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Rings of constants for k-derivations in $k[x_1, \ldots, x_n]$

By

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In this note we give several remarks on the rings of constants for a family D of k-derivations in the rings of polynomials over a field k.

1. Preliminaries.

Let us recall at first ([1]) that if $k[x_1,\ldots, x_n]$ is the ring of polynomials over a commutative ring k and $f_1,\ldots, f_n \in k[x_1,\ldots, x_n]$ then there exists a unique k-derivation d of $k[x_1,\ldots, x_n]$ such that $d(x_1)=f_1,\ldots, d(x_n)=f_n$. This derivation d is defined by

$$d(h)=(\partial h/\partial x_1)f_1+\cdots+(\partial h/\partial x_n)f_n,$$

for $h\in k[x_1,\ldots, x_n]$.

Let k be a field, A a commutative k-algebra with 1, and D a family of k-derivations of A. We denote by A^D the set of constants of A with respect to D, that is,

$$A^D=\{a\in A;\ d(a)=0 \text{ for any } d\in D\}.$$

If D has only one element d then we write A^d instead of $A^{\{d\}}$. It is clear that $A^D=\bigcap_{d\in D} A^d$.

The set A^D is a k-subalgebra of A containing k. If A is a field then A^D is a subfield of A containing k.

Assume now that A has no zero divisors and A_0 is the field of quotients of A. Denote by $\bar{D}$ the set $\{\bar{d};\ d\in D\}$, where $\bar{d}$ is the k-derivation of A_0 defined by

$$\bar{d}(a/b)=(d(a)b-ad(b))b^{-2},$$

for a, $b\in A$ and $b\neq 0$. In this situation we have two subfields of A_0:

$(A^D)_0$=the field of quotients of A^D,

$(A_0)^{\bar{D}}$=the field of constants of A_0 with respect to $\bar{D}$.

The following example shows that these subfields could be different

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Example 1.1. *Let* $\operatorname{char}(k)=0$ *and let* d *be the* k*-derivation of* $A=k[x, y]$ *such that* $d(x)=x$ *and* $d(y)=y$. *Then* $(A^d)_0 \neq (A_0)^{\bar{d}}$.

Proof. It is easy to show that $(A^d)_0=k$ and $x/y \in (A_0)^{\bar{d}} \setminus k$.

Proposition 1.2. *If* D *is a family of* k*-derivations in a* k*-domain* A *then*

$$k \subseteq A^D \subseteq (A^D)_0 \subseteq (A_0)^{\bar{D}} \subseteq A_0, \tag{1}$$

$$(A^D)_0 \cap A = (A_0)^{\bar{D}} \cap A = A^D. \tag{2}$$

The proof is straightforward.

2. The case char(k)=0.

In this section k is always a field of characteristic zero.

Lemma 2.1. *If* D *is a family of* k*-derivations in a* k*-domain* A *then the ring* A^D *is integrally closed in* A.

Proof. Let $a \in A$ be an integral element over A^D and let

$$a^n + c_1 a^{n-1} + \cdots + c_{n-1} a + c_n = 0,$$

where $c_1, \ldots, c_n \in A^D$ and n is minimal. If $d \in D$ then

$$0 = d(0) = ud(a),$$

where $u = na^{n-1} + (n-1)c_1 a^{n-2} + \cdots + c_{n-1}$. Since $u \neq 0$ (because n is minimal and $\operatorname{char}(k)=0$), $d(a)=0$ and hence, $a \in \bigcap_{d \in D} A^d = A^D$.

As an immediate consequence of Lemma 2.1 we obtain

Proposition 2.2. *If* D *is a family of* k*-derivations of* $A=k[x_1, \ldots, x_n]$, *where* k *is a field of characteristic zero, then the ring* A^D *is integrally closed in* A. *In particular* A^D *is normal.*

Note the following well known ([3] p. 177)

Lemma 2.3. *Let* $L \subseteq K$ *be a separable algebraic extension of fields. If* d *is an* L*-derivation of* K *then* $d=0$.

This lemma implies

Proposition 2.4. *If* D *is a non-zero family of* k*-derivations of* $A=k[x_1, \ldots, x_n]$, *where* k *is a field of characteristic zero, then* $\operatorname{tr.deg}_k(A^D) \leqslant n-1$.

Proof. Let $s = \operatorname{tr.deg}_k(A^D)$, $K = k(x_1, \ldots, x_n)$ and $L = (A_0)^{\bar{D}}$. It is clear that $s \leqslant n$. Suppose now that $s=n$. Then $L \subseteq K$ is a separable algebraic field extension. If $d \in D$ then $\bar{d}$ is an L-derivation of K so, by Lemma 2.3, $\bar{d}=0$ and hence $d=0$; that is, $D=0$ and we have a contradiction to our assumption.

Now let us recall a result due to Zariski ([9], see [5] p. 41)

Zariski's Theorem 2.5. *Let* k *be a field and let* L *be a subfield of* $k(x_1, \ldots, x_n)$

containing k. If $\mathrm{tr.deg}_k(L)\leqslant 2$ *then the ring* $L\cap k[x_1,\dots, x_n]$ *is finitely generated over* k.

As a consequence of the Zariski's Theorem, Propositions 2.4 and 1.2(2) we obtain the following

Theorem 2.6. *Let D be a family of k-derivations of the polynomial ring $k[x_1,\dots, x_n]$ over a field k of characteristic zero. If $n\leqslant 3$ then there exest polynomials $f_1,\dots, f_s\in k[x_1,\dots, x_n]$ such that $k[x_1,\dots, x_n]^D=k[f_1,\dots, f_s]$.*

The next result is due to Zaks ([8], see also [2]).

Zaks' Theorem 2.7. *Let k be a field. If R is a Dedekind subring of $k[x_1,\dots, x_n]$ containing k then there exist a polynomial $f\in k[x_1,\dots, x_n]$ such that $R=k[f]$.*

By Zaks' Theorem and Theorem 2.6 we have

Thorem 2.8. *Let* $\mathrm{char}(k)=0$ *and let D be a non-zero family of k-derivations of $k[x, y]$. Then there exists a polynomial $f\in k[x, y]$such that $k[x, y]^D=k[f]$.*

Proof. Let $R=k[x, y]^D$ and $s=\mathrm{tr.deg}_k(R)$. We know, by Proposition 2.4, that $s\leqslant 1$. If $s=0$ then $R=k$, so $R=k[f]$, where for example $f=1$. If $s=1$ then, by Proposition 2.2 and Theorem 2.6, R is a Dedekind subring of $k[x, y]$ containing k and hence, by Zaks' Theorem, $R=k[f]$, for some $f\in k[x, y]$.

3. Closed polynomials in characteristic zero.

Consider the following family $\mathcal{M}$ of subrings in $k[x_1,\dots, x_n]$:

$$\mathcal{M}=\{k[f]\,;\, f\in k[x_1,\dots, x_n]\smallsetminus k\}.$$

If $\mathrm{char}(k)=0$ and $k[f]\subsetneqq k[g]$, for some polynomials f, $g\in k[x_1,\dots, x_n]\smallsetminus k$, then $\deg(f)>\deg(g)$ and hence, we see that in the family $\mathcal{M}$ there exist maximal elements.

We shall say that a polynomial $f\in k[x_1,\dots, x_n]\smallsetminus k$ is *closed* if the ring $k[f]$ is integrally closed in $k[x_1,\dots, x_n]$.

Lemma 3.1. *Let* $\mathrm{char}(k)=0$ *and $f\in k[x_1,\dots, x_n]\smallsetminus k$. Then f is closed if and only if the ring $k[f]$ is a maximal element in $\mathcal{M}$.*

Proof. Let f be closed and assume that $k[f]\subseteq k[g]$ for some $g\in k[x_1,\dots, x_n]$. Then $f\in k[g]$, that is,

$$f=a_sg^s+\cdots+a_1g+a_0,$$

for some $a_0,\dots, a_s\in k$ with $a_s\neq 0$. Hence

$$g^s+a_s^{-1}a_{s-1}g^{s-1}+\cdots+a_s^{-1}a_1g+(a_s^{-1}a_0-f)=0$$

and hence g is integral over $k[f]$. Since $k[f]$ is integrally closed in $k[x_1,\dots, x_n]$, $k[f]=k[g]$ and we see that $k[f]$ is maximal in $\mathcal{M}$.

Assume now that $k[f]$ is a maximal element in $\mathcal{M}$ and denote by E the integral closure of $k[f]$ in $k[x_1,\dots, x_n]$. Then E is a Dedekind subring of

$k[x_1,\dots, x_n]$ containing k so, by Theorem 2, 7, $E=k[h]$, for some $h\in k[x_1,\dots, x_n]$. Now, by the maximality of $k[f]$ in $\mathcal{M}$, $k[f]=k[h]=E$ and so, f is closed.

Proposition 3.2. *Let D be a family of k-derivations in $A=k[x_1,\dots, x_n]$, where k is a field of characteristic zero. If the ring A^D is finitely generated over k (for example, if $n\leqslant 3$) then $A^D=k$ or there exist closed polynomials $f_1,\dots,f_s\in A$ such that $A^D=k[f_1,\dots,f_s]$.*

Proof. Assume that $A^D\neq k$ and let $A^D=k[h_1,\dots, h_s]$ for some $h_1,\dots, h_s\in A-k$. Let $f_1,\dots, f_s$ be polynomials in $A\setminus k$ such that $k[h_i]\subseteq k[f_i]$ and $k[f_i]$ is a maximal element in $\mathcal{M}$, for $i=1,\dots, s$. Then there exist polynomials $u_1(t),\dots, u_s(t)\in k[t]$ such that $h_i=u_i(f_i)$, for $i=1,\dots, s$. We may assume that the polynomials $u_1(t),\dots, u_s(t)$ have minimal degrees. Now, using the same argument as in the proof of Lemma 2.1, we see that $f_1,\dots, f_s\in A^D$. Hence $k[f_1,\dots, f_s]\subseteq A^D=k[h_1,\dots, h_s]\subseteq k[f_1,\dots, f_s]$, that is, $A^D=k[f_1,\dots, f_s]$ and, by Lemma 3.1, $f_1,\dots, f_s$ are closed.

Proposition 3.3. *Let D be a non-zero family of k-derivations in $k[x, y]$, where k is a field of characteristic zero. Denote $R=k[x, y]^D$. If $f\in R\setminus k$, then R is the intergral closure of the ring $k[f]$ in $k[x, y]$.*

Proof. If $f\in R\setminus k$ then $R\neq k$ and, by Theorem 2.8 and Proposition 3.2, $R=k[h]$, for some closed polynomial $h\in k[x, y]$. Hence $k[f]\subseteq k[h]$, $k[h]$ is integrally closed in $k[x, y]$ and $k[h]$ is integral over $k[f]$. This means that $R=k[h]$ is the integral closure of $k[f]$ in $k[x, y]$.

Theorem 3.4. *Let k be a field of characteristic zero and let A be a subring of $k[x, y]$ containing k, such that A is integrally closed in $k[x, y]$. If* Krull-dim$(A)\leqslant 1$ *then there exists a k-derivation d of $k[x, y]$ such that $A=k[x, y]^d$.*

Proof. Let $s=$Krull-dim(A). If $s=0$ then $A=k$ and we have $A=k[x, y]^d$, where, for example, d is such k-derivation of $k[x, y]$ that $d(x)=x$ and $d(y)=y$.

Assume that $s=1$. Then A is a Dedekind subring of $k[x, y]$ containing k (see [2] Theorem 1) hence, by Theorem 2.7, $A=k[h]$ for some closed polynomial $h\in k[x, y]\setminus k$. Consider k-derivation d of $k[x, y]$ such that $d(x)=\partial h/\partial y$, $d(y)=-\partial h/\partial x$. Then $h\in k[x, y]^d\setminus k$ and we see, by Proposition 3.3, that $A=k[x, y]^d$.

4. The case char$(k)=p>0$.

Throughout this section k is a field of characteristic $p>0$.

Denote $A=k[x_1,\dots, x_n]$, $R=k[x_1^p,\dots, x_n^p]$. It is well known that A is a free R-module on the basis (p-basis)

$$\{x_1^{i_1}\cdots x_n^{i_n};\ i_1<p,\dots,\ i_n<p\}$$

and hence, in particular, A is a noetherian R-module.

If D is a family of k-derivations of A then $R\subseteq A^D$ and so, A^D is an R-submodule of A. Therefore we have

Proposition 4.1. *If D is a family of k-derivations of $A=k[x_1,\dots, x_n]$, where k is a field of characteristic $p>0$, then there exist polynomials $f_1,\dots, f_s\in A$ such that*

$$A^D=k[x_1^p,\dots,\ x_n^p,\ f_1,\dots,\ f_s].$$

If $\mathrm{char}(k)=2$ and $n=2$ then the following proposition shows (if $D\neq 0$ and s is as in Proposition 4.1) that $s=1$.

Proposition 4.2. *Let k be a field of characteristic two and let D be a non-zero family of k-derivations in $k[x, y]$. Then there exists a polynomial $f\in k[x, y]$ such that $k[x, y]^D=k[x^2, y^2, f]$.*

Proof. If $k[x, y]^D=k[x^2, y^2]$, then $k[x, y]^D=k[x^2, y^2, f]$, where $f=1$. Assume that $k[x, y]^D\neq k[x^2, y^2]$. Let $f_1,\dots, f_s$ be as in Proposition 4.1, and let $f_i=a_ix+b_iy+c_ixy+u_i$, where $a_i,\ b_i,\ c_i,\ u_i\in k[x^2, y^2]$, for $i=1,\dots, s$.

We may assume that

(1) $f_1,\dots, f_s$ do not belong to $k[x^2, y^2]$,

(2) $u_1=\cdots=u_s=0$.

Moreover, we may assume that

(3) there is no elements $v_i\in k[x^2, y^2]\setminus k$ such that $v_i\mid f_i$, for $i=1,\dots, s$.

In fact, if for example $f_1=vg$, where $v\in k[x^2, y^2]\setminus k$ and $g\in k[x, y]$, then for any $d\in D$, $0=d(f_1)=vd(g)$, that is, $d(g)=0$ and hence $g\in k[x,y]^D$ and we have $k[x, y]^D=k[x^2, y^2, g, f_2,\dots, f_s]$.

Denote by L the field $k(x^2, y^2)[f_1,\dots, f_s]$ and let $m=[L : k(x^2, y^2)]$. Then $m=4$, 2 or 1. If $m=4$ then $L=k(x, y)$ and we have a contradiction to the assumption that $D\neq 0$. If $m=1$, then $k[x, y]^D=k[x^2, y^2]$.

Assume now that $m=2$. Then $L=k(x^2, y^2)[f_i]$, for some $i=1,\dots, s$ (since $k(x^2, y^2)[f_i]$ is a two-dimensional subspace of L over $k(x^2, y^2)$), and, in particular, we have $af_1=bf_2+c$, where a and b are non-zero elements in $k[x^2, y^2]$ and $c\in k[x^2, y^2]$. But $c=0$, by (2), hence $af_1=bf_2$. Let $u=\gcd(a, b)$, $a=ua'$, $b=ub'$, for a', $b'\in k[x, y]$. Then $a'f_1=b'f_2$, $\gcd(a', b')=1$ and it is easy to show that $k\setminus\{0\}$. $k[x^2, y^2]$. This implies that $a'\mid f_2$, $b'\mid f_1$ so, by (3), a' and b' belong to a', $b'\in$Therefore $f_1=cf_2$, for some $c\in k\setminus\{0\}$ and we have

$$k[x, y]^D=k[x^2, y^2,\ f_2, f_3,\dots, f_s].$$

Repeating the above argument we see that $k[x, y]^D=k[x^2, y^2, f_s]$.

If $\mathrm{char}(k)=p>2$ then the assertion of Proposition 4.2 is not true, in general.

Example 4.3. *Let $\mathrm{char}(k)=p>2$ and let d be the k-derivation of $k[x, y]$ such that $d(x)=x$, and $d(y)=y$. Then there is no polynomial $f\in k[x, y]$ such that $k[x, y]^d=k[x^p, y^p, f]$.*

Proof. Suppose that $k[x, y]^d=k[x^p, y^p, f]$, for some $f\in k[x, y]$, and consider the monomials $x^{p-1}y$ and xy^{p-1}. We see that these monomials belong to $k[x, y]^d$. Therefore

$$x^{p-1}y=u(f)\ \text{and}\ xy^{p-1}=v(f),$$

for some polynomials $u(t)$, $v(t)\in k[x^p, y^p][t]$, and we have

$$-x^{p-2}y=(\partial/\partial x)(x^{p-1}y)=u'(f)(\partial f/\partial x)$$

$$x^{p-1}=(\partial/\partial y)(x^{p-1}y)=u'(f)(\partial f/\partial y)$$

$$y^{p-1}=(\partial/\partial x)(xy^{p-1})=v'(f)(\partial f/\partial x)$$

$$-xy^{p-2}=(\partial/\partial y)(xy^{p-1})=v'(f)(\partial f/\partial y),$$

where $u'(t)$, $v'(t)$ are derivatives of $u(t)$ and $v(t)$, respectively. This implies, in particular, that $u'(f)=ax^{p-2}$, for some $a\in k\setminus\{0\}$. Hence $x^{p-2}\in k[x^p, y^p, f]=k[x, y]^d$. But it is a contradiction, because $d(x^{p-2})=-2x^{p-2}\neq 0$.

Observe that if $n=2$ and $p=2$ then, by Proposition 4.2, every ring of constants is a free $k[x^p, y^p]$-module. Now we shall show that it is also true for an arbitrary $p>0$ and $D=\{d\}$.

Theorem 4.4. *Let k be a field of characteristic $p>0$ and* d *a k-derivation of $k[x, y]$. Then the ring $k[x, y]^d$ is a free $k[x^p, y^p]$-module.*

Before the proof of Theorem 4.4 we recall a few facts for M-sequences in regular local rings (see [6]).

Let R be a commutative ring and M a non-zero R-module. We denote by $\mathrm{hd}(M)$ the projective dimension of M. An element $r\in R$ is called a zero divisor with respect to M if there is a nonzero element m of M such that $rm=0$.

Assume now that R is a regular local ring with the maximal ideal $\mathfrak{m}$ and $M\neq 0$ is a finitely generated R-module.

We say that a sequence $t_1,\dots, t_n$ of elements of $\mathfrak{m}$ is an M-sequence if t_i is not a zero divisor with respect to $M/\sum_{j=1}^{i-1} t_jM$, for each $i=1,\dots, n$. It is known (see [6] p.97) that all maximal M-sequences have the same length, this length we denote by $s(M)$.

Note the following theorem which is due to Auslander, Buchsbaum and Serre (see [6] p.98)

Theorem 4.5. *Let $(R, \mathfrak{m})$ be a regular local ring and M a finitely generated R-module different from zero. Then*

$$\mathrm{hd}(M)=\text{Krull-dim}(R)-s(M).$$

Proof of Theorem 4.4. Denote $R=k[x^p, y^p]$, $A=k[x, y]$, $M=d(A)$, $K=k[x, y]^d$ and consider the following exact sequence of R-modules:

$$0 \longrightarrow K \longrightarrow A \xrightarrow{d} M \longrightarrow 0. \tag{1}$$

Let $\mathfrak{m}$ be a maximal ideal of R. Then the sequence (1) induces the exact sequence of $R_{\mathfrak{m}}$-modules:

$$0 \longrightarrow K_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}} \xrightarrow{d_{\mathfrak{m}}} M_{\mathfrak{m}} \longrightarrow 0. \tag{2}$$

Since $R_{\mathfrak{m}}$ is a regular local ring and $M_{\mathfrak{m}}$ is a finitely generated $R_{\mathfrak{m}}$-module different from zero, we have (by Theorem 4.5)

$$\mathrm{hd}(M_{\mathfrak{m}})=2-s(M_{\mathfrak{m}}).$$

But $M_{\mathfrak{m}}$ is contained in the ring $A_{\mathfrak{m}}$ which is an integral domain and so, $s(M_{\mathfrak{m}}) \geqslant 1$. Therefore $hd(M_{\mathfrak{m}}) \leqslant 1$ and hence, by the sequence (2) (since $A_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module), $hd(K_{\mathfrak{m}})=0$ and we have $\mathrm{hd}(K)=\sup_{\mathfrak{m}} \mathrm{hd}(M_{\mathfrak{m}})=0$. This implies that K is a projective R-module and hence, by [7] (see [4]), $K=k[x, y]^d$ is a free $k[x^p, y^p]$-module.

The next example shows that if $n \geqslant 3$ then the assertion of Theorem 4.4 is not true, in general.

Example 4.6. *Let* $\mathrm{char}(k)=p>0$, $n \geqslant 3$, *and let* d *be the* k*-derivation of* $k[x_1,\dots, x_n]$ *such that* $d(x_i)=x_i^p$, *for* $i=1,\dots, n$. *Then the ring* $k[x_1,\dots, x_n]^d$ *is not a free* $k[x_1^p,\dots, x_n^p]$*-module.*

Proof. Denote $R=k[x_1^p,\dots, x_n^p]$, $A=k[x_1,\dots, x_n]$, $M=d(A)$ and $K=A^d$. Let $\mathfrak{m}$ be the maximal ideal of R generated by $x_1^p,\dots, x_n^p$ and consider the exact sequences (1) and (2) as in the proof of Theorem 4.4. We shall show that $s(M_{\mathfrak{m}})=1$.

Let $t_1=x_1^p/1,\dots, t_n=x_n^p/1$. The elements $t_1,\dots, t_n$ generate the maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$. Observe that t_1 is not a zero divisor with respect to $M_{\mathfrak{m}}$, and $t_1 \in M_{\mathfrak{m}} \setminus t_1 M_{\mathfrak{m}}$ (since $1 \notin M$). If u is an arbitrary element of $\mathfrak{m}R_{\mathfrak{m}}$, then $u=a_1t_1+\dots+a_nt_n$, for some $a_1,\dots, a_n \in R_{\mathfrak{m}}$ and we have

$$\begin{aligned} u &= a_1 d_{\mathfrak{m}}(x_1/1)+\dots+a_n d_{\mathfrak{m}}(x_n/1) \\ &= d_{\mathfrak{m}}(a_1(x_1/1)+\dots+a_n(x_n/1)), \end{aligned}$$

that is, $u \in M_{\mathfrak{m}}$ and hence, $t_1 u \in t_1 M_{\mathfrak{m}}$.

Therefore t_1 is a maximal $M_{\mathfrak{m}}$-sequence and hence (since all maximal $M_{\mathfrak{m}}$-sequences have the same length), $s(M_{\mathfrak{m}})=1$. Now, by Theorem 4.5, $\mathrm{hd}(M_{\mathfrak{m}})=n-1 \geqslant 2$ and hence, $\mathrm{hd}(K_{\mathfrak{m}}) \geqslant 1$. This implies that $\mathrm{hd}(K)=\sup \mathrm{hd}(K_{\mathfrak{m}}) \geqslant 1$, that is, $K=k[x_1,\dots, x_n]^d$ is not a free $k[x_1^p,\dots, x_n^p]$-module.

Remaek 4.7. Using the same argument as in the proof of Example 4.6 we may prove that if $n \geqslant 3$ and $d(x_i)=x_{v(i)}^p$, where v is a permutation of $\{1,\dots, n\}$, then the ring $k[x_1,\dots, x_n]^d$ is not a free $k[x_1^p,\dots, x_n^p]$-module.

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