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Mixed problems for the wave equation, IV The existence and the exponential decay of solutions

By

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§ 1. Introduction

The purpose of this paper is to generalize and improve the results presented in the preceding papers [3] and [4]. In [3] the existence of solutions of mixed problems is considered on the condition that the coefficients of the boundary operator are independent of t. We will generalize this existence theorem to the case of a boundary operator with coefficients depending on t by using the idea of Soga [9] and Tsuji [10]. And concerning the exponential decay of solutions a results corresponding to Theorem 1 of [4] may be shown for boundary operators inhomogeneous in t on some suppositions about the behavior of the coefficients near $t=\infty$.

If we impose once more the restriction on boundary operators that all the coefficients are independent of t Theorems 1 and 2 of [4] can be made better in some directions. The essential means for this improvement is a new estimate of $\mathscr{N}^{(\delta)}(p)$, which is presented in Theorem 3.2.

We will explain the problem and state the theorems:

Let $\mathcal{O}$ be a bounded object in $\boldsymbol{R}^3$ with sufficiently smooth boundary Γ. Let us set

$$\Omega=\boldsymbol{R}^3-\mathcal{O}-\Gamma$$

and

$$B=\sum_{j=1}^{3} b_j(x, t)\frac{\partial}{\partial x_j}+c(x, t)\frac{\partial}{\partial t}+\frac{1}{2}\sum_{j=1}^{3}\partial(b_j(x, t)-n_j(x))/\partial x_j$$
$$+\frac{1}{2}\partial c(x, t)/\partial t+d(x, t)$$

where b_j, c and d are functions belonging to $C^\infty(\Gamma\times[0, \infty))$ and $n(x)=(n_1(x), n_2(x), n_3(x))$ denotes the unit outer normal of Γ at $x\in\Gamma$.

We consider a mixed problem

$$
(\mathrm{P})\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{3} \dfrac{\partial^2 u}{\partial x_j^2} = 0 & \text{in } \Omega \times (0, \infty) \\ Bu = 0 & \text{on } \Gamma \times (0, \infty) \\ u(x, 0) = u_0(x) & \\ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) & \end{cases}
$$

on the following assumptions:

(A-I) The Gaussian curvature of Γ is bounded away from zero.

(A-II) b_j, $j=1, 2, 3$ and c are real valued.

(A-III) $\sum_{j=1}^{3} b_j(x, t) n_j(x) = 1$ on $\Gamma \times [0, \infty)$.

Theorem 1. *In order that (P) is C^∞ well posed and has a finite propagation speed it must and it suffices to hold*

$$
(1.1)\qquad c(x, t) < 1 \quad on \quad \Gamma \times [0, \infty).
$$

The following four theorems are concerned with the exponential decay of solutions of (P).

For $u(x, t) \in C^\infty(\bar{\Omega} \times [0, \infty))$ we set

$$
E_m(u, r, t) = \sum_{|\alpha| \leq m} \int_{\Omega_r} |D^\alpha_{x,t} u(x, t)|^2 dx
$$

$$
\Omega_r = \{x; x \in \Omega, |x| < r\}.
$$

We denote $(12e \times \text{the diameter of } \mathcal{O})^{-1}$ by δ_0.

Theorem 2. *Suppose that all the coefficients of B belong to $\mathcal{B}^\infty(\Gamma \times [0, \infty))$ and satisfy*

$$
(1.2)\qquad \sup_{(x,t) \in \Gamma \times (0,\infty)} c(x, t) < 1,
$$

$$
(1.3)\qquad \liminf_{t \to \infty} \mathrm{Re}(-d(x, t)) \geq d_0
$$

$$
(1.4)\qquad \lim_{t \to \infty} \left(\sum_{j=1}^{3} \left| \frac{\partial b_j(x, t)}{\partial t} \right| + \left| \frac{\partial c(x, t)}{\partial t} \right| \right) = 0,
$$

where d_0 is a constant determined by Ω alone. Then the solution $u(x, t)$ of (P) for $u_0, u_1 \in C_0^\infty(\bar{\Omega})$ decays exponentially,

$$
(1.5)\qquad E_m(u, r, t) \leq C_m \exp\{3\delta_0(r + 2\kappa)\} \cdot \exp\left(-\frac{\delta_0 t}{2}\right) \cdot E_{m+2}(u, \infty, 0),
$$

where κ denotes the diameter of $\bigcup_{j=0}^{1}$ supp u_j and C_m is a constant independent of u_0, u_1.

Theorem 3. *Consider the case of the boundary condition of the third kind with time independent coefficients, i.e.*

$$b_j(x,t)=n_j(x), \quad j=1,2,3,$$
$$c(x,t)=0$$
$$d(x,t)=d(x).$$

If $d(x)$ *is real valued and satisfies*

$$(1.6) \qquad d(x)<|x-Q|^{-2}\sum_{j=1}^{3}(x_j-q_j)\,n_j(x) \quad \text{for all } x\in\Gamma$$

for some point $Q=(q_1,q_2,q_3)\in\mathcal{O}$,[1] *the solution* $u(x,t)$ *of* (P) *for initial data* $u_0, u_1\in C_0^\infty(\bar{\Omega})$ *satisfies the following inequality*

$$(1.7) \qquad E_m(u,r,t)\le C_m\exp\{3\delta_0(r+2\kappa)\}\cdot\exp(-\gamma t)\cdot E_{m+1}(u,\infty,0)$$

where γ *is a positive constant independent of* u_0, u_1, *and* κ *denotes the diameter of* $\bigcup_{j=0}^{1}\operatorname{supp} u_j$.

Theorem 4. *Suppose that the coefficients of* B *are independent of* t. *Denote by* $H(x)$ *the mean curvature of* Γ *at* x *with respect to* $-n(x)$. *When* $d(x)$ *satisfies*

$$(1.8) \qquad d(x)<\min\Big(|x-Q|^{-2}\sum_{j=1}^{3}(x_j-q_j)\,n_j(x),\ H(x)\Big), \quad \forall x\in\Gamma,$$

there exists a consant $a>0$, *which depends on* Ω *and* $d(x)$, *such that if*

$$(1.9) \qquad \sum_{j=1}^{3}|b_j(x)-n_j(x)|+|c(x)|+\frac{1}{2}\sum_{j=1}^{3}\left|\frac{\partial(b_j(x)-n_j(x))}{\partial x_j}\right|<a$$

holds the solution of (P) *for initial data in* $C_0^\infty(\bar{\Omega})$ *decays exponentially in the form* (1.7).

Theorem 5. *Suppose that the coefficients of* B *are independent of* t. *For each* $\eta>0$ *there exists a constant* C_η *such that the condition*

$$\operatorname{Re} d(x)\le H(x)-2\eta$$
$$|\operatorname{Im} d(x)|\ge C_\eta$$

implies the exponential decay of the form (1.7) *of solutions of* (P) *for initial data in* $C_0^\infty(\bar{\Omega})$.

§ 2. Existence of solutions (proof of Theorem 1)

The notation and terminology of the preceding papers will be used freely.

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[1] This condition was introduced in Asakura [1] as a sufficient condition for the local energy decay of solutions of the problem for a star-shaped obstacle.

We explain some spaces of functions other than the ones used in the previous papers.

$H^m(\Gamma\times\boldsymbol{R}^1)$: the space of Sobolev of order m on $\Gamma\times\boldsymbol{R}^1$. Denote the scalar product and the norm by $\langle\cdot,\cdot\rangle_m$ and $[\cdot]_m$ respectively.

$H^m(\Omega\times\boldsymbol{R}^1)$: the space of Sobolev of order m on $\Omega\times\boldsymbol{R}^1$. Denote the scalar product and the norm by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_m$ and $[[\cdot]]_m$ respectively.

$H^m_\mu(\Gamma\times\boldsymbol{R}^1)$: the space of functions $u(x,t)$ defined on $\Gamma\times\boldsymbol{R}^1$ such that $e^{-\mu t}u(x,t)\in H^m(\Gamma\times\boldsymbol{R}^1)$. And for u, $v\in H^m_\mu(\Gamma\times\boldsymbol{R}^1)$ define the scalar product and the norm by

$$\langle u,v\rangle_{m,\mu}=\langle e^{-\mu t}u, e^{-\mu t}v\rangle_m,\ [u]_{m,\mu}=[e^{-\mu t}u]_m\,.$$

$H^m_\mu(\Omega\times\boldsymbol{R}^1)$: the space of functions $u(x,t)$ defined on $\Omega\times\boldsymbol{R}^1$ such that $e^{-\mu t}u(x,t)\in H^m(\Omega\times\boldsymbol{R}^1)$. Define for u, $v\in H^m_\mu(\Omega\times\boldsymbol{R}^1)$ the scalar product and the norm by

$$\langle\!\langle u,v\rangle\!\rangle_{m,\mu}=\langle\!\langle e^{-\mu t}u, e^{-\mu t}v\rangle\!\rangle_m$$
$$[[u]]_{m,\mu}=[[e^{-\mu t}u]]_m\,.$$

First we prove Theorem 1 on the following additional condition on B:

(2.1) $\begin{cases}\text{The coefficients of } B \text{ are defined on } \Gamma\times\boldsymbol{R}^1 \text{ and there exists a constant } T>0 \text{ and a boundary operator } B^0 \text{ with coefficients independent of } t \text{ such that}\\ \qquad B=B^0\quad\text{for } |t|>T\,.\end{cases}$

Let $\varphi(x)$ be a real valued function in $\mathscr{B}^\infty(\boldsymbol{R}^3)$ satisfying

$$\sup_{x\in\boldsymbol{R}^3}|\nabla\varphi(x)|<1\,, \tag{2.2}$$

and let us set

$$A_\varphi\Big(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\Big)=(1-(\nabla\varphi)^2)\frac{\partial^2}{\partial t^2}-2\nabla\varphi\cdot\nabla\frac{\partial}{\partial t}-\Delta-\Delta\varphi\cdot\frac{\partial}{\partial t}$$

$$B_\varphi\Big(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\Big)=\sum_{j=1}^3 b_{j\varphi}(x,t)\frac{\partial}{\partial x_j}+\Big(c_\varphi(x,t)+\frac{\partial\varphi}{\partial n}\Big)\frac{\partial}{\partial t}+d_\varphi(x,t)$$

where

$$b_{j\varphi}(x,t)=b_j(x,t-\varphi(x))$$

$$c_\varphi(x,t)=\Big[c(x,t')+\sum_{j=1}^3 b_j(x,t')\frac{\partial\varphi}{\partial x_j}(x)-\frac{\partial\varphi}{\partial n}(x)\Big]_{t'=t-\varphi(x)}$$

$$d_\varphi(x,t)=\Big[\frac{1}{2}\{\sum_{j=1}^3\partial(b_j(x,t')-n_j(x))/\partial x_j+\partial c(x,t')/\partial t'\}$$
$$+d(x,t')\Big]_{t'=t-\varphi(x)}\,.$$

From (A-III) and (2.1) we have

$$\sum_{j=1}^{3} b_{j\varphi}(x,t)\,n_j(x)=1 \quad \text{on } \Gamma\times\boldsymbol{R}^1, \tag{2.3}$$

$$\begin{cases} b_{j\varphi},\ c_\varphi,\ d_\varphi \text{ are independent of } t \text{ for} \\ \qquad |t|\geq T+\sup_{x\in\Gamma}|\varphi(x)|. \end{cases} \tag{2.4}$$

Consider the existence and the uniqueness of solutions of a mixed problem

$$(\mathrm{P}_\varphi)\qquad \begin{cases} A_\varphi\left(\dfrac{\partial}{\partial t},\dfrac{\partial}{\partial x}\right)u=f & \text{in } \Omega\times(0,\infty) \\ B_\varphi\left(\dfrac{\partial}{\partial t},\dfrac{\partial}{\partial x}\right)u=h & \text{on } \Gamma\times(0,\infty) \\ u(x,0)=u_0(x) & \\ \dfrac{\partial u}{\partial t}(x,0)=u_1(x). & \end{cases}$$

To this end we note some results obtained in [3]. A boundary value problem with a complex parameter $p=\mu+ik$

$$\begin{cases} A_\varphi\left(p,\dfrac{\partial}{\partial x}\right)v(x)=0 & \text{in } \Omega \\ v(x)=g & \text{on } \Gamma \end{cases} \tag{2.5}$$

has a unique solution in $\bigcap_{m\geq 1} H^m(\Omega)$ for every $g\in C^\infty(\Gamma)$ when $\mu\geq\mu_\varphi$, where μ_φ is a constant determined by φ. Denote this $v(x)$ by $U_\varphi(p,g:x)$. Then we have an estimate

$$\begin{aligned} \mu|||U_\varphi(p,g:x)|||_m^2+\left\|\frac{\partial U_\varphi}{\partial n}(p,g;x)\right\|_{m-1}^2 \\ \leq C_m\|g\|_m^2, \qquad m=1,2,\cdots. \end{aligned} \tag{2.6}$$

Define an operator $\mathscr{N}_\varphi(p)$ by

$$\mathscr{N}_\varphi(p)g=\left(\frac{\partial}{\partial n}+\frac{\partial\varphi}{\partial n}p\right)U_\varphi(p,g:x)\Big|_\Gamma$$

and the theorem 1 of [3] shows that

$$-\mathrm{Re}\,(\mathscr{N}_\varphi(\mu+ik)g,g)_m\geq(\mu c_1(\varphi)-C_m)\|g\|_m^2 \tag{2.7}$$

holds for all $g\in C^\infty(\Gamma)$ and $\mu\geq\mu_\varphi$, where

$$c_1(\varphi)=\inf_{x\in\Gamma}\sqrt{1-(\varphi_s)^2}, \qquad \varphi_s=\nabla\varphi-(\nabla\varphi\cdot n)\,n\,.$$

Consider a problem

$$(2.8)\qquad \begin{cases} A_\varphi\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}\right) u(x,t)=0 & \text{in } \Omega\times\boldsymbol{R}^1 \\ B^0_\varphi\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}\right) u(x,t)=h & \text{on } \Gamma\times\boldsymbol{R}^1 \\ \operatorname{supp} u\subset\bar{\Omega}\times[t_0,\infty) \end{cases}$$

for $h\in H^m_\mu(\Gamma\times\boldsymbol{R}^1)$ such that $\operatorname{supp} h\subset\Gamma\times[t_0,\infty)$. Theorem 2 of [3] says that there exists a solution $u(x,t)$ in $H^m_\mu(\Omega\times\boldsymbol{R}^1)$ uniquely and that an estimate

$$[[u]]_{m,\mu}\le C_m[h]_{m,\mu}$$

holds for $\mu\ge\tilde{\mu}_\varphi$, where $\tilde{\mu}_\varphi$ is a constant determined by Ω and B^0.

To prove Theorem 1 of this paper it is essential to show

Proposition 2.1. *Suppose* (2.1). *Let* φ *satisfy*

$$(2.9)\qquad \inf_{x\in\Gamma}\sqrt{1-(\varphi_s)^2}>\sup_{(x,t)\in\Gamma\times\boldsymbol{R}^1}(|\varphi_s(x)|v(x,t)+c(x,t))$$

where

$$v(x,t)=\Big(\sum_{j=1}^{3}(b_j(x,t)-n_j(x))^2\Big)^{1/2}.$$

Then for $h\in H^m_\mu(\Gamma\times\boldsymbol{R}^1)$, $m\ge1$, *such that*

$$\operatorname{supp} h\subset\Gamma\times[t_1,\infty),\quad t_1\ge0\,,$$

there exists a solution $u(x,t)$ *in* $H^m_\mu(\Omega\times\boldsymbol{R}^1)$ *uniquely of the problem*

$$(2.10)\qquad \begin{cases} A_\varphi\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}\right) u(x,t)=0 & \textit{in } \Omega\times\boldsymbol{R}^1 \\ B_\varphi\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}\right) u(x,t)=h & \textit{on } \Gamma\times\boldsymbol{R}^1 \\ u(x,t)=0 \quad \textit{for } t\le0\,, \end{cases}$$

when $\mu\ge\tilde{\mu}_\varphi$. *Moreover* $u(x,t)$ *satisfies*

$$(2.11)\qquad \operatorname{supp} u(x,t)\subset\bar{\Omega}\times[t_1,\infty)$$

$$(2.12)\qquad [[u(x,t)]]_{m,\mu}\le C_m[h]_{m,\mu}\,.$$

We show this proposition in the following. Let $\mu\in\boldsymbol{R}$. For any $g(x,t)\in C_0^\infty(\Gamma\times\boldsymbol{R}^1)$ there exists uniquely a solution $w(x,t)\in C^\infty(\bar{\Omega}\times\boldsymbol{R}^1)$ of the problem

$$\begin{cases} A_\varphi\left(\dfrac{\partial}{\partial t}+\mu, \dfrac{\partial}{\partial x}\right) w(x,t)=0 & \text{in } \Omega\times\boldsymbol{R}^1 \\ w(x,t)=g(x,t) & \text{on } \Gamma\times\boldsymbol{R}^1 \\ w(x,t)=0 \quad \forall t\le -t_0 \text{ for some } t_0\,. \end{cases}$$

We denote this solution by $\mathcal{W}_\varphi(\mu, g; x, t)$. Then it can be represented as

$$(2\cdot 13)\qquad \begin{cases} \mathcal{W}_\varphi(\mu, g; x, t) = \int_{-\infty}^{\infty} e^{ikt} U_\varphi(\mu+ik, \hat{g}(\cdot, ik); x)\, dk \\ \hat{g}(x, ik) = \int_{-\infty}^{\infty} e^{-ikt} g(x, t)\, dt\,. \end{cases}$$

Remark that it follows from (2. 6) that

$$(2\cdot 14)\qquad \mu[[\mathcal{W}_\varphi(\mu, g; x, t)]]_m^2 + \left[\frac{\partial}{\partial n}\mathcal{W}_\varphi(\mu, g; x, t)\right]_{m-1}^2 \leq C_m[g]_m^2\,, \qquad m=1, 2, 3\,.$$

Define an operator $\boldsymbol{B}_\varphi(\mu)$ by

$$\boldsymbol{B}_\varphi(\mu) g = B_\varphi\Big(\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\Big)\mathcal{W}_\varphi(\mu, g; x, t)\,|_{\Gamma\times \boldsymbol{R}^1}$$

and it is a continuous mapping from $C_0^\infty(\Gamma\times \boldsymbol{R}^1)$ into $C^\infty(\Gamma\times \boldsymbol{R}^1)$. In the same way we define $\boldsymbol{N}_\varphi(\mu)$ and $\boldsymbol{B}_\varphi^0(\mu)$ for $N_\varphi = \frac{\partial}{\partial n} + \frac{\partial\varphi}{\partial n}\frac{\partial}{\partial t}$ and B_φ^0 respectively. Then it holds that

$$\boldsymbol{N}_\varphi(\mu) g = \int_{-\infty}^{\infty} e^{ikt}\mathcal{N}_\varphi(\mu+ik)\hat{g}(x, ik)\, dk\,.$$

Note that we have

$$[\boldsymbol{N}_\varphi(\mu) g]_m \leq C_m[g]_{m+1}\,,$$

which follows from an estimate

$$\|\mathcal{N}_\varphi(\mu+ik)u\|_m \leq C_m\|u\|_{m+1} \quad \text{for all } u\in C^\infty(\Gamma)\,.$$

Since it follows from (A-III) that

$$(2.\,15)\qquad \boldsymbol{B}_\varphi(\mu) = \boldsymbol{N}_\varphi(\mu) + \text{a differential operator of first order on } \Gamma\times \boldsymbol{R}^1\,,$$

we have

$$(2.\,16)\qquad [\boldsymbol{B}_\varphi(\mu) g]_m \leq C_m[g]_{m+1}, \quad \forall g\in C_0^\infty(\Gamma\times \boldsymbol{R}^1)\,.$$

Therefore $\boldsymbol{B}_\varphi(\mu)$ can be extended to a continuous operator from $H^{m+1}(\Gamma\times \boldsymbol{R}^1)$ into $H^m(\Gamma\times \boldsymbol{R}^1)$. With the aid of (2. 7) we have

$$(2.\,17)\qquad -\mathrm{Re}\,\langle \boldsymbol{N}_\varphi(\mu) g, g\rangle = \int_{-\infty}^{\infty} -\mathrm{Re}\,(\mathcal{N}_\varphi(\mu+ik)\hat{g}(\cdot, ik), \hat{g}(\cdot, ik))\, dk$$
$$\geq (c_1(\varphi) - C_0)\int_{-\infty}^{\infty}\|\hat{g}(\cdot, ik)\|_0^2 dk$$
$$= (c_1(\varphi) - C_0)\,[g]_0^2\,.$$

Taking account that an estimate holds for any $l\in \boldsymbol{R}$

$$|([\mathscr{N}_\varphi(ik+\mu),(|D|^2+k^2)^{1/2}](|D|^2+k^2)^{-1/2}u,u)|$$
$$\leq C_l\{-\mathrm{Re}(\mathscr{N}_\varphi(\mu+ik)u,u)+C'\|u\|^2\},\quad \forall u\in C^\infty(\Gamma)$$

it follows from (2.17) that for any $m\in\boldsymbol{R}$

$$-\mathrm{Re}\langle \boldsymbol{N}_\varphi(\mu)g,g\rangle_m\geq(c_1(\varphi)\mu-C_m)[g]_m^2. \tag{2.18}$$

Note that in the representation (2.15) all the coefficients of $\boldsymbol{B}_\varphi(\mu)-\boldsymbol{N}_\varphi(\mu)-\mu c_\varphi(x,t)$ are independent of μ and those of the principale part are real valued. Therefore it follows that

$$|\mathrm{Re}\langle(\boldsymbol{B}_\varphi(\mu)-\boldsymbol{N}_\varphi(\mu)-\mu c_\varphi(x,t))g,g\rangle_m\leq C_m[g]_m^2.$$

By combining the above estimate to (2.18) we have

$$-\mathrm{Re}\langle \boldsymbol{B}_\varphi(\mu)g,g\rangle_m\geq(c_2(\varphi)\mu-C_m)[g]_m^2$$

where

$$c_2(\varphi)=c_1(\varphi)-\sup c_\varphi$$
$$\geq\inf\sqrt{1-|\varphi_s|^2}-\sup(c(x,t)+|v||\varphi_s|).$$

Thus we have

Lemma 2.2. *Suppose that* φ *satisfies* (2.9). *For any* $m\in\boldsymbol{R}$ *it holds that*

$$[\boldsymbol{B}_\varphi(\mu)g]_m\geq[g]_m \quad \textit{for all } g\in H^{m+1}(\Gamma\times\boldsymbol{R}^1) \tag{2.19}$$

when $\mu\geq\mu_{\varphi,m}$, *where* $\mu_{\varphi,m}$ *is a constant depending on* φ *and* m.

By taking account of (2.15) and the properties of $\boldsymbol{N}_\varphi(\mu)$ we have

$$[g]_{m+1}^2\leq C_m\left\{[\boldsymbol{B}_\varphi(\mu)g]_m^2+\left[\frac{\partial g}{\partial t}\right]_m^2+[g]_m^2\right\}.$$

And by using (2.19) it follows immediately that

$$[g]_{m+1}^2\leq C\left\{[\boldsymbol{B}_\varphi(\mu)g]_m^2+\left[\frac{\partial g}{\partial t}\right]_m^2\right\}\qquad \forall g\in C_0^\infty(\Gamma\times\boldsymbol{R}^1) \tag{2$\cdot$20}$$

if $\mu\geq\mu_{\varphi,m}$.

Let us set

$$B'\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)=\sum_{j=1}^3 b_j'(x,t)\frac{\partial}{\partial x_j}+c(x,t)\frac{\partial}{\partial t}+\overline{d(x,t)}$$
$$+\frac{1}{2}\{\sum_{j=1}^3\partial(n_j(x)-b_j(x,t))/\partial x_j+\partial c(x,t)/\partial t\},$$

$$b_j'(x,t)=2n_j(x)-b_j(x,t).$$

it can be verified easily that

$$(2.21)\quad \left\langle\!\left\langle A_\varphi\left(\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\right)u, v\right\rangle\!\right\rangle - \left\langle\!\left\langle u.\ A_{-\varphi}\left(-\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\right)v\right\rangle\!\right\rangle$$

$$=\left\langle B_\varphi\left(\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\right)u, v\right\rangle - \left\langle u, B'_{-\varphi}\left(-\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\right)v\right\rangle$$

holds for any pair of functions $u, v\in C^\infty(\bar{\Omega}\times \boldsymbol{R}^1)$ such that $\operatorname{supp} u\cap \operatorname{supp} v$ is compact in $\bar{\Omega}\times \boldsymbol{R}^1$. Denote by $\mathcal{W}_\varphi^-(\mu, h; x, t)$ the solution for a boundary data $h\in C_0^\infty(\Gamma\times \boldsymbol{R}^1)$ of the problem

$$\begin{cases} A_{-\varphi}\left(-\dfrac{\partial}{\partial t}+\mu, \dfrac{\partial}{\partial x}\right)w=0 & \text{in } \Omega\times \boldsymbol{R}^1 \\ w=h & \text{on } \Gamma\times \boldsymbol{R}^1 \\ \operatorname{supp} w\subset\bar{\Omega}\times(-\infty, t_0] & \text{for some } t_0<\infty\,. \end{cases}$$

Then it turns out

$$\mathcal{W}_\varphi^-(\mu, h; x, t)=\mathcal{W}_{-\varphi}(\mu, \tilde{h}; x, -t)$$

where $\tilde{h}(x, t)=h(x, -t)$.

Let us set

$$(\boldsymbol{B}_\varphi'^-(\mu)h)(x, -t)=B'_{-\varphi}\left(\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\right)\mathcal{W}_\varphi^-(\mu, h; x, t)\,|_{\Gamma\times\boldsymbol{R}^1}\,.$$

We have from the above remark

$$(\boldsymbol{B}_\varphi'^-(\mu)h)(x, -t)=B'_{-\varphi}\left(\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\right)\mathcal{W}_{-\varphi}(\mu, \tilde{h}; x, t)$$

$$=(\boldsymbol{B}'_{-\varphi}(\mu)\tilde{h})(x, t)\,.$$

For $g, h\in C_0^\infty(\Gamma\times \boldsymbol{R}^1)$ set

$$u(x, t)=\mathcal{W}_\varphi(\mu, g; x, t),\quad v(x, t)=\mathcal{W}_\varphi^-(\mu, h; x, t)$$

and we have

$$(2.22)\qquad \langle \boldsymbol{B}_\varphi(\mu)g, h\rangle=\langle g, \boldsymbol{B}_\varphi^{-\prime}(\mu)h\rangle$$

by substituting these u and v into (2.21) because $\operatorname{supp} u\cap\operatorname{supp} v$ is compact. Since B' satisfies the condition (A-II) and (A-III) it holds that for $\mu\geq\mu_{m,\varphi}$

$$(2.23)\qquad \langle \boldsymbol{B}_\varphi'^-(\mu)h\rangle_m=\langle \boldsymbol{B}'_{-\varphi}(\mu)\tilde{h}\rangle_m\geq\langle\tilde{h}\rangle_m=\langle h\rangle_m,\quad \forall h\in C^\infty(\Gamma\times\boldsymbol{R}^1).$$

Now we have

Lemma 2.3. *Let $m\in\boldsymbol{R}$ be fixed. When μ is large to some extend the equation*

$$(2.24)\qquad \boldsymbol{B}_\varphi(\mu)g=h$$

has a unique solution in $H^m(\Gamma\times\boldsymbol{R}^1)$ *for every* $h\in H^m(\Gamma\times\boldsymbol{R}^1)$, *and*

$$(2.25)\qquad \mathrm{supp}\, h\subset\Gamma\times[t_0,\infty)$$

implies

$$(2.26)\qquad \mathrm{supp}\, g\subset\Gamma\times[t_0,\infty).$$

Proof. First let us show that

$$(2.27)\qquad \{\boldsymbol{B}_\varphi(\mu)g;\, g\in C_0^\infty(\Gamma\times\boldsymbol{R}^1)\}\ \text{is dense in}\ H^m(\Gamma\times\boldsymbol{R}^1)$$

when $\mu\geq\mu_{-m-1,\varphi}$. Suppose that there exists $f\in H^m(\Gamma\times\boldsymbol{R}^1)$ such that

$$\langle\boldsymbol{B}_\varphi(\mu)g, f\rangle_m=0 \qquad \text{for all}\ g\in C_0^\infty(\Gamma\times\boldsymbol{R}^1).$$

Then we have for all $g\in C_0^\infty(\Gamma\times\boldsymbol{R}^1)$

$$\begin{aligned}0&=\langle(|D_x|^2+|D_t|^2+1)^m\boldsymbol{B}_\varphi(\mu)g, f\rangle\\&=\langle g, \boldsymbol{B}_\varphi'^-(\mu)(|D_x|^2+|D_t|^2+1)^m f\rangle,\end{aligned}$$

which implies

$$\boldsymbol{B}_\varphi'^-(\mu)(|D_x|^2+|D_t|^2+1)^m f=0 \qquad \text{in}\ \mathcal{D}'(\Gamma\times\boldsymbol{R}^1).$$

On the other hand $(|D_x|^2+|D_t|^2+1)^m f\in H^{-m}(\Gamma\times\boldsymbol{R}^1)$. Applying Lemma 2.2 we have $(|D_x|^2+|D_t|^2+1)^m f=0$ if $\mu\geq\mu_{-m-1,\varphi}$, from which $f=0$ follows. This shows that (2.27) holds.

Then for $h\in H^m(\Gamma\times\boldsymbol{R}^1)$ there exists a sequence $g_j\in C_0^\infty(\Gamma\times\boldsymbol{R}^1)$, $j=1, 2, 3\cdots$ such that

$$\boldsymbol{B}_\varphi(\mu)g_j\to h \ \text{in}\ H^m(\Gamma\times\boldsymbol{R}^1).$$

By using (2.19) we have

$$\langle g_j-g_l\rangle_m\leq\langle\boldsymbol{B}_\varphi(\mu)g_j-\boldsymbol{B}_\varphi(\mu)g_l\rangle_m\to0 \quad \text{as}\ j, l\to\infty,$$

which shows that g_j converges to some $g\in H^m(\Gamma\times\boldsymbol{R}^1)$ when $j\to\infty$. This implies that $\boldsymbol{B}_\varphi(\mu)g_j$ converges to $\boldsymbol{B}_\varphi(\mu)g$ in $H^{m-1}(\Gamma\times\boldsymbol{R}^1)$. Therefore we have

$$\boldsymbol{B}_\varphi(\mu)g=h.$$

Suppose that (2.25). $\boldsymbol{B}_\varphi^0(\mu)g=h+(\boldsymbol{B}_\varphi^0(\mu)-\boldsymbol{B}_\varphi(\mu))g=h'\in H^{m-1}(\Gamma\times\boldsymbol{R}^1)$ and

$$\mathrm{supp}\, h'\subset\Gamma\times[t_1,\infty)$$

where $t_1=\min(-T, t_0)$. From the consideration on (2.8) we have

$$\operatorname{supp} g \subset \Gamma \times [t_1, \infty).$$

Then for all $\eta \geq 0$ $e^{-\eta(t-t_0)} g \in H^m(\Gamma \times \boldsymbol{R}^1)$. From the definition of $\boldsymbol{B}_\varphi(\mu)$ it follows

$$\boldsymbol{B}_\varphi(\mu+\eta)\,(e^{-\eta(t-t_0)} g) = e^{-\eta(t-t_0)} \boldsymbol{B}_\varphi(\mu)\, g = e^{-\eta(t-t_0)} h\,.$$

Applying (2.19) to $e^{-\eta(t-t_0)} g$ we obtain

$$\langle e^{-\eta(t-t_0)} g \rangle_m \leq \langle e^{-\eta(t-t^0)} h \rangle_m, \quad \forall \eta > 0\,.$$

Here the right-hand side of the inequality rests bounded for all $\eta > 0$ since (2.25) holds. Then we have

$$\langle e^{-\eta(t-t_0)} g \rangle_m \leq C \qquad \text{for all } \eta > 0\,,$$

from which (2.26) follows. Q.E.D.

Proposition 2.1 follows immediately from the above lemmas. Indeed, by using Lemma 2.3 for $h \in H^m(\Gamma \times \boldsymbol{R}^1)$ there exists uniquely a function satisfying

$$\boldsymbol{B}_\varphi(\mu)\, g = e^{-\mu t} h$$

when μ is large to some extend. And $\operatorname{supp} h \subset \Gamma \times [t_1, \infty)$ implies $\operatorname{supp} g \subset \Gamma \times [t_1, \infty)$. It is very easy to verify that a function

$$u(x, t) = e^{\mu t} \mathcal{W}_\varphi(\mu, g; x, t)$$

belongs to $H^m_\varphi(\Omega \times \boldsymbol{R}^1)$ and satisfies (2.10) and (2.11). Lemma 2.2 shows

$$\langle g \rangle_m \leq \langle e^{-\mu t} h \rangle_m = \langle h \rangle_{m,\mu}\,.$$

Applying (2.14) to $\mathcal{W}_\varphi(\mu, g; x, t)$ we have (2.12) from the above inequality. Thus Proposition 2.1 is proved.

We have immediately

Proposition 2.4. *On the condition* (2.1) *the mixed problem* (*P*) *is* C^∞*-well posed and has a finite propagation speed.*

Indeed, we may show the finiteness of propagation speed of (P) through the same reasoning done in § 2 of [3]. On the other hand Proposition 2.1 assures the existence of solution $u(x, t) \in H^m_\mu(\Omega \times (0, \infty))$ of the problem (P_φ) for $\varphi = 0$ for any given data $u_0, u_1 \in H^{m+2}(\Omega)$, $f \in H^{m+2}_\mu(\Omega \times (0, \infty))$ and $h \in H^{m+2}_\mu(\Gamma \times (0, \infty))$ satisfying the compatibility condition. The finiteness of propagation speed and the existence of solution in $H^m_\mu(\Omega \times (0, \infty))$ implies the well posedness in the sense of C^∞.

We set about to prove Theorem 1 with the aid of Proposition 2.4. Let $\chi(t)$ be a real valued function in $C^\infty(\boldsymbol{R}^1)$ verifying

$$\chi(t)=\begin{cases}1 & \text{for } t\leq 1\\ 0 & \text{for } t\geq 2.\end{cases}$$

For $a(x, t)\in C^{\infty}(\Gamma\times \boldsymbol{R}^1)$ we set

$$a^{(N)}(x, t)=a(x, 0)+(a(x, t)-a(x, 0))\chi(|t|/N).$$

Evidently we have $a^{(N)}\in C^{\infty}(\Gamma\times \boldsymbol{R}^1)$ and

$$a^{(N)}(x, t)=\begin{cases}a(x, t) & \text{for } |t|\leq N\\ a(x, 0) & \text{for } |t|\geq 2N.\end{cases}$$

Let us denote by B^N an operator made by replacing the coefficients b_j, c, d of B by $b_j^{(N)}$, $c^{(N)}$, $d^{(N)}$ respectively. Consider the problem

$$(\mathrm{P}^N)\qquad \begin{cases}\Box u=f & \text{in } \Omega\times(0, \infty)\\ B^N u=g & \text{on } \Gamma\times(0, \infty)\\ u(x, 0)=u_0 & \\ \dfrac{\partial u}{\partial t}(x, 0)=u_1. & \end{cases}$$

Since B^N satisfies the condition (2.1) (P^N) is well posed in the sense of C^{∞}. Suppose that given data (u_0, u_1, f, g) satisfy the compatibility condition for (P). Then they satisfy also the compatibility condition for (P^N) for all $N\geq 1$. Therefore Proposition 2.4 assures the existence of the solution $u^N(x, t)\in C^{\infty}(\bar{\Omega}\times[0, \infty))$ of the problem (P^N).

Let $M>N$ and set $v(x, t)=u^M(x, t)-u^N(x, t)$. It satisfies

$$\begin{cases}\Box v=0 & \text{in } \Omega\times(0, \infty)\\ v(x, 0)=0 & \\ \dfrac{\partial v}{\partial t}(x, 0)=0. & \end{cases}$$

And we have $B^M v=-(B^M-B^N)u^N=g_N\in C^{\infty}(\Gamma\times[0, \infty))$ and

$$g_N=0 \quad \text{for } t\leq N$$

since B^N and B^M coinside with for $|t|\leq N$. The finiteness of propagation speed shows that $v=0$ for $t\leq N$, i.e.

$$u^M=u^N \quad \text{for } t\leq N.$$

Then we may define a function $u(x, t)$ by

$$u(x, t)=\lim_{N\to\infty} u^N(x, t),$$

which belongs to $C^{\infty}(\bar{\Omega}\times[0, \infty))$ and $u(x, t)=u^N(x, t)$ for $t\leq N$. Therefore it holds that

$$\begin{cases} \square u = f & \text{in } \Omega \times (0, N) \\ Bu = g & \text{on } \Gamma \times (0, N) \\ u(x, 0) = u_0 \\ \dfrac{\partial u}{\partial t}(x, 0) = u_1 \end{cases}$$

for any N. Then $u(x, t)$ is the required solution of (P). The continuity of the mapping $(u_0, u_1, f, g) \to u(x, t)$ from $C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega} \times [0, \infty)) \times C^\infty(\Gamma \times [0, \infty))$ into $C^\infty(\bar{\Omega} \times [0, \infty))$ follows immediately from that of $(u_0, u_1, f, g) \to u^N(x, t)$. And the finiteness of propagation speed of (P) is also derived from that of (P^N). Thus Theorem 1 is proved.

§ 3. Preparations for proofs of the exponential decay of solutions

In order to consider the exponential decay of solutions we prepare some results. Let the origin 0 of $\boldsymbol{R}^3 \in \mathcal{O}$. Set for $\delta > 0$

$$\Delta_\delta = \Delta + 2\delta \frac{\partial}{\partial |x|} + \delta^2 + \frac{2\delta}{|x|}$$

$$N^{(\delta)} = \frac{\partial}{\partial n} + \delta \frac{1}{|x|} \sum_{j=1}^{3} n_j(x) x_j$$

and we have

$$e^{-\delta|x|} \Delta u(x) = \Delta_\delta (e^{-\delta|x|} u)$$

$$e^{-\delta|x|} \frac{\partial u}{\partial n}\bigg|_\Gamma = N^{(\delta)}(e^{-\delta|x|} u).$$

A boundary value problem for a data $g \in C^\infty(\Gamma \times \boldsymbol{R}^1)$ satisfying $\operatorname{supp} g \subset \Gamma \times [a, \infty)$

$$\begin{cases} \left(\left(\dfrac{\partial}{\partial t} + \mu\right)^2 - \Delta_\delta\right) w(x, t) = 0 & \text{in } \Omega \times \boldsymbol{R}^1 \\ w(x, t) = g(x, t) & \text{on } \Gamma \times \boldsymbol{R}^1 \\ \operatorname{supp} w \subset \bar{\Omega} \times [a, \infty) \end{cases}$$

has a unique solution in $C^\infty(\bar{\Omega} \times \boldsymbol{R}^1)$. Denote this solution by $\mathcal{W}^{(\delta)}(\mu, g; x, t)$. For every $g \in C^\infty(\Gamma \times \boldsymbol{R}^1)$ an estimate

$$\text{(3.1)} \qquad |||\mathcal{W}^{(\delta)}(\mu, g; \cdot, t)|||_m \leq \frac{C_m}{\delta_0 + \mu} [g]_m, \qquad \forall t \in (-\infty, \infty)$$

holds if $\delta \geq \delta_0 + 1$, $\mu > -\delta_0$, which is a consequence of considerations of Morawets in [5] and follows immediately from Proposition 3.2 of [4].

Hereafter we fix $\delta = \delta_0 + 1$. Let us set

$$\boldsymbol{N}^{(\delta)}(\mu) g = N^{(\delta)} \mathcal{W}^{(\delta)}(\mu, g; x, t)|_{\Gamma \times \boldsymbol{R}^1}$$

$$\boldsymbol{B}^{(\delta)}(\mu)g = B^{(\delta)}(\mu)\mathcal{W}^{(\delta)}(\mu, g; x, t)|_{\Gamma\times\boldsymbol{R}^1}$$

where

$$B^{(\delta)}(\mu) = B\Big(\frac{\partial}{\partial t}+\mu, \frac{\partial}{\partial x}\Big)+\delta\frac{1}{|x|}\sum_{j=1}^{3} b_j(x, t)x_j .$$

From (3.1) the operators $\boldsymbol{N}^{(\delta)}(\mu)$ and $\boldsymbol{B}^{(\delta)}(\mu)$ may be extended to continuous mappings from $H^{m+1}(\Gamma\times\boldsymbol{R}^1)$ into $H^m(\Gamma\times\boldsymbol{R}^1)$. And by using (3.1) once more we can define for each $h\in C^\infty(\Gamma)$ an $H^m(\Omega)$-valued analytic function $U^{(\delta)}(p, h; x)$ in $\mathrm{Re}\, p>-\delta_0$ such that

$$\begin{cases} (p^2-\Delta_\delta)U^{(\delta)}(p, h; x) = 0 & \text{in } \Omega \\ U^{(\delta)}(p, h; x) = h & \text{on } \Gamma \end{cases}$$

and an estimate

$$|||U^{(\delta)}(p, h; x)|||_m \leq \frac{C_m}{\mathrm{Re}\, p+\delta_0}\|h\|_m ,$$

holds for all $h\in C^\infty(\Gamma)$ and $\mathrm{Re}\, p>-\delta_0$.[2)]

Define an operator $\mathcal{N}^{(\delta)}(p)$ by

$$\mathcal{N}^{(\delta)}(p)h = N^{(\delta)}U^{(\delta)}(p, h; x)|_\Gamma$$

and when all the coefficients of B are independent of t, $\mathcal{B}^{(\delta)}(p)$ by

$$\mathcal{B}^{(\delta)}(p)h = \Big(B\Big(ik+\mu, \frac{\partial}{\partial x}\Big)+\delta\frac{1}{|x|}\sum_{j=1}^{3} b_j(x)x_j\Big)U^{(\delta)}(p, h; x)|_\Gamma .$$

Then $\mathcal{N}^{(\delta)}(p)$ and $\mathcal{B}^{(\delta)}(p)$ are defined in $\mathrm{Re}\, p>-\delta_0$ and they are continuous mapping from $H^{m+1}(\Gamma)$ into $H^m(\Gamma)$.

Proposition 3.1. *Consider an equation in $\Gamma\times\boldsymbol{R}^1$ with a parameter μ*

$$\boldsymbol{B}^{(\delta)}(\mu)g = h . \tag{3.2}$$

If there exists $\mu<0$ such that for any $h\in C_0^\infty(\Gamma\times\boldsymbol{R}^1)$ verifying $\mathrm{supp}\, h \subset\Gamma\times(0, \infty)$ (3.2) *has a unique solution in* $\bigcap_{m\geq1} H^m(\Gamma\times\boldsymbol{R}^1)$ *satisfying* $\mathrm{supp}\, g\subset\Gamma\times[0, \infty)$ *and for some $l>0$*

$$[g]_m \leq C_m[h]_{m+l} , \tag{3.3}$$

the solution of (P) decays exponentially, i.e. when $u_0, u_1\in C_0^\infty(\bar{\Omega})$ and $\bigcup \mathrm{supp}\, u_j\subset\{x; |x|\leq\kappa\}$, *the solution $u(x, t)$ of (P) for initial data u_0, u_1 satisfies the inequality*

$$E_m(t, u, R)\leq C_m e^{2\delta R}\cdot e^{2\kappa(-\mu)}\cdot e^{2\mu t}\{|||u_0|||^2_{m+l+1}+|||u_1|||^2_{m+l}\}$$

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2) See §2 of [4].

for all $t \geq 0$, $m=1, 2, \cdots$.

Proof. Take $v_0, v_1 \in C_0^\infty(\boldsymbol{R}^3)$ in such a way

$$u_j(x) = v_j(x) \qquad \text{in } \Omega,\ j=0, 1\,.$$

Let $v(x, t)$ be the solution of

$$\begin{cases} \Box v = 0 & \text{in } \boldsymbol{R}^3 \times (0, \infty) \\ v(x, 0) = v_0(x) \\ \dfrac{\partial v}{\partial t}(x, 0) = v_j(x)\,. \end{cases}$$

Set

$$h(x, t) = -Bv(x, t)\,|_{\Gamma \times \boldsymbol{R}^1}$$

and we have

$$h \in C_0^\infty(\Gamma \times \boldsymbol{R}^1)\,,\ \operatorname{supp} h \subset \Gamma \times [0, \infty)$$

since $v_j(x)$ has compact support. Then

$$e^{-\mu t} e^{-\delta|x|} h(x, t) \in C_0^\infty(\Gamma \times \boldsymbol{R}^1)\,.$$

From the assumption on the equation (3. 2) we have

$$g(x, t) \in \bigcap_{m \geq 1} H^m(\Gamma \times \boldsymbol{R}^1)\,,\ \operatorname{supp} g \subset \Gamma \times [0, \infty)$$

satisfying

$$\boldsymbol{B}^{(\delta)}(\mu) g = e^{-\mu t} e^{-\delta|x|} h(x, t)\,.$$

By combining (3. 3) and (3. 1) we have that a function $\tilde{w}(x, t) = \mathcal{W}^{(\delta)}(\mu, g; x, t)$ satisfies

$$\text{(3. 4)} \qquad |||\tilde{w}(\cdot, t)|||_{m-1}^2 + \left\|\frac{\partial \tilde{w}}{\partial t}(\cdot, t)\right\|_m^2 \leq \left(\frac{C_m}{\delta_0 + \mu}\right)^2 [e^{-\mu t} e^{-\delta|x|} h]_{m+l}^2\,.$$

On the other hand it follows directly from the definitions of $\mathcal{W}^{(\delta)}$ and $\boldsymbol{B}^{(\delta)}$ that

$$w(x, t) = e^{\mu t} e^{\delta|x|} \tilde{w}(x, t)$$

verifies

$$\begin{cases} \Box w(x, t) = 0 & \text{in } \Omega \times \boldsymbol{R}^1 \\ Bw(x, t) = -h & \text{on } \Gamma \times \boldsymbol{R}^1 \\ \operatorname{supp} w \subset \bar{\Omega} \times [0, \infty)\,. \end{cases}$$

Now the form of w and the inequality (3. 4) give an estimate

$$(3.5)\quad E_{m+1}(w, R, t) = \sum_{|\alpha|\leq m+1}\int_{\Omega_R}|D_x{}^{\alpha}w(x,t)|^2dx+\sum_{|\alpha|\leq m}\int_{\Omega_R}|D_tD_x{}^{\alpha}w(x,t)|^2dx$$
$$\leq e^{2\mu t}e^{2\delta R}\left(|||\tilde{w}(x,t)|||^2_{m+1}+\left|\left|\left|\frac{\partial\tilde{w}}{\partial t}(x,t)\right|\right|\right|^2_m\right)$$
$$\leq e^{2\mu t}e^{2\delta R}[e^{-\mu t}e^{-\delta|x|}h]^2_{m+l+1}.$$

The condition $\bigcup_{j=0}^{1}\operatorname{supp} u_j\subset\{x;|x|<\kappa\}$ implies that

$$\operatorname{supp} h\subset\Gamma\times[0,\kappa+\delta_0]$$

because

$$(3.6)\qquad v(x,t)=0\quad\text{for } |x|\leq t-\kappa.$$

Therefore

$$(3.7)\qquad [e^{-\mu t}e^{-\delta|x|}h]_{m+l+1}$$
$$\leq C_m(|||u_0|||_{m+l+2}+|||u_1|||_{m+l+1})e^{-\mu(\kappa+\delta_0)}.$$

Note that the solution u of (P) is represented as

$$u=v+w.$$

Then from (3.5), (3.6) and (3.7) we obtain the desired energy estimate.

Q.E.D.

Next consider the case where the coefficients of B are independent of t. Admit the following theorem, whose proof will be given in § 6.

Theorem 3.2. *For any $\varepsilon>0$ and s real there exists a constant $C_{s,\varepsilon}$ such that an estimate*

$$(3.8)\qquad -\operatorname{Re}(e^{2\delta|x|}\mathcal{N}^{(\delta)}(p)g,g)_s$$
$$\geq(e^{2\delta|x|}(\mu+H(x)-\varepsilon)g,g)_s-\frac{C_{s,\varepsilon}}{|p|+1}\|e^{\delta|x|}g\|^2_s$$

holds for all $g\in C^\infty(\Gamma)$ and $\operatorname{Re}p>-\delta_0$.

Proposition 3.3. *Suppose that all the coefficients of B are independent of t. If there exists $\gamma>0$ such that*

$$(3.9)\qquad \|e^{\delta|x|}\mathcal{B}^{(\delta)}(p)g\|_0\geq c_0\|e^{\delta|x|}g\|_0,\quad c_0>0$$

holds for all $g\in C^\infty(\Gamma)$ and $\operatorname{Re}p>-\gamma$, *the solution of (P) has an estimate*

$$(3.10)\qquad E_m(t,u,R)\leq C_me^{2\delta\kappa}\cdot e^{2\delta R}\cdot e^{-2\gamma t}E_{m+1}(0,u,\infty).$$

Proof. As we see it in [3] and [4], $\mathscr{B}^{(\delta)}(p)$ is a pseudodifferential operator in Γ with the following property:

Let I_σ be a open neighborhood of the origin of $\boldsymbol{R}^2$ and Γ_0 is a neighborhood of $s_0 \in \Gamma$, and $I_\sigma \ni (\sigma_1, \sigma_2) \rightarrow s(\sigma) \in \Gamma_0$ be a local coordinate patch. Denote the symbol of $\mathscr{B}^{(\delta)}(p)$ by $\mathscr{B}^{(\delta)}(s, \xi; p)$ and we have

$$|\mathscr{B}^{(\delta)}(s, \xi; p)| \geq C|\xi| \quad \text{for all } |\xi| \geq 2|p|,$$

where $\xi = (\xi_1, \xi_2)$ are dual variables of σ. Therefore it holds that

$$\|g\|_{s+1} \leq c\|\mathscr{B}^{(\delta)}(p) g\|_s + C_{p,s}\|g\|_s, \quad \forall g \in C^\infty(\Gamma) \tag{3.11}$$

where c is a positive constant independent of p and s. With the aid of an estimate

$$|\mathrm{Re}\,(e^{2\delta|x|}\mathscr{B}^{(\delta)}(p) g, g)_s - \mathrm{Re}\,(e^{2\delta|x|}\mathscr{N}^{(\delta)}(p) g, g)_s| \leq C_s\|g\|_s^2$$

it follows from Theorem 3.2

$$-\mathrm{Re}\,(e^{2\delta|x|}(\mathscr{B}^{(\delta)}(p) - \lambda) g, g)_s \geq \|e^{\delta|x|}g\|_s^2 \tag{3.12}$$

for a large positive number λ. This shows that, for each s and p fixed $(\mathscr{B}^{(\delta)}(p) - \lambda)^{-1}$ exists as an operator $H^s(\Gamma) \rightarrow H^s(\Gamma)$ when λ is sufficiently large. Moreover taking account of (3.11) we see that $(\mathscr{B}^{(\delta)}(p) - \lambda)^{-1}$ is a continuous mapping from $H^s(\Gamma)$ into $H^{s+1}(\Gamma)$. Since Γ is compact $(\mathscr{B}^{(\delta)}(p) - \lambda)^{-1}$ is a completely continuous operator in $H^s(\Gamma)$. By the theory of Riesz-Schauder in order to show the existence of $\mathscr{B}^{(\delta)}(p)^{-1}$ it suffices to verify that $I + \lambda(\mathscr{B}^{(\delta)}(p) - \lambda)^{-1}$ is injective. Suppose that there exists $g \in H^s(\Gamma)$ verifying

$$\{I + \lambda(\mathscr{B}^{(\delta)}(p) - \lambda)^{-1}\} g = 0.$$

Then we have $g \in \bigcap_{m \geq 1} H^m(\Gamma)$ and $\mathscr{B}^{(\delta)}(p) g = 0$. From the assumption (3.9) $\mathrm{Re}\,p \geq -\gamma$ implies $g = 0$, which assures the existence of $\mathscr{B}^{(\delta)}(p)^{-1}$ in $H^s(\Gamma)$ for all s. From (3.9) we have $\|\mathscr{B}^{(\delta)}(p)^{-1}\| \leq c_0$ for $\mathrm{Re}\,p \geq -\gamma$. Now by using (3.11) and (3.12) we have for all positive integer m

$$\|\mathscr{B}^{(\delta)}(p)\|_{\mathscr{L}(H^m(\Gamma), H^m(\Gamma))} \leq C_m, \quad \mathrm{Re}\,p \geq -\gamma. \tag{3.13}$$

Let $h(x, t) \in H^m(\Gamma \times \boldsymbol{R}^1)$, $\mathrm{supp}\,h \subset \Gamma \times [0, \infty)$. Set

$$\hat{h}(x, ik) = \int_{-\infty}^{\infty} e^{-ikt} h(x, t)\, dt$$

and we have

$$\sum_{j \leq m} \int_{-\infty}^{\infty} |k|^{2j} \|\hat{h}(\cdot, ik)\|_{m-j}^2 dk < +\infty.$$

Define $g(x, t)$ by

$$g(x, t) = \int_{-\infty}^{\infty} e^{ikt}\mathscr{B}^{(\delta)}(-\gamma + ik)^{-1}\hat{h}(x, ik)\, dk.$$

Then $g\in H^m(\Gamma\times \boldsymbol{R}^1)$ and $\mathrm{supp}\, g\subset\Gamma\times[0,\infty)$. Moreover it holds that

$$\boldsymbol{B}^{(\delta)}(-\gamma)g=h$$

and $[g]_m\leq C_m[h]_m$. Thus we may apply Proposition 3.1 to this case by taking $l=0$ and (3.10) follows.

Corollary. *Let B_0 be a boundary operator with coefficients independent of t. Suppose that B_0 satisfies the assumption of Proposition* 3.3 *and that $B=B_0$ for $|t|\geq T$. Then the equation*

$$\boldsymbol{B}^{(\delta)}(-\gamma)g=h$$

for $h\in C_0^\infty(\Gamma\times(0,\infty))$ has a solution $g\in\bigcap_{m\geq 1}H^m(\Gamma\times\boldsymbol{R}^1)$ satisfying

$$\mathrm{supp}\, g\subset\Gamma\times[0,\infty)$$

and

$$[g]_m\leq C_m[h]_{m+1},\ m=1,2,\cdots.$$

Proof. Let us decompose h as $h=h_1+h_2$, $\mathrm{supp}\, h_1\subset\Gamma\times[0,T+1]$ and $\mathrm{supp}\, h_2\subset\Gamma\times[T,\infty)$. Consider the solution of the problem

$$\begin{cases}\Box u=0 & \text{in } \Omega\times(0,\infty)\\ Bu=e^{\delta|x|}\cdot e^{-\gamma t}h_1 & \text{on } \Gamma\times(0,\infty)\\ u(x,0)=0 & \\ \dfrac{\partial u}{\partial t}(x,0)=0\,. & \end{cases}$$

From Theorem 1 we have always $u(x,t)$ in $C^\infty(\bar{\Omega}\times\boldsymbol{R}^1)$ and there exists a constant R independent of h such that

$$\mathrm{supp}\, u(\cdot,T+1)\cup\mathrm{supp}\frac{\partial u}{\partial t}(\cdot,T+1)\subset\bar{\Omega}_R\,. \tag{3.14}$$

Take $v_j\in C^\infty(\boldsymbol{R}^3)$, $j=0,1$ in such a way

$$v_0(x)=u(x,T+1)\qquad\text{for } x\in\Omega$$

$$v_1(x)=\frac{\partial u}{\partial t}(x,T+1)\quad\text{for } x\in\Omega$$

From the consideration in §2 it holds that

$$|||v_0(x)|||_{m+1}+|||v_1(x)|||_m\leq C_m[h_1]_{m+1}\leq C_m[h]_{m+1}\,. \tag{3.15}$$

Let us denote by $v(x,t)$ the solution of

$$\begin{cases} \square v(x, t) = 0 & \text{in } \boldsymbol{R}^3 \times (T+1, \infty) \\ v(x, T+1) = v_0(x) \\ \dfrac{\partial v}{\partial t}(x, T+1) = v_1(x). \end{cases}$$

It follows from (3.14) that

$$v(x, t) = 0 \quad \text{for } |x| \leq t - T - 1 - R,$$

which implies

$$Bv(x, t) = 0 \quad \text{for } t \geq T + 2R + 1 + \rho_0.$$

Set

$$w(x, t) = \begin{cases} u(x, t) & \text{for } (x, t) \in \bar{\Omega} \times [0, T+1] \\ v(x, t) & \text{for } (x, t) \in \bar{\Omega} \times (T+1, \infty). \end{cases}$$

We see that $w(x, t) \in C^\infty(\bar{\Omega} \times [0, \infty))$ and

$$Bw = \begin{cases} e^{\delta|x|} \cdot e^{-\gamma t} h_1 & \text{for } t \geq T+1 \\ 0 & \text{for } t \geq T + 1 + \delta_0 + R. \end{cases}$$

If we set $g_1(x, t) = e^{-\delta|x|} \cdot e^{\gamma t} w(x, t)|_{\Gamma \times \boldsymbol{R}^1}$ we have

$$e^{-\delta|x|} \cdot e^{\gamma t} \cdot Bw = \boldsymbol{B}^{(\delta)}(-\gamma) g_1.$$

Set

$$h_3 = -\boldsymbol{B}^{(\delta)}(-\gamma) g_1 + h$$

and we have $h_3 \in C_0^\infty(\Gamma \times \boldsymbol{R}^1)$, $\operatorname{supp} h_3 \subset \Gamma \times (T, \infty)$ and

$$[h_3]_m \leq C_m [h]_{m+1}. \tag{3.16}$$

From the assumption on B_0 the equation

$$\boldsymbol{B}_0^{(\delta)}(-\gamma) g_2 = h_3$$

has a solution $g_2 \in \bigcap_{m \geq 1} H^m(\Gamma \times \boldsymbol{R}^1)$ satisfying

$$\operatorname{supp} g_2 \subset \Gamma \times [T, \infty) \tag{3.17}$$

and

$$[g_2]_m \leq C_m [h_3]_m. \tag{3.18}$$

Let us set

$$g(x, t) = g_1(x, t) + g_2(x, t).$$

Evidently $g \in \cap H^m(\Gamma \times \boldsymbol{R}^1)$ and $\operatorname{supp} g \subset \Gamma \times [0, \infty)$. Taking account that $\boldsymbol{B}^{(\delta)}(-\gamma) g_2 = \boldsymbol{B}_0^{(\delta)} g_2$ we have

$$\boldsymbol{B}^{(\delta)}(-\gamma)g=\boldsymbol{B}^{(\delta)}(-\gamma)g_1+\boldsymbol{B}_0^{(\delta)}(-\gamma)g_2=h\,.$$

The required estimate for g follows immediately from (3.15), (3.16) and (3.17).

§ 4. Proof of Theorem 2

Taking account that B may be expressed as

$$B=\frac{\partial}{\partial n}+\frac{1}{2}\sum_{j=1}^{3}\left\{\frac{\partial}{\partial x_j}(b_j-n_j)+(b_j-n_j)\frac{\partial}{\partial x_j}\right\}$$
$$+\frac{1}{2}\left\{\frac{\partial}{\partial t}c(x,t)+c(x,t)\frac{\partial}{\partial t}\right\}+d(x,t),$$

we have for all $g\in C_0^\infty(\Gamma\times\boldsymbol{R}^1)$

$$-\mathrm{Re}\,\langle e^{2\delta|x|}\boldsymbol{B}^{(\delta)}(\mu)g,g\rangle_0=-\mathrm{Re}\,\langle e^{2\delta|x|}\boldsymbol{N}^{(\delta)}(\mu)g,g\rangle_0$$
$$-\mu\langle e^{2\delta|x|}c(x,t)g,g\rangle_0-\mathrm{Re}\,\langle e^{2\delta|x|}d(x,t)g,g\rangle_0\,.$$

Moreover we have

$$-\mathrm{Re}\,\langle e^{2\delta|x|}\boldsymbol{N}^{(\delta)}(\mu)g,g\rangle_0$$
$$=\int_{-\infty}^{\infty}\{-\mathrm{Re}\,(e^{2\delta|x|}\mathscr{N}^{(\delta)}(\mu+ik)\hat{g}(\cdot,ik),\hat{g}(\cdot,ik))\}dk$$
$$\geq(\mu-C)\int_{-\infty}^{\infty}\|\hat{g}(\cdot,ik)\|^2dk=(\mu-C)\,[g]_0^2\,.$$

Then by setting

$$d_1=\inf(-\mathrm{Re}\,d(x,t)),\quad c_1=\sup c(x,t)$$

we have an estimation

$$-\mathrm{Re}\,\langle\boldsymbol{B}^{(\delta)}(\mu)g,g\rangle\geq\{(1-c_1)\mu+(d_1-C)\}\,[g]_0^2\,. \tag{4.1}$$

Lemma 4.1. *Let us denote by M a bounded subset of $\mathscr{B}^\infty(\Gamma)$. For a pair of M and a positive constant γ we can choose positive constant α_m and C_m, $m=1,2,\cdots$ with the following properties: for every B satisfying*

$$\{b_j(\cdot,t),c(\cdot,t);t\in\boldsymbol{R}^1\}\subset M \tag{4.2}$$

and

$$\sum_{j=1}^{3}\left|\frac{\partial b_j}{\partial t}\right|+\left|\frac{\partial c}{\partial t}\right|\leq\alpha_m\qquad\text{for all }\ (x,t)\in\boldsymbol{\Gamma}\times\boldsymbol{R}^1 \tag{4.3}$$

an estimate

$$(4.4) \qquad [g]_m \leq C_m [\boldsymbol{B}^{(\delta)}(\mu) g]_m, \quad \forall g \in C_0^\infty(\Gamma \times \boldsymbol{R}^1)$$

holds for all μ *verifying*

$$(4.5) \qquad (1-c_1)\mu + d_1 - C \geq \gamma .$$

Proof. For $m=0$ (4.4) follows from (4.1) on the condition (4.5). Note that when B satisfies (4.2) we have estimates

$$(4.6) \qquad [g]_{m+1} \leq C_{M,m}\left\{[\boldsymbol{B}^{(\delta)}(\mu) g]_m + \left[\frac{\partial g}{\partial t}\right]_m + [g]_m\right\}, \qquad \forall g \in C_0^\infty(\Gamma \times \boldsymbol{R}^1)$$

for all $\mu > -\delta_0$ and $m=0, 1, 2, \cdots$, which are derived by the same process as (3.11). For any $g \in C_0^\infty(\Gamma \times \boldsymbol{R}^1)$ it holds that

$$\boldsymbol{B}^{(\delta)}(\mu)\frac{\partial g}{\partial t} = \frac{\partial}{\partial t}(\boldsymbol{B}^{(\delta)}(\mu) g) - \sum_{j=1}^{3} \frac{\partial (b_j - n_j)}{\partial t} \frac{\partial g}{\partial x_j}$$
$$- \frac{\partial c}{\partial t} \cdot \frac{\partial g}{\partial t} - \left\{\sum_{j=1}^{3}\left(\frac{\partial^2 b_j}{\partial t \partial x_j} + \delta \frac{\partial^2 b_j}{\partial t \partial x_j} \frac{x_j}{|x|}\right) - \frac{\partial d}{\partial t}\right\} g ,$$

from which we have

$$\left[\boldsymbol{B}^{(\delta)}(\mu)\frac{\partial g}{\partial t}\right]_m \leq \left[\frac{\partial}{\partial t}\boldsymbol{B}^{(\delta)}(\mu) g\right]_m + C'\alpha [g]_{m+1} + C''[g]_m$$

where $\alpha = \sup\left(\sum_{j=1}^{3}\left|\frac{\partial b_j}{\partial t}\right| + \left|\frac{\partial c}{\partial t}\right|\right)$. Suppse that (4.4) holds when $\alpha \leq \alpha_m$. Then it follows that

$$\left[\frac{\partial g}{\partial t}\right]_m \leq C_m \left[\boldsymbol{B}^{(\delta)}(\mu)\frac{\partial g}{\partial t}\right]_m$$
$$\leq C_m \left\{\left[\frac{\partial}{\partial t}(\boldsymbol{B}^{(\delta)}(\mu) g\right]_m + C'\alpha [g]_{m+1} + C''[g]_m\right\}.$$

The substitution this inquality into (4.6) gives

$$[g]_{m+1} \leq C'_m \{[\boldsymbol{B}^{(\delta)}(\mu) g]_{m+1} + \alpha [g]_{m+1} + [\boldsymbol{B}^{(\delta)}(\mu) g]_m\}.$$

Therefore we see that (4.4) holds for $m+1$ if α is sufficinetly small. Since (4.4) holds for $m=0$ on the condition (4.5) Lemma is proved by the induction. Q.E.D.

Corollary. *Let* M *be a bounded subset of* $C^\infty(\Gamma)$. *Suppose that* B *satisfies* (4.2) *and*

$$\sum_{j=1}^{3} |\partial b_j(x, t)/\partial t| + |\partial c(x, t)/\partial t| \leq \alpha_m , \quad \forall (x, t) \subset \Gamma \times [T_m, \infty),$$

where T_m, $m=1, 2, \cdots$ *are positive constants such that* $T_m \leq T_{m+1}$. *Then* *for μ satisfying*

$$(1-c_1)\mu+\tilde{d}_1-C\geq\gamma, \quad \tilde{d}_1=\inf_{(x,t)\in\Gamma\times[T_1,\infty)}(-\mathrm{Re}\, d(x,t))$$

we have an estimate

$$[g]_m\leq C_m[\boldsymbol{B}^{(\delta)}(\mu)g]_{m+1}, \quad \forall g\in C_0^\infty(\Gamma\times\boldsymbol{R}^1). \tag{4.7}$$

Proof. Let us set $\boldsymbol{B}^{(\delta)}(\mu)g=h$. Following the process of Corollary of Proposition 3.3 we can decompose g as $g=g_1+g_2$ in such a way that

$$[g_1]_m\leq C_m[h]_m \tag{4.8}$$

and $\boldsymbol{B}^{(\delta)}(\mu)g_1=h$ for $t\leq T_m+1$, and $h_3=-\boldsymbol{B}^{(\delta)}(\mu)g_1+h$ verifies

$$[h_3]_m\leq C_m[h]_{m+1} \tag{4.9}$$

and

$$\mathrm{supp}\, h_3\subset\Gamma\times[T_m,\infty).$$

Let us denote by $\tilde{B}$ an operator satisfying $\tilde{B}=B$ for $t\geq T_m$ and the condition on the coefficients of Lemma 4.1 for all $t\in\boldsymbol{R}^1$. Then by Lemma 4.1 we have g_2 such that

$$\tilde{\boldsymbol{B}}^{(\delta)}(\mu)g_2=h_3 \tag{4.10}$$

$$\mathrm{supp}\, g_2\subset\Gamma\times[T_m,\infty)$$

$$[g_2]_m\leq C_m[h_3]_m\,. \tag{4.11}$$

Since $\tilde{\boldsymbol{B}}^{(\delta)}(\mu)g_2=\boldsymbol{B}^{(\delta)}(\mu)g_2$, we have that $g=g_1+g_2$ from the uniqueness of solutions of (4.10). Now (4.7) follows from (4.8) and (4.11). Q.E.D.

Lemma 4.2. *Suppose that B satisfies* (1.4). *For any μ satisfying*

$$(1-c_1)\mu+\tilde{d}_1-C>0\,, \tag{4.12}$$

the equation

$$\boldsymbol{B}^{(\delta)}(\mu)g=h$$

for $h\in H^{m+1}(\Gamma\times\boldsymbol{R}^1)$ such that $\mathrm{supp}\, h\subset\Gamma\times[0,\infty)$ *has a solution uniquely in $H^m(\Gamma\times\boldsymbol{R}^1)$ and which verifies*

$$[g]_m\leq C_m[h]_{m+1} \tag{4.13}$$

$$\mathrm{supp}\, g\subset\Gamma\times[0,\infty).$$

Proof. let B^N be the operator introduced in § 2. Since the coefficients of B^N are independent of t for $t\geq 2N$ we can apply Corollary of Proposition 3.3 and obtain g_N verifying

$$\boldsymbol{B}^{N(\delta)}(\mu)\, g_N = h$$

and $g_N \in H^m(\Gamma \times \boldsymbol{R}^1)$, supp $g_N \subset \Gamma \times [0, \infty)$. We see immediately that

$$\{b_{Nj}(\cdot, t), c_N(\cdot, t); t \in \boldsymbol{R}^1\} \text{ is a bounded set in } C^\infty(\Gamma)$$

and that for any $\alpha > 0$ it holds

$$\sum_{j=1}^{3} |\partial b_{Nj}(x\ t)/\partial t| + |\partial c_N(x, t)/\partial t| < \alpha, \quad {}^\forall t \geq T_\alpha$$

where T_α is independent of N. Hence applying Corollary of Lemma 4. 1 we have

$$[g_N]_m \leq C_m [h]_{m+1}, \tag{4. 14}$$

where C_m is independent of N. Let $N' > N$. Then

$$\boldsymbol{B}^{N'(\delta)}(\mu)\,(g_{N'} - g_N) = -(\boldsymbol{B}^{N'(\delta)}(\mu) - \boldsymbol{B}^{N(\delta)}(\mu))\, g_N,$$

and since supp $(\boldsymbol{B}^{N'(\delta)}(\mu) - \boldsymbol{B}^{N(\delta)}(\mu))\, g_N \subset \Gamma \times [N, \infty)$ we have

$$\text{supp}\,(g_{N'} - g_N) \subset \Gamma \times [N, \infty)$$

namely

$$g_{N'} = g_N \quad \text{for } t \leq N. \tag{4. 15}$$

Now (4. 14) and (4. 15) imply that g_N converges to some element in $H^m(\Gamma \times \boldsymbol{R}^1)$ when N tends to infinity. Let us denote it by g. Then it holds that

$$g = g_N \quad \text{for } t \leq N$$

$$[g]_m \leq C_m [h]_{m+1}.$$

We have

$$\boldsymbol{B}^{(\delta)}(\mu) = h$$

because

$$\boldsymbol{B}^{(\delta)}(\mu)\, g - h = \boldsymbol{B}^{(\delta)}(g - g_N) + (\boldsymbol{B}^{(\delta)} - \boldsymbol{B}^{N(\delta)})\, g_N$$

and the right-hand side equals zero for $t \leq N$. Thus Lemma 4. 2 is proved. Q.E.D.

If d_0 is choosen in such a way that (1. 3) implies (4. 12) for some $\mu < 0$ and T_1, the exponential decay of solutions of (P) follows from Lemma 4. 2 and Proposition 3. 1. Thus Theorem 2 is proved.

§ 5. Proofs Theorems 3, 4 and 5

Lemma 5. 1. *Suppose that an estimate*

$$|(e^{2\delta|x|}\mathscr{B}^{(\delta)}(p)g, g)| \geq \left(c - \frac{C}{|p|+1}\right)\|e^{\delta|x|}g\|^2 \tag{5.1}$$

holds for all $\operatorname{Re} p \geq -\gamma$ *and* $g \in C^\infty(\Gamma)$, *where* c *and* γ *are positive constants. If*

$$(e^{2\delta|x|}\mathscr{B}^{(\delta)}(p)g, g) \neq 0 \quad \text{for all } \operatorname{Re} p \geq 0 \tag{5.2}$$

for any $g \neq 0$, *there exist* γ' *and* c' *positive constants such that*

$$|(e^{2\delta|x|}\mathscr{B}^{(\delta)}(p)g, g)| \geq c'\|e^{\delta|x|}g\|^2, \quad \operatorname{Re} p \geq -\gamma' \tag{5.3}$$

holds for all $g \in C^\infty(\Gamma)$.

Proof. It follows from (5.1) that there exists $A>0$ such that

$$|(e^{2\delta|x|}\mathscr{B}^{(\delta)}(p)g, g)| \geq \frac{c}{2}\|e^{\delta|x|}g\|^2 \tag{5.4}$$

when $\operatorname{Re} p \geq -\gamma$ and $|p| \geq A$. Using the consideration in the proof of Proposition 3.3 we obtain from (5.2) the existence of $\mathscr{B}^{(\delta)}(p)^{-1}$ in $L^2(\Gamma)$. And it turns out that for every $\operatorname{Re} p \geq 0$ $\mathscr{B}^{(\delta)}(p)^{-1}$ is a continuous mapping from $L^2(\Gamma)$ into $H^1(\Gamma)$. Note that $\mathscr{B}^{(\delta)}(p)$ depends continuously on p as $\mathscr{L}(H^1(\Gamma), L^2(\Gamma))$-valued function. Therefore we have

$$\alpha_0 = \inf\{\|\mathscr{B}^{(\delta)}(p)g\|_0;\ \|g\|_1 = 1, \quad \operatorname{Re} p \geq 0, |p| \leq A\} > 0\,.$$

Then there exists $\gamma' > 0$ such that

$$\|\mathscr{B}^{(\delta)}(p)g\| \geq \alpha_0/2 \quad \text{for all } \|g\|_1 = 1 \tag{5.5}$$

holds for all $\operatorname{Re} p \geq -\gamma'$, $|p| \leq A$. Combining (5.4) and (5.5) we have (5.3).
Q.E.D.

Let us set

$$\mathscr{B}_0^{(\delta)}(p) = \mathscr{N}^{(\delta)}(p) + \frac{1}{2}\sum_{j=1}^{3}\left\{(b_j - n_j)\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}(b_j - n_j)\right\}$$
$$+ \delta \sum_{j=1}^{3}(b_j - n_j)x_j/|x| + c(x)p\,.$$

Denote by $\Lambda(g)$ a set of complex numbers $z = a + ib$ satisfying the following properties: for b such that $\{k; b = -\operatorname{Im}(e^{2\delta|x|}\mathscr{B}_0^{(\delta)}(ik)g, g)\} = \phi$, all $a \in \boldsymbol{R}$. For b such that $\{k; b = -\operatorname{Im}(e^{2\delta|x|}\mathscr{B}_0^{(\delta)}(ik)g, g)\} = S \neq \phi$,

$$a < \inf_{k \in S}\{-\operatorname{Re}(e^{2\delta|x|}\mathscr{B}_0^{(\delta)}(ik)g, g)\}\,.$$

Lemma 5.2. *Suppose that* $d(x) \in C^\infty(\Gamma)$ *satisfies*

$$(e^{2\delta|x|}d(x)g, g) \in \Lambda(g), \quad \forall g \in C^\infty(\Gamma),\ \|g\| = 1 \tag{5.6}$$

and that $\mathscr{B}^{(\delta)}(p)=\mathscr{B}_0^{(\delta)}(p)+d(x)$ *verifies the condition* (5. 1). *Then* (3. 9) *holds.*

Proof. Let $g\in C^\infty(\Gamma)$ and $g\neq 0$. Set for $\lambda\geq 0$

$$F_\lambda(p)=(e^{2\delta|x|}(\mathscr{B}_0^{(\delta)}(p)+d(x)-\lambda)g, g).$$

Then $F_\lambda(p)$ is analytic in $\mathrm{Re}\, p>-\delta_0$ for all $\lambda\geq 0$. (5. 1) implies that for some $A>0$

$$F_\lambda(p)\neq 0 \qquad \text{for all } |p|\geq A \ \mathrm{Re}\, p\geq -\gamma\,.$$

From the definition of $\Lambda(g)$ we have for all $\lambda\geq 0$

$$(e^{2\delta|x|}(d(x)-\lambda)g, g)\in\Lambda(g),$$

which shows for all $\lambda\geq 0$

$$F_\lambda(ik)\neq 0 \qquad \text{for all } k\in\boldsymbol{R}\,.$$

On the other hand we have from (3. 12) that

$$F_{\lambda_0}(p)\neq 0, \quad \forall \mathrm{Re}\, p\geq -\gamma \tag{5.7}$$

for λ_0 sufficiently large. Then by taking account of the continuity with respect to λ we see that the number of zeros of $F_\lambda(p)$ is same for all $\lambda\geq 0$. And it follows from (5. 7) that this number is zero. Then

$$F_0(p)\neq 0\,, \quad \mathrm{Re}\, p\geq 0\,,$$

which is nothing but (5. 2). By the previous lemma (5. 1) and (5. 2) imply (3. 9). Q.E.D.

Proof of Theorem 5. Suppose that $d(x)$ verifies for some $\varepsilon>0$

$$\mathrm{Re}\, d(x)\leq H(x)-2\varepsilon\,.$$

Then we have from Theorem 3. 2 that for some $A>0$

$$-\mathrm{Re}(e^{2\delta|x|}\mathscr{B}_0^{(\delta)}(p)g, g)\geq\varepsilon\|e^{\delta|x|}g\|^2 \qquad \mathrm{Re}\, p\geq-\frac{\varepsilon}{2},\ |p|>A\,. \tag{5.8}$$

Due to Lemma 5. 2, in order to apply Proposition 3. 3 it suffices to verify (5. 6).

Note that an estimate

$$|\mathrm{Im}\{(\mathscr{A}_3(k)f, f)+(\mathscr{A}_4(k)f, f)\}|\leq C(\|X_3f\|^2+\|X_4f\|^2)$$

follows from the consideration on § 5 of [4]. Then we have

$$|\mathrm{Im}(\mathscr{A}(k)f, f)|\leq|k|(\|X_1f\|^2+\|X_2f\|^2)+C(\|X_3f\|^2+\|X_4f\|^2)$$

$$\leq C|k|\|f\|^2,$$

which implies that

$$|\mathrm{Im}\,(e^{2\delta|x|}\mathcal{N}^{(\delta)}(ik)g, g)|\leq C|k|\|e^{\delta|x|}g\|^2.$$

Since

$$\mathrm{Im}\,(e^{2\delta|x|}\mathcal{B}_0^{(\delta)}(ik)g, g)=\mathrm{Im}\,(e^{2\delta|x|}\mathcal{N}^{(\delta)}(ik)g, g)$$
$$+\mathrm{Im}\Big(\sum_{j=1}^{3}(b_j-n_j)\frac{\partial}{\partial x_j}(e^{\delta|x|}g), e^{\delta|x|}g\Big)+k(e^{2\delta|x|}c(x)g, g)$$

we have for $|k|\leq A$

$$|\mathrm{Im}\,(e^{2\delta|x|}\mathcal{B}_0^{(\delta)}(ik)g, g)|\leq(\sup|b-n|)\|X_4e^{\delta|x|}g\|_{1/2}^2+CA\|e^{\delta|x|}g\|^2.$$

Suppose that

$$|\mathrm{Im}\,(e^{2\delta|x|}\mathcal{B}_0^{(\delta)}(ik)g, g)|\geq C_\varepsilon\|e^{\delta|x|}g\|^2. \tag{5.9}$$

Then it is necessary to hold

$$(\sup|b-n|)\|X_4e^{\delta|x|}g\|_{1/2}^2\geq(C_\varepsilon-CA)\|e^{\delta|x|}g\|^2. \tag{5.10}$$

(5.18) of [4] shows

$$-\mathrm{Re}\,(e^{2\delta|x|}\mathcal{N}^{(\delta)}(ik)g, g)\geq\|X_4e^{\delta|x|}g\|_{1/2}^2-C|k|\|e^{\delta|x|}g\|^2. \tag{5.11}$$

(5.10) and (5.11) imply that

$$-\mathrm{Re}\,(e^{2\delta|x|}\mathcal{N}^{(\delta)}(ik)g, g)\geq(\sup|b-n|)^{-1}(C_\varepsilon-2CA)\|e^{\delta|x|}g\|^2.$$

Therefore if we choose C_ε sufficiently large (5.9) implies

$$-\mathrm{Re}\,(e^{2\delta|x|}\mathcal{B}_0^{(\delta)}(ik)g, g)>(e^{2\delta|x|}H(x)g, g)\qquad\text{for all }|k|\leq A.$$

Thus (5.6) is shown.

Proof of Theorem 3. First let us show that

$$\mathrm{Im}\,(e^{2\delta|x|}\mathcal{N}^{(\delta)}(ik)g, g)\neq 0$$

holds for any $k\neq 0$ and $g\neq 0$. Suppose that for some g and $k\neq 0$

$$\mathrm{Im}\,(e^{2\delta|x|}\mathcal{N}^{(\delta)}(ik)g, g)=0. \tag{5.12}$$

Set

$$u(x)=e^{\delta|x|}U^{(\delta)}(ik, g; x).$$

Then we see that

$$(\Delta+k^2)u(x)=0$$
$$u|_\Gamma=e^{\delta|x|}g$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma}=e^{\delta|x|}\mathcal{N}^{(\delta)}(ik)g$$

and that it satisfies the radiation condition of Sommerfeld. Then

$$0=\int_{\Omega\cap\{|x|<R\}}\left\{(\Delta+k^2)u\cdot\bar{u}-u\overline{(\Delta+k^2)u}\right\}dx$$

$$=\int_{\Gamma}\left(-\frac{\partial u}{\partial n}\bar{u}+u\frac{\partial\bar{u}}{\partial n}\right)dS-\int_{|x|=R}\left(-\frac{\partial u}{\partial n}\bar{u}+u\frac{\partial\bar{u}}{\partial n}\right)dS.$$

Note that $\int_{\Gamma}\frac{\partial u}{\partial n}\bar{u}dS=(e^{2\delta|x|}\mathcal{N}^{(\delta)}(ik)g,g)$ is real. The radiation condition of Sommerfeld implies

$$\int_{|x|=R}\left(\frac{\partial u}{\partial n}\bar{u}-u\frac{\partial\bar{u}}{\partial n}\right)dS=2ik\int_{|x|=R}|u|^2dS+o(1)\quad\text{as } R\to\infty.$$

Since $k\neq0$ we have

$$\int_{|x|=R}|u|^2dS=o(1)\qquad\text{as } R\to\infty.$$

By Rellich's uniqueness theorem $u=0$ holds. This is a contradiction. Thus (5.12) is proved.

Next consider for $k=0$.

$$u(x)=e^{\delta|x|}U^{(\delta)}(0,g;x)$$

satisfies $\Delta u=0$ and $|x|u(x)$ and $|x|^2\frac{\partial u}{\partial x_j}$ are bounded in Ω. Then it follows that

$$0=\int_{\Omega}\Delta u\bar{u}dx=-\int_{\Gamma}\frac{\partial u}{\partial n}\bar{u}dS-\int_{\Omega}|\Delta u|^2dx.$$

Then

$$(e^{2\delta|x|}\mathcal{N}^{(\delta)}(0)g,g)=\int_{\Gamma}\frac{\partial u}{\partial n}\bar{u}\,dS=-\int_{\Omega}|\nabla u|^2dx.$$

By Asakura's result

$$\int_{\Omega}|\nabla u|^2dx\geq\int_{\Gamma}\sigma_0|u|^2dS=(e^{2\delta|x|}\sigma_0(x)g,g)$$

where

$$\sigma_0(x)=\langle x-Q,n(x)\rangle|x-Q|^{-2}.$$

Therefore if $d(x)<\sigma_0(x)$ for all $x\in\Gamma$ we have

$$(e^{2\delta|x|}d(x)g,g)\in\Lambda(g),\quad g\neq0.$$

Thus (5.6) holds for $d(x)$ verifying (1.6).

Proof of Theorem 4.
Suppose that

$$d(x) \leq \min(\sigma_0(x), H(x)) - 2\varepsilon$$

holds. Then it follows that

$$-\mathrm{Re}(e^{2\delta|x|}\mathscr{B}^{(\delta)}(p)g, g) = -\mathrm{Re}(e^{2\delta|x|}(\mathscr{N}^{(\delta)}(p) + d(x))g, g)$$
$$\geq \varepsilon\|e^{\delta|x|}g\|^2$$

for all $\mathrm{Re}\,p \geq -\varepsilon/2$, $|p| \geq A$, where the constant A is independent of $b(x)$ satisfying (A-II) and (A-III). Then we have

$$\|\mathscr{N}^{(\delta)}(p)g\| \geq \varepsilon\|e^{\delta|x|}g\| \quad \text{for } \mathrm{Re}\,p \geq -\varepsilon/2, |p| \geq A. \tag{5.13}$$

On the other hand for $|p| \leq A$, $\mathrm{Re}\,p > -\gamma$ $(\mathscr{N}^{(\delta)}(p) + d(x))^{-1}$ exists and uniformly bounded. Then by using the estimate (3.11)

$$\|(\mathscr{N}^{(\delta)}(p) + d)^{-1}g\|_1 \leq CA\|g\|_0$$

for all $\mathrm{Re}\,p > -\gamma$, $|p| \leq A$. When the a of (1.9) is not so large it holds that

$$\left\|\frac{1}{2}\sum_{j=1}^{3}\left\{(b_j - n_j)\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}(b_j - n_j)\right\} + pc(x) + \delta\sum_{j=1}^{3}\frac{(b_j - n_j)x_j}{|x|}\right\|_{\mathscr{L}(H^1(\Gamma), L^2(\Gamma))}$$
$$< \|(\mathscr{N}^{(\delta)}(p) + d)\|_{\mathscr{L}(H^1(\Gamma), L^2(\Gamma))}$$

Since the left-hand side is equal to $\|\mathscr{B}^{(\delta)}(p) - (\mathscr{N}^{(\delta)}(p) + d)\|_{\mathscr{L}(H^1(\Gamma), L^2(\Gamma))}$ the above inequality implies

$$\|e^{\delta|x|}\mathscr{B}^{(\delta)}(p)g\| \geq c\|e^{\delta|x|}g\|, \quad \forall g \in C^\infty(\Gamma). \tag{5.14}$$

By combiming (5.13) and (5.14) we have (3.9).

Remark. Till now we showed the existence and uniform boundedness of $\mathscr{B}^{(\delta)}(p)^{-1}$ for $\mathrm{Re}\,p \geq -\gamma$, $\gamma > 0$, which imply the exponential decay of solutions of (P). Suppose that $\mathscr{B}^{(\delta)}(p)^{-1}$ exists and satisfies

$$\|\mathscr{B}^{(\delta)}(p)^{-1}\| < C \quad \text{for all } \mathrm{Re}\,p \geq -\gamma.$$

Then we see easily that for any $\tilde{d}(x)$ such that

$$\sup|\tilde{d}(x)| < \frac{1}{C}$$

$(\mathscr{B}^{(\delta)}(p) + \tilde{d}(x))^{-1}$ also exists and uniformly bounded in $\mathrm{Re}\,p \geq -\gamma$. Then we have also the exponential decay of solutions for anther boundary operator $B + \tilde{d}$.

§ 6. Proof of theorem 3.2

We showed in [4] an inequality

$$-\mathrm{Re}\,(e^{2\delta|x|}\mathcal{N}^{(\delta)}(p)g,g)\geq(\mu-C)\|e^{\delta|x|}g\|^2,\quad \mathrm{Re}\,p>-\delta_0.$$

Examing the estimates used in [4] we see that in order to obtain (3.8) it suffices to show an estimate

$$(6.1)\qquad -\mathrm{Re}\Big(e^{2\delta|x|}\frac{e^{-\delta|x|}}{\widehat{m}(p)}\int_{-\infty}^{\infty}e^{-ikt}\frac{\partial \mathcal{W}_1}{\partial n}(\mu, m(t')e^{-\mu t'}e^{\delta|x|}g;x,t)\,dt,g\Big)_m$$
$$\geq(e^{2\delta|x|}(\mu+H(x)-\varepsilon)X_1g,X_1g)_m-\frac{C_{m,\varepsilon}}{1+|\mu+ik|}\|e^{\delta|x|}g\|_m^2$$

for all $g\in C_0^\infty(\Gamma_0)$ when Γ_0 is choosen sufficiently small. After this we show (6.1). Let $s(\sigma)$ be a mapping

$$I_\sigma=-[\sigma_{10},\sigma_{10}]\times[-\sigma_{20},\sigma_{20}]\ni\sigma=(\sigma_1,\sigma_2)\to s(\sigma)\in\Gamma_0$$

such that $s(\sigma)=(\sigma_1,\sigma_2,\varphi(\sigma_1,\sigma_2))$,

$$(6.2)\qquad \frac{\partial\varphi}{\partial\sigma_j}(0)=0,\qquad j=1,2.$$

Set

$$h^{lj}(\sigma)=\Big(\frac{\partial s}{\partial\sigma_l}\cdot\frac{\partial s}{\partial\sigma_j}\Big)\Big(\Big(\frac{\partial s}{\partial\sigma_1}\Big)^2\Big(\frac{\partial s}{\partial\sigma_2}\Big)^2-\Big(\frac{\partial s}{\partial\sigma_1}\cdot\frac{\partial s}{\partial\sigma_2}\Big)^2\Big)^{-1}$$

Remark that it follows from (6.2) that

$$\frac{\partial h^{lj}}{\partial\sigma_k}\Big|_{\sigma=0}=0,\qquad \forall j,l,k.$$

Let us denote by $\tilde f(\sigma,\eta)$ a solution of

$$(6.3)\qquad \begin{cases}\displaystyle\sum_{l,j=1}^{2}h^{lj}(\sigma)\frac{\partial\tilde f}{\partial\sigma_l}\frac{\partial\tilde f}{\partial\sigma_j}=1\\ \displaystyle\frac{\partial\tilde f}{\partial\sigma_j}\Big|_{\sigma=0}=\eta_j\\ \tilde f(0,\eta)=0\end{cases}$$

where $\eta=(\eta_1,\eta_2)\in\Sigma=\{(\eta_1,\eta_2);\eta_1^2+\eta_2^2=1\}$. Let Γ_0 be a small neighborhood in Γ of $s_0=0$ and $\theta(s,\eta)$ be a function belonging to $C^\infty(\Gamma_0)$ defined by $\theta(s(\sigma),\eta)=\tilde f(\sigma,\eta)$. For fixed $0<\beta_0<1$ consider an equation

$$\beta\{\theta(s(\sigma),\eta)-\theta(s(\sigma'),\eta)\}=\alpha\langle\sigma-\sigma',\xi'\rangle$$

$\eta,\xi'\in\Sigma$, $\beta_0<\beta<1$, σ, $\sigma'\in I_\sigma$. If we choose I_σ sufficiently small there exist C^∞-functions $\alpha(\sigma,\sigma',\beta,\eta)$ and $\xi'(\sigma,\sigma',\beta,\eta)$ satisfying the above equality for all $\sigma, \sigma' \in I_\sigma, \beta_0 < \beta < 1, \eta \in \Sigma$ and

$$\alpha(0, 0, \beta, \eta) = \beta$$

$$\xi'(0, 0, \beta, \eta) = \eta .$$

Further more we have from (6.3)

$$(6.4) \qquad \begin{cases} \dfrac{\partial \tilde{f}}{\partial \sigma_1} \dfrac{\partial \alpha}{\partial \sigma_1} + \dfrac{\partial \tilde{f}}{\partial \sigma_2} \dfrac{\partial \alpha}{\partial \sigma_2} = 0 & \text{at } \sigma = \sigma' = 0 \\ \dfrac{\partial \tilde{f}}{\partial \sigma_1} \dfrac{\partial \xi_j}{\partial \sigma_1} + \dfrac{\partial \tilde{f}}{\partial \sigma_2} \dfrac{\partial \xi_j}{\partial \sigma_2} = 0 & \text{at } \sigma = \sigma' = 0 \qquad \text{for } j = 1, 2 . \end{cases}$$

Remark that $\theta(s, \eta)$ introduced in the above varies a little from $\theta(s, \eta)$ used in [4]. But concerning the estimates for $\mathcal{W}_j$, $j = 1, 2, 3$ we do not have to change the process. We will use the same notations as [4].

$$\begin{aligned} (\mathcal{V}_1 h)(s(\sigma), t) &= \omega(s(\sigma), t) \int_{\boldsymbol{R}^1} dk \int_0^\infty d\alpha \int_\Sigma d\xi' \int_{I_\sigma} d\sigma' \int_{I_t} dt' \\ &\quad \times \exp\{ik(t - t' + \alpha\langle \sigma - \sigma', \xi' \rangle)\} \chi_1(\alpha)^2 k^2 \alpha \tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t') \\ &= \omega(s(\sigma), t) \int_{\boldsymbol{R}^1} dk \int_0^{\alpha_0} d\alpha \cdots + \omega(s(\sigma), t) \int_{\boldsymbol{R}^1} dk \int_{\alpha_0}^\infty d\alpha \\ &= (\mathcal{V}_{10} h)(s(\sigma), t) + (\mathcal{V}_{11} h)(s(\sigma), t), \end{aligned}$$

where α_0 is a positive constant determined later. By using a change of variables

$$\begin{aligned} (\mathcal{V}_{11} h)(s(\sigma), t) &= \omega(s(\sigma), t) \int_{\boldsymbol{R}^1} dk \int_{\beta_0}^\infty d\beta \int_\Sigma d\eta \int_{I_\sigma} d\sigma' \int_{I_t} dt' \\ &\quad \times \exp\{ik(\beta(\theta(s(\sigma), \eta) - \theta(s(\sigma'), \eta)) + t - t')\} \chi_1(\alpha)^2 \alpha k^2 \\ &\quad \times \frac{D(\xi', \alpha)}{D(\eta, \beta)} \tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t), \end{aligned}$$

where β_0 depends on σ, σ', η but there exists $\beta_1 > 0$ such that $\beta_0 \geq \beta_1$ for all σ, σ', η if I_σ is small.

For the purpose of the construction of $\mathcal{W}_1$ consider an asymptotic solution of the equation

$$\left(\left(\frac{\partial}{\partial t} + \mu\right)^2 - \Delta\right) u = 0 \qquad \text{in } \Omega \times \boldsymbol{R}^1$$

with an oscillatory value

$$u(x, t) = v(s, t) \exp\{ik(t + \beta\theta(s, \eta))\} \qquad \text{on } \Gamma \times \boldsymbol{R}^1$$

for

$$(6.5) \qquad 0 < \beta < 1 - \varepsilon_0, \quad \varepsilon_0 > 0 .$$

Let $\psi(x, \eta, \beta)$ be a solution of

$$(6.6)\qquad \begin{cases} \psi(s, \eta, \beta)=\beta\theta(s, \eta) & \text{on } \Gamma \\ (\nabla_x\psi)^2=1 & \text{in } \Omega \\ \dfrac{\partial\psi}{\partial n}<0\,. & \end{cases}$$

Note that $\psi(x, \eta, \beta)$ verifying (6.6) is a C^∞-function of $x\in\bar{\Omega}$, $\eta\in\Sigma$ and β of (6.5) and all derivatives with respect to x, η and β are bounded in $\bar{\Omega}$, Σ and β of (6.5). We set $p=\mu+ik$ and ask for a solution u in the form

$$(6.7)\qquad u(x, t; p)=\exp\{ik(t+\psi(x, \eta, \beta))\}\cdot G(x, t; p)\,.$$

$$\begin{aligned} &\exp\{-ik(t+\psi)\}\cdot\left(\left(\frac{\partial}{\partial t}+\mu\right)^2-\Delta\right)u \\ &=\{(ik+\mu)^2-(ik\nabla\psi)^2\}\cdot G+2(ik+\mu)\left(\frac{\partial G}{\partial t}+\mu G\right) \\ &\quad -2ik\nabla\psi\cdot\nabla G-ik\Delta\psi\cdot G+\left(\left(\frac{\partial}{\partial t}+\mu\right)^2-\Delta\right)G \\ &=ik\left(2\frac{\partial G}{\partial t}-2\nabla\psi\cdot\nabla G-\Delta\psi\cdot G+2\mu G\right) \\ &\quad +\left(\left(\frac{\partial}{\partial t}+\mu\right)^2-\Delta\right)G+2\mu\left(\frac{\partial G}{\partial t}+\mu G\right)+\mu^2 G\,. \end{aligned}$$

Restricted to $\mu_0\geq\mu\geq-\delta_0$ construct $G(x, t; p)$ vanishing the right-hand side of the equality asymptotically. Set

$$G(x, t; p)\sim\sum_{j=0}^{\infty}(ik)^{-j}G_j(x, t; \mu)$$

and determine G_j successively. G_0 is required to verify

$$(6.8)_0\qquad \begin{cases} 2\dfrac{\partial G_0}{\partial t}-2\nabla\psi\nabla G_0-(\Delta\psi-2\mu)G_0=0 & \text{in } \Omega\times\boldsymbol{R}^1 \\ G_0(x, t)=v(x, t) & \text{on } \Gamma\times\boldsymbol{R}^1 \end{cases}$$

and $G_j, j\geq1$ must verify

$$(6.8)_j\qquad \begin{cases} 2\dfrac{\partial G_j}{\partial t}-2\nabla\psi\cdot\nabla G_j-(\Delta\psi-2\mu)G_j \\ \qquad =-\left(\left(\dfrac{\partial}{\partial t}+\mu\right)^2-\Delta\right)G_{j-1}-2\mu\left(\dfrac{\partial G_{j-1}}{\partial t}+\mu G_{j-1}\right)-\mu^2 G_{j-1} \\ G_j=0 \qquad \text{in } \Gamma\times\boldsymbol{R}^1 \end{cases}$$

The solution $G_j, j=0, 1, 2, 3, \cdots$ of $(6.8)_j$ are determined uniquely for given function $v(x, t)$, therefore there exists $G(x, t; p)$ with required properties.

Construct $G(x, t; p)$ following the above process for $v(s, t)$ with parameters σ', η, β

$$v(s(\sigma), t) = \omega(s(\sigma), t)\chi_1(\alpha(\sigma, \sigma', \eta, \beta))^2\alpha(\sigma, \sigma', \eta, \beta)\frac{D(\xi', \alpha)}{D(\eta, \beta)}$$

and denote this $G(x, t; p)$ by $G(x, t; \sigma', \eta, \beta, p)$. Using this G

$$\begin{aligned}\mathcal{W}_{11}(\mu, h: x, t) = \int_{\boldsymbol{R}^1} dk \int_{\beta_0} d\beta \int_{\Sigma} d\eta \int d\sigma' \int dt' \\ \times \exp\{ik(\psi(x, \beta, \eta) - \beta\theta(s(\sigma'), \eta) + t - t')\} \\ \times G(x, t; \sigma', \eta, \beta, p) k^2\tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t')\end{aligned}$$

Evidently

$$\mathcal{W}_{11}(\mu, h: x, t)|_{\Gamma\times\boldsymbol{R}^1} = \mathcal{V}_{11}h. \tag{6.9}$$

Next consider $\partial\mathcal{W}_{11}/\partial n$

$$\begin{aligned}\frac{\partial\mathcal{W}_{11}(\mu, h: x, t)}{\partial n} \\ = \int\cdots\int \exp[ik\{\beta(\theta(s(\sigma), \eta) - \theta(s(\sigma'), \eta)) + t - t'\}] \\ \times\left(ik\frac{\partial\psi}{\partial n}G + \frac{\partial G}{\partial n}\right)k^2\tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t').\end{aligned} \tag{6.10}$$

Let us set

$$Ih = \int\cdots\int\exp[\ ]\cdot ik\frac{\partial\psi}{\partial n}Gk^2\tilde{\omega}h$$

$$IIh = \int\cdots\int\exp[\ ]\frac{\partial G}{\partial n}k^2\tilde{\omega}h.$$

Let h be of the form $g(s)e^{-t}m(t)$.

$$\begin{aligned}Ih = \omega\int_{\boldsymbol{R}^1} dk\int_{\beta_0}^{\infty} d\beta\int_{\Sigma} d\eta\int_{I_\sigma} d\sigma'\int_{I_t} dt' \\ \times\exp[ik\{\beta(\theta(s(\sigma), \eta) - \theta(s(\sigma'), \eta)) + t - t'\}] \\ \times ik\frac{\partial\psi}{\partial n}(s(\sigma), \beta, \eta)\chi_1(\alpha)^2\alpha k^2\frac{D(\xi', \alpha)}{D(\eta, \beta)}\tilde{\omega}g(s(\sigma'))e^{-\mu t'}m(t') \\ = \int\cdots\int - (1-\omega)\int\cdots\int = I_1h + I_2h.\end{aligned}$$

Since $\operatorname{supp}(1-\omega)\cap\operatorname{supp}\tilde{\omega} = \phi$, I_2 is considered as a pseudodifferential operator in $\Gamma\times\boldsymbol{R}^1$ of the class $S^{-\infty}$. Then for any m, m' there exists a constant $C_{m,m'}$ such that

(6.11) $$\left\| \int_{-\infty}^{\infty} e^{-ik't} I_2 h dt \right\|_m \leq C_{m,m'} (1+k'^2)^{-m'} \|g\|_0 \left(\int_{-\infty}^{\infty} |e^{-\mu t} m(t)|^2 dt \right)^{1/2}.$$

Let $\mathcal{A}_{11}(k)$ be an operator defined for $g \in C_0^\infty(\Gamma_0)$ by

$$\mathcal{A}_{11}(k) g = \int_{\beta_0}^{\infty} d\beta \int_{\Sigma} d\eta \int d\sigma' \cdot \exp\{ik\beta(\theta(s(\sigma), \eta) - \theta(s(\sigma'), \eta))\}$$
$$\times ik \frac{\partial \psi}{\partial n}(s(\sigma), \eta, \beta) \cdot \chi_1(\alpha)^2 k^2 \alpha \frac{D(\xi', \alpha)}{D(\eta, \beta)} g(s(\sigma')).$$

Then $I_1 h$ be represented as

$$(I_1 h)(s(\sigma), t) = \int_{-\infty}^{\infty} e^{ikt} \mathcal{A}_{11}(k) g \widehat{m}(\mu + ik) dk,$$

which implies

(6.12) $$\int_{-\infty}^{\infty} e^{-ikt} (I_1 h)(s(\sigma), t) dt = \mathcal{A}_{11}(k) g \cdot \widehat{m}(\mu + ik).$$

Since $\theta(s, \eta)$ satisfies

$$\sum_{j,l=1}^{3} h^{jl}(\sigma) \frac{\partial}{\partial \sigma_j}(s(\sigma), \eta) \frac{\partial}{\partial \sigma_l}(s(\sigma), \eta) = 1$$

we have

$$\frac{\partial \psi}{\partial n}(s(\sigma), \eta, \beta) = -\sqrt{1-\beta^2}, \qquad \text{for all } \sigma \in I_\sigma.$$

By taking account of the form of the equation which $\alpha(\sigma, \sigma', \eta, \beta)$ and $\xi'(\sigma, \sigma', \eta, \beta)$ satisfy we see immediately the relations

$$\begin{cases} \alpha(\sigma, \sigma', \eta, \beta) = \alpha(\sigma', \sigma, \eta, \beta) \\ \xi'(\sigma, \sigma', \eta, \beta) = \xi'(\sigma', \sigma, \eta, \beta). \end{cases}$$

Then we have that $\dfrac{D(\xi', \alpha)}{D(\eta, \beta)}(\sigma, \sigma', \eta, \beta)$ is also symmetric with respect to σ and σ'. Therefore we have

$$\left(\frac{1}{i} \mathcal{A}_{11}(k) g, g\right) = \int d\sigma \int d\beta \int d\eta \int d\sigma' \cdot \exp\{ik(\theta(s(\sigma), \eta) - \theta(s(\sigma'), \eta))\}$$
$$\times k(-\sqrt{1-\beta^2}) \chi_1(\alpha)^2 k^2 \alpha \frac{D(\xi', \alpha)}{D(\eta, \beta)} g(s(\sigma')) \overline{g(s(\sigma))}$$
$$= \left(g, \frac{1}{i} \mathcal{A}_{11}(k) g\right),$$

namely

(6.13) $$\mathrm{Re}(\mathcal{A}_{11}(k) g, g) = 0.$$

Combining (6.11), (6.12) and (6.13) it holds that

$$\text{(6.14)}\qquad \left|\mathrm{Re}\left(e^{2\delta|x|}\frac{e^{-\delta|x|}}{\widehat{m}(p)}\int_{-\infty}^{\infty} e^{-ikt}I(e^{\delta|x|}g\cdot e^{-\mu t}m(t))\,dt,\ g\right)_m\right|$$

$$\leq C_m\cdot\frac{1}{1+|k|}\|e^{\delta|x|}g\|_m^2 .$$

In the next place consider IIh. From the relation which satisfies G we have

$$-\sqrt{1-\beta^2}\frac{\partial G}{\partial n}\Big|_{\Gamma\times\boldsymbol{R}^1}=\frac{\partial G}{\partial t}-2(\nabla\psi)_s(\nabla G)_s-\left(\frac{1}{2}\Delta\psi-\mu\right)G$$

$$+\frac{1}{ik}\left\{\left(\left(\frac{\partial}{\partial t}+\mu\right)^2-\Delta\right)G+2\mu\left(\frac{\partial G}{\partial t}+\mu G\right)+\mu^2 G\right\}\quad (\mathrm{mod}\ k^{-\infty})$$

Then

$$IIh=\int\cdots\int\left(-\frac{1}{\sqrt{1-\beta^2}}\right)\frac{\partial G}{\partial t}k^2\tilde{\omega}h+\int\cdots\int\frac{1}{\sqrt{1-\beta^2}}(\nabla\psi)_s(\nabla G)_s k^2\tilde{\omega}h$$

$$+\int\cdots\int\frac{1}{\sqrt{1-\beta^2}}\left(\frac{1}{2}\Delta\psi-\mu\right)Gk^2\tilde{\omega}h+\int\cdots\int\frac{1}{\sqrt{1-\beta^2}}\frac{1}{ik}\{\ \}k^2\tilde{\omega}h$$

$$=II_1h+II_2h+II_3h+II_4h .$$

Concerning II_1h, since

$$\frac{\partial G}{\partial t}=\frac{\partial\omega}{\partial t}\chi_1^2\alpha\frac{D(\xi',\alpha)}{D(\eta,\beta)}\quad\text{and}\quad \mathrm{supp}\,\frac{\partial\omega}{\partial t}\cap\mathrm{supp}\,\tilde{\omega}=\phi$$

it follows that

$$\text{(6.15)}\qquad \left\|\int_{-\infty}^{\infty} e^{-ik't}II_1h\,dt\right\|_m\leq\frac{C_{m,m'}}{(1+|k'|)^{m'}}\|g\|_0\left(\int_{-\infty}^{\infty}|e^{-\mu t}m(t)|^2dt\right)^{1/2}.$$

Consider II_2h.

$$(\nabla\psi)_s\cdot(\nabla G)_s(s(\sigma))=\sum_{j,l=1}^{2}h^{jl}(\sigma)\frac{\partial\theta(s(\sigma),\eta)}{\partial\sigma_j}\frac{\partial G(s(\sigma))}{\partial\sigma_l}$$

$$=\sum_{j,l=1}^{2}h^{jl}\frac{\partial\tilde{f}(\sigma,\eta)}{\partial\sigma_j}\frac{\partial}{\partial\sigma_l}\left(\omega\chi_1(\alpha)^2\alpha\frac{D(\xi',\alpha)}{D(\eta,\beta)}\right).$$

From (6.4) and the fact $h^{jl}(0)=\delta_{jl}$ it follows that

$$\text{(6.16)}\qquad (\nabla\psi)_s\cdot(\nabla G)_s(s_0)=0 .$$

Using $\mathrm{supp}(1-\omega)\cap\mathrm{supp}\,\tilde{\omega}=\phi$ II_2h is represented as

$$II_2h=\int_{-\infty}^{\infty}e^{ikt}\widehat{m}(\mu+ik)\left\{\int d\beta\int d\eta\int d\sigma'\right.$$

$$\left.\cdot\exp\{ik\beta(\theta(s(\sigma),\eta)-\theta(s(\sigma'),\eta))\}a(\sigma,\sigma',\beta,\eta)k^2g(s(\sigma'))\right\}dk$$

$$+ (\text{operator of class } S^{-\infty})\, h\,,$$

where $a(0,0,\beta,\eta)=0$. therefore for any $\varepsilon>0$ we have

$$\left\| \frac{1}{\widehat{m}(\mu+ik')} \int_{-\infty}^{\infty} e^{-ik't} II_2 h\, dt \right\|_m \leq \frac{\varepsilon}{2} \|g\|_m + C_{m,m'}(1+|k'|^2)^{-m'} \|g\|_0 \tag{6.17}$$

for all $g(s) \in C_0^\infty(\Gamma_0)$ with support contained in a sufficiently small neighborhoof of s_0.

Concerning $II_4 h$ we have immediately

$$\left\| \frac{1}{\widehat{m}(\mu+ik')} \int_{-\infty}^{\infty} e^{-ik't} II_4 h\, dt \right\|_m \leq \frac{C_m}{1+k'} \|g\|_m . \tag{6.18}$$

Consider $II_3 h$. First check up on the value of $\Delta\psi$. According to [7] and [8] we know that the value of $\Delta\psi$ at x of ψ verifing $(\nabla\psi)^2=1$ is equal to two times of the mean curvature of a surface $\{y; \psi(y)=\psi(x)\}$ at x with respect to $(-\nabla\psi)$. Now, when $\psi(x,\beta,\eta)$ satisfies (6.6), the mean curvature at $y=0$ with respect to $-\nabla\psi$ of the surface $\{y; \psi(y,\beta,\eta)=\psi(0,\beta,\eta)\}$ is given by

$$\frac{1}{2}\left\{(\eta_2^2\kappa_1+\eta_1^2\kappa_2)+\frac{1}{\sqrt{1-\beta^2}}(\eta_1^2\kappa_1+\eta_2^2\kappa_2)\right\},$$

where κ_1 and κ_2 denote the principale curvature with respect to $-n$ of Γ at $x=0$. Then we have for all $0\leq\beta<1$

$$-\Delta\psi(0,\beta,\eta) \geq \kappa_1+\kappa_2 = 2H. \tag{6.19}$$

Define $\mathcal{A}_{12}(k)$ by

$$\mathcal{A}_{12}(k) g = \int_{\beta_0}^{\infty} d\beta \int_{\Sigma} d\eta \int d\sigma' \cdot \exp\{ik\beta(\theta)(s(\sigma),\eta) - \theta(s(\sigma'),\eta))\}$$
$$\times (1-\beta^2)^{-1/2}\left(\frac{1}{2}\Delta\psi(s(\sigma),\beta,\eta)-\mu\right)\chi_1(\alpha)^2 \alpha k^2 \frac{D(\xi',\alpha)}{D(\eta,\beta)} g(s(\sigma')).$$

Then we have

$$II_3 h = \int_{-\infty}^{\infty} e^{ikt}\widehat{m}(\mu+ik)\mathcal{A}_{12}(k) g dk + (\text{operator of class } S^{-\infty})\, h\,,$$

which implies

$$\left\| \int_{=\infty}^{\infty} e^{-ik't} II_3 h dt - \widehat{m}(\mu+ik')\mathcal{A}_{12}(k') g \right\|_m \leq C_{m,m'}(1+|k'|^2)^{-m'} \|g\|_0 \left(\int_{-\infty}^{\infty} |e^{-\mu t} m(t)|^2 dt\right)^{1/2}. \tag{6.20}$$

On the other hand by a change of variables

$$\mathcal{A}_{12}(k)\,g=\int_{|\xi|\geq\alpha_0|k|}d\xi\int d\sigma'\exp\langle\sigma-\sigma',\xi\rangle$$

$$\cdot(1-\beta^2)^{-1/2}\Big(\frac{1}{2}(\Delta\psi)(s(\sigma),\beta,\eta)-\mu\Big)\chi_1(|\xi|/k)^2g(s(\sigma'))\Big).$$

Taking account of (6.19), for any $\varepsilon>0$ we have

$$-\mathrm{Re}(\mathcal{A}_{12}(k)g,g)_m\geq\Big(H(0)+\mu-\frac{\varepsilon}{2}\Big)\|X_{11}g\|_m^2-\frac{C_m}{1+|k|}\|X_{11}g\|_m^2 \tag{6.21}$$

for all $g\in C_0^\infty(\Gamma_0)$ with small support, where

$$(X_{11}g)(s(\sigma))=\int_{|\xi|\geq\alpha_0|k|}d\xi\int d\sigma\exp\langle\sigma-\sigma',\xi\rangle\chi_1(|\xi|/k)^2g(s(\sigma'))\,.$$

From the estimates (6.15), (6.17), (6.18), (6.20) and (6.21) it follows that

$$-\mathrm{Re}\Big(e^{2\delta|x|}\int_{-\infty}^{\infty}e^{-ikt}II(e^{\delta|x|}g\cdot e^{-\mu t}u)\,dt,\,g\Big)_m \tag{6$\cdot$22}$$

$$\geq(H(0)+\mu-\varepsilon)\|X_{11}g\|_m^2-\frac{C_m}{1+|k|}\|g\|_m^2\,.$$

Then (6.14) and (6.22) imply

$$-\mathrm{Re}\Big(e^{2\delta|x|}\frac{e^{-\delta|x|}}{\widehat{m}(p)}\int_{-\infty}^{\infty}e^{-ikt}\frac{\partial\mathcal{W}_{11}}{\partial n}(\mu,e^{\delta|x|}g\cdot e^{-\mu t}m;x,t)\,dt,\,g\Big)_m \tag{6.23}$$

$$\geq(H(0)+\mu-\varepsilon)\|X_{11}e^{\delta|x|}g\|_m^2-\frac{C_m}{1+|k|}\|g\|_m^2$$

for all $g\in C_0^\infty(\Gamma_0)$ if we take Γ_0 sufficiently small neighborhood of s_0.

Secondly consider an asymptotic solution $\mathcal{W}_{10}$ for boundary data $\mathcal{V}_{10}h$. If we choose α_0 and Γ_0 sufficiently small there exists a solution of

$$\begin{cases}\psi(s(\sigma),\xi',\alpha)=\alpha\langle\sigma,\xi'\rangle & \text{for } s(\sigma)\in\Gamma_0\\(\nabla\psi)^2=1 & \text{in } \bar{\Omega}\\\dfrac{\partial\psi}{\partial n}<0\end{cases}$$

and $\psi(x,\xi',\alpha)$ depends smoothly on α and ξ'. Then by the same process we may construct an asymptotic solution

$$\mathcal{W}_{10}(\mu,h;x,t)=\int_{R^1}dk\int_0^{\alpha_0}d\alpha\int_\Sigma d\xi'\int_{I_\sigma}d\sigma'$$

$$\cdot\exp\{ik(\psi(x,\xi',\alpha)-\alpha\langle\sigma',\xi'\rangle+t-t')\}G(x,t;\xi',\alpha,p)$$

$$\cdot\alpha k^2\tilde{\omega}(s(\sigma'),t')h(s(\sigma'),t')\,.$$

Note that $\frac{1}{2}\mathcal{J}\psi(0, \xi', \alpha) - H(0)$ becomes arbitrary small by choosing α_0 small. Then we have

$$(6.24) \quad -\mathrm{Re}\left(e^{2\delta|x|}\frac{e^{-r|x|}}{\widehat{m}(\mu+ik)}\int_{-\infty}^{\infty} e^{-ikt}\frac{\partial \mathcal{W}_{10}}{\partial n}(\mu, e^{\delta|x|}g\cdot e^{-\mu t}m; x, t)\,dt, g\right)_m$$
$$\geq (H(0)+\mu-\varepsilon)\,\|X_{10}e^{\delta|x|}g\|_m^2 - \frac{C_m}{1+|k|}\|g\|_m^2$$

where

$$(X_{10}g)\,(s(\sigma)) = \int_{|\xi|\leq \alpha_0|k|} d\xi \int d\sigma'\cdot \exp\langle \sigma-\sigma', \xi\rangle\cdot g(s(\sigma')).$$

Set $\mathcal{W}_1(\mu, h; x, t) = \mathcal{W}_{10}(\mu, h; x, t) + \mathcal{W}_{11}(\mu, h; x, t)$. Evidently

$$\mathcal{W}_1(\mu, h; x, t)\,|_{\Gamma\times \boldsymbol{R}_1} = \mathcal{V}_1 h\,.$$

Let us denote by $\mathcal{Z}_1(\mu, h; x, t)$ the solution of

$$\begin{cases} \left(\left(\dfrac{\partial}{\partial t}+\mu\right)-\Delta\right)z(x, t) = -\left(\left(\dfrac{\partial}{\partial t}+\mu\right)^2-\Delta\right)\mathcal{W}_1(\mu, h; x, t) \\ z(x, t) = 0 \qquad \text{on } \Gamma\times \boldsymbol{R}^1. \end{cases}$$

Then by the same consideration on $\mathcal{Z}_1$ in [4] we have

$$(6.25) \quad \sum_{l=0}^{m}\int_{-\infty}^{\infty}\|D_t^l\mathcal{Z}_1(\mu, h; x, t)\,|_\Gamma\|_m^2\,dt \leq C_m\int_{-\infty}^{\infty}\|h(s, t)\|_0^2\,dt\,.$$

Therefore we obtain (6.1) from (6.23) and (6.24).

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