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Minkowskian product of Finsler spaces and Berwald connection

Dedicated to Professor Dr. Makoto Matsumoto on the occasion of his sixtieth birthday

By

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In the present paper, we shall first show Theorem 1: The Berwald connection of a Finsler space is the *h-connection* [1], [3] which is uniquely determined from the fundamental function by four axioms. A hint as to this theorem has been got from J. Grifone [2]. Next, using Theorem 1 we shall prove Theorem 2: The Berwald connection of any *Minkowskian product* of Finsler spaces coincides with that of the *Euclidean product* of these Finsler spaces. Finally, we shall obtain Theorem 3 on geodesics of the Minkowskian product of Finsler spaces.

The terminology and notations in the present paper are referred to M. Matsumoto's monograph [1].

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§ 1. Berwald connection

Let M be a C^∞ manifold and TM the tangent bundle of M. And let (x^i), $i=1, 2,\dots, n$, be local coordinates of M and (x^i, y^i) the canonical local coordinates of TM. We denote by $s_0(TM)$ the zero section of TM and $(TM)_0 \equiv TM - s_0(TM)$. A positively homogeneous domain is defined as a subdomain D of TM such that if $y \in D$ then $ty \in D$ for any $t \in \boldsymbol{R}^+ - (0)$, where $\boldsymbol{R}^+$ is the non-negative real line.

Definition 1. A *Finsler space* (M, F) is a pair of a C^∞ manifold M and a C^1 function F: $TM \to \boldsymbol{R}^+$ such that

(A) $F(y)=0$ if and only if $y \in s_0(TM)$,

(B) $F(ty)=t^2F(y)$ for $t \in \boldsymbol{R}^+$

(C) F is C^∞ on a positively homogeneous domain D_1,

(D) $g_{ij} \equiv \dfrac{\partial^2 F}{\partial y^i \partial y^j}$ is nondegenerate on a positively homogeneous domain $D_2 (\subset D_1)$.

We call F the *fundamental function* of Finsler space (M, F).

Remark 1. It is obvious that if $D_1 = TM$ then (M, F) is a Riemann space. In §1 we may take $D_1 = D_2 = (TM)_0$, but in §2 and §3 those domains should be restricted as occasion calls. (See Remark 5 and 8)

Coefficients G^i_{kj} of the *Berwald connection* of a Finsler space (M, F) are given on D_2 by

$$G^i = \frac{1}{2} g^{ik} \left(\frac{\partial^2 F}{\partial x^r \partial y^k} y^r - \frac{\partial F}{\partial x^k} \right),$$

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \qquad G^i_{kj} = \frac{\partial G^i_j}{\partial y^k}.$$

Theorem 1. *The Berwald connection of a Finsler space* (M, F) *is the h-connection which is uniquely determined on* D_2 *by the following four axioms*:

(a) *The h-covariant derivatives of the fundamental function* $F_{;i} = 0$.

(b) *The deflection tensor* $D^i_j \equiv y^i_{;j} = 0$.

(c) *The (v)hv-torsion tensor* $P^i_{jk} = 0$.

(d) *The (h)h-torsion tensor* $T^i_{jk} = 0$.

Proof. In terms of canonical local coordinates, we denote coefficients of the above h-connection by (F^i_{jk}, F^i_j). Then these axioms are written as

(a) $F_{;i} = \dfrac{\partial F}{\partial x^i} - \dfrac{\partial F}{\partial y^l} F^l_i = 0,$

(b) $D^i_j = -F^i_j + y^l F^i_{lj} = 0,$

(c) $P^i_{jk} = \dfrac{\partial F^i_j}{\partial y^k} - F^i_{kj} = 0,$

(d) $T^i_{jk} = F^i_{jk} - F^i_{kj} = 0.$

It is well-known that the Berwald connection (G^i_{jk}, G^i_j) satisfies these axioms.

Now, from (c) and (d) we have

$$F^i_{jk} = F^i_{kj} = \frac{\partial F^i_k}{\partial y^j} = \frac{\partial F^i_j}{\partial y^k}. \tag{1.1}$$

The partial differentiation of (b) by y^k and (1.1) lead us to

$$\frac{\partial F^i_{lj}}{\partial y^k} y^l = 0. \tag{1.2}$$

Differentiating (a) by y^j and putting $g_{jl} = \dfrac{\partial^2 F}{\partial y^j \partial y^l}$, we get

$$\frac{\partial g_{jl}}{\partial x^i} y^l - g_{jl} F^l_i - g_{ls} y^s F^l_{ji} = 0. \tag{1.3}$$

Moreover, the differentiation of (1.3) by y^k leads us to

$$(1.4) \qquad \frac{\partial g_{jk}}{\partial x^i} - 2g_{jkl}F^l_i - g_{jl}F^l_{ki} - g_{kl}F^l_{ji} - g_{ls}y^s\frac{\partial F^l_{ji}}{\partial y^k} = 0,$$

where we put $g_{jkl} = \frac{1}{2}\frac{\partial g_{jk}}{\partial y^l}$. Contracting (1.4) by y^i and using (1.1), (1.2) and (b), we obtain

$$(1.5) \qquad \frac{\partial g_{jk}}{\partial x^i}y^i - 2g_{jkl}F^l_iy^i - g_{jl}F^l_k - g_{kl}F^l_j = 0.$$

Applying the Christoffel process [1] to (1.3) and (1.5), by (d) we see

$$\{i, k, j\}y^i - g_{jkl}F^l_iy^i - g_{kl}F^l_j = 0,$$

where $\{i, k, j\} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$. Therefore

$$(1.6) \qquad F^k_j = \{{}^{\,k}_{ij}\}y^i - g^k_{jl}F^l_iy^i,$$

where $\{{}^{\,k}_{ij}\} = g^{kl}\{i, l, j\}$ and $g^k_{jl} = g^{ks}g_{jsl}$. In the following, the suffix 0 means contraction by y. Contracting (1.6) by y^j, we get

$$(1.7) \qquad F^k_0 = \{{}^{\,k}_{00}\} = 2G^k.$$

Substituting (1.7) into (1.6), we have

$$(1.8) \qquad F^k_j = \{{}^{\,k}_{0j}\} - g^k_{lj}\{{}^{\,l}_{00}\} = \frac{1}{2}\frac{\partial\{{}^{\,k}_{00}\}}{\partial y^j} = G^k_j.$$

Hence, from (c) we obtain

$$(1.9) \qquad F^k_{ij} = \frac{1}{2}\frac{\partial^2\{{}^{\,k}_{00}\}}{\partial y^i\partial y^j} = G^k_{ij}.$$

These three equations (1.7), (1.8) and (1.9) complete the proof.

Remark 2. Differentiating (1.5) by y^i and using (1.4), we see

$$(1.10) \qquad 2g_{ijk;0} + g_{l0}\frac{\partial F^l_{ji}}{\partial y^k} = 0.$$

Therefore, from (1.4) we obtain

$$(1.11) \qquad g_{jk;i} = -2g_{ijk;0}.$$

This is a well-known equation, showing that the Berwald connection is not *h-metrical* in general although (a) is satisfied.

Remark 3. [M. Matsumoto] The equations (1.7) are also got directly as follows. Differentiating (a) by y^j, we see

$$(1.3)' \qquad \frac{\partial^2 F}{\partial x^i\partial y^j} - \frac{\partial^2 F}{\partial y^k\partial y^j}F^k_i - \frac{\partial F}{\partial y^k}\frac{\partial F^k_i}{\partial y^j} = 0.$$

Contracting (1.3)′ by y^i and using (b), (c) and (d), we have

$$\frac{\partial^2 F}{\partial x^0 \partial y^j} - \frac{\partial^2 F}{\partial y^k \partial y^j} F^k_0 - \frac{\partial F}{\partial y^k} F^k_j = 0. \tag{1.12}$$

The last equations and (a) imply

$$g_{kj} F^k_0 = \frac{\partial^2 F}{\partial x^0 \partial y^j} - \frac{\partial F}{\partial x^j}. \tag{1.13}$$

Thus we obtain

$$F^k_0 = g^{kj}\left(\frac{\partial^2 F}{\partial x^0 \partial y^j} - \frac{\partial F}{\partial x^j}\right) = 2G^k. \tag{1.7}$$

§ 2. Minkowskian product of Finsler spaces

Let Ψ be a C^0 function: $\boldsymbol{R}^+ \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that

(α) $\Psi(q^1, q^2) = 0$ if and only if $(q^1, q^2) = (0, 0)$,

(β) $\Psi(tq^1, tq^2) = t\Psi(q^1, q^2)$ for $t \in \boldsymbol{R}^+$,

(γ) Ψ is C^∞ on $\boldsymbol{R}^+_0 \times \boldsymbol{R}^+_0$, where $\boldsymbol{R}^+_0$ is the positive real line,

(δ) $\Psi_A \equiv \dfrac{\partial \Psi}{\partial q^A} \neq 0$ on $\boldsymbol{R}^+_0 \times \boldsymbol{R}^+_0$, $A = 1, 2$,

(ε) $\Delta \equiv \Psi_1 \Psi_2 - 2\Psi\Psi_{12} \neq 0$ on $\boldsymbol{R}^+_0 \times \boldsymbol{R}^+_0$, where $\Psi_{12} = \dfrac{\partial^2 \Psi}{\partial q^1 \partial q^2}$.

Remark 4. $\tilde{\Psi}(p^1, p^2) \equiv \Psi(\frac{1}{2}(p^1)^2, \frac{1}{2}(p^2)^2)$ is C^∞ on $\boldsymbol{R}_0 \times \boldsymbol{R}_0$ if and only if $\Psi(q^1, q^2)$ is C^∞ on $\boldsymbol{R}^+_0 \times \boldsymbol{R}^+_0$, where $\boldsymbol{R}_0$ is the real line except 0. Putting $G_{AB} = \dfrac{\partial^2 \tilde{\Psi}}{\partial p^A \partial p^B}$, we have

$$\Delta = G_{11} G_{22} - (G_{12})^2$$

on $\boldsymbol{R}_0 \times \boldsymbol{R}_0$. Moreover, putting $\Phi(p^1, p^2) = \sqrt{2\tilde{\psi}(p^1, p^2)}$, we get

$$\Delta = -\frac{\Phi^3}{p^1 p^2} \frac{\partial^2 \Phi}{\partial p^1 \partial p^2}$$

on $\boldsymbol{R}_0 \times \boldsymbol{R}_0$. Therefore, $\Delta = 0$ identically if and only if Φ is linear homogeneous with respect to p^1 and p^2 on $\boldsymbol{R}_0 \times \boldsymbol{R}_0$. For example, $\Psi = ((q^1)^r + (q^2)^r)^{\frac{1}{r}}$ satisfies all conditions $(\alpha) \sim (\varepsilon)$ provided $r \neq \frac{1}{2}$, but it does not satisfy (ε) if $r = \frac{1}{2}$.

Definition 2. A Finsler space (M, F) is called the *Minkowskian product* of Finsler spaces $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ with respect to a *product function* Ψ if $M = M_1 \times M_2$ and $F = \Psi(\overset{1}{F}, \overset{2}{F})$. Especially, in case of $\Psi = q^1 + q^2$ it is called the *Euclidean product* of these Finsler spaces.

Let (x^a) (resp. (x^α)) be local coordinates of M_1 (resp. M_2) and (x^a, y^a) (resp.

(x^α, y^α)) the canonical local coordinates of TM_1 (resp. TM_2). We shall refer to the local coordinates $(x^i)=(x^a, x^\alpha)$ adapted to the product manifold $M_1 \times M_2 = M$ and the canonical local coordinates $(x^i, y^i)=(x^a, x^\alpha, y^a, y^\alpha)$ adapted to $T(M_1 \times M_2)= TM$. Throughout the remainder of the present paper, the ranges of indices are as follows: $a, b, c, \ldots = 1, 2, \ldots, r$; $\alpha, \beta, \gamma, \ldots = r+1, r+2, \ldots, r+s\ (=n)$; $i, j, k, \ldots = 1, 2, \ldots, n$; $A, B = 1, 2$.

Assuming that $\overset{A}{F}$ is C^∞ on $(TM_A)_0$ and putting $\overset{1}{g}_{ab}=\dfrac{\partial^2 \overset{1}{F}}{\partial y^a \partial y^b}$, $\overset{2}{g}_{\alpha\beta}=\dfrac{\partial^2 \overset{2}{F}}{\partial y^\alpha \partial y^\beta}$, the metric tensor g_{ij} of the Minkowskian product (M, F) of Finsler spaces $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ with respect to Ψ is written

$$
(2.1)\qquad \begin{cases} g_{ab}=\lambda_a^c \overset{1}{g}_{cb}, \\ g_{a\beta}=\lambda_a^\gamma \overset{2}{g}_{\gamma\beta}=\lambda_\beta^c \overset{1}{g}_{ca}, \\ g_{\alpha\beta}=\lambda_\alpha^\gamma \overset{2}{g}_{\gamma\beta}, \end{cases}
$$

on $(TM_1)_0 \times (TM_2)_0$, where

$$
(2.2)\qquad \begin{cases} \lambda_a^c=\Psi_1 \delta_a^c+\Psi_{11} \overset{1}{y}_a y^c, & \lambda_a^\gamma=\Psi_{12} \overset{1}{y}_a y^\gamma, \\ \lambda_\alpha^c=\Psi_{12} \overset{2}{y}_\alpha y^c, & \lambda_\alpha^\gamma=\Psi_2 \delta_\alpha^\gamma+\Psi_{22} \overset{2}{y}_\alpha y^\gamma, \end{cases}
$$

$$
\overset{1}{y}_a=\frac{\partial \overset{1}{F}}{\partial y^a}, \quad \overset{2}{y}_\alpha=\frac{\partial \overset{2}{F}}{\partial y^\alpha}, \quad \Psi_{AB}=\frac{\partial^2 \Psi}{\partial q^A \partial q^B}.
$$

If conditions (δ) and (ε) are satisfied, then the reciprocal (ω_j^k) of (λ_i^j) is given by

$$
(2.3)\qquad \begin{cases} \omega_b^c=\dfrac{1}{\Psi_1}\delta_b^c+\Psi^{11} \overset{1}{y}_b y^c, & \omega_b^\gamma=\Psi^{12} \overset{1}{y}_b y^\gamma, \\ \omega_\beta^c=\Psi^{12} \overset{2}{y}_\beta y^c, & \omega_\beta^\gamma=\dfrac{1}{\Psi_2}\delta_\beta^\gamma+\Psi^{22} \overset{2}{y}_\beta y^\gamma, \end{cases}
$$

on $(TM_1)_0 \times (TM_2)_0$, where

$$
(2.4)\qquad \Psi^{11}=\frac{-\Psi_{11}\Psi_2}{\Delta \Psi_1}, \quad \Psi^{12}=\frac{-\Psi_{12}}{\Delta}, \quad \Psi^{22}=\frac{-\Psi_{22}\Psi_1}{\Delta \Psi_2}.
$$

Moreover, if $(\overset{1}{g}_{ab})$ (resp. $(\overset{2}{g}_{\alpha\beta})$) is nondegenerate on $(TM_1)_0$ (resp. $(TM_2)_0$) and its reciprocal is denoted by $(\overset{1}{g}{}^{bc})$ (resp. $(\overset{2}{g}{}^{\beta\gamma})$), then the reciprocal (g^{jk}) of (g_{ij}) is written as

$$
(2.5)\qquad \begin{cases} g^{ac}=\overset{1}{g}{}^{ab}\omega_b^c, \\ g^{a\gamma}=\overset{1}{g}{}^{ab}\omega_b^\gamma=\overset{2}{g}{}^{\gamma\beta}\omega_\beta^a, \\ g^{\alpha\gamma}=\overset{2}{g}{}^{\alpha\beta}\omega_\beta^\gamma \end{cases}
$$

on $(TM_1)_0 \times (TM_2)_0$.

Remark 5. It is obvious [6] that if the fundamental function F of the Euclidean product (M, F) of Finsler spaces $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ is C^2 on $(TM)_0$, then all (M, F), $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ are Riemann spaces. In case of Minkowskian product, it is not true in general. For example, when the fundamental function $\overset{1}{F}=\frac{1}{2}\sqrt{u_{abcd}(x^a)y^a y^b y^c y^d}$ (resp. $\overset{2}{F}=\frac{1}{2}\sqrt{v_{\alpha\beta\gamma\delta}(x^\alpha)y^\alpha y^\beta y^\gamma y^\delta}$) and the product function $\Psi=\sqrt{(q^1)^2+(q^2)^2}$, the fundamental function $F=\Psi(\overset{1}{F}, \overset{2}{F})$ of Minkowskian product of $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ is C^2 on $(TM)_0$, but then (g_{ij}) is degenerate on $s_0(TM_1)\times(TM_2)_0 \cup (TM_1)_0 \times s_0(TM_2)$. (See Remark 8)

Now, we shall show that the Berwald connection of the Minkowskian product space is independent of Ψ.

Theorem 2. *The Berwald connection of the Minkowskian product of Finsler spaces $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ with respect to any product function Ψ coincides with the Berwald connection of the Euclidean product of these Finsler spaces on $(TM_1)_0\times(TM_2)_0$ provided that $\overset{1}{g}_{ab}$ (resp. $\overset{2}{g}_{\alpha\beta}$) exists and is nondegenerate on $(TM_1)_0$ (resp. $(TM_2)_0$).*

Proof. In terms of canonical local coordinates, let $\overset{1}{G}{}^a_{bc}$ (resp. $\overset{2}{G}{}^\alpha_{\beta\gamma}$) be coefficients of the Berwald connection of $(M_1, \overset{1}{F})$ (resp. $(M_2, \overset{2}{F})$). Next, in terms of the adapted canonical local coordinates, let G^i_{jk} be coefficients of the Berwald connection $B\Gamma^e$ of the Euclidean product of $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$. Then it is obvious that on $(TM_1)_0\times(TM_2)_0$

$$(2.6)\qquad \begin{cases} G^a_{bc}=\overset{1}{G}{}^a_{bc}, \quad G^\alpha_{\beta\gamma}=\overset{2}{G}{}^\alpha_{\beta\gamma}, \\ G^a_{\beta\gamma}=G^\alpha_{bc}=G^a_{\beta c}=G^\alpha_{\beta c}=0. \end{cases}$$

Now, $B\Gamma^e$ is an h-connection on M which satisfies three axioms (b), (c) and (d) in in Theorem 1. As for (a), the h-covariant derivatives with respect to $B\Gamma^e$ of the fundamental function $F=\Psi(\overset{1}{F}, \overset{2}{F})$ of the Minkowskian product space are written as

$$(2.7)\qquad \begin{cases} F_{;a}=\dfrac{\partial\Psi}{\partial q^1}\left(\dfrac{\partial \overset{1}{F}}{\partial x^a}-\dfrac{\partial \overset{1}{F}}{\partial y^b}G^b_{ca}y^c\right)-\dfrac{\partial\Psi}{\partial q^2}\dfrac{\partial \overset{2}{F}}{\partial y^\beta}G^\beta_{ka}y^k, \\ F_{;\alpha}=\dfrac{\partial\Psi}{\partial q^2}\left(\dfrac{\partial \overset{2}{F}}{\partial x^\alpha}-\dfrac{\partial \overset{2}{F}}{\partial y^\beta}G^\beta_{\gamma\alpha}y^\gamma\right)-\dfrac{\partial\Psi}{\partial q^1}\dfrac{\partial \overset{1}{F}}{\partial y^b}G^b_{k\alpha}y^k. \end{cases}$$

Therefore $F_{;i}=0$ are satisfied, because the h-covariant derivatives of the fundamental function $\overset{1}{F}$ (resp. $\overset{2}{F}$) with respect to the Berwald connection $(\overset{1}{G}{}^a_{bc})$ (resp. $(\overset{2}{G}{}^\alpha_{\beta\gamma})$) of $(M_1, \overset{1}{F})$ (resp. $(M_2, \overset{2}{F})$) vanish and $G^b_{k\alpha}=G^\beta_{ka}=0$. Hence, from uniqueness of the Berwald connection in Theorem 1 this connection $B\Gamma^e=(G^i_{jk})$ is nothing but the Berwald connection of Finsler space (M, F), $M=M_1\times M_2$, $F=\Psi(\overset{1}{F}, \overset{2}{F})$.

A Finsler space is called a *Berwald space* iff its Berwald connection is linear, or the coefficients of the Berwald connection do not depend on y^i. Concerning Berwald spaces, the following are immediately deduced from Theorem 2.

Corollary 1. [7] *If both Finsler spaces* $(M_1, \overset{1}{F})$ *and* $(M_2, \overset{2}{F})$ *are Berwald spaces, then the Minkowskian product of these spaces with respect to any* Ψ *is a Berwald space.*

In particular we have

Corollary 2. *If both* $(M_1, \overset{1}{F})$ *and* $(M_2, \overset{2}{F})$ *are Riemann spaces, then the Minkowskian product of these spaces with respect to any* Ψ *is a Berwald space and its Berwald connection is Levi-Civita's connection on the product Riemann manifold* $M_1 \times M_2$.

§3. Geodesics of Minkowskian product space

A *geodesic* $x^i(t)$ *with an affine parameter* t of a Finsler space (M, F), or an extremal of the energy integral $\int F\left(x^i(t), \frac{dx^i}{dt}\right)dt$, satisfies Euler-Lagrange differential equations $e_i=0$, where

$$e_i = \frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x_i}, \qquad y^i = \frac{dx^i}{dt}.$$

That t is an affine parameter is implied from

Lemma 1. $\frac{dF}{dt} = e_i y^i$.

Proof. $e_i y^i = \frac{d}{dt}\left(\frac{\partial F}{\partial y^i} y^i\right) - \left(\frac{\partial F}{\partial y^i}\frac{dy^i}{dt} + \frac{\partial F}{\partial x^i} y^i\right) = \frac{dF}{dt}.$

Now, let (M, F) be the Minkowskian product of Finsler spaces $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ with respect to Ψ. Preparing for the following theorem, stronger conditions for Ψ than (γ), (δ) and (ε) must be intrdoduced as follows;

$(s\gamma)$ Ψ is C^∞ on $(\boldsymbol{R}^+ \times \boldsymbol{R}^+)_0 \equiv \boldsymbol{R}^+ \times \boldsymbol{R}^+ - (0, 0)$,

$(s\delta)$ $\Psi_A \neq 0$ on $(\boldsymbol{R}^+ \times \boldsymbol{R}^+)_0$, $A=1, 2$,

$(s\varepsilon)$ $\Delta \equiv \Psi_1\Psi_2 - 2\Psi\Psi_{12} \neq 0$ on $(\boldsymbol{R}^+ \times \boldsymbol{R}^+)_0$.

Putting

$$\overset{1}{e}_a = \frac{d}{dt}\left(\frac{\partial \overset{1}{F}}{\partial y^a}\right) - \frac{\partial \overset{1}{F}}{\partial x^a}, \qquad \overset{2}{e}_\alpha = \frac{d}{dt}\left(\frac{\partial \overset{2}{F}}{d y^\alpha}\right) - \frac{\partial \overset{2}{F}}{\partial x^\alpha},$$

we have

Lemma 2. (1) *Under the condition* $(s\gamma)$, *if* $\overset{1}{e}_a=0$ *and* $\overset{2}{e}_\alpha=0$, *then* $e_i=0$. (2) *Under the conditions* $(s\gamma)$, $(s\delta)$ *and* $(s\varepsilon)$, *if* $e_i=0$, *then* $\overset{1}{e}_a=0$ *and* $\overset{2}{e}_\alpha=0$.

Proof. From $F=\Psi(\overset{1}{F}, \overset{2}{F})$, we see

$$(3.1)\qquad \begin{cases} e_a=\Psi_1\overset{1}{e}_a+\left(\Psi_{11}\dfrac{d\overset{1}{F}}{dt}+\Psi_{12}\dfrac{d\overset{2}{F}}{dt}\right)\overset{1}{y}_a, \\ e_\alpha=\Psi_2\overset{2}{e}_\alpha+\left(\Psi_{21}\dfrac{d\overset{1}{F}}{dt}+\Psi_{22}\dfrac{d\overset{2}{F}}{dt}\right)\overset{2}{y}_\alpha. \end{cases}$$

By Lemma 1 $d\overset{1}{F}/dt=\overset{1}{e}_a y^a$ and $d\overset{2}{F}/dt=\overset{2}{e}_\alpha y^\alpha$; therefore we have

$$(3.2)\qquad \begin{cases} e_a=\lambda_a^b\overset{1}{e}_b+\lambda_a^\beta\overset{2}{e}_\beta, \\ e_\alpha=\lambda_\alpha^b\overset{1}{e}_b+\lambda_\alpha^\beta\overset{2}{e}_\beta \end{cases}$$

on $(TM)_0$ provided $(s\gamma)$ is satisfied (See (2.2)). Moreover, the last equations lead us to

$$(3.3)\qquad \begin{cases} \overset{1}{e}_b=\omega_b^c e_c+\omega_b^\gamma e_\gamma, \\ \overset{2}{e}_\beta=\omega_\beta^c e_c+\omega_\beta^\gamma e_\gamma \end{cases}$$

on $(TM)_0$ provided $(s\gamma)$, $(s\delta)$ and $(s\varepsilon)$ are satisfied (See (2.3)). (3.2) and (3.3) complete the proofs of (1) and (2) respectively.

Remark 6. $\Psi=((q^1)^r+(q^2)^r)^{\frac{1}{r}}$, $r=2, 3,\ldots$, satisfies $(s\gamma)$, but not $(s\delta)$ and $(s\varepsilon)$. Following Ψ's are examples satisfying all $(s\gamma)$, $(s\delta)$ and $(s\varepsilon)$.

(1) $\Psi=q^1+q^2$ (Euclidean)

(2) $\Psi=\frac{1}{2}\{q^1+q^2+((q^1)^r+(q^2)^r)^{\frac{1}{r}}\}$, $r=2, 3,\ldots$, [7]

(3) $\Psi=((q^1)^2+3q^1q^2+(q^2)^2)^{\frac{1}{2}}$

(4) $\Psi=(2q^1+3q^2)^2/(q^1+q^2)$

(5) $\Psi=((q^1)^2+3q^1q^2+(q^2)^2)/(q^1+q^2)$

Concerning geodesics of the Minkowskian product of two Finsler spaces, we obtain from Lemma 2 the following theorem, which is a generalization of the result [8] showed in case of the Minkowskian product of two Riemann spaces.

Theorem 3. *Let a Finsler space (M, F) be the Minkowskian product of Finsler spaces $(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ with respect to a product function Ψ.*

(a) *Under the condition $(s\gamma)$, any geodesic of $(M_1, \overset{1}{F})$ (resp. $(M_2, \overset{2}{F})$) is a geodesic of (M, F). That is to say, the Finsler space $(M_1, \overset{1}{F})$ (resp. $(M_2, \overset{2}{F})$) is a totally geodesic subspace of the Finsler space (M, F).*

(b) *Under the conditions $(s\gamma)$, $(s\delta)$ and $(s\varepsilon)$, the projection of any geodesic of (M, F) into M_1 (resp. M_2) is a geodesic of $(M_1, \overset{1}{F})$ (resp. $(M_2, \overset{2}{F})$).*

Proof (a) As $\overset{2}{F}$ is C^1 on TM_2 and positively homogeneous of degree 2 with respect to y^α, we see $\dfrac{\partial \overset{2}{F}}{\partial y^\alpha}=0$ on $s_0(TM_2)$. Therefore, we have $\overset{2}{e}_\alpha=0$ if $y^\alpha(t)=0$ identically.

A geodesic $(x^a(t), x^\alpha)$ (x^α being constant) with an affine parameter t of the subspace $(M_1, \overset{1}{F})$ satisfies $\overset{1}{e}_a=0$, and $\dfrac{dx^\alpha}{dt}=0$ imply $\overset{2}{e}_\alpha=0$ by the above notice. Hence, by Lemma 2 (1) we obtain $e_i=0$. Thus the curve $(x^a(t), x^\alpha)$ is a geodesic with an affine parameter t of (M, F). As to geodesic of $(M_2, \overset{2}{F})$, it is proved similarly.

(b) We consider a geodesic $(x^a(t), x^\alpha(t))$ with an affine parameter of (M, F). By Lemma 2 (2) $e_i=0$ imply $\overset{1}{e}_a=0$. Hence, the curve $(x^a(t), x^\alpha)$ (x^α being constant) projected into M_1 is a geodesic with an affine parameter t of $(M_1, \overset{1}{F})$. The proof as to the curve projected into M_2 is done similarly.

Remark 7. If $\overset{A}{F}$ is C^2 on $(TM_A)_0$, $A=1, 2$, then we have

$$
(3.4)\qquad \begin{cases} \overset{1}{e}_a=\overset{1}{g}_{ab}\dfrac{dy^b}{dt}+2\overset{1}{G}_a & \text{on}\quad (TM_1)_0, \\[2ex] \overset{2}{e}_\alpha=\overset{2}{g}_{\alpha\beta}\dfrac{dy^\beta}{dt}+2\overset{2}{G}_\alpha & \text{on}\quad (TM_2)_0, \end{cases}
$$

where

$$
2\overset{1}{G}_a=\frac{\partial^2\overset{1}{F}}{\partial x^b\partial y^a}y^b-\frac{\partial\overset{1}{F}}{\partial x^a},\quad 2\overset{2}{G}_\alpha=\frac{\partial^2\overset{2}{F}}{\partial x^\beta\partial y^\alpha}y^\beta-\frac{\partial\overset{2}{F}}{\partial x^\alpha}.
$$

The equations (3.4) together with (3.2) and (2.1) lead us to

$$
(3.5)\qquad \begin{cases} G_a=\lambda_a^b\overset{1}{G}_b+\lambda_a^\beta\overset{2}{G}_\beta, \\ G_\alpha=\lambda_\alpha^b\overset{1}{G}_b+\lambda_\alpha^\beta\overset{2}{G}_\beta, \end{cases}
$$

where

$$
2G_i=\frac{\partial^2F}{\partial x^j\partial y^i}y^j-\frac{\partial F}{\partial x^i}.
$$

Moreover, if $(\overset{1}{g}_{ab})$ (resp. $(\overset{2}{g}_{\alpha\beta})$) is nondegenerate on $(TM_1)_0$ (resp. $(TM_2)_0$), then from (3.5) and (2.5) we obtain

$$
(3.6)\qquad G^a=\overset{1}{G}{}^a,\quad G^\alpha=\overset{2}{G}{}^\alpha
$$

on $(TM_1)_0\times(TM_2)_0$, where

$$
G^i=g^{ij}G_j,\quad \overset{1}{G}{}^a=\overset{1}{g}{}^{ab}\overset{1}{G}_b,\quad \overset{2}{G}{}^\alpha=\overset{2}{g}{}^{\alpha\beta}\overset{2}{G}_\beta.
$$

This is another proof of Theorem 2.

Remark 8. If Ψ satisfies $(s\gamma)$, $(s\delta)$ and $F=\Psi(\overset{1}{F}, \overset{2}{F})$ is C^2 on $(TM)_0$, then both

$(M_1, \overset{1}{F})$ and $(M_2, \overset{2}{F})$ are Riemann spaces. Hence, by Corollary 2 of Theorem 2 the Minkowskian product (M, F) of these spaces is Berwald space with Levi-Civita's connection.

Because $\overset{1}{y}_a \equiv \dfrac{\partial \overset{1}{F}}{\partial y^a} = \dfrac{1}{\Psi_1} \dfrac{\partial F}{\partial y^a}$ $\left(\text{resp. } \overset{2}{y}_\alpha \equiv \dfrac{\partial \overset{2}{F}}{\partial y^\alpha} = \dfrac{1}{\Psi_2} \dfrac{\partial F}{\partial y^\alpha}\right)$

are C^1 on $(TM)_0$ under the above conditions. Therefore, $\overset{1}{y}_a$ (resp. $\overset{2}{y}_\alpha$) are C^1 on TM_1 (resp. TM_2). Thus $\overset{1}{F}$ (resp. $\overset{2}{F}$) is C^2 on TM_1 (resp. TM_2).

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