J. Math. Kyoto Univ. (JMKYAZ)
22-4 (1983) 767-777

Curves with negative self intersection on rational surfaces

By

N. Mohan KUMAR* and M. Pavaman MURTHY

(Communicated by Prof. M. Nagata Sept. 18, 1981)

§0. Introduction

Here we study normal local rings A of rational surfaces with their completion of the type $K[[X^n, X^{n-1}Y, X^iY^{n-i},\dots, Y^n]]$. It is well known that such local rings are obtained by blowing down non-singular rational curves with self intersection $-n$. So the problem is to study Zariski neighbourhoods of such curves. We introduce Logarithmic Kodaira dimension for embeddings of curves (following Iitaka) for classification of such embeddings. For instance logarithmic Kodaira dimension is $-\infty$ would precisely mean that $A\cong K[X^n, X^{n-1}Y,\dots, X^iY^{n-i},\dots, Y^n]_{\mathfrak{m}}$, ($\mathfrak{m}$ = origin.) We would show that for $n\leqslant 3$, Kodaira dimension is $-\infty$. The classification of local rings with Kodaira dimension $-\infty$ is done in §2 and those with Kodaira dimension zero or one is done in §3. These results are related to a theorem of Coolidge which gives necessary and sufficient condition for a plane curve to be transformed into a straight line by a Cremona transformation. We reproduce his proof for completeness and recover the same result in our set up. In §4, we will also see that the class groups of some of these local rings are smaller than one would expect!

§1. Here we shall be working over an algebraically closed field k. Surface, unless otherwise mentioned, would mean a non-singular rational projective surface.

Let D be a reduced projective curve and X, X', smooth projective surfaces. Let $i: D\to X$ and $i': D\to X'$ be two closed embeddings. We say that i and i' are equivalent if there exist Zariski neighbourhoods U, U' in X and X' of $i(D)$ and $i'(D)$ and an isomorphism $f: U\to U'$ such that $f\circ i=i'$. We are interested in studying the equivalence classes of such embeddings.

Remarks. 1) If (i, D, X) and (i', D, X') are equivalent then the normal bundles $N_{D/X}$ and $N_{D/X'}$ are isomorphic. In particular the intersection matrices of D in X and X' are the same.

2) If D is an irreducible curve in X and X' with isomorphic formal neighbour-

* Both the authors were partially supported by NSF grants.

hoods and $D^2>0$, then the embeddings are equivalent [H]. This is trivially false if $D^2<0$. For instance, if P, Q are points on a smooth surface X such that the local rings $\mathcal{O}_{X,P} \not\approx \mathcal{O}_{X,Q}$, then consider the exceptional curves of the blowing up of P and Q.

3) Let $E \hookrightarrow X$ and $E' \hookrightarrow X'$ be two exceptional curves of the first kind on smooth rational surfaces. Then the two embeddings are equivalent. This immediately follows from the fact that if P is a smooth point on a rational surface then $\mathcal{O}_{X,P} \simeq k[X, Y]_{(X,Y)}$. [N]

4) If $D \simeq \boldsymbol{P}^1$, $D^2=-n$, $n \geqslant 4$ then there exist inequivalent embeddings of D in smooth rational surfaces.
[See §3].

Kodaira dimension of germs of embeddings:

Let (i, D, X) be an embedding of D in X. Let K denote a canonical divisor of X. We define the Kodaira dimension of (i, D, X) to be the dimension of the image of $X \to \boldsymbol{P}(H^0(m(D+K)))$ for $m \gg 0$. By convention we say that Kodaira dimension of $(i, D, X) = -\infty$ if $|m(D+K)| = \emptyset\ \forall m > 0$. We denote the Kodaira dimension by $\kappa(i, D, X)$.

Lemma 1.1. *If $D \hookrightarrow X$ and $X' = dil_P(X)$, $P \notin D$, then $\kappa(i, D, X) = \kappa(i', D, X')$.*

Proof. The natural map, $H^0(m(D+K_X)) \to H^0(m(D+K_{X'}))$ is an isomorphism for every m.

Corollary 1.2. *(i, D, X) depends only on the equivalence class of embeddings of D.*

When there is no confusion, we will denote $\kappa(i, D, X)$ by $\kappa(D)$.

Lemma 1.3. *Let $D \to X$ and $P \in D$ be a smooth point of D. Let $X' = dil_P(X)$ and $D' =$ the proper transform of D. Then $\kappa(D) = \kappa(D')$.*

Proof. The pull back of $m(D+K_X) = m(D'+K_{X'})$.

Definition. (i, D, X) is a relatively minimal embedding, if every exceptional curve of the first kind on X meets D.

Remark. It is clear that every embedding (i, D, X) 'dominates' a relatively minimal embedding.

Proposition 1.4. *If (i, D, X) is an embedding with $\kappa(D) \geqslant 0$, then there exists a unique relatively minimal embedding in this class.*

Proof. Let (i, D, X) and (i, D, X') be two relatively minimal embeddings in the same equivalence class. Then we have by definition a rational map $f\colon X \to X'$ which is an isomorphism in a neighbourhood of D in X and X'. Since neither f nor f^{-1} is a morphism, there exists a curve C in X such that $f(C) = a$ point C becomes an exceptional curve of the first kind after a few blowing ups. Since $C \cap D = \emptyset$, C cannot be an exceptional curve of the first kind and hence $C^2 \geqslant 0$. So $C \cdot K < 0$. Hence $C \cdot (D+K) < 0$. Since $C^2 \geqslant 0$, we get $|m(D+K)| = \emptyset\ \forall m > 0$, which is a contradiction.

Remark. This is analogous to the Castelnuovo-Enriques-Zariski theorem that all the classes of surfaces except the classes of ruled surfaces have absolute minimal models.

Proposition 1.5. *$D \subset X$ and assume D has only ordinary normal crossings. Suppose $X \backslash D$ has an algebraic family of complete rational curves, possibly singular and* char $k=0$. *Then $\kappa(D, X)=-\infty$.*

Proof. By assumption, we have a cycle $\Delta \subset X \times C$, where C parametrises the family of rational curves on $X \backslash D$. We may assume that C is a non-singular curve (not complete) and Δ to be irreducible. Desingularising Δ and choosing a small enough irreducible open set in it, we may assume that there exists a curve Γ such that $\boldsymbol{P}^1 \times \Gamma$ dominates Δ. (char $k=0$ is used in this process) Since under the first projection Δ maps into $X \backslash D$, by hypothesis, we see that $\boldsymbol{P}^1 \times \Gamma$ dominates $X \backslash D$. Hence by [I], $\kappa(D, X)=-\infty$.

Lemma 1.6. *$D \hookrightarrow X$, reduced curve. Suppose there exists uncountably many complete rational curves (possibly singular) in $X \backslash D$. Then there exists an algebraic family of complete rational cuves in $X \backslash D$.*

Proof. This follows easily from considering C_d=Chow family of rational curves of 'degree' d inside $X \backslash D$.

Corollary 1.7. *If* char $k=0$, *k is uncountable, D has only ordinary normal crossings then the hypothesis in lemma* $1.6 \Rightarrow \kappa(D, X)=-\infty$.

§2. Kodaira dimension $-\infty$

From now on we assume that the surfaces are rational and $D \approx \boldsymbol{P}^1$ with $(i(D))^2<0$. These assumptions are motivated by Remark 2, §1. Note that such a D can always be blown down to a normal point and the formal neighbourhood of such a D is completely determined by the (negative) number $D^2=-n$, and in fact, the completion of this normal local ring is isomorphic to $K[\![X^n, X^{n-1}Y, \ldots, X^iY^{n-i}, \ldots, Y^n]\!]$.

We denote by D_n' the unique section of $\boldsymbol{F}_n=\boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$ with $D_n^2=-n$.

Theorem 2.1. *Let $D \hookrightarrow X$, $D^2=-n$ with $n>0$. Then the following are equivalent.*

a) *(D, X) is equivalent to $(D_n, \boldsymbol{F}_n)$.*
b) *There exists $\sigma: Y \rightarrow X$ a birational morphism such that $\sigma|_{\sigma^{-1}(D)}: \sigma^{-1}(D) \rightarrow D$ is an isomorphism and $Y \backslash \sigma^{-1}(D)$ contains an open set of the form $U \times \boldsymbol{P}^1$, U a curve.*
c) $|mK+nD|=\emptyset \ \forall m>0$.
d) $\kappa(D)=-\infty$.

Proof. a)$\Rightarrow$b)

By hypothesis we have a rational map, $p: X \rightarrow \boldsymbol{F}_n$ which is an isomorphism between a neighbourhood of D and that of D_n. By blowing up outside D, we get $\sigma: Y \to X$ and $p \circ \sigma$ is a morphism, $Y \to \boldsymbol{F}_n$. Consider $G \in |D_n + nF|$ irreducible, where F is a fibre of $\boldsymbol{F}_n$. Then G is a non-singular curve with $h^0(\boldsymbol{F}_n, G) \geqslant 2$ and $G.D_n = 0$. Since $|G|$ has no base points, we may further assume that G does not pass through the base points of the birational map, $(p \circ \sigma)^{-1}$ and then $(p \circ \sigma)^{-1}(G) = G'$ is a non-singular rational curve on Y with $G'^2 \geqslant 0$ and $G' \cap \sigma^{-1}(D) = \emptyset$. Now blowing up further on G' we may assume that $G'^2 = 0$ and then b) is clear.

b)$\Rightarrow$c)

Without loss of generality, we may assume that we have a morphism $\pi: X \to \boldsymbol{P}^1$ with general fibre $\boldsymbol{P}^1$ and D is contained in fibre. But for any fibre C of π, $|mK + nC| = \emptyset$, $m \geqslant 1$ and hence $|mK + nD| = \emptyset$ $m \geqslant 1$.

c)$\Rightarrow$d) Obvious

d)$\Rightarrow$a)

We start with the

Remark. Let $X' = dil_p(X)$, with $p \in D$. D' = proper transform of D in X'. Then (D, X) is equivalent to $(D_n, \boldsymbol{F}_n)$ if and only if (D', X') is equivalent to $(D_{n+1}, \boldsymbol{F}_{n+1})$.

We prove d)$\Rightarrow$a) by induction on n and the proof follows closely the proofs in [M$-$S]. If $n = 1$, result is clear from Remark 3, §1. If there exists an exceptional curve of the first kind, which meets D once, by blowing it down, we are done. If X is relatively minimal, again we are done. So we may assume that there exists at least one exceptional curve E of the first kind on X and for any such E, $E \cdot D \geqslant 2$. If E is such a curve, then $l(E + K + D) = E \cdot D - 1 \geqslant 1$ by Riemann-Roch.

Case 1. $(D+K)^2 \leqslant -2$.

By Fujita's theorem [F], choose $m > 0$ such that $|E + m(D+K)| \neq \emptyset$ and $|E + (m+1)(D+K)| = \emptyset$. Then, $(E + m(D+K)) \cdot K = -1 + m[(K+D)^2 + 2] \leqslant -1$. So $E + m(D+K) \neq 0$ and if C is any irreducible curve contained in $|E + m(D+K)|$, then C is a non-singular rational curve and $C \cdot D \leqslant 1$, because, $|C + D + K| = \emptyset$. If $C^2 \geqslant 0$ and $C \cdot D = 1$, then by blowing up on C outside D, we get an exceptional curve of the first kind meeting D once. By blowing this down, we are done by induction. If $C \cdot D = 0$, then there exists a morphism after blowing up on C, if necessary, from X to $\boldsymbol{P}^1$, such that D is part of a fibre. Now, since this fibre contains an exceptional curve, we may blow it down and eventually assume that there is one which meets D once and hence we are done.

If $C^2 < 0$ for every C in $|E + m(D+K)|$, choose one with $C \cdot K < 0$, since $K \cdot (E + m(D+K)) \leqslant -1$. But then C is an exceptional curve of the first kind and $C \cdot D \leqslant 1$, which is a contradiction.

Case 2. $(D+K)^2 \geqslant -1$.

Since $|2K + D| = \emptyset$, by Riemann Roch, we get that $|-D-K| \neq \emptyset$. But since $E \cdot D \geqslant 2$ for any exceptional curve of the first kind, we get $|E + D + K| \neq \emptyset$. But then $E = (E + D + K) + (-D - K)$ and since $l(E) = 1$ we see that $-D - K = E$ because $-D - K \neq 0$. Thus we see that there exists a unique exceptional curve of the first kind E on X, $E \cdot D = 2$, $E + D + K = 0$. Now, if F is any curve with $F^2 < 0$, $F \neq E$ or D, then $F \cdot K = -F \cdot E - F \cdot D \leqslant 0$ and hence F is a non-singular rational curve. Since F is not an exceptional curve, we get that $F^2 = -2$ and $F \cdot E = F \cdot D = 0$.

If there is no such curve F, then when we blow down E, we get a model which has at most one curve of negative self intersection and this curve is singular. Then the image must be $\boldsymbol{P}^2$. But then $X = \boldsymbol{F}_1$ and this is not possible. Again, if there is such an F, blowing E down we get a minimal model and hence must be $\boldsymbol{F}_2$, and there is a unique F. Let $D' =$ image of D in $\boldsymbol{F}_2$. Then $D \cap D_2 = \emptyset$ because D_2 is the image of F. Thus, $D' \sim r(D_2 + 2f)$, where f is a general fibre of $\boldsymbol{F}_2$. But $D'^2 = 4 - n = 2r^2$. Thus $n = 2$ or 4. If $n = 2$ then $r = 1$ and hence $p_a(D') = 0$, which is not possible. If $n = 4$, $r = 0$, which is also not possible. This proves the theorem.

Lemma 2.2. *Suppose* $(D+K)^2 \leqslant -2$ *and* $|m(D+K)| \neq \emptyset$ *for some* $m \geqslant 1$. *Then there exists an exceptional curve of the first kind such that* $E \cap D = \emptyset$.

Proof. Write $m(D+K) = rD + \Delta$, with $r > 0$ and Δ does not have D as a component. So $-2m \geqslant m(D+K)^2 = -2r + \Delta \cdot (D+K)$. $2(r-m) \geqslant \Delta \cdot (D+K)$. Since $|mK| = \emptyset$, $r < m$. Thus $\Delta \cdot (D+K) < 0$. So there exists an irreducible curve C in Supp $|\Delta|$ such that $C \cdot (D+K) < 0$. So $C \cdot \Delta < 0$ and hence $C^2 < 0$. Also $C \cdot K < 0$. So C is exceptional of the first kind. Now $C \cdot (D+K) < 0$ implies, $C \cdot D - 1 < 0 \Rightarrow C \cdot D = 0$.

Corollary 2.3. *If* $D \hookrightarrow X$ *is a minimal embedding and* $\kappa(D) \geqslant 0$, *then* $(D+K)^2 \geqslant -1$.

Corollary 2.4. *The following are equivalent*:

a) $\kappa(D) = -\infty$.

b) $|2K + D| = \emptyset$.

c) $|2(K+D)| = \emptyset$.

Proof. a)$\Rightarrow$b)$\Leftrightarrow$c) are obvious.

So assume b) and suppose $\kappa(D) \geqslant 0$. We may assume that the embedding is minimal. So $(D+K) \geqslant {}^2 -1$ by Corollary 2.3. So by Riemann-Roch, $l(-D-K) \geqslant 1$. So $m(D+K) = 0$, which is impossible, since $D(D+K) = -2$.

Corollary 2.5. *If* $D^2 \geqslant -3$, *then* $\kappa(D) = -\infty$.

Proof. Easy to see that $D.(D+2K) < 0$ and hence $l(D+2K) = l(2K) = 0$.

Remarks. 1) This result is true even for $D^2 \geqslant 0$ from Nagata's Theorem [N].

2) Let V be an affine, normal rational surface with one isolated singularity P, which is obtained by blowing down a non-singular rational curve D with self-intersection $-n$ and $\kappa(D) = -\infty$. Then every vector bundle on V is a direct sum of a trivial bundle and a line bundle. This easily follows since by Theorem 2.1, there exists sufficiently many rational curves on V, not passing through P and hence for any simple point $x \in V$, $k(x)$ is zero in $K_0(V)$.

We now make some remarks about a theorem of Coolidge [C] about plane curves transformable into straight lines. Let C be any irreducible curve on a smooth rational surface Y. Let $F: X \to Y$ be a birational morphism such that the proper tramsform D of C is smooth. We define $\bar{\kappa}(C, Y) = \kappa(D, X)$. It is easily checked that $\bar{\kappa}(C, Y)$ is independent of F.

Theorem 2.6. (Coolidge) *Let $C \hookrightarrow \boldsymbol{P}^2$ be an irreducible rational curve. Then there exists a Cremona transformation σ of $\boldsymbol{P}^2$ such that $\sigma(C)$ is a line if and only if $\bar{\kappa}(C, \boldsymbol{P}^2) = -\infty$.*

Proof. Suppose such a σ exists. Then, since $\kappa(L, \boldsymbol{P}^2) = -\infty$ for a line L, it easily follows that $\bar{\kappa}(C, \boldsymbol{P}^2) = -\infty$. Suppose now that $\bar{\kappa}(C, \boldsymbol{P}^2) = -\infty$. By Theorem 2.1, there exists a birational map $f: \boldsymbol{P}^2 \to \boldsymbol{F}_n$ for some n, such that $f(C) = D_n$. It is well known that there is a birational map $g: \boldsymbol{F}_n \to \boldsymbol{P}^2$ such that $g(D_n)$ is a line.

We now reproduce Coolidge's proof of the above theorem. The plan is to reduce the degree of C by quadratic transformations. A curve D of degree $n-3s$ is said to be an s^{th} adjoint of C if at any point $P \in C$, $m_P(D) \geqslant m_P(C) - s$, where for infinitely near points P of C, C and D are interpreted as the proper transforms. It is easy to check that $\bar{\kappa}(C, \boldsymbol{P}^2) = -\infty$ if and only if C has no adjoints.

Let $P_1 \in C$ be a point of largest multiplicity say r_1. Let $n - r_1 = 2s$. The case when $n - r_1$ is odd can be treated similarly. We claim that $r_1 > s$. If not, $n - 3r_1 = 2(s - r_1) > 0$. Then any curve of degree $n - 3r_1$ is an r_1^{th} adjoint. Suppose C has only one singular point P_1. Consider $(n-3)L$, where L is a line through P_1. The curve $(n-3)L$ is not an adjoint of C. Hence $n - 3 < r_1 - 1$. i.e. $r_1 \geqslant n - 1 \Rightarrow r_1 = n - 1$. Apply a standard quadratic transformation with cetres P_1, P_2, P_3 where P_2 and P_3 are suitable points on C to reduce the degree of C. So we may assume that C has at least two singular points. Assume C has exactly two singular points, P_1 and P_2. Then $(n-3s)L$, where L is the line joining P_1 and P_2 can easily seen to be an s^{th} adjoint. So we may assume that C has at least three singular points. Let $P_1, \dots, P_k$, $(k \geqslant 3)$ be the singular points of C, with multiplicities, $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_k$ respectively. We claim that $r_2 > s$. If not, $r_i \leqslant s$ for $i \geqslant 2$. Take $(n-3s)L$ where L is a line passing through P_1. It is immediate that this is an s^{th} adjoint of C.

We know claim that $r_1 + r_2 + r_3 > n$. If not, let L be the line joining P_1 and P_2. Then $(n - 3r_3)L$ is an r_3^{th} adjoint. If P_1, P_2, P_3 can be chosen to be the centres of a quadratic transformation, then we can reduce the degree of C. The only case where we have a problem is when P_2 and P_3 are both in the first order neighbourhood of P_1. Let P_j be any point with $r_j > s$. If P_j is not infinitely near to P_1, then by taking the image of P_j in $\boldsymbol{P}^2$, we have a point P_l in $\boldsymbol{P}^2$, $l \neq 1$ and $r_l > s$. Then P_1, P_2 and P_l form possible centres for a quadratic transformation which reduce the degree of C, since $r_1 + r_2 + r_l > n$. So we may assume that all such P_j's are infinitely near to P_1. If P_j is not in the first order neighbourhood of P_1, then by taking its images, we have, P_l, P_m two points, P_l is in the first order neighbourhood of P_m, P_m is in the first order neighbourhood of P_1 and r_l, $r_m > s$. Thus these three points P_1, P_l and P_m can be taken as the centres of a quadratic transformation and since $r_1 + r_l + r_m > n$, the degree of C will drop. So we may assume that any point P_j with $r_j > s$ lies on the first order neighbourhood of P_1 unless it is P_1 itself. Let $P_1, \ldots, P_j$ be the points with multiplcity bigger than s. Let L_i be the line joining P_1 and P_i, $2 \leqslant i \leqslant j$. We have $r_1 > r_2 + \cdots + r_j$ and hence $n - 3s > \sum_{i=2}^{j} r_i - s \geqslant \sum_{i=2}^{j} (r_i - s)$. Consider $\sum_{i=2}^{j} (r_i - s)L_i + M$ where M is an $n - 3s - \sum_{i=2}^{j} (r_i - s)$ degree curve passing through P_1. Then it is easy to see that this is an s^{th} adjoint, giving us the required contradiction.

§ 3. The case when Kodaira dimension is bigger than or equal to zero

As before let $D \hookrightarrow X$; $D \approx \boldsymbol{P}^1$ and $D^2 < 0$. Note that if $\kappa(D) \geqslant 0$ then $|-D-K| = \emptyset$.

Theorem 3.1. *The following are equivalent.*

a) $D^2 = -n$, $l(2K+D) = 1$ *and every exceptional curve of the first kind meets* D *at least twice.*

b) $(D+K)^2 = -1$, $\kappa(D) \geqslant 0$ *and every exceptional curve meets* D *at least twice.*

c) $K^2 = -1$, $l(-K) = 0$, $n = 4$.

Proof. a)$\Rightarrow$b)

Riemann-Roch gives, $l(2K+D) + l(-D-K) \geqslant (K+D)^2 + 2$. Therefore, $(D+K)^2 \leqslant -1$. By lemma 2.2, we see that $(D+K)^2 = -1$. We claim that, if $\kappa(D) \geqslant 0$ then $K^2 < 0$. If $K^2 \geqslant 0$, then by Riemann-Roch, $|-K| \neq \emptyset$. Also by Corollary 2.5 $D^2 \leqslant -4$ and hence $D \cdot (-K) < 0$. So D is part of $|-K|$, which implies, $|-K-D| \neq \emptyset$, contradicting the earlier remark.

b)$\Rightarrow$c)

Since $\kappa(D) \geqslant 0$, $|2K+D| \neq \emptyset$ by Corollary 2.4. If $(2K+D)^2 < 0$, then there exists $C \in \operatorname{Supp}|2K+D|$ such that $C \cdot (2K+D) < 0$. Since $|2K| = \emptyset$, $D \notin \operatorname{Supp}|2K+D|$ and hence $C \neq D$. But then $C^2 < 0$ and $C \cdot K < 0$. So C is an exceptional curve of the first kind and hence $0 > (2K+D) \cdot C = C \cdot D - 2$ which is a contradiction to our hypothesis. So, $(2K+D)^2 \geqslant 0$. $0 \leqslant (2K+D)^2 = K^2 + 2K \cdot (K+D) + (K+D)^2 = K^2 + 3(K+D)^2 - 2D(K+D) = K^2 - 3 + 4 = K^2 + 1$. Thus $K^2 \geqslant -1$ and hence $K^2 = -1$. Since $\kappa(D) \geqslant 0$, $|-K| = \emptyset$. Also $-1 = (K+D)^2 = K^2 + 2(D \cdot K) + D^2 = -1 + 2(-2+n) - n$. Therefore $n = 4$.

c)$\Rightarrow$a)

By Riemann-Roch, $l(2K) - h^1(2K) + l(-K) = K^2 + 1 = 0$. Thus $h^1(2K) = 0$. The exact sequence,

$$0 \longrightarrow \mathcal{O}(2K) \longrightarrow \mathcal{O}(2K+D) \longrightarrow \mathcal{O}_D \longrightarrow 0 \quad \text{gives,}$$

$l(2K+D) = 1$. So $\kappa(D) \geqslant 0$. If there exists an exceptional curve meeting D at most once, then blowing it down, we still have $\kappa(\text{Image of } D) \geqslant 0$ but $K^2 = 0$, contradicting our earlier remark. This proves the theorem.

Corollary 3.2. $\kappa(D) = 0$ *or* 1 *if and only if* $l(2K+D) = 1$.

Proof. By proof of Lemma 1.1, we may assume that every exceptional curve of the first kind meet D at least twice. First we will show that, if $\kappa(D) \geqslant 0$ and $(D+K)^2 \geqslant$

0 then $\kappa(D)=2$. If $D^2=-n$, it is easy to see that, $l(mn(D+K))=l(mn(D+K)-mD)$. Now, $(n(D+K)-D)^2=n^2(D+K)^2+4n-n=(D+K)^2n^2+3n>0$ if $(D+K)^2\geqslant 0$. Thus, $l(mn(D+K)-mD)$ is a quadratic polynomial in m, (notice that $\kappa(D)\geqslant 0$ is used to say that $H^2(mn(D+K)-mD)=0$) and hence $\kappa(D)=2$.

So, if $\kappa(D)=0$ or 1, $(D+K)^2\leqslant -1$ and by Cor. 2.3, $(D+K)^2=-1$. Now by Theorem 3.1, we see that $l(2K+D)=1$.

If $l(2K+D)=1$, then by Theorem 3.1, we have $(2K+D)^2=0$. Also, easy to see that $2K+D$ is a numerically effective divisor. If $\kappa(D)=2$, then since, $l(2m(D+K))=l(2mK+mD)$, $|m(2K+D)|$ gives a map to a surface. So for any ample divisor H, $m(2K+D)-H$ is effective for $m\gg 0$. But this implies, $H\cdot(2K+D)\leqslant 0$ a contradiction. Thus $\kappa(D)=0$ or 1.

Theorem 3.3. *Assume any equivalent conditions of Theorem* 3.1. *Then there exists an exceptional curve* E_0 *such that* $E_0\cdot D=2$. *Also*:

a) $|3K+D|=\emptyset\Leftrightarrow\kappa(D)=0$ *and then we have a birational morphism* f: $X\to \boldsymbol{P}^2$ *such that* $f(D)=$ *a sextic with ten double points, (possibly infinitely near) and in this case,* D *is linearly equivalent to* $-2K$. *Also if* E *is any exceptional curve of the first kind, the linear system* $|D+2E|$ *is an irreducible pencil of elliptic curves.*

b) $|3K+D|\neq\emptyset\Leftrightarrow\kappa(D)=1$. *Also in this case, there exists a birational morphism* f: $X\to\boldsymbol{P}^2$, *such that* $f(D)=$ *a curve of degree* $3m$, $m\geqslant 3$ *with nine m-tuple points and one double point (possibly infintely near.) Also the linear system* $|D+2E_0|$ *is an irreducible pencil of elliptic curves and this* E_0 *is unique.*

Proof. If $|3K+D|=\emptyset$, then, $l(-2K-D)\geqslant 1$ by Riemann-Roch and hence $2K+D=0$. So every exceptional curve meets D twice. If $|3K+D|\neq\emptyset$, then since $(3K+D)^2=-1$, there exists an irreducible curve $E\in\operatorname{Supp}|3K+D|$ such that $E\cdot(3K+D)<0$. So $E^2<0$ and $E\cdot K<0$, implying E is exceptional curve of the first kind. Also, then $E\cdot D<3$ implying $E\cdot D=2$. To show uniqueness of such an exceptional curve, first observe that they are all fixed components of $|3K+D|$. If there are two such, then since D meet them twice each, $4\leqslant D\cdot(3K+D)=-4+6=2$, a contradiction.

So we have found our E_0, and it is unique in the case $|3K+D|\neq\emptyset$.

Claim: $l(D+2E_0)\geqslant 2$.

$$D+2E_0=(D+E_0+K)+(E_0-K).$$

$l(D+E_0+K)\geqslant 1$ and $l(E_0-K)\geqslant 1$, by Riemann-Roch. Since $|D+K|=\emptyset$, $E_0\notin\operatorname{Supp}|D+E_0+K|$. Since $l(-K)=0$, E is not in $\operatorname{Supp}|E_0-K|$. So this curve in, $(D+E_0+K)+(E_0-K)$ is not equal to $D+2E_0$ and hence $l(D+2E_0)\geqslant 2$.

Using the exact sequence, $0\to\mathcal{O}(2E_0)\to\mathcal{O}(2E_0+D)\to\mathcal{O}_D\to 0$, one gets, $l(2E_0+D)\leqslant 2$ and thus $l(2E_0+D)=2$. So neither D nor E_0 is a fixed component of $|D+2E_0|$. But $(D+2E_0)^2=0$ says that $|D+2E_0|$ is a pencil of curves without base points. Also, $p_a(D+2E_0)=1$. Since $D+2E_0$ is a connected member of this system, which is reduced at some points, we see that the general member is irreducible of arithmetic genus 1. This gives the required elliptic fibration. Finally, we separately analyse the two cases to prove the Theorem.

a) $|3K+D|=\emptyset$. We know that $D+2K\sim 0$, and hence $\kappa(D)=0$. Observe that if X is any smooth surface with a curve D such that $D+2K_X\sim 0$ and if f: $X\to Y$ any birational morphism, then $f(D)+2K_Y\sim 0$. So we need only show that X dominates $\boldsymbol{P}^2$ birationally. Let now $Y=\boldsymbol{F}_n$, where f: $X\to\boldsymbol{F}_n$ is a birational morphism. We claim that $n\leqslant 2$. Let $\bar{D}=f(D)$ and F a general fibre. We have, $\bar{D}\sim 4D_n+2(n+2)F$. In particular, $\bar{D}\neq D_n$ and hence $\bar{D}\cdot D_n\geqslant 0$. This gives $n\leqslant 2$. If $n=0$, X dominates some blow up of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and hence $\boldsymbol{F}_1$. If $n=2$, then $\bar{D}\cdot D_n=0$ (i.e. $\bar{D}\cap D_n=\emptyset$) and hence X dominates a surface got by blowing up $\boldsymbol{F}_2$ away from D_n and hence $\boldsymbol{F}_1$. So in any case X dominates $\boldsymbol{P}^2$. Let f: $X\to\boldsymbol{P}^2$ be this morphism. Then since $f(D)+2K_{\boldsymbol{P}^2}\sim 0$, $f(D)$ is a rational sextic. Using $D+2K\sim 0$, it is also clear that it has only double points and hence exactly ten of them, may be infinitely near.

b) $|3K+D|\neq\emptyset$.

We first show that $\kappa(D)=1$. By Corollary 3.2, it suffices to show that $\kappa(D)\neq 0$. E_0 is a fixed component of $|3K+D|$. If E_0 is a component of $|2K+D|$, then $|2K+D-E_0|\neq\emptyset$. But $D\cdot(2K+D-E_0)=-2$ and hence $|2K-E_0|\neq\emptyset$, which is not possible. So E_0 is not a component of $|2K+D|$. Let $A\in|2K+D|$ and $B\in|3K+D|$. Then $3A+3D\sim 6(D+K)\sim 2B+4D$. But, $3A+3D\neq 2B+4D$ and hence $l(6(K+D))\geqslant 2$. Hence $\kappa(D)>0$.

Let $\bar{X}$ be the contraction of E_0 and $\bar{D}$, the image of D in $\bar{X}$. Since $|D+2K|\neq\emptyset$, afortiori, $|\bar{D}+2\bar{K}|\neq\emptyset$. Now choose $m\geqslant 2$ such that $|\bar{D}+m\bar{K}|\neq\emptyset$ and $|\bar{D}+(m+1)\bar{K}|=\emptyset$. We also have $\bar{D}^2=0$ and $\bar{K}^2=0$. By Riemann-Roch, it easily follows that $l(-\bar{D}-m\bar{K})\neq 0$, and hence $\bar{D}+m\bar{K}\sim 0$. Now as in the proof of a), one easily sees that there is a birational morphism $\bar{f}$: $\bar{X}\to\boldsymbol{P}^2$, such that $\bar{f}(\bar{D})$ is a rational curve of degree 3 m with nine m-tuple points and a double point. This completes the proof of Theorem 3.3.

Remark 3.4. If $\kappa(D)\geqslant 0$ and every exceptional curve meets D at least twice, then $(D+K)^2\geqslant 0$ implies $\kappa(D)=2$.

Proposition 3.5. *Let $D\hookrightarrow X$ and $D'\hookrightarrow X'$ be as in Theorem* 3.1 *and f: $X\to X'$ be a birational map such $f(D)=D'$. Then f is an isomorphism.*

Proof. Let g: $Y\to X$ be a birational morphism so that $h=f\circ g$: $Y\to X'$ is also a morphism. We may assume that there exists no exceptional curves of the first kind, which is in the fibre of g as well as h. Let E_0, E_0' be exceptional curves as in Theorem 3.3. Then there exists m, $m'(\geqslant 2)$ such that $|D+mK|=\{(m-2)E_0\}$ and $|D'+mK'|=\{(m'-2)E_0\}$ where K, K' are canonical divisors of X and X' respecitvely. Notice that m and m' are largest integers such that $|D+mK|\neq\emptyset$ and $|D'+m'K'|\neq\emptyset$. These numbers are invariant under blowing ups and hence comparing them in Y, we get $m=m'$. Also note that $l(D+mK)=l(D'+m'K')=1$ and this property also is invariant under blowing ups. In other words if $\tilde{D}$ is the proper transform of D (and hence of D') on Y, then $l(\tilde{D}+m\tilde{K})=1$ where $\tilde{K}$ is the canonical divisor of Y. By successively checking at each blowing ups one sees that, $|\tilde{D}+m\tilde{K}|=\{\sum_{i=0}^{p} m_iE_i\}$, where E_i $(i\geqslant 1)$ are all the irreducible fibres of g and $m_i>0$ for $i\geqslant 1$.

Similarly, looking at h, we have, $|\tilde{D}+m\tilde{K}|=\{\sum_{i=0}^{p} m_i' E_i'\}$ with $m_i'>0$, $i\geqslant 1$. (If $f\colon A\to B$ is the blowing up of a point of B, where A and B are smooth surfaces, then $\operatorname{rk}\operatorname{Pic} A=\operatorname{rk}\operatorname{Pic} B+1$ and $K_B^2-1=K_A^2$. In other words, if $f\colon A\to B$ is a composite of blowing ups, then $\operatorname{rk}\operatorname{Pic} A=\operatorname{rk}\operatorname{Pic} B+K_B^2-K_A^2$ and thus f is the composite of $K_B^2-K_A^2$ blowing ups. So in the above situation, Y is obtained as the composite of $K_X^2-K_Y^2$ blowing ups from X via g. Similarly for h. But $K_X^2=K_{X'}^2$. Thus the number of exceptional fibres of g and h are same).

Let E_k be an exceptional of the first kind on Y such that $g(E_k)$ is a point. Then $E_k=E_0'$, by our assumption that $h(E_k)\neq$ point. Since $E_k^2=-1$, h^{-1} has no fundamental points on E_0'. Hence $E_k.\tilde{D}=E_0'.D'=2$, which is a contradiction, since D is non-singular. Hence f is an isomorphism.

Corollary 3.6. *Let $D\hookrightarrow X$, $D'\hookrightarrow X'$ be embeddings of $\boldsymbol{P}^1$, such that $D^2=D'^2=-4$ and with $\kappa(D)=\kappa(D')=0$ or 1. Let $f\colon X\to\boldsymbol{P}^2$ and $f'\colon X'\to\boldsymbol{P}^2$ be birational morphisms as in Proposition 3.3. Let C, C' be images of D and D' in $\boldsymbol{P}^2$. Then the embeddings $D\hookrightarrow X$ and $D'\hookrightarrow X'$ are equivalent if and only if there is a Cremona transformation $\sigma\colon\boldsymbol{P}^2\to\boldsymbol{P}^2$ such that $\sigma(C)=C'$.*

§4. Class groups

Let X be a projective normal rational surface with isolated singularity $P\in X$. We define $\kappa(\mathcal{O}_{X,P})$ to be $\bar{\kappa}(X-P)$ (in the sense of Iitaka [I].). It is easy to see that it depends only on $\mathcal{O}_{X,P}$. For a ring A, we denote by $C(A)$, its class group.

Proposition 4.1. *Let A be a normal geometric local ring such that $\hat{A}\cong k[\![X^n, X^{n-1}Y,\dots, X^iY^{n-i},\dots, Y^n]\!]$. Further assume that the quotient field of A is $k(X, Y)$. Then,*

a) $\kappa(A)=-\infty \Longrightarrow C(A)=C(\hat{A})=\boldsymbol{Z}/n\boldsymbol{Z}$

b) $\kappa(A)=0 \Longrightarrow \begin{cases} C(A)=\boldsymbol{Z}/2\boldsymbol{Z},\ C(\hat{A})=\boldsymbol{Z}/4\boldsymbol{Z} & \text{if } n=4. \\ C(A)=\boldsymbol{Z}/n\boldsymbol{Z}=C(\hat{A}) & \text{if } n>5. \end{cases}$

Proof. Let U be the punctured spectrum of A.

Let $f\colon X\to\operatorname{Spec} A$ be a minimal desingularisation of $\operatorname{Spec} A$. It is well known that f^{-1} (closed point) $=D\approx\boldsymbol{P}^1$ and $D^2=-n$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{Z}\cdot D & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{Pic} U & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow\theta & & \downarrow & & \\
0 & \longrightarrow & \boldsymbol{Z} & \xrightarrow{n} & \boldsymbol{Z} & \longrightarrow & \boldsymbol{Z}/n\boldsymbol{Z} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\theta(F)=F\cdot D$. [[L], page 225]. To compute $C(A)=\operatorname{Pic} U$, it is sufficient to compute G.

a) $\kappa(A)=-\infty$. In this case, by Theorem 2.1, using a fibre of $\boldsymbol{F}_n$, we see that there exists a divisor F on X such that $F\cdot D=1$ and hence θ is surjective. Hence $C(A)\cong \boldsymbol{Z}/n\boldsymbol{Z}=C(\hat{A})$.

b) $\kappa(A)=0$.

By Theorem 3.1, if $n\geqslant 5$, there exists an exceptional curve which meets D once and hence θ is surjective. Thus $C(A)=\boldsymbol{Z}/n\boldsymbol{Z}$.

If $n=4$, by Theorem 3.3, $D\sim -2K$ and therefore $C\cdot D$ is even for all curves C. Since there is an E_0 such that $E_0\cdot D=2$, we get $G=\boldsymbol{Z}/2\boldsymbol{Z}$ and hence $C(A)\cong\boldsymbol{Z}/2\boldsymbol{Z}$.

We now consider the case when $\kappa(A)=1$ with $\hat{A}\cong K[\![X^n, X^{n-1}Y,\dots, X^iY^{n-i},\dots, Y^n]\!]$. Let Y be a complete rational normal surface with isolated singularity $y\in Y$, such that $\mathcal{O}_{Y,y}=A$. Let $f\colon X\to Y$ be the minimal desingularisation of Y. Then $f^{-1}(y)=D\approx \boldsymbol{P}^1$ and $D^2=-n$. By Proposition 1.4 and Theorem 3.3, there is a birational morphism $g\colon X\to\boldsymbol{P}^2$ such that $g(D)$ is a curve of degree $3m(m>2)$ and m depends only on A.

Proposition 4.2. *With the notations as above, assume $\kappa(A)=1$. Then,*

a) *If $n\geqslant 5$, then $C(A)\simeq\boldsymbol{Z}/n\boldsymbol{Z}\cong C(\hat{A})$.*

i) *$n=4$, m odd, then $C(A)=\boldsymbol{Z}/4\boldsymbol{Z}=C(\hat{A})$.*

ii) *$n=4$, m even, then $C(A)=\boldsymbol{Z}/2\boldsymbol{Z}$.*

Proof. When $n\geqslant 5$, the proof is exactly as in Proposition 4.1, b). Assume $n=4$. Let L be a generic line in $\boldsymbol{P}^2$. Then $g^{-1}(L)\cdot D=3m$. The image of θ: Pic $X\to\boldsymbol{Z}$ (as in the Proof of Proposition 3.1) contains $3m$ and 2 (note $E_0\cdot D=2$). Hence if m is odd θ is surjective. i.e. $G=0$, giving us $C(A)=\boldsymbol{Z}/4\boldsymbol{Z}$. Suppose m is even. Then $D+mK\sim(m-2)E_0$. Hence $D\cdot C$ is even for any curve C. Thus Image $\theta=2\cdot\boldsymbol{Z}$. Thus $C(A)=\boldsymbol{Z}/2\boldsymbol{Z}$.

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL
RESEARCH BOMBAY 400 005

References

[H] R. Hartshorne, Curves with high self intersection on algebraic surfaces, Publ. Math IHES, **36** (1969), 111–125.

[N] M. Nagata, On rational surfaces, I. Mem. Coll. Sci. Univ. Kyoto, **32** (1960).

[I] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, Complex analysis and algebraic Geometry, Iwanami Shoten, (1977), 175–189.

[F] T. Fujita, On Zariski Problem, Proc. Japan. Acad., **55**, Ser. A. (1799), 106–110

[C] J. L. Coolidge, A treatise on Algebraic Plane curves, Dover. (1959).

[L] J. Lipman, Rational singularities......... Publ. Math. IHES, **36** (1969), 195–279.

[M–S] M. Miyanishi and T. Sugie, Affine surfaces containing cylinderlike open sets, J. Math. Kyoto. Univ., **20** (1980), 11–42.