i+p} l_{2j}} & (p+1 \leqq i \leqq p+m,\ 1 \leqq j \leqq p), \\ \varepsilon_{i+p} \delta_{i+p}^{\varepsilon_{i+p} l_{j+p}} & (p+1 \leqq i,\ j \leqq p+m), \end{cases} \qquad \cdots\cdots(19)$$

and when $n=2p+1$,

$$\tilde{\kappa}(h, P)=(-1) \sum_{\varepsilon_0} 2^p \det (c_i'{}^j)_{1 \leqq i, j \leqq p} \det (d_i'{}^j)_{1 \leqq i, j \leqq m+p+1},$$

where

$$c_i'{}^j=\delta_{2i}^{-l_{2j}}-\delta_{2i}^{l_{2j}}$$

and

$$d_i'{}^j=\begin{cases} \delta_{2i-1}^{-l_{2j-1}} & (1 \leqq i,\ j \leqq p+1), \\ \delta_{2i-1}^{-l_{j+p}} & (1 \leqq i \leqq p+1,\ p+2 \leqq j \leqq p+m+1), \\ \varepsilon_{p+i} \delta_{p+i}^{\varepsilon_{p+i} l_{2j-1}} & (1 \leqq j \leqq p+1,\ p+2 \leqq i \leqq p+m+1), \\ \varepsilon_{p+i} \delta_{p+i}^{\varepsilon_{p+i} l_{p+j}} & (p+2 \leqq i,\ j \leqq p+m+1). \end{cases} \qquad \cdots(19')$$

Here $\varepsilon_0=(\varepsilon_{n+1}, \cdots, \varepsilon_{n+m})$ and $\varepsilon_i=\pm 1$ and h is an element in $A^{(0)}(P)$ of form:

$$h=\operatorname{diag}(e^{h_1}, \cdots, e^{h_n}, e^{-h_1}, \cdots, e^{-h_n}, 1, K_{n+1}, \cdots, K_{n+m}),$$

where

$$K_i=\begin{pmatrix} \cos h_i & \sin h_i \\ -\sin h_i & \cos h_i \end{pmatrix}.$$

For any integer $k=1, 2, \cdots, p$, we get that

$$\Sigma_R(A^{(k)})=\begin{Bmatrix} \pm e_i \pm e_j & (1 \leqq i \leqq j \leqq 2k \text{ or } 2k+1 \leqq i<j \leqq n), \\ \pm e_i & (2k+1 \leqq i \leqq n) \end{Bmatrix}.$$

So it is of type $D_{2k} \times B_{n-2k}$. We introduce an order on $\Sigma_R(A^{(k)})$ by restricting that on $\Sigma_R(\mathfrak{h}^{(0,0)})$ defined before. Then the collection $\{E_0, E_1, \cdots, E_{p-k}\}$ is the totality of the standard maximal orthogonal system in $\Sigma_R(A^{(k)})$ and for any i, $P(E_i)=E_i^*\cup\{e_{2k+1}, e_{2k+2}, \cdots, e_n\}$ in this case.

Following the way in § 3 step by step, we get the concrete expression of $\tilde{\kappa}(h, P)$ on $A^{(k)}(P)$ for $k=1, 2, \cdots, p$. That is, by changing the signs of δ_i's for $i=1, 2, \cdots, 2k$ in (19) or (19'), $\tilde{\kappa}(h, P)$ has the expression on $A^{(k)}(P)$ as on $A^{(0)}(P)$.

We define the ordered sets of positive roots in $\Sigma_R(\mathfrak{h}^{(0,0)})$, $E^{(k-q,q)}$ and $E^{(k-q,q)'}$, as follows:

$$E^{(k-q,q)}=(e_1+e_2, e_1-e_2, \cdots, e_{2k-2q-1}+e_{2k-2q}, e_{2k-2q-1}-e_{2k-2q},$$
$$e_{2k-2q+1}+e_{2k-2q+2}, \cdots, e_{2q-1}+e_{2q}) \quad \text{for } 0\leqq q\leqq k\leqq p,$$

$$E^{(k-q,q)'}=(e_1+e_2, e_1-e_2, \cdots, e_{2k-2q-1}+e_{2k-2q}, e_{2k-2q-1}-e_{2k-2q},$$
$$e_{2k-2q+1}+e_{2k-2q+2}, \cdots, e_{2q-1}+e_{2q}, e_n) \quad \text{for } 0\leqq q\leqq k\leqq [(n-1/2].$$

Then any two elements in $E^{(k-q,q)}$ or $E^{(k-q,q)'}$ are strongly orthogonal and any such subset of positive roots in $\Sigma_R(\mathfrak{h}^{(0,0)})$ is conjugate to some $E^{(k-q,q)}$ or some $E^{(k-q,q)'}$ under the action of Weyl group of $\Sigma_R(\mathfrak{h}^{(0,0)})$. Put $\mathfrak{h}^{(k-q,q)}=\nu(E^{(k-q,q)})(\mathfrak{h}_c^{(0,0)})\cap\mathfrak{g}$ and $\mathfrak{h}^{(k-q,q)'}=\nu(E^{(k-q,q)'})(\mathfrak{h}_c^{(0,0)})\cap\mathfrak{g}$. Then the collection $\{\mathfrak{h}^{(k-q,q)}(0\leqq q\leqq k\leqq p), \mathfrak{h}^{(k-q,q)'}(0\leqq q\leqq k\leqq[(n-1)/2])\}$ is the totality of Cartan subalgebras of $\mathfrak{g}$ modulo conjugacy under inner automorphisms. Let $H^{(k-q,q)}$ and $H^{(k-q,q)'}$ be the Cartan subgroups corresponding to $\mathfrak{h}^{(k-q,q)}$ and $\mathfrak{h}^{(k-q,q)'}$ respectively.

Put

$$A(j)=\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_j \end{pmatrix},\quad B(j)=\begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_j \end{pmatrix},\quad C(j)=\begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_j \end{pmatrix},\quad D(j)=\begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_j \end{pmatrix},$$

$$L(i,j)=\begin{pmatrix} \cos\theta_{i+1}1_2 & & 0 \\ & \ddots & \\ 0 & & \cos\theta_j 1_2 \end{pmatrix},\quad L'(i,j)=\begin{pmatrix} \sin\theta_{i+1}I_2 & & 0 \\ & \ddots & \\ 0 & & \sin\theta_j I_2 \end{pmatrix},$$

where $\theta_i=(h_{2i-1}+h_{2i})/2$ and A_i, B_i, C_i, D_i denote the same matrices in the preceding sections. Then $\mathfrak{h}^{(k-q,q)}$ and $\mathfrak{h}^{(k-q,q)'}$ are of the following forms:

$\mathfrak{h}_+^{(k-q,q)}=\mathfrak{h}^{(k-q,q)}\cap\mathfrak{k}$:

$$Y=\begin{pmatrix} A(k-q) & & (-1)^n B(k) & & & \\ & 0_{n-2k+2p} & & 0 & & 0 \\ (-1)^n B(k) & & A(k-q) & & & \\ & 0 & & 0_{n-2k+2q+1} & & \\ & & & & A_{n+1} & \\ & 0 & & & & \ddots \\ & & & & & A_{n+m} \end{pmatrix},$$

$\mathfrak{h}_-^{(k-q,q)}=\mathfrak{h}^{(k-q,q)}\cap\mathfrak{p}$:

$$X=\operatorname{diag}(0_{2k-2q}, F_{k-q+1}, \cdots, F_k, h_{2k+1}, \cdots, h_n, 0_{2k-2q},$$
$$-F_{k-q+1}, \cdots, -F_k, -h_{2k+1}, \cdots, -h_n, 0_{2m+1}),$$

$\mathfrak{h}_+^{(k-q,q)\prime}=\mathfrak{h}^{(k-q,q)\prime}\cap\mathfrak{k}$:

$$Y=\left(\begin{array}{cccccc}
A(k-q) & & (-1)^n B(k) & & \left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\}\cdots n-1 & \\
 & 0_{n-2k+2q} & & 0 & & \\
 & & & & h_n/\sqrt{2} & \\
(-1)^n B(k) & & A(k-q) & & \left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\}\cdots n-1 & 0 \\
 & 0 & & 0_{n-2k+2q} & & \\
 & & & & -h_n/\sqrt{2} & \\
\multicolumn{4}{c}{\underbrace{0,\cdots,0}_{n-1}, -h_n/\sqrt{2}, \underbrace{0,\cdots,0}_{n-1}, h_n/\sqrt{2}} & 0 & \\
 & & 0 & & & \begin{matrix}A_{n+1} & \\ & \ddots \\ & & A_{n+m}\end{matrix}
\end{array}\right)$$

$\mathfrak{h}_-^{(k-q,q)\prime}=\mathfrak{h}^{(k-q,q)\prime}\cap\mathfrak{p}$:

$$X=\operatorname{diag}(0_{2k-2q}, F_{k-q+1}, \cdots, F_k, h_{2k+1}, \cdots, h_{n-1}, 0_{2k-2q+1},$$
$$-F_{k-q+1}, \cdots, -F_k, -h_{2k+1}, \cdots, -h_{n-1}, 0_{2m+2}).$$

Therefore we see that $H_K^{(k-q,q)}=\bigcup_{\rho} h_\rho \exp(\mathfrak{h}_+^{(k-q,q)})$ (disjoint), where $\rho=(\rho_{2k+1}, \rho_{2k+2}, \cdots, \rho_n)$ with $\rho_i=\pm 1$, $\prod_{i=2k+1}^{n}\rho_i=1$ and $h_\rho=\operatorname{diag}(1_{2k}, \rho_{2k+1}, \rho_{2k+2}, \cdots, \rho_n, 1_{2k}, \rho_{2k+1}, \cdots, \rho_n, 1_{2m+1})$, and moreover that $H_K^{(k-q,q)\prime}=\bigcup_{\rho} h'_\rho \exp(\mathfrak{h}_+^{(k-q,q)\prime})$ (disjoint), where $\rho=(\rho_{2k+1}, \rho_{2k+2}, \cdots, \rho_{n-1})$ with $\rho_i=\pm 1$, $\prod_{i=2k+1}^{n-1}\rho_i=1$ and $h'_\rho=\operatorname{diag}(1_{2k}, \rho_{2k+1}, \rho_{2k+2}, \cdots, \rho_{n-1}, 1_{2k+1}, \rho_{2k+1}, \cdots, \rho_{n-1}, 1_{2m+2})$. Moreover $H^{(k-q,q)}=H_K^{(k-q,q)}\cdot\exp\mathfrak{h}_-^{(k-q,q)}$ and $H_0^{(k-q,q)}=\exp\mathfrak{h}_+^{(k-q,q)}\cdot\exp\mathfrak{h}_-^{(k-q,q)}$, and similarly for $H^{(k-q,q)\prime}$.

We consider the formula of $\tilde{\kappa}(h, P)$ on $H_0^{(k-q,q)}$and $H_0^{(k-q,q)\prime}$. We denote the root $\nu(E^{(k-q,q)})(\alpha)$ (resp. $\nu(E^{(k-q,q)\prime})(\alpha)$) of $\mathfrak{h}^{(k-q,q)}$ (resp. $\mathfrak{h}^{(k-q,q)\prime}$) again by α for $\alpha\in\Sigma(\mathfrak{h}^{(0,0)})$. Then

$$\Sigma_R(\mathfrak{h}^{(k-q,q)})=\left\{\begin{array}{l}\pm(e_{2i-1}-e_{2i})\quad(k-q+1\leqq i\leqq k),\\ \pm e_i\pm e_j\quad(2k+1\leqq i<j\leqq n),\\ \pm e_i\quad(2k+1\leqq i\leqq n)\end{array}\right\},$$

$$\Sigma_R(\mathfrak{h}^{(k-q,q)\prime})=\left\{\begin{array}{l}\pm(e_{2i-1}-e_{2i})\quad(k-p+1\leqq i\leqq k),\\ \pm e_i\pm e_j\quad(2k+1\leqq i<j\leqq n-1),\\ \pm e_i\quad(2k+1\leqq i\leqq n-1)\end{array}\right\}.$$

For the standard maximal orthogonal systems in the positive roots of $\Sigma_R(\mathfrak{h}^{(k-q,q)})$, we can apply Lemma 3, 4, and 5 when $n-2k$ is even and Lemma 3′ and 4′ when $n-2k$ is odd. For those of $\Sigma_R(\mathfrak{h}^{(k-q,q)\prime})$ we apply Lemma 3′

and 4′ when $n-2k$ is even and lemma 3, 4, and 5 when $n-2k$ is odd. Therefore for

$$E_i=(e_{2k-2q+1}-e_{2k-2q+2},\cdots,e_{2k-1}-e_{2k},e_{2k+1}+e_{2k+2},e_{2k+1}-e_{2k+2},$$
$$\cdots,e_{2p-2i-1}+e_{2p-2i},e_{2p-2i-1}-e_{2p-2i},e_{2p-2i+1},\cdots,e_{2p},[e_n]),$$

we get the following:
for $h=\exp Y\exp X$ $(Y\in\mathfrak{h}_+^{(k-q,q)}, X\in\mathfrak{h}_-^{(k-q,q)})$,

$$Z(h,E_i,\Lambda)=$$
$$2^{p+k-i-1}\sum_{s\in S}\sum_{\varepsilon}\{\operatorname{sgn}(s)\prod_{j=1}^{2k-2q}\varepsilon_j\delta_j^{\varepsilon_j l_{s(j)}}\prod_{j=n+1}^{n+m}\varepsilon_j\delta_j^{\varepsilon_j l_{s(j)}}\}$$
$$\times\prod_{j=k-q+1}^{k}\Big(\sum_{\substack{\varepsilon_{2j}=\pm1\\ \tau_j=0,1}}(-1)^{\tau_j}\delta_{2j-1+\tau_j}^{-l_{s(2j-1)}}\delta_{2j-\tau_j}^{\varepsilon_{2j}l_{s(2j)}}\Big)$$
$$\times\Big[\sum_{\eta}\sum_{u\in W^i}\Big(\prod_{j=k+1}^{p-i}\delta_{u(2j-1)}^{-l_{s(2j-1)}}\prod_{j=k+1}^{p-i}\delta_{u(2j)}^{-\eta_j l_{s(2j)}}\prod_{j=2p-2i+1}^{n}\delta_{u(j)}^{-l_{s(j)}}\Big]\times[2].$$

(The last facter $\times[2]$ appears only when n is odd.) Here δ_i's are given by the coordinates of Y and X as follows:

$$\delta_j=e^{-\sqrt{-1}h_j}\quad(1\leqq j\leqq 2k-2q \text{ or } n+1\leqq j\leqq n+m),$$
$$\delta_j=e^{h_j}\quad(k+1\leqq j\leqq n),\quad \delta_{2j-1}=e^{r_j+\sqrt{-1}\theta_j},$$
$$\delta_{2j}=e^{r_j+\sqrt{-1}\theta_j}\quad(k-q+1\leqq j\leqq k).$$
$$\varepsilon=(\varepsilon_1,\cdots,\varepsilon_{2k-2q},\varepsilon_{n+1},\cdots,\varepsilon_{n+m}),$$
$$\eta=(\eta_{k+1},\cdots,\eta_{p-i}),\ \eta_j=\pm1,\ \varepsilon_j=\pm1,$$

and u runs over the following subset W^i of the permutation group $S(2k+1, 2k+2,\cdots,n)$:

$$W^i=\left\{\begin{matrix}u;\,u(2k+1)<u(2k+2)<\cdots<u(2p-2i),\,u(2j-1)\ \text{ is odd and}\\ u(2j)=u(2j-1)+1\ \text{ for } p-i+1\leqq i\leqq p,\ (u(n)=n \text{ if } n=2p+1)\end{matrix}\right\}.$$

The alternating sum of $Z(h,E_i,\Lambda)$'s $(0\leqq i\leqq p-k)$ is reduced to that of the last parts in the brackets $[\cdots]$. For $H_0^{(k-q,q)\prime}$, the the situation is completely parallel. Therefore we get the following theorem which includes the theorems in the preceding sections.

Theorem 6. *Let* $\Lambda=(l_1,l_2,\cdots,l_{n+m})$ *be an element in* $\mathfrak{b}_B^*$ *such that all* l_i*'s are integers and* $l_1>l_3>\cdots>l_{2p-1}>l_2>l_4>\cdots>l_{2p}>l_{n+1}>\cdots>l_{n+m}>0$ *(when* $n=2p$*) or* $l_1>l_3>\cdots>l_{2p-1}>l_n>l_{n+1}>\cdots>l_{n+m}>l_2>l_4>\cdots>l_{2p}>0$ *(when* $n=2p+1$*). Then for* $h=\exp(Y)\exp(X)\in H_0^{(k-q,q)}(P)$*, when* $n=2p$,

$$\tilde{\kappa}(h,P)=(-1)^q 2^{p-k-1}\sum_{(a,b)}\sum_{\varepsilon}\det(c_i^j)\det(d_i^j)$$
$$\times\operatorname{sgn}(a,b)\{\sum_{\tau}(-1)^{\Sigma\tau_i}\det(x_i^j)\det(y_i^j)\},\qquad\cdots\cdots(20)$$

and when $n=2p+1$,

$$\tilde{\kappa}(h, P)=(-1)^{q+1}2^{p-k}\sum_{(a,b)}\sum_{\varepsilon}\det(c_i'^{j})\cdot\det(d_i'^{j})$$
$$\times\operatorname{sgn}(a,b)\{\sum_{\tau}(-1)^{\Sigma\tau_i}\det(x_i^j)\det(y_j^i)\} \qquad \cdots\cdots(20)'$$

Here

$$c_i^j=\begin{cases}\varepsilon_{2i-1}\delta_{2i-1}^{\varepsilon_{2i-1}l(a'(j))} & (1\leqq i\leqq k-q,\ 1\leqq j\leqq p-q),\\ \delta_{2i+2q-1}^{-l(a'(j))} & (k-q+1\leqq i\leqq p-q,\ 1\leqq j\leqq p-q),\end{cases}$$

$$d_i^j=\begin{cases}\varepsilon_{2i}\delta_{2i}^{\varepsilon_{2i}l(b'(j))} & (1\leqq i\leqq k-q,\ 1\leqq j\leqq p-q+m),\\ \delta_{2i+2q}^{-l(b'(j))}-\delta_{2i+2q}^{l(b'(j))} & (k-q+1\leqq i\leqq p-q,\ 1\leqq j\leqq p-q+m),\\ \varepsilon_{i+n-p+q}\delta_{i+n-p+q}^{\varepsilon_{i+n-p+q}l(b'(j))} & (p-q+1\leqq i\leqq p-q+m,\ 1\leqq j\leqq p-q+m)\end{cases}$$

$$x_i^j=\delta_{2k-2q+2i-1+\tau_{k-q+i}}^{-l(a(j))},$$

$$y_i^j=\varepsilon_{2k-2q+2i}\delta_{2k-2q+2i-\tau_{k-q+i}}^{\varepsilon_{2k-2q+2i}l(b(j))} \quad (1\leqq i,\ j\leqq q),$$

$$c_i'^{j}=\begin{cases}c_i^{\,j} & (1\leqq i\leqq p-q,\ 1\leqq j\leqq p-q+m+1),\\ \delta_{2p+1}^{-l(a'(j))} & (i=p-q+1,\ 1\leqq j\leqq p-q+m+1),\\ \varepsilon_{i+n-p+q-1}\delta_{i+n-p+q+1}^{\varepsilon_{i+n-p+q+1}l(a'(j))} & \\ \qquad (p-q+2\leqq i\leqq p-q+m+1,\ 1\leqq j\leqq p-q+m+1),\end{cases}$$

$$d_i'^{j}=d_i^j \quad (1\leqq i,j\leqq p-q),$$

$$\varepsilon=(\varepsilon_1,\cdots,\varepsilon_{2k-2q},\varepsilon_{2k-2q+2},\cdots,\varepsilon_{2k},\varepsilon_{n+1},\varepsilon_{n+2},\cdots,\varepsilon_{n+m}),$$

$$\tau=(\tau_{k-q+1},\cdots,\tau_k)\quad \varepsilon_j=\pm1,\ \tau_j=0 \text{ or } 1.$$

When $n=2p$, $a=(a(1),\cdots,a(q))$ runs over the subsets consisting of q numbers in $\{1,3,\cdots,2p-1\}$ and $b=(b(1),b(2),\cdots,b(q))$ runs over the subsets of q numbers in $\{2,4,\cdots,2p,n,n+1,\cdots,n+m\}$. When $n=2p+1$, $a=(a(1),\cdots,a(q))$ runs over those in $\{1,3,\cdots,2p-1,n,n+1,\cdots,n+m\}$ and b does over those in $\{2,4,\cdots,2p\}$. In both cases, a (resp. b) and its complementary set $a'=(a'(1),\cdots,a'(q'))$ (resp. $b'=(b'(1),\cdots,b'(q'))$) are arranged in order and let f be the number of $a(i)$'s (when $n=2p$) or $b(i)$'s (when $n=2p+1$) that are greater than or equal to $n+1$. Then $\operatorname{sgn}(a,b)=(-1)^N$ with

$$2N=\sum_{i=1}^{q}(a(i)+b(i))+(\sum_{i=f+1}^{q}b(i))+q-2p(q-f)\quad(\text{when } n=2p),$$
$$=\sum_{i=1}^{q}(a(i)+b(i))+(\sum_{i=f+1}^{q}a(i))+f-2q(q-f)\quad(\text{when } n=2p+1).$$

Note. We have a similar expression of $\tilde{\kappa}(h,P)$ on $H_0^{(k-q,q)\prime}$. The signs $(\tau_{p-q+},\cdots,\tau_p)$ runs under the condition that $(-1)^{\Sigma\tau_j}=\varepsilon_1\varepsilon_3\cdots\varepsilon_{2p-2q+1}$ when $n=2k=2p$. On the other connected component of $H^{(k-q,q)}$, after changing the necessary signs of δ_i's in (20) and (20′) among $\{\delta_{2k+1}, \cdots, \delta_n\}$, we can also see that $\tilde{\kappa}(h, P)$ has the same expression as above on these connected components.

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Acknowledgement: The author wishes to express his hearty thanks to Professor T. Hirai for his kind suggestions.

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