J. Math. Kyoto Univ. (JMKYAZ)
28-4 (1988) 579-592

A theorem of characterization of residual transcendental extensions of a valuation

By

Victor ALEXANDRU, Nicolae POPESCU and Alexandru ZAHARESCU

Let K be a field and v a valuation on K. The *r.t.* (residual transcendental) extensions of v to $K(X)$ have been considered by Nagata [7] in connection with some problems in field theory. Also in [7] it is conjectured that if w is an *r.t.* extension of v then k_w, the residue class field of w, is a simple transcendental extension of a finite algebraic extension of k_v, the residue class field of v. Althoug hthis problem has been affirmatively solved in [9] and independently in [11], there exist many interesting questions on *r.t.* extensions. Some questions about *r.t.* extensions have been considered by Ohm in [8] and [10]. Particularly in [10] three conjectures relative to some natural numbers like ramification index and residual degree are stated.

The main result of this work is Theorem 2.1 which gives a characterization of *r.t.* extensions w of a valuation v using the notion of minimal pair of definition. As a consequence of our theorem, Nagata's conjecture, all Ohm's conjectures and also some interesting consequences given in Section 3 result in a natural way.

Finally we remark that in [1] it is given a description of *r.t.* extensions using the so called "pair of definition". Another description of *r.t.* extensions (based on the obvious existence of minimal pair) is derived in this work (Corollary 2.4) and it seems that this description is very satisfactory. However, in contrast with pairs of definition of an *r.t.* extension, which are easy to indicate, we do not have yet a criterion to recognize if a pair of definition is a minimal one.

This question and some related problems shall make the object of a future work.

1. Notations and definitions.

Let K be a field and v a valuation on K. Denote by k_v the residue field, by Γ_v the value group and by O_v the valuation ring of v. If $x \in O_v$, denote by x^* the image of x into k_v. We refer the reader to [5], [6], [12] or [13], Vol. II, for general notions and definitions.

Let K'/K be an extension of fields. A valuation v' on K' will be called an *extension* of v if $v'(x) = v(x)$ for all $x \in K$. If v' is an extension of v, we shall identify

Communicated by Prof. Nagata, April 15, 1987

canonically k_v to a subfield of $k_{v'}$ and Γ_v to a subgroup of $\Gamma_{v'}$. We shall say that K'/K is an immediate extension if $k_v=k_{v'}$ and $\Gamma_v=\Gamma_{v'}$ (see [12, Ch. II]).

Let $K(X)$ be the field of rational functions of an indeterminate X. A valuation w on $K(X)$ will be called a *residual transcendental* (*r. t.*) *extension* of v if it is an extension of v and k_w/k_v is a transcendental extension. It is well known that tr.deg. $k_w/k_v=1$ (see [5, Ch. VI, § 10]). Then there exists element $r\in O_w$, such that r^* is transcendental over k_v.

For any $r\in K(X)$, $r\notin K$, define $\deg r=[K(X)\colon K(r)]$. Denote $\deg(w/v)=$least n such that there exists $r\in O_w$ of degree n such that r^* is transcendental over k_v.

It is also easy to see that $[\Gamma_w\colon \Gamma_v]<\infty$; the number $[\Gamma_w\colon \Gamma_v]$ will be denoted by $e(w/v)$.

Let k be the algebraic closure of k_v in k_w; it is easy to see that $[k\colon k_v]<\infty$, and the number $[k\colon k_v]$ will be denoted by $f(w/v)$.

In what follows (see Corollary 2.2) we shall prove that generally one has:

$$e(w/v)f(w/v)\leq \deg(w/v)\,.$$

Let us denote by $\bar{K}$ a fixed algebraic closure of K and by $\bar{v}$ a fixed extension of v to $\bar{K}$. If w is an extension of v to $K(X)$, then there exists an extension $\bar{w}$ of w to $\bar{K}(X)$ such that $\bar{w}$ is also an extension of $\bar{v}$. Let us denote

$$M_w = \{w(X-a)\,|\,a\in K\}\subseteq\Gamma_w\,,$$
$$M_{\bar{w}} = \{\bar{w}(X-\alpha)\,|\,\alpha\in\bar{K}\}\subseteq\Gamma_{\bar{w}}\,.$$

Let Γ be an ordered group which contains Γ_v as an ordered subgroup and let $\gamma\in\Gamma$ and $a\in K$. If $f(X)\in K[X]$ one has the Taylor's expansion:

$$f(X) = a_0+a_1(X-a)+\cdots+a_n(X-a)^n\,.$$

Let us define:

$$w(f(X)) = \inf_i\,(v(a_i)+i\gamma)\,.$$

It is easy to check (see [5, Ch. VI, § 10]) that w is a valuation on $K[X]$, which may be canonically extended to a valuation on $K(X)$. We shall say that w is the *valuation on* $K(X)$ *defined by* inf, v, a, and γ. Also it is easy to see that w is an *r.t.* extension of v if and only if γ has a finite order over Γ_v.

Proposition 1.1. *Let w be an extension of v to $K(X)$. The following assertions are equivalent*:

a) *w is an r.t. extension of v;*
b) *$\bar{w}$ is an r.t. extension of $\bar{v}$;*
c) *$\Gamma_{\bar{v}}=\Gamma_{\bar{w}}$, the set $M_{\bar{w}}$ is bounded in $\Gamma_{\bar{w}}$ and contains its upper bound.*

Proof. The equivalence a)$\Leftrightarrow$b) is obvious.
b)$\Rightarrow$c). Let $\bar{w}$ be an *r.t.* extension of $\bar{v}$. According to [1, Proposition 2], $\bar{w}$ is

defined by inf, $\bar{v}$, $\alpha \in \bar{K}$ and $\delta \in \Gamma_{\bar{v}} = \Gamma_{\bar{w}}$, Moreover one has $\bar{w}(X-\alpha)=\delta$. Then $\delta = \sup M_{\bar{w}}$. Indeed, if $\beta \in \bar{K}$, then $\bar{w}(X-\beta)=\bar{w}(X-\alpha+\alpha-\beta)=\inf(\delta, \bar{v}(\alpha-\beta)) \leq \delta$.

c)$\Rightarrow$b). Let $\alpha \in \bar{K}$ be such that $\bar{w}(X-\alpha)=\delta=\sup M_{\bar{w}}$. The equality $\Gamma_{\bar{v}}=\Gamma_{\bar{w}}$ shows that there exists an element $d \in \bar{K}$ such that $\bar{w}(X-\alpha)=\bar{v}(d)=\delta$. Hence $\bar{w}((X-\alpha)/d)=0$. We assert that $t=((X-\alpha)/d)^*$ is transcendental over $k_{\bar{v}}$. Indeed, if t is algebraic then $t \in k_{\bar{v}}$ since $k_{\bar{v}}$ is algebraically closed (because $\bar{K}$ is algebraically closed by hypothesis). Hence there exists an element $a \in \bar{K}$ such that $\bar{v}(a)=0$ and $a^*=t$. But then $\bar{w}((X-\alpha)/d-a)>0$ and so $\bar{w}(X-(\alpha+ad))>\bar{v}(d)=\delta$, a contradiction.

Remark 1.2. According to the hypothesis made above, M_w is also a bounded set. Conversely, even if M_w is a bounded set, $M_{\bar{w}}$ is not necessarily bounded, although $\Gamma_{\bar{v}}=\Gamma_{\bar{w}}$. Indeed, let $\boldsymbol{Q}$ be the field of rational numbers, p a suitable prime number, $\boldsymbol{Q}_p$ the field of p-adic numbers and v the p-adic valuation on both $\boldsymbol{Q}$ and $\boldsymbol{Q}_p$. Denote by t a unit of $\boldsymbol{Q}_p$ such that t is transcendental over $\boldsymbol{Q}$. Let X be a root of the polynomial Y^2-pt^2. Then X is also transcendental over $\boldsymbol{Q}$. Let v_1 be the unique extension of v to $\boldsymbol{Q}_p(X)$, and let w be the restriction of v_1 to $\boldsymbol{Q}(X)$. It is clear that w is an extension of v to $\boldsymbol{Q}(X)$, but w is not an *r.t.* extension of v. However, M_w is bounded since $X \notin \boldsymbol{Q}_p$.

Let $\bar{\boldsymbol{Q}}$ be an algebraic closure of $\boldsymbol{Q}$, $\bar{v}$ an extension of v to $\bar{\boldsymbol{Q}}$, and $\bar{w}$ an extension of w to $\bar{\boldsymbol{Q}}(X)$ such that $\bar{w}$ induces $\bar{v}$ on $\bar{\boldsymbol{Q}}$. We assert that $M_{\bar{w}}$ is not bounded in $\Gamma_{\bar{w}}=\Gamma_{\bar{v}}$. Indeed, let $\{a_n\}_n$ be a Cauchy sequence of rational numbers such that $\lim_n a_n=t$. Then $\{\sqrt{p}\,a_n\}_n$ is a Cauchy sequence of $\bar{\boldsymbol{Q}}$ (relative to $\bar{v}$) and $X=\lim_n \sqrt{p}\,a_n$. It is now clear that $M_{\bar{w}}$ is not bounded.

The above proof can be adapted to an arbitrary field K as follows:

Proposition 1.3. *Let w be an r.t. extension of v to $K(X)$. The following assertions are equivalent*:

a) *w is defined by inf, v, $a \in K$ and $\delta \in \Gamma_v$*;

b) *$e(w/v)=f(w/v)=1$ and the set M_w is bounded and contains its upper bound.*

Corollary 1.4. *Let v be a rank one and discrete valuation on K and w an r.t. extension of v to $K(X)$ such that $e(w/v)=f(w/v)=1$. Then w is defined by inf, v, $a \in K$, and $\delta \in \Gamma_v$.*

Proof. According to Proposition 1.3, it will be enough to show that M_w is bounded and contains its upper bound. Indeed, since $K(X)/K$ is not an immediate extension, then M_w is bounded and since Γ_v is discrete and rank one, then M_w contains its upper bound.

According to [5] (see also [6]), a valuation v on K is said to be *Henselian* if, for every algebraic extension L/K, v has a unique extension to L.

2. The representation theorem for r.t. extensions.

We preserve all notation made in previous section.

If w is an *r.t.* extension of v to $K(X)$, then $\bar{w}$ is an *r.t.* extension of $\bar{v}$ to $\bar{K}(X)$, and moreover there exist an element $\alpha \in \bar{K}$ and an element $\delta \in \Gamma_{\bar{v}}$ such that $\bar{w}$ is defined by inf, $\bar{v}$, α and δ ([1, Proposition 2]). In particular, one has $\bar{w}(X-\alpha)=\delta$. Therefore any *r.t.* extension $\bar{w}$ of $\bar{v}$ to $K(X)$ is well defined by a pair $(\alpha, \delta) \in \bar{K} \times \Gamma_{\bar{v}}$, called a *pair of definition for* $\bar{w}$. Sometimes $\bar{w}$ is called the *valuation defined by the pair* (α, δ).

In [1, Proposition 3], it is shown that two pairs (α_1, δ_1) and (α_2, δ_2) define the same valuation $\bar{w}$ if and only if:

$$(1) \qquad \delta_1 = \delta_2 \quad \text{and} \quad \bar{v}(\alpha_1-\alpha_2) \geq \delta_1 .$$

By *minimal pair* (*of definition*) for $\bar{w}$ we mean a pair of definition (α, δ) such that $[K(\alpha): K]$ is minimal. Now it is clear that every *r.t.* extension $\bar{w}$ of w has a minimal pair, and if (α, δ), (α', δ) are two minimal pairs, then $[K(\alpha): K]=[K(\alpha'): K]$.

If $K \subseteq K_1 \subseteq \bar{K}$, and $\gamma \in \Gamma_{\bar{v}}$, denote by $e(\gamma, K_1)$ the smallest natural number e such that $e\gamma \in \Gamma_{v_1}$, where v_1 is the restriction of $\bar{v}$ to K_1.

We shall prove the following result.

Theorem 2.1. *Let v be a valuation on K and let w be an r.t. extension of v to $K(X)$. Then there exists a pair of definition (α, δ) for $\bar{w}$, $\alpha \in \bar{K}$, and $\delta \in \Gamma_{\bar{v}}$ such that:*

a) *If we denote $[K(\alpha): K]=n$, then for every polynomial $g(X)$ of $K[X]$, such that* $\deg g(X)<n$, *one has*

$$w(g(X)) = \bar{v}(g(\alpha)) .$$

b) *For the monic minimal polynomial $f(X)$ of α over K, let $\gamma=w(f(X))$ and $e=e(\gamma, K(\alpha))$. Then there exists $l(X) \in K[X]$ with $\deg l<n$ such that for $r=f^e/l$ one has $w(r)=0$, and r^* is transcendental over k_v.*

c) *If v_1 is the restriction of $\bar{v}$ to $K(\alpha)$, then*

$$\deg(w/v) = n \cdot e(\gamma, K(\alpha)); \quad e(w/v) = e(v_1/v)e(\gamma, K(\alpha)) .$$

d) *The field k_{v_1} can be canonically identified with the algebraic closure of k_v in k_w and*

$$f(w/v) = f(v_1/v) .$$

Proof. Let (α, δ) be a minimal pair of definition of $\bar{w}$. Denote

$$f(X) = N_{K(\alpha)(X)/K(X)}(X-\alpha)$$

It is easy to see that $f(X)$ is the minimal polynomial of α over K. Moreover $f(X)$ is monic.

a) Let $g(X) \in K[X]$, $m=\deg g(X)<n$. Let also $\beta_1, \cdots, \beta_m$ be all roots of $g(X)$ in $\bar{K}$. Then one has

$$g(X) = a \prod_{i=1}^{m} (X-\beta_i) .$$

Now since $[K(\beta_i): K] \leq m<n$, then for every i, $1 \leq i \leq m$, one has:

$$\bar{v}(\alpha-\beta_i)<\delta\,. \tag{2}$$

Indeed, if $\bar{v}(\alpha-\beta_i)\geq\delta$, then according to (1), (β_1, δ) is also a pair of definition of $\bar{w}$, contradicting the minimality of (α, δ).

Then by (2) one has $\bar{w}(X-\beta_i)=\inf(\delta, \bar{v}(\alpha-\beta_i))=\bar{v}(\alpha-\beta_i)$, and so:

$$\begin{aligned} w(g(X)) = \bar{w}(g(X)) &= \bar{v}(a)+\sum_i \bar{w}(X-\beta_i) \\ &= \bar{v}(a)+\sum_i \bar{v}(\alpha-\beta_i) = \bar{v}(a\prod_{i=1}^{m}(\alpha-\beta_i)) = \bar{v}(g(\alpha))\,. \end{aligned}$$

b) Now since $e\gamma=w(f^e)\in\Gamma_{v_1}$, there exists $l(X)\in K[X]$, $\deg l<n=[K(\alpha): K]$, such that $e\gamma=\bar{v}(l(\alpha))=w(l(X))$. Hence $w(f^e/l)=0$. Now we show that $t=(f^e/l)^*$ is transcendental over k_v; let

$$f(X)=\prod_{i=1}^{n}(X-\alpha_i)\,,\quad \alpha_1=\alpha$$

be the decomposition of f in $\bar{K}(X)$, and let $d_i\in\bar{K}$ be such that $\bar{w}(X-\alpha_i)=\bar{v}(d_i)$, $i=1,\cdots,n$. Let $d=d_1\cdots d_n$. Then $\bar{w}((X-\alpha_i)/d_i)=0$ for all i, and so $w(f/d)=0$. Now since $((X-\alpha)/d_1)^*$ is transcendental over $k_{\bar{v}}$ (see Proposition 1.1, c)$\Rightarrow$b)), it follows that $(f/d)^*$ and also $(f^e/d^e)^*$ are transcendental over k_v. Therefore

$$r^*=(f^e/l)^*=((f^e/d^e)/(l/d^e))^*=(f^e/d^e)^*/(l/d^e)^*$$

is also transcendental over k_v since $(l/d^e)^*$ is obviously algebraic (see the proof of a)).

c) Firstly, let $g\in K[X]$ be a polynomial such that $\deg g<ne$. Then we may write:

$$g=g_0+g_1f+\cdots+g_{e-1}f^{e-1} \tag{3}$$

where $g_i\in K[X]$ and $\deg g_i<n$, for all $i=0, 1, \cdots, e-1$. Moreover, one has

$$w(g)=\inf_i w(g_if^i) \tag{4}$$

since, according to the definition of $e=e(\gamma, K(\alpha))$, any two terms in the right of the equality (3) are of distinct values.

Now let $u=g/h$ be an element of $K(X)$ such that $\deg u<ne$. This means that both polynomials g and h are of degree smaller than ne, and so one has

$$u=\frac{g}{h}=\frac{g_0+g_1f+\cdots+g_{e-1}f^{e-1}}{h_1+h_0f+\cdots+h_{e-1}f^{e-1}}$$

where $\deg g_i<n$, $\deg h_i<n$, $i=0, 1, \cdots, e-1$. Let us assume that $w(u)=0$. But according to (4) one has:

$$w(u)=w(g)-w(h)=\inf_{0\leq i<e}(w(g_if^i))-\inf_{0\leq j<e}(w(h_jf^j))=0$$

and so, according to the definition of e there exists only one index i_0, $0\leq i_0\leq e-1$ such that

(5) $$w(g)=w(g_{i_0}f^{i_0})=w(h)=w(h_{i_0}f^{i_0})$$

Therefore one has:

$$u=\frac{g_{i_0}}{h_{i_0}}\,\frac{\dfrac{g_0}{g_{i_0}f^{i_0}}+\cdots+1+\cdots+\dfrac{g_{e-1}f^{e-1}}{g_{i_0}f^{i_0}}}{\dfrac{h_0}{h_{i_0}f^{i_0}}+\cdots+1+\cdots+\dfrac{h_{e-1}f^{e-1}}{h_{i_0}f^{i_0}}}$$

and so

$$w(u)=w(g_{i_0}/h_{i_0})=0\,,\quad\text{and}\quad u^*=(g_{i_0}/h_{i_0})^*\,.$$

Furthermore, we check that u^* is algebraic over k_v. Indeed, in $\bar{K}[X]$ one has:

$$g_{i_0}(X)=a\prod_i(X-\beta_i)\,,\quad h_{i_0}(X)=b\prod_j(X-\varepsilon_j)\,,\qquad \beta_i,\ \varepsilon_j\in\bar{K}\,.$$

There exist elements d_i, $p_j\in\bar{K}$ such that

$$\bar{w}(X-\beta_i)=\bar{v}(d_i)\,,\quad \bar{w}(X-\varepsilon_j)=\bar{v}(p_j)\,.$$

Denote $d=\prod_i d_i$, $p=\prod_j p_j$. Since $\deg g_{i_0}<n$, $\deg h_{i_0}<n$, then according to the choice of α (see a)) and (1), it follows that for all i and j the elements

$$((X-\beta_i)/d_i)^*\quad\text{and}\quad((X-\varepsilon_j)/p_j)^*$$

are algebraic over k_v. But one has: $u^*=(g_{i_0}/h_{i_0})^*=((d/p)(g_{i_0}/d)(p/h_{i_0}))^*=(d/p)^*\prod_i((x-\beta_i)/d_i)^*\prod_j(p_j/(x-\varepsilon_j))^*$. Therefore u^* is also algebraic over k_v. In conclusion, it follows that

$$\deg(w/v)=ne=\deg r\,.$$

Now consider the extension of degree ne:

$$K(r)\to K(X)\,.$$

If u is an element of $K(X)$ we may write:

(6) $$u=u_0(r)+u_1(r)X+\cdots+u_{ne-1}(r)X^{ne-1}$$

where $u_i(r)\in K(r)$. Let

$$u_i(r)=g_i(r)/h(r)\,,\quad g_i(r),\ h(r)\in K[r]\,.$$

Then (6) can be written

$$u=((g_0(r)+g_1(r)X+\cdots+g_{ne-1}(r)X^{en-1})/h(r))$$

and if we consider the numerator of u as a polynomial of X one has

(7) $$u=((t_0(X)+t_1(X)r+\cdots+t_s(X)r^s)/h(r))$$

where $\deg t_i(X)<ne$ for all $0\le i\le s$. We assert that

$$w(t_0+t_1r+\cdots+t_sr^s)=\inf_i(w(t_i)) \tag{8}$$

This is the case if there exists only one index i_0 such that $w(t_{i_0})=\inf(w(t_i))$. Otherwise we assume that there exists at least two indices $i_0<i_1$ such that

$$w(t_{i_0})=w(t_{i_1})=\inf_i(w(t_i))$$

but (8) is not true. Then by (7) we may write:

$$hut_{i_0}^{-1}=(t_0/t_{i_0})+(t_1/t_{i_0})r+\cdots+r^{i_0}+\cdots+(t_{i_1}/t_{i_0})r^{i_1}+\cdots+(t_s/t_{i_0})r^s$$

and since $w(hut_{i_0}^{-1})>0$ (we have assumed that (8) is not true) one has:

$$(t_0/t_{i_0})^*+\cdots+(r^*)^{i_0}+\cdots+(t_{i_1}/t_{i_0})^*(r^*)^{i_1}+\cdots=0\,.$$

But then according to above considerations all $(t_i/t_{i_0})^*$ are algebraic over k_v, and $(t_{i_1}/t_{i_0})^*\neq 0$. This shows that r^* is algebraic over k_v, a contradiction. Hence:

$$w(u)=\inf_i(w(t_i))-w(h(r))$$

and so, according to (4) we may derive that $w(u)\in\Gamma_{v_1}+\boldsymbol{Z}\gamma$, hence $\Gamma_w\subseteq\Gamma_{v_1}+\boldsymbol{Z}\gamma$ and since the reverse inclusion is obvious,

$$e(w/v)=e(v_1/v)e(\gamma,\ K(\alpha))$$

d) Let $q=e(\delta,\ K(\alpha))$ and $b\in K(\alpha)$ such that $\bar{v}(b)=q\delta$. Let β be a root of the polynomial X^q-b. It is easy to see that $[K(\alpha,\beta)\colon K(\alpha)]=q$ and $\bar{v}(\beta)=\delta$. Let w_2 be the restriction of $\bar{w}$ to $K(\alpha,\beta)(X)$ and v_2 the restriction of $\bar{v}$ to $K(\alpha,\beta)$. Since (α,δ) is a pair of definition of w, the assertion a) of Proposition 1.3 is valid relative to w_2, v_2, $\alpha\in K(\alpha,\beta)$ and $\delta\in\Gamma_{v_2}$. Hence according to Proposition 1.3 b), k_{v_2} is algebraically closed in k_{w_2}. Now by the commutative diagram canonically defined:

$$\begin{array}{ccc} k_v & \longrightarrow & k_w \\ \downarrow & & \downarrow \\ k_{v_1}=k_{v_2} & \longrightarrow & k_{w_2} \end{array}$$

we may derive that k, the algebraic closure of k_v in k_w, is included in k_{v_1}.

Now we shall show the reverse inclusion: $k_{v_1}\subseteq k$. It will be enough to show (see a) above) that for every $h(X)\in K[X]$, such that $\deg h(X)<n$ and $\bar{v}(h(\alpha))=0$, one has $h(\alpha)^*\in k$. But according to a) one has: $w(h(X))=\bar{v}(h(\alpha))=v_1(h(\alpha))=0$. We assert that

$$h(X)^*=h(\alpha)^* \tag{9}$$

Indeed, let $h(X)=\prod_{j=1}^{m}(X-\beta_j)$, $m<n$. Since (α,δ) is a minimal pair of definition of $\bar{w}$, it follows that $\bar{w}(X-\beta_i)=\bar{v}(\alpha-\beta_i)<\bar{w}(X-\alpha)=\delta$ and so:

$$\bar{w}((X-\beta_i)/(\alpha-\beta_i)-1)=\bar{w}((X-\alpha)/(\alpha-\beta_i))>0\,.$$

Hence

$$((X-\beta_i)/(\alpha-\beta_i))^* = 1$$

and consequently $(h(X)/h(\alpha))^*=1$, therefore (9) is true, i.e. $k_{v_1}\subseteq k$, as claimed. In particular,

$$f(w/v) = f(v_1/v) .$$

The proof of Theorem 2.1 is complete.

Now we list some direct consequences of Theorem 2.1. We preserve hypotheses and notation used in Theorem 2.1.

Corollary 2.2. (*see also* [10, 1.2])

$$\deg (w/v) \geq f(w/v)e(w/v) .$$

This follows immediately from c) and d) in Theorem 2.1.

Corollary 2.1. (*Nagata's conjecture* [7]; *see also* [9] *and* [11]) *One has*:

$$k_w = k_{v_1}(r^*) .$$

The proof follows from the considerations made in the proof of c) and d).

Corollary 2.4. *The valuation w is defined as follows*:
i) *If* $h(r)=a_0+a_1r+\cdots+a_mr^m\in K[r]$, *then*

$$w(h(r)) = \inf_i (v(a_i)) .$$

ii) *If* $g(X)\in K[X]$ *and* $\deg g(X)<n$, *then*

$$w(g(X)) = \bar{v}(g(\alpha)) .$$

iii) *If* $g(X)\in K[X]$ *is such that* $\deg g<ne$, *then we have the unique representation*:

$$g(X) = g_0(X)+g_1(X)f(X)+\cdots+g_{e-1}(X)f^{e-1}(X) , \quad \deg g_i(X)<n,\ 0\leq i<e$$

and

$$w(g(X)) = \inf (\bar{v}(g_i(\alpha))+i\gamma) .$$

iv) *If* $u\in K(X)$ *and if we represent u according to* (6) *and* (7), *then*:

$$w(u) = \inf_i (w(t_i(X)))-w(h(r)) .$$

The proof is contained in the proof of Theorem 2.1.

Corollary 2.5. (*See* [10, *Conjecture* 0.3]) *If v is Henselian and char* $k_v=0$, *then*:

$$\deg (w/v) = f(w/v)e(w/v)$$

Proof. Using notations of Theorem 2.1, one has:

$$\deg(w/v) = ne(\gamma, K(\alpha)) = [K(\alpha): K]e(\gamma, K(\alpha)) .$$

Now, according to [2, Corollary, p. 63], or to [5, Ch. VI, § 8, Exercise 9, a)] it follows that $[K(\alpha): K]=n=f(v_1/v)e(v_1/v)$, and so:

$$\deg(w/v) = f(v_1/v)e(v_1/v)e(\gamma, K(\alpha)) = f(w/v)e(w/v) .$$

Corollary 2.6. (*See* [10, *Conjectures* 0.1 *and* 0.4]) *The equality*:

$$\deg(w/v) = f(w/v)e(w/v)$$

is true if:

a) v *is of rank one, and char* $k_v=0$.

b) v *is of rank one and discrete.*

Proof. Let v be of rank one; then w is also of rank one. Let $K(X)$ be the completion of $\widetilde{K(X)}$ (see [12, Ch. II], or [5, Ch. VI, § 5]) relative to w, and w'' the canonical extension of w to $\widetilde{K(X)}$. Since $[\Gamma_w: \Gamma_v]<\infty$, then Γ_v is a cofinal subset of Γ_w and so $\tilde{K}$ the adherence of K in $\widetilde{K(X)}$, is the topological completion of K relative to v. Let $\tilde{v}$ be the restriction of w'' to $\tilde{K}$. Now, since $\tilde{v}$ is an immediate extension of v (see [12, Ch. II]), then, it follows that X is also transcendental over $\tilde{K}$. Let us denote by $\tilde{w}$ the restriction of w'' to $\tilde{K}(X)$. Now it is easy to see that $\tilde{w}$ is an *r.t.* extension of $\tilde{v}$ to $K(X)$ and that

$$k_v = k_{\tilde{v}} , \quad k_w = k_{\tilde{w}} , \quad \Gamma_v = \Gamma_{\tilde{v}} , \quad \Gamma_w = \Gamma_{\tilde{w}} . \tag{10}$$

According to [12, Ch. II], $\tilde{v}$ is Henselian.

We assert that in conditions a) and b) (in fact the statement is generally valid without restriction on the rank of v) one has:

$$\deg(w/v) = \deg(\tilde{w}/\tilde{v}) . \tag{11}$$

Indeed, the inequality $\deg(w/v)\geq\deg(\tilde{w}/\tilde{v})$ is obvious. On the other hand, if $u=g(X)/h(X)$ is an element of $\tilde{K}(X)$ such that $\tilde{w}(u)=0$, and if u^* is transcendental, then in a canonical way we may define two sequences $\{g_n(X)\}_n$ and $\{h_n(X)\}_n$ of polynomials of $K(X)$ such that:

$$\deg g_n(X) = \deg g(X); \ \deg h_n(X) = \deg h(X) , \quad \text{for all } n ,$$

and

$$\tilde{w}(g-g_n) \to \infty, \quad \tilde{w}(h-h_n) \to \infty .$$

Thus it is easy to see that for n large enough:

$$\tilde{w}(u-u_n)>0$$

where $u_n=g_n/h_n$. Therefore $w(u_n)=0$, and $u_n^*=u^*$ is also transcendental over $k_v=k_{\tilde{v}}$. Hence $\deg(w/v)\leq\deg(\tilde{w}/\tilde{v})$ and so (11) is proved.

Now by (10) it follows that:

$$f(w/v) = f(\tilde{w}/\tilde{v}) \quad \text{and} \quad e(w/v) = e(\tilde{w}/\tilde{v})\,.$$

Finally, the equality

$$\deg(w/v) = \deg(\tilde{w}/\tilde{v}) = f(\tilde{w}/\tilde{v})e(w/v) = f(w/v)e(w/v)$$

follows in the case a) by Corollary 2.5, and in the case b) by the general theory of discrete rank one and complete valuations (see [2] or [5]).

3. Condition $e(w/v)=f(w/v)=1$.

As usual, let v be a valuation on K and w an *r.t.* extension of v to $K(X)$. We use the same symbols as in previous sections. If $K\subseteq K_1\subseteq \bar{K}$ is an intermediate subfield we assume tacitly that K_1 is endowed with restriction v_1 of $\bar{v}$. Then K_1/K is an immediate extension if and only if $e(v_1/v)=f(v_1/v)=1$.

Now we shall consider the case where

$$e(w/v) = f(w/v) = 1\,. \tag{12}$$

Condition (12) is fulfilled if w is defined by inf, v, $\alpha\in K$ and $\delta\in\Gamma_v$. There exists also some cases where (12) is fulfilled but w is not defined by inf, any $\alpha\in K$ and $\delta\in\Gamma_v$. Precisely one has the following result.

Proposition 3.1. *The following assertions are equivalent*:

a) $e(w/v)=f(w/v)=1$.

b) If (α, δ) *is a minimal pair of definition of* $\bar{w}$, *then* $K(\alpha)/K$ *is an immediate extension and* $\deg(w/v)=[K(\alpha)\colon K]$.

c) There exists a minimal pair (α, δ) *of definition of* $\bar{w}$ *such that* $K(\alpha)/K$ *is an immediate extension and* $\deg(w/v)=[K(\alpha)\colon K]$.

Proof. a)$\Rightarrow$b). Let (α, δ) be a minimal pair of definition of $\bar{w}$ and v_1 the restriction of $\bar{v}$ to $K(\alpha)$. According to Theorem 2.1, one has:

$$e(v_1/v) = f(v_1/v) = 1$$

i.e. $K(\alpha)/K$ is an immediate extension. Moreover if $f(X)$ is the minimal polynomial of α over K, then condition $e(w/v)=1$ shows that $\gamma=w(f(X))\in\Gamma_v$ and so $e(\gamma, K(\alpha))=1$, i.e. $\deg(w/v)=[K(\alpha)\colon K]$.

The other implications follow, according to Theorem 2.1, in an obvious manner.

Remark 3.2. Let w be an *r.t.* extension of v such that condition (12) is accomplished, and let (α, δ) be a minimal pair of definition of w. Let also $f(X)$ be the minimal polynomial of α relative to K:

a) For $g(X)\in K[X]$ expand

$$g(X) = g_0(X)+g_1(X)f+\cdots+g_s(X)f^s$$

where deg $g_i(X)<\deg f$, $0\leq i\leq s$. Then according to Corollary 2.4, one has:

$$w(g(X)) = \inf_{0\leq i\leq s} (\bar{v}(g_i(\alpha))+iw(f)) . \tag{13}$$

b) Let v_1 be the restriction of $\bar{v}$ to $K(\alpha)$ and w_1 the restriction of $\bar{w}$ to $K(\alpha)(X)$. Also denote $e=e(\delta, K(\alpha))$ and $e\delta=v_1(d)$, $d\in K(\alpha)$. Then:

$$e(w_1/v_1) = e\ , \quad f(w_1/v_1) = 1\ . \tag{14}$$

Indeed, (α, δ) is also a minimal pair of definition of $\bar{w}_1=\bar{w}$ and thus (13) follows by Theorem 2.1 c) and d), since $w_1(X-\alpha)=\delta$. Moreover if $g(x)\in K(\alpha)$ $[X]$, we may write:

$$g(x) = \sum_{j=0}^{s}\sum_{i=0}^{e-1} a_{ij}(X-\alpha)^i\ ((X-\alpha)^e/d)^j$$

and thus:

$$w_1(g(X)) = \inf_j (\inf_i v_1(a_{ij})+i\delta)) .$$

Now we shall consider the following question: Assume that condition (12) is accomplished. Under what conditions w is defined by inf, v, $\alpha\in K$ and $\delta\in\Gamma_v$?

Before answering (partially) to this question we shall make some useful remarks. To derive the equality (11), we shall use the same notations and considerations as in the proof of Corollary 2.6. We point out that the valuation considered in the present case is not necessarily of rank one. As usual $\tilde{K}$ is the completion in the sense of [5, Chap. VI, Par. 5, N$^{\text{O}}$ 3] of K relative to v.

Let $f\in K[X]$ be such that $w(f)=0$. Now since $f\in\tilde{K}(X)$, then f^*, the residue of f in k_w, is the same as the residue of f considered as an element of $\tilde{K}(X)$. Hence, if for example w is defined by inf, v, $\alpha\in K$ and $\delta\in\Gamma_v$, then $\tilde{w}$ is also defined by inf, v, α and δ.

Now let $f\in\tilde{K}[X]$ be such that $\tilde{w}(f)=0$. Then there exists a polynomial $f_1\in K[X]$, of the same degree, such that $w(f_1)=0$ and $\tilde{w}(f-f_1)>0$, i.e. $f^*=f_1^*$. Therefore, if for example $\tilde{w}$ is defined by inf, v, $\alpha\in\tilde{K}$ and $\delta\in\Gamma_{\tilde{v}}=\Gamma_v$, then w is also defined by inf, v, a suitable $\alpha_1\in K$ and δ.

According to these considerations, the study of the set of all polynomials g over K such that $w(g)=0$ and g^* is transcendental over k_v is equivalent to the study of the set of all polynomials g over $\tilde{K}$ such that $\tilde{w}(g)=0$ and g^* is transcendental over $k_{\tilde{v}}=k_v$. Therefore in what follows we may assume that $K=\tilde{K}$ and $w=\tilde{w}$.

Theorem 3.3. *Let K be a field and v a valuation on K. The following assertions are equivalent*:

a) If w is an r.t. extension of v to $K(X)$ such that $e(w/v)=f(w/v)=1$, then w is defined by inf, v, $\alpha\in K$ and $\delta\in\Gamma_v$.

b) $\tilde{K}$ does not admit immediate finite extensions relative to $\tilde{v}$.

c) $\tilde{K}$ is algebraically closed in a maximally complete extension of $\tilde{K}$ relative to $\tilde{v}$.

Proof. The equivalence b)$\Leftrightarrow$c) is obvious.

b)$\Rightarrow$a) Let w be such that $e(w/v)=f(w/v)=1$ and let (α, δ) be a minimal pair of definition of w. If $\alpha\notin K$ (i.e. a) is not true), then according to Proposition 3.1, $K(\alpha)/K$ is an immediate extension and condition $\alpha\notin K$ shows that $\alpha\notin\tilde{K}$. Hence $\tilde{K}(\alpha)/\tilde{K}$ is an immediate extension relative to v, a contradiction.

a)$\Rightarrow$b) Let us assume that $\tilde{K}$ has an immediate algebraic extension $\tilde{K}(\beta)/\tilde{K}$ relative to $\tilde{v}$ and let v_1 be the corresponding extension of $\tilde{v}$ to $\tilde{K}(\beta)$. Obviously, we assume that $\beta\notin\tilde{K}$. Then the set

$$M(\beta)=\{v_1(\beta-a)\,|\,a\in\tilde{K}\}$$

is bounded in $\Gamma_{\tilde{v}}=\Gamma_{v_1}$. Let $d_1\in\tilde{K}$ be such that $v_1(d_1)>\gamma$ for all $\gamma\in M(\beta)$. Let us denote by w_1 the valuation on $\tilde{K}(\beta)(X)$ defined by inf, v_1, β, and $\delta_1=v_1(d_1)$ and let w' be the restriction of w_1 to $\tilde{K}(X)$. It is clear that w' is an *r.t.* extension of $\tilde{v}$ and $\Gamma_{\tilde{v}}=\Gamma_{v_1}=\Gamma_{w_1}=\Gamma_{w'}$, hence $e(w'/\tilde{v})=e(w_1/v_1)=1$. Moreover, since $k_{\tilde{v}}=k_{v_1}$ and $f(w_1/v_1)=1$, it follows that $f(w'/\tilde{v})=1$. Let w be the restriction of w' to $K(X)$. Since obviously $e(w/v)=f(w/v)=1$, by hypothesis, it follows that w is defined by inf, v, $\alpha\in K$ and $\delta\in\Gamma_v$. Then one has: $w(X-\alpha)=\delta$. Let $d\in K$ be such that $v(d)=\delta$. Consider the equality:

$$(X-\alpha)/d=((X-\beta)/d_1)\cdot(d_1/d)+(\beta-\alpha)/d\,. \tag{15}$$

If $\delta_1=v(d_1)>v(d)=\delta$, then by (15) it follows that the image $(X-\alpha)/d$ in the residue field is an element of k_v, a contradiction.

If $v_1(d_1)=v(d)$, then by (15) it follows that $v_1(\beta-\alpha)\geq v_1(d_1)$, which contradicts the choice of d_1.

Finally, assume that $v(d)>v_1(d_1)$, and let $b\in K$ be such that $v(\alpha-b)>v(d)$. Then one has:

$$X-b=((X-\alpha)/d)d+(\alpha-b)\,,$$
$$X-b=((X-\beta)/d_1)d_1+(\beta-b)\,.$$

Since w is the restriction of w_1 to $K(X)$, one has $w_1(X-b)=w(X-\beta)$. But:

$$w(X-b)=\inf(v(d),\ v(\alpha-b))=v(d)\,,$$
$$w_1(X-b)=\inf(v_1(d_1),\ v_1(\beta-b))<v(d)\,.$$

We were considering the case where $v(d)>v_1(d_1)$, and we have a contradiction. The proof is complete.

Corollary 3.4. *Let v be a valuation on K. The equivalent conditions of Theorem 3.3 are accomplished if $\tilde{K}$ is maximally complete relative to $\tilde{v}$. This is the case if v is of rank one and discrete or K is maximally complete relative to v.*

Other cases, where the conditions of Theorem 3.3 are verified, are given in the following:

Proposition 3.5. *Let K be a field and v a valuation on K such that:*

a) $\tilde{v}$ *is Henselian and char* $k_v=0$

or

b) v *is of rank one and K is perfect of characteristic $p>0$.*

Then $\tilde{K}$ *does not admit nontrivial finite and immediate extensions relative to* v.

Proof. According to [12, Ch. II, Theorem 4], in the case b) $\tilde{K}$ is Henselian relative to $\tilde{v}$. Also it is easy to check that $\tilde{K}$ is perfect.

Let $\bar{K}$ be an algebraic closure of $\tilde{K}$ and let $\bar{v}$ be the unique extension of $\tilde{v}$ to $\bar{K}$. Suppose $\tilde{K}(\alpha)$ is a finite and immediate extension of $\tilde{K}$ relative to $\tilde{v}$, such that $\alpha\notin\tilde{K}$. Let also $\Delta(\alpha)=\inf \bar{v}(\alpha-\alpha')$ where α' runs over all conjugate elements of α. Then, according to [1, Section 2, Proposition 2'], there exists an element $a\in\tilde{K}$ such that $\bar{v}(\alpha-a)=\Delta(\alpha)$. It is easy to see that:

$$\bar{v}(\alpha-a)=\Delta(\alpha)=\sup\{\bar{v}(\alpha-b)\,|\,b\in\tilde{K}\}\ . \tag{16}$$

Now since the extension $\tilde{K}(\alpha)/\tilde{K}$ is immediate we can assume that $\bar{v}(\alpha)=0$ and so $\bar{v}(\alpha-a)>0$. Let $d\in K$ be such that $\tilde{v}(d)=\bar{v}(\alpha-a)$. Then $\bar{v}((\alpha-a)/d)=0$ and since the extension $\tilde{K}(\alpha)/\tilde{K}$ is immediate, there exist a_1, $d_1\in\tilde{K}$ such that $\bar{v}(((\alpha-a)/d)-a_1)=\tilde{v}(d_1)>0$. But then $\bar{v}(\alpha-a-a_1d)=\tilde{v}(dd_1)>\bar{v}(\alpha-a)$, which contradicts (16).

Remarks 3.6. a) Let v be a valuation on K such that the equivalent assertions of Theorem 3.3 are accomplished. Then $\tilde{v}$ is necessarily Henselian. Indeed, let $K_1/\tilde{K}$ be an algebraic extension, and let $K_2/\tilde{K}$ be an immediate extension, such that K_2 is maximally complete and that the condition c) of Theorem 3.3 is accomplished. Then K_2K_1/K_2 is an algebraic extension. Now if v_2 is the extension of $\tilde{v}$ to K_2, then since v_2 is Henselian (see [12, Ch. II, Theorem 7]) it follows that $\tilde{v}$ has a unique extension to K_1 i.e. $\tilde{v}$ is Henselian.

b) According to Corollary 2.5, if v is Henselian and char $k_v=0$, and the condition (12) is accomplished, then w is defined by inf, v, $\alpha\in K$ and $\delta\in\Gamma_v$. Therefore according to Theorem 3.3, $\tilde{K}$ does not admit immediate extensions relative to $\tilde{v}$. Moreover, it can be proved that $\tilde{v}$ is also Henselian.

Proposition 3.7. *Let K be a field and v a valuation on K. The following assertions are equivalent:*

a) If w is an r.t. extension of v to $K(X)$, then $e(w/v)=f(w/v)=1$.

b) Every extension $\bar{v}$ of v to $\bar{K}$ is an immediate extension.

Proof. a)$\Rightarrow$b) Let $\bar{v}$ be an extension of v to $\bar{K}$. Firstly we shall prove that $\Gamma_v=\Gamma_{\bar{v}}$. Indeed, let us assume there exists $\beta\in\bar{K}$ such that $\delta=\bar{v}(\beta)$ does not belong to Γ_v. Let w be the valuation on $K(X)$ defined by inf, v, a suitable $a\in K$ and δ. Then $w(X-a)=\delta\in\Gamma_w$, and $\delta\notin\Gamma_v$, a contradiction.

Now we shall prove that $k_v=k_{\bar{v}}$. Indeed, assume that $k_v\neq k_{\bar{v}}$ and let $\varepsilon\in k_{\bar{v}}\backslash k_v$. Let $\alpha\in\bar{K}$ be such that $\bar{v}(\alpha)=0$ and $\alpha^*=\varepsilon$. Let w be the valuation on $K(X)$ defined by inf, v, α and $\delta>0$. Then one has $w(X-\alpha)=\delta>0$, and so $w(X)=w(\alpha)=$

$\bar{v}(\alpha)=0$. But then $X^*=\alpha^*=\varepsilon\notin k_v$. Hence $f(w/v)\neq 1$ again, a contradiction. Thus a)$\Rightarrow$b) is proved.

b)$\Rightarrow$a) Indeed $k_v=k_{\bar{v}}$ is algebraically closed and $\Gamma_v=\Gamma_{\bar{v}}$ is divisible. This means that $e(w/v)=f(w/v)$ for every *r.t.* extension of v to $K(X)$ and the proof is complete.

The conditions of Proposition 3.7 are verified for the field $\boldsymbol{R}$ of real numbers relative to every nonarchimedean valuation and also for the field $\boldsymbol{K}$ generated by all roots of unity over the rational number field $\boldsymbol{Q}$.

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS
STR. ACADEMIEI 14, 70109 BUCHAREST, ROMANIA
DEPARTMENT OF MATHEMATICS, INCREST
BDUL PACII 220, 79622 BUCHAREST, ROMANIA
LICEUL AGROINDUSTRIAL CODLEA
2252 CODLEA
JUDETUL BRASOV, ROMANIA

References

[1] V. Alexandru et N. Popescu, Sur une classe de prolongements à $K(X)$ d'une valuation sur un corps K (to appear).
[2] E. Artin, Algebraic Numbers and Algebraic Functions, Gordon and Breach, New York, 1967.
[3] J. Ax, Zeros of polynomials over local fields—the Galois action, J. Algebra, **15** (1970), 417–428.
[4] N. Bourbaki, Théorie des Ensembles, Ch. III, Hermann, Paris, 1956.
[5] N. Bourbaki, Algèbre commutative, Ch. V, VI, Hermann, Paris, 1964.
[6] O. Endler, Valuation Theory, Springer, Berlin-Heidelberg-New York, 1972.
[7] M. Nagata, A theorem on valuation rings and its applications, Nagoya Math. J., **29** (1967), 85–91.
[8] J. Ohm, Simple transcendental extensions of valued fields, J. Math. Kyoto Univ., **22** (1982), 201–221.
[9] J. Ohm, The ruled residue theorem for simple transcendental extensions of valued field, Proc. Amer. Math. Soc., **89** (1983), 16–18.
[10] J. Ohm, Simple transcendental extensions of valued fields, II: A fundamental inequality, J. Math. Kyoto Univ., **25** (1985), 583–596.
[11] N. Popescu, On a problem of Nagata in valuation theory, Rev. Roumaine Math. Pures Appl., **31** (1986), 639–641.
[12] O. F. G. Schilling, The theory of valuations, A.M.S. Math. Surveys, Nr. 4. Providence, Rhode Island, 1950.
[13] P. Samuel and O. Zariski, Commutative Algebra, I, II, Van Nostrand, Princeton, 1958, 1960.