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K-ring of the orbit spaces of spheres by finite free actions

Dedicated to professor Hirosi Toda on his 60th birthday

By

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1. Introduction

If a finite group G operates freely on a sphere S^n and the operation is piecewise linear, then its quotient space S^n/G is a PL-manifold and K-ring $K(S^n/G)$ is defined. The purpose of this paper is to give a complete description of $K(S^n/G)$ by terms of the representation ring of G, $R(G)$. This is obtained by showing that Atiyah's conjecture in [1] is true for the Artin-Tate groups (with periodic cohomology). We may consider the Atiyah's conjecture in two steps.

(A) For p-group G, universal cycles in the Atiyah's spectral sequence $H^*(G) \Rightarrow \widehat{R(G)}$ are generated by Chern classes of representations of G.

(B) If p-group G_p satisfies (A), then for every group G which contains G_p as a p-Sylow subgroup, p-primary component in universal cycles in the spectral sequence $H^*(G) \Rightarrow \widehat{R(G)}$ is generated by Chern classes of representations of G. I believe that (A) is still an unsolved problem. We prove that (B) is true for G_p which is isomorphic to a cyclic group or a generalized quaternion group but (B) is not true in general.

2. Filtrations in K-theory

For any finite CW-complex X, there are two impotant filtrations in $K(X)$. One is called the ordinary filtration; the subgroup $K_q(X)$ is defined as the kernel of the restriction homomorphism $K(X) \to K(X_{q-1})$ where X_{q-1} is the $(q-1)$-skeleton of X. Another is called the γ-filtration; $K^{\gamma}_{2q}(X)$ is defined as the subgroup generated by the monomials,

$$\gamma^{n_1}(\tau_1)\cdot\gamma^{n_2}(\tau_2)\cdot\cdots\cdot\gamma^{n_k}(\tau_k),\ \tau_i \in \tilde{K}(X)$$

and $\sum n_i \geqq q$ where γ^{n_i} is the γ-operation on $K(X)$. It is well known that the both filtrations make $K(X)$ filtered rings and satisfy the relation; $K^{\gamma}_{2q}(X) \subseteq K_{2q}(X)$. We want to get a sufficient condition being $K^{\gamma}_{2q}(X) = K_{2q}(X)$. For this, we shall proceed to describe the geometric meaning of γ^i-operation. Let $B_{U(n)}$ be the classifying space of n-dimensional complex vector bundles and Γ^n be the universal

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bundle over $B_{U(n)}$. Let T^n be the maximal torus in $U(n)$. Let $\pi_i: B_{T^n} = B_{S^1} \times \cdots \times B_{S^1} = CP^\infty \times \cdots \times CP^\infty \to CP^\infty$ be the projection on i-th factor and $\xi_i = \pi_i^!(\xi)$ where ξ is the canonical line bundle over CP^∞. Let $i: B_{T^n} \to B_{U(n)}$ be the map induced by the natural inclusion map $T^n \to U(n)$. Then $i^!(\Gamma^n) = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$, and for the i-th Chern class $c_i(\Gamma^n)$ of Γ^n, $i^*c_i(\Gamma^n)$ is the i-th elementary symmetric function of $c_1(\xi_i)$. Put $\sigma_i = \xi_i - 1$, then $\gamma^0(\sigma_i) = 1$, $\gamma^1(\sigma_i) = \sigma_i$ and $\gamma^j(\sigma_i) = 0$ for $j > 1$. Hence the sum formula of γ-operation implies $\gamma^i(\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n - n) = \gamma^i(\sigma_1 + \sigma_2 + \cdots + \sigma_n) = i$-th elementary symmetric function of σ_j, $j = 1, 2, \cdots, n$. Since $H^{2i+1}(B_{T^n}) = 0$, the Atiyah-Hirzebruch spectral sequence $H^*(B_{T^n}) \Rightarrow K(B_{T^n})$ is trivial, (Note that $K(B_{T^n})$ is considerd as a K-ring of a sufficient large skeleton of B_{T^n}), and so we have an isomorphism; $\varphi: H^{2i}(B_{T^n}) \to K_{2i}(B_{T^n})/K_{2i+2}(B_{T^n})$. Clearly $\varphi(c_1(\xi_i)) = \sigma_i$ and $i^*: H^*(B_{U(n)}) \to H^*(B_{T^n})$ is injective, we obtain $\varphi(c_i(\Gamma^n)) = [\gamma^i(\Gamma^n - n)] \in K_{2i}(B_{U(n)})/K_{2i+2}(B_{U(n)})$ by the naturality of the spectral sequence, Chern classes and γ^i-operations. It follows that for a vector bundle η over X, $c_i(\eta)$ is an universal cycle in the Atiyah-Hirzebruch spactral sequence and $\varphi(c_i(\eta)) = \gamma_i(\eta - \dim \eta)$. This shows the following proposition.

Proposition 1. If $H^{\text{even}}(X) = \sum_i H^{2i}(X)$ is generated by Chern classes of vector bundle over X, then $K_{2q}^\gamma(X) = K_{2q}(X)$.

Let G be a finite group. Let $\rho: G \to U(n)$ be a representation of G and M_ρ be its representation module. Let B_G be a classifying space of G and $E_G \to B_G$ be the universal G-bundle. Then the associated vector bundle $E_G \times_G M_\rho \to B_G$ is written by $\alpha(\rho)$. Recall that the i-th Chern class of ρ, $c_i(\rho) \in H^{2i}(G)$ is defined to be the i-th Chern class of $\alpha(\rho)$ by identifying $H^*(B_G) = H^*(G)$ where $H^*(G)$ is the cohomology ring of G with the coefficient group Z on which G operates trivially. In view of the Whitney's sum formula, Chern class may be extended naturally for elements of the representation ring $R(G)$.

Let $f: G \times S^{2n-1} \to S^{2n-1}$ be a free PL-action and $S^{2n-1}/G, f$ be the quotient space defined by f. Let $\alpha_f: R(G) \to K(S^{2n-1}/G, f)$ be a homomorphism obtained by assigning to each representation module M, its associated vector bundle $S^{2n-1} \times_G M \to S^{2n-1}/G, f$.

To state the following lemma, we recall the γ-filtration in $R(G)$ defined as in the case of $K(X)$. Let $I(G)$ be the kernel of the augmentation homomorphism $R(G) \to Z$. Then $R_{2n}^\gamma(G)$ is the ideal generated by monomials; $\gamma^{n_1}(\sigma_1) \cdots \gamma^{n_k}(\sigma_k)$ with $\sigma_i \in I(G)$ and $\sum_{n_i} \geqq n$.

Lemma 2. *Let notations be as above and assumme that $H^{\text{even}}(G)$ is generated by Chern classes of representations of G. Then a ring homomorphism α_f induces an isomorphism; $R(G)/R_{2n}^\gamma(G) \cong K(S^{2n-1}/G, f)$.*

Proof. G is necessary a Artin-Tate group and so $H^*(G) = H^{\text{even}}(G)$. S^{2n-1}/G, f may be seen as a $(2n-1)$-skeleton of B_G and so $H^{2i}(S^{2n-1}/G, f) \cong H^{2i}(G)$ for i

$\leqq n-1$. If $c_i(\rho_j)$ generate $H^*(G)$, then $c_i(\alpha_f(\rho_j))$ generate $H^{\mathrm{even}}(S^{2n-1}/G, f)$. It follows that using Atiyah-Hirzebruch spectral sequence, $\alpha_f(\gamma^i(\rho_j - \dim. \rho_j))$ generate $K(S^{2n-1}/G, f)$. Thus we have showed that α_f is surjective. Since α_f induces an isomorphism $R^{\gamma}_{2q}(G)/R^{\gamma}_{2q+2}(G) \cong K_{2q}(S^{2n-1}/G, f)/K_{2q+2}(S^{2n-1}/G, f)$ $(= H^{2q}(G))$ for $q \leqq n-1$, we have Ker. $\alpha_f = R^{\gamma}_{2n}(G)$.

In the view of the lemma 2, we review the cohomology of finite groups. $H^n(G)$ is annihilated by the order of G, $|G|$, for all $n > 0$. For each prime p, we denote by $H^n(G, p)$ the p-primary component of $H^n(G)$. Then $H^n(G)$ admits a primary decomposition; $H^n(G) = \sum_p H^n(G, p)$ where p ranges over the prime deviding $|G|$. Let G_p be a p-Sylow subgroup of G and $\mathrm{res}^G_{G_p}\colon H^n(G) \to H^n(G_p)$ the homomorphism induced by the restriction. It is well-known that $\mathrm{res}^G_{G_p}$ maps $H^n(G, p)$ isomorphically onto the set of G-invariant elements in $H^n(G_p)$ and the image is a direct summand of $H^n(G_p)$. Recall that an element $x \in H^n(G_p)$ is called G-invariant if for each $g \in G$ it satisfies the relation;

$$\mathrm{res}^{G_p}_{G_p \cap gG_pg^{-1}}(x) = \mathrm{res}^{gG_pg^{-1}}_{G_p \cap gG_pg^{-1}}(f^*_g(x)),$$

where $f_g\colon gG_pg^{-1} \to G_p$ is defined by $f_g(h) = g^{-1}hg$ for $h \in gG_pg^{-1}$.

In the following two sections, we shall show that $H^n(G, p)$ is generated by Chern classes of representations of G if G_p is isomorphic to either a cyclic group or a generalized quaternion group.

3. Case for cyclic p-Sylow subgroups

Let C_q the cyclic group of order q and a be a generator, i.e. $\langle a \rangle = C_q$. Let $\rho_q\colon C_q \to U(1)$ be the representation defined by $\rho_q(a) = \exp(2\pi i/q)$. Then the irreducible unitary representation of C_q is one of the i-time tensor product of ρ_q, ρ^i_q, $i = 0, 1, \dots, q-1$ where ρ^0_q is the trivial representation written ordinarily by 1. If r divides q, C_q has unique subgroup of order r. i.e. $C_r = \langle a^{q/r} \rangle$. We denote by $\mathrm{res}^{C_q}_{C_r}\colon R(C_q) \to R(C_r)$ the restriction homomorphism. Clearly we have $\mathrm{res}^{C_q}_{C_r}(\rho_q) = \rho_r$. Put $x = c_1(\rho_q) \in H^2(C_q) \cong Z/q$, then x generates multiplicatively $\tilde{H}^*(C_q) = \boldsymbol{Z}/q(x)$. Note that for a finite group G, $c_1\colon \mathrm{Hom}(G, U(1)) \to H^2(G)$ is an isomorphism being $\mathrm{Hom}(G, U(1))$ the group defined by the tensor product. By the naturality of the Chern class, $c_1(\mathrm{res}^{C_q}_{C_r}(\rho_q)) = c_1(\rho_r) = \mathrm{res}^{C_q}_{C_r}(c_1(\rho_q))$, and therefore $\mathrm{res}^{C_q}_{C_r}\colon H^*(C_q) \to H^*(C_r)$ is the natural projection.

Lemma 3. *For every group G which contains C_{p^n} as a normal p-Sylow subgroup, $H^*(G, p)$ is generated by the Chern class of the representation of G.*

Proof. $H^*(G, p)$ is isomrophic to the subring $H^*(C_{p^n})^{G/C_{p^n}}$ in $H^*(C_{p^n})$ generated by G-invariant elements. Let C be the centralizer of C_{p^n} in G, then obviously $H^*(C_{p^n})^{G/C_{p^n}} = H^*(C_{p^n})^{G/C}$. Since the automorphism group of C_{p^n} is isomorphic to the cyclic group of order $(p-1)p^n$ for odd prime and has an order of 2-power for $p = 2$, G/C is a cyclic group of the order s dividing $(p-1)$. Let $\bar{g}$

be a generator of G/C and $g^{-1}ag = a^r$. Then we have $(r, p) = 1$, $r^i \not\equiv 1 \bmod (p)$ for $i < s$ and $r^s \equiv 1 \bmod (p)$. Put $x = c_1(\rho_{p^n})$, then $f_g^*(x) = f_g^*(c_1(\rho_{p^n})) = c_1(f_g^!(\rho_{p^n})) = c_1(\rho_{p^n}^r) = r \cdot c_1(\rho_{p^n}) = r \cdot x$. It follows that x^m is stable if and only if $r^m \equiv 1 \bmod (p)$ and so $m = k \cdot s$ for some k. At all, $H^*(C_{p^n})^{G/C}$ is the subring generated by x^s. We see from the theorem of Burnside that there exists a normal subgroup H of C such that C is isomorphc to $C_{p^n} \times H$. Let μ be the representation of C defined by $\mu(a, h) = \exp(2\pi i/_{p^n})$ and consider the induced representation κ of μ i.e. $\kappa = \mathrm{ind}_C^G(\mu)$. Then $\mathrm{res}_{C_{pn}}^G(\kappa) = \rho_{p^n} \oplus \rho_p^{r_n} \oplus \cdots \oplus \rho_{p^n}^{r^{s-1}}$ and its Chern class is given by

$$\begin{aligned} c(\mathrm{res}_{C_{pn}}^G(\kappa) &= c(\rho_{p^n})\, c(\rho_p^r n) \cdots\cdots c(\rho_{p^n}^{r^{s-1}}) \\ &= (1+x)(1+rx)\cdots\cdots(1+r^{s-1}x) \\ &= 1 + r^{s(s-1)/2}x^s = 1 + x^s. \end{aligned}$$

Thus, $c_s(\mathrm{res}_{C_{pn}}^G(\kappa))$ generates $H^*(C_{p^n})^{G/C}$ and therefore $c_s(\kappa)$ generates $H^*(G, p)$.

We proceed to the general case. Let C_{p^n} be a p-Sylow subgroup of G and N, C denote the normalizer and centralizer of C_{p^n}, respectively. By the lemma 3, we have a representation κ of N which satisfies $c(\mathrm{res}_{C_{pn}}^N(\kappa)) = 1 + x^s$, $s = [N; C]$. Consider the induced representation of κ, $\mathrm{ind}_N^G(\kappa)$. Then Mackey's decomposition theorem implies;

$$\mathrm{res}_{C_{pn}}^G \mathrm{ind}_N^G(\kappa) = \bigoplus_{g\in E} \mathrm{ind}_{C_{p^n}\cap gNg^{-1}}^{C_{p^n}} \cdot \mathrm{res}_{C_{p^n}\cap gNg^{-1}}^{gNg^{-1}}(f_g^!(\kappa)).$$

where E is a set of representatives for the double cosets $C_pn \cdot g \cdot N$. Let E_i be the subset of E which satisfies $C_{p^n} \cap gNg^{-1} = C_{p^i}$.

Lemma 4. $\mathrm{res}_{C_{p^n}\cap gNg^{-1}}^{gNg^{-1}}(f_g^!(\kappa))$ *doesn't depend to* $g \in E_i$ *for a fixed* i. *That is,* $\mathrm{res}_{p^n\cap gNg^{-1}}^{gNg^{-1}}(f_g^!(\kappa)) = \mathrm{res}_{C_{p^i}}^N(\kappa) = \rho_{p^i} \oplus \cdots \oplus \rho_{p^i}^{r^{s-1}}$.

Proof. Let D be the centralizer of $C_{p^n} \cap gC_{p^n}g^{-1}$ in G. Since C_{p^n}, $gC_{p^n}g^{-1} \subset D$ and C_{p^n} is a p-Sylow subgroup of G, these are also p-Sylow subgroups of D. Therefore there is a $h \in D$ such that $h(gC_{p^n}g^{-1})h^{-1} = C_{p^n}$ and so $hg \in N$. Now, for $y \in C_{p^n} \cap gNg^{-1}$, we have $\mathrm{res}_{C_{p^n}gNg^{-1}}^{gNg^{-1}}(f_g^!(\kappa))(y) = \kappa(g^{-1}yg) = \kappa(g^{-1}h^{-1}ygh) = \kappa((hg)^{-1}y(hg)) = \kappa(y) = \mathrm{res}_{C_{p^i}}^N(\kappa)(y)$.

Theorem 1. *For every* G *which contains* C_{p^n} *as a* p*-Sylow subgroup,* $H^*(G, p)$ *is generated by Chern classes of representations of* G.

Proof. Since $\mathrm{res}_{C_{pn}}^G(H^*(G)) \subset \mathrm{res}_{C_{pn}}^N(H^*(N))$, in order to prove our theorem, it is enough to show that $\mathrm{res}_{C_{p^n}}^N(H^*(N))$ is generated by Chern classes of representations of N which are the restriction of representations of G. We put $\sigma = \mathrm{ind}_N^G(\kappa)$, $\sigma' = \mathrm{ind}_N^G(s)$, and we shall show that $\mathrm{res}_{C_{pn}}^G(c_s(\sigma - \sigma'))$ generates $\mathrm{res}_{C_{pn}}^N(H^*(N))$. For this, we must to see that $\mathrm{res}_{C_{pn}}^G(c_s(\sigma - \sigma'))$ is a generator of $H^{2s}(C_{p^n})$. Cleraly for above, it is sufficient to see that $\mathrm{res}_{C_p}^G(c_s(\sigma - \sigma'))$ is a generator of $H^{2s}(C_p)$. We make a computation by using the Mackey's decomposition theorem.

$$\begin{aligned}\operatorname{res}^G_{C_p}(\sigma - \sigma') &= \operatorname{res}^G_{C_p} \operatorname{ind}^G_N(\kappa - s) \\ &= \bigoplus_{g \in E} \operatorname{ind}^{C_p}_{C_p \cap gNg^{-1}} \operatorname{res}^{gNg^{-1}}_{C_p \cap gNg^{-1}} (f^!_g(\kappa - s))\end{aligned}$$

where E is a set of representatives for the double cosets C_pgN. Putting $E_0 = \{g \in E | C_p \cap gNg^{-1} = \{e\}\}$, $E_1 = \{g \leftarrow E | C_p \cap gNg^{-1} = C_p\}$, we have $\operatorname{res}^G_{C_p}(\sigma - \sigma') = \bigoplus_{g \in E_1} \operatorname{ind}^{C_p}_{C_p} \operatorname{res}^{gNg^{-1}}_{C_p}(\kappa - s) = \bigoplus_{g \in E_1} \operatorname{res}^N_{C_p}(\kappa - s)$, where the last equation follows from the lemma 4. Let α_i be the number of elements in E_i for $i = 0, 1$. Since the number of left cosets of C_p in C_pgN is $[N: N \cap g^{-1}C_pg]$, we have $|G| = |N| \cdot p \cdot \alpha_0 + [N: C_p] \cdot p \cdot \alpha_1$ and so $[G: N] = p \cdot \alpha_0 + \alpha_1$. It follows α_1 is prime to p. We are now ready to compute the Chern class.

$$\begin{aligned} c(\operatorname{res}^G_{C_p}(\sigma - \sigma')) &= c(\alpha_1(\kappa - s)) = c(\kappa - s)^{\alpha_1} \\ &= c(\rho_p + \rho_p^r + \cdots + \rho_p^{r^{s-1}} - s)^{\alpha_1} \\ &= (1 + x^s)^{\alpha_1} = 1 + \alpha_1 \cdot x^s + \text{higher terms}\end{aligned}$$

where $x = c_1(\rho_p)$. It follows that $c_s(\sigma - \sigma')$ generates $H^*(G, p)$.

4. Case for generalized quaternion groups.

The generalized quaternion group of order 2^m, $m \geqq 3$, has a presentation;

$$Q_{2^m} = \langle a, b;\, a^{2^{m-2}} = b^2,\ \text{bad}^{-1} = a^{-1} \rangle.$$

Q_{2^m} has four one-dimensional irreducible representations, $\xi_0 = 1$, ξ_1, ξ_2, ξ_3, defined by $\xi_1(a) = 1$, $\xi_1(b) = -1$; $\xi_2(a) = -1$, $\xi_2(b) = 1$; $\xi_3(a) = -1$, $\xi_3(b) = -1$. Other irreducible representations have the dimension 2 and given by

$$\zeta_r(a) = \begin{bmatrix} \exp(2\pi ri/2^{m-1}) & 0 \\ 0 & \exp(-2\pi ri/2^{m-1}) \end{bmatrix}, \quad \zeta_r(b) = \begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix}$$

$r = 1, 2, \ldots, 2^{m-2} - 1$. Put $c_1(\xi_1) = x$, $c_1(\xi_2) = y$, $c_2(\zeta_1) = z$. Then $c(\zeta_1) = 1 + z$ because $c_1(\zeta_1) = c_1(\det. \zeta_1) = c_1(\xi_0) = 0$. For simplicity we use the notation Q instead of Q_{2^m}. The cohomology of Q is as follows; $H^2(Q) = \boldsymbol{Z}/2 + \boldsymbol{Z}/2$, $H^4(Q) = \boldsymbol{Z}/2^m$, $H^{2i+1}(Q) = 0$, and a generator of $H^4(Q)$ gives the periodicity, i.e. $H^{i+4}(Q)$. Since $\operatorname{res}^Q_{\langle a \rangle}(\xi_1) = \rho^0_{2^{m-1}}$, $\operatorname{res}^Q_{\langle b \rangle}(\xi_1) = \rho^2_4$, $\operatorname{res}^Q_{\langle a \rangle}(\xi_2) = \rho^{2^{m-2}}_{2^{m-1}}$, $\operatorname{res}^Q_{\langle b \rangle}(\xi_2) = \rho^0_4$, $H^2(Q) = \boldsymbol{Z}/2 + \boldsymbol{Z}/2$ is generated by x, y. $\operatorname{res}^Q_{\langle b^2 \rangle}(z) = \operatorname{res}^\alpha_{\langle b^2 \rangle}(c_2(\zeta_1)) = c_2(\operatorname{res}^Q_{\langle b^2 \rangle}(\zeta_1)) = c_2(\rho_2 + \rho_2) = c_1(\rho_2)^2$ is a generator of $H^4(C_2)$ and hence z is a generator of $H^4(Q)$. Making a similar cmputation, we obtain the relations $x^2 = y^2 = 0$, $xy = 2^{m-1}z$.

Theorem 2. *If a 2-Sylow subgroup of G is isomorphic to the generalized quaternion group, then $H^*(G, 2)$ is generated by Chern classes of representations of G.*

Proof. $H^2(G)$ is always generate by Chern classes of 1-dimensional representations of G. We consider $H^4(G)$ and prove that $H^4(G, 2) = \boldsymbol{Z}/_{2^m}$, and it is generated by Chern classes. Now $H^*(G, 2)$ has the periodicity with the cup product by the generator of $H^4(G, 2)$, and our theorem will be proved. In order to prove above assertion, it is sufficient to costruct a representation η of G such that $\mathrm{res}_Q^G c_2(\eta)$ is a generator of $H^4(Q)$. Furthermore, for this, it is enough to show that $\mathrm{res}^G_{\langle a^{2^{m-1}}\rangle} c_2(\eta)$ is a non-zero element in $H^4(C_2)$ because $H^4(G, 2)$ is a direct summand of $H^4(Q)$. Let N be the normalizer of $\langle a\rangle$ in G. Then N contains Q and there is a normal subgroup T of N such that $Q = N/T$, [2]. We define a representation η' of N by the composition $\zeta_1 \circ \pi$: $N \to N/T = Q \to U(2)$ where π is the natural projection. We put $\eta = \mathrm{ind}_N^G(\eta' - 2)$. By the Mackey's decomposition theorem, we have

$$\begin{aligned}\mathrm{res}^G_{C_2}(\eta) &= \mathrm{res}^G_{C_2}(\mathrm{ind}^G_N(\eta' - 2)) \\ &= \bigoplus_{g\in E} \mathrm{ind}^{C_2}_{C_2\cap gNg^{-1}} \mathrm{res}^{gNg^{-1}}_{C_2\cap gNg^{-1}} (f_g^!(\eta' - 2)) \\ &= \bigoplus_{g\in E_1} \mathrm{res}^{gNg^{-1}}_{C_2} (f_g^!(\eta' - 2))\end{aligned}$$

where E is a set of representatives for the double cosets C_2gN and $E_1 = \{g\in E | C_2 \cap gNg^{-1} = C_2\}$. As in the proof of the theorem 1, the number α_1 of elements in E_1 is prime to 2. Since an automorphism or C_2 is trivial, we have

$$\begin{aligned}c(\mathrm{res}^G_{C_2}(\eta)) &= c(\gamma_1(\mathrm{res}^Q_{C_2}(\zeta_1 - 2))) = c(\rho_2 + \rho_2 - 2)^{\alpha_1} \\ &= 1 + \alpha_i c_1(\rho_2)^2 + \text{higher terms.}\end{aligned}$$

From this $c_2(\mathrm{res}^G_{C_2}(\eta))$ is a generator of $H^4(C_2)$ and the theorem is proved.

5. Main theorem and examples

It is well-known that if a group G operates freely on a sphere, each p-Sylow subgroup of G is either cyclic or is a generalized quoternion group. Then from lemma 2, theorem 1 and theorem 2, we have

Theorem 3. *Let f: $G \times S^{2n-1} \to S^{2n-1}$ be a free PL-action of a finite group G on a S^{n-1}. Then α_f: $R(G) \to K(S^{2n-1}/G, f)$ induces an isomorphism; $R(G)/R^{\gamma}_{2n}(G) \cong K(S^{2n-1}/G, f)$.*

If a sphere has an even dimension, then because of well-known fixed point theorem, every element $g\in G$, $g \neq e$, must reverse the orientation of the sphere. Therefore non-trivial such group G is only C_2. For this case, the methods proving theorem 3 is applicable, and we have $K(S^{2n}/C_2) \cong R(C_2)/R^{\gamma}_{2(n+1)}(C_2) \cong R(C_2)/I(C_2)^{n+1} \cong K(RP^{2n})$.

Proposition 5. *For an Artin-Tate group G, we have an isomorphism; $H^{2i}(G) \cong R^{\gamma}_{2i}(G)/R^{\gamma}_{2(i+1)}(G)$.*

Proof. This is also a corollary of proposition 1, theorem 1 and 2.

As an application of the theorem 3, we compute some K-groups of orbit space of sphere. Let $\boldsymbol{Z}_{p,q}$ be the metacyclic group defined by

$$\boldsymbol{Z}_{p,q} = \langle a,\ b \mid a^p = b^q = e,\ bab^{-1} = a^s \rangle$$

where p, q are odd primes and $(s-1, p) = 1$ and s is a primitive q-th root of 1 mod p. $\boldsymbol{Z}_{p,q}$ cannot operate linealy on a sphere without fixed point. But T. Petrie proved in [4] that $\boldsymbol{Z}_{p,q}$ can operate on S^{2q-1} freely and differentiably. Let $f\colon \boldsymbol{Z}_{p,q} \times S^{2q-1} \to S^{2q-1}$ be its operation. Then considering S^{2nq-1} as n-times join of S^{2q-1}, an operation $f_n\colon \boldsymbol{Z}_{p,q} \times S^{2nq-1} \to S^{2nq-1}$ is defined by

$$f_n(g,\ t_1x_1 \oplus t_2x_2 \oplus \cdots\cdots \oplus t_nx_n) = t_1f(g,\ x_1) \oplus t_2f(g,\ x_2) \oplus \cdots\cdots \oplus t_nf(g,\ x_n)$$

where $\sum t_i = 1$, $g \in \boldsymbol{Z}_{p,q}$, $x_i \in S^{2q-1}$.

This operatin f_n is free and piecewise linear. We want to determine the group extension of $\tilde{K}(S^{2nq-1}/\boldsymbol{Z}_{p,q}, f_n)$. Clearly it is the direct sum of p and q-primary component. The q-primary component is isomorphic to $\tilde{K}(S^{2nq-1}/C_q)$ from the fact that $\mathrm{res}_{C_q}^{Z_{p,q}}\colon H^*(\boldsymbol{Z}_{p,q}) \to H^*(C_q)$ is an isomorphism for q-primary component and its group extension was determined in [3]. Lets consider the p-primary component. Put

$$\tau = \mathrm{res}_{C_p}^{Z_{p,q}}\, \mathrm{ind}_{C_p}^{Z_{p,q}}(\rho_p) - q.$$

Proposition 6. *p-primary component of $\tilde{K}(S^{2nq-1}/\boldsymbol{Z}_{p,q})$ is generated by $\gamma^q(\tau)^i$, $i = 1, 2, \ldots, t = (p-1)/q$ and isomorphic to*

$$(\boldsymbol{Z}/p^{s+1})^r + (\boldsymbol{Z}/p^s)^{t-r}$$

where $n - 1 = st + r$, $0 \leqq r < t$.

Proof. $H^*(\boldsymbol{Z}_{p,q}, p)$ is generated by $c_q(\tau + q)$, and so p-primary component of $R^\gamma_{2q}(\boldsymbol{Z}_{p,q})/R^\gamma_{(q+1)}(Z_{p,q}) = Z/p$ is generated by $\gamma^q(\tau)$ and hence $\gamma^q(\tau)^i$, $i = 1, 2, \ldots$ generate p-primary component of $\tilde{K}(S^{2nq-1}/\boldsymbol{Z}_{p,q})$ and $\gamma^q(\tau)^n = 0$. Now in the Atiyah-Hirzebruch spactral sequence, $\varphi(c_{p-1}(\sum_{1=0}^{p-1} \rho_p^i - p)) = (-1)^{p-2}(p-2)!$ $[\sum_{i=0}^{p-1} \rho_p^i - p] \in R_{2(p-1)}(C_p)/R_{2p}(C_p)$ and $c_{p-1}(\sum_{i=0}^{p-1} \rho_p^i - p) = -c_1(\rho_p)^{p-1}$. It follows $\gamma^q(\tau)^t = -(p-2)!(\sum_{i=0}^{p-1} \rho_p^i - p) \bmod \gamma^q(\tau)^{t+1}$. Multiplying $\gamma^q(\tau)$ on this equation, we have $\gamma^q(\tau)^{t+1} = (p-2)!\ p\ \gamma^q(\tau) \bmod \gamma^q(\tau)^{t+2}$. It follows from the induction augument that $\gamma^q(\tau)^i$, $i = 1, 2, \ldots, t$ generate additively p-primary component of $\tilde{K}(S^{2nq-1}/Z_{p,q})$ which has the order p^{n-1}. It is clear that the order of $\gamma^q(\tau)^i \leqq$ the order of $\sigma^{qi} = (\rho_p - 1)^{qi}$. The order of σ^{qi} is from [3] p^{s+1} for $i \leqq r$ and p^s for $i > r$. Now it is easy to see that the order of $\gamma^q(\tau)^i$ must be just one of σ^{qi} and the extension is determined.

6. A counterexample of odd primary component for Atiyah's conjecture

The Atiysh's filtration conjecture in [1] is equivarent to the fact that in the Atiyah's spectral sequence $H^*(G) \Rightarrow \widehat{R(G)}$, universal cycles are generated by Chern classes of representations of G. Theorem 1 and 2 imply specially that his conjecture consists for the Artin-Tate groups. In [6], E. weiss gave a counterexample examining 2-Sylow subgroup of the alternating group A_4. C. B. Thomas extended his augument for 2-Sylow subgroup of projective special linear groups. In this section, we show that 2 is not special prime for the Atiyah's conjecture.

As seen in [1], for an odd prime p, $H^*(C_p \times C_p) = p(x, y) \otimes E(z)$, $\deg. x = deg. y = 2, \deg. z = 3$. We see easily x, y being first Chern classes of representations of $C_p \times C_p$ and therefore $H^{\text{even}}(C_p \times C_p)$ is generated by Chern classes of representations.

Proposition 7. *There exists a group G which contains $C_3 \times C_3$ as 3-Sylow subgroup and $H^*(G)$ has an element which is not expressible by Chern classes but an universal cycle in the Atiyah's spectral sequence.*

Proof. We define a group G by the following presentation,

$$G = \langle a, b, c \mid a^3 = b^3 = c^4 = e,\ ab = ba,\ cac^{-1} = ab,\ cac^{-1} = ab^{-1} \rangle$$

3-Sylow subgroup of G is normal and isomrophic to $C_3 \times C_3$. Let σ_1, σ_2 be irreducible representations of $C_3 \times C_3$ defined by $\sigma_1(a) = \omega$, $\sigma_1(b) = 1$, $\sigma_2(a) = 1$, $\sigma_2(b) = \omega$ where $\omega = \exp(2\pi i/3)$. Then an irreducible representation of $C_3 \times C_3$ is one of $\sigma_1^i \otimes \sigma_2^j$, $i, j = 1, 2, 3$. G has four 1-dimensional representation defined by the composite of the natural projection $G \to G/C_3 \times C_3 = C_4$ and one of C_4. G has two 4-dimensional irreducible representations $\mathrm{ind}^G_{C_3 \times C_3}(\sigma_1) = \kappa_1$, $\mathrm{ind}^G_{C_3 \times C_3}(\sigma_2) = \kappa_2$. On the other hand, we have

$$H^*(G) = H^*(C_3 \times C_3)^{C_4} \oplus H^*(C_4)$$

where the first component implies the subring of $H^*(C_3 \times C_3)$ generated by invariant elements for the action of C_4. Put $x = c_1(\sigma_1)$ and $y = c_1(\sigma_2)$. Since $f_c^!(\sigma_1)(a) = \sigma_1(cac^{-1}) = \sigma_1(ab) = \omega$, $f_c^!(\sigma_1)(b) = \sigma_1(ab^{-1}) = \omega$, we have $f_c^!(\sigma_1) = \sigma_1 \otimes \sigma_2$. Similarly we have $f_c^!(\sigma_2) = \sigma_1 \otimes \sigma_2^{-1}$. It follows $f_c^*(x) = f_c^*(c_1(\sigma_1)) = c_1(f_c^!(\sigma_1)) = c_1(\sigma_1 \otimes \sigma_2) = c_1(\sigma_1) + c_1(\sigma_2) + x + y$ and similarly $f_c^*(y) = x - y$. Therefore, $H^*(C_3 \times C_3)^{C_4}$ consists of polynomials $p(x, y)$ satisfing the relation $p(x, y) = p(x + y, x - y)$. As easily seen,

$$\kappa_1(a) = \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^{-1} & 0 \\ 0 & 0 & 0 & \omega^{-1} \end{bmatrix} \quad \kappa_2(b) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^{-1} \end{bmatrix}$$

Then we have $\mathrm{res}^G_{C_3\times C_3}(\kappa_1)=\sigma_1+\sigma_1\otimes\sigma_2+\sigma_1^{-1}+\sigma_1^{-1}\otimes\sigma_2^{-1}$. Similarly, $\mathrm{res}^G_{C_3\times C_3}(\kappa_2)=\sigma_2+\sigma_1\otimes\sigma_2^{-1}+\sigma_1^{-1}+\sigma_2^{-1}+\sigma_1^{-1}\otimes\sigma_2$. Therefore 3-primary component of the total Chern class of κ_1, κ_2 are as follows;

$$\begin{aligned} c(\kappa_1) &= (1+x)(1+x+y)(1-x)(1-x-y) \\ &= 1+(x^2+xy-y^2)+(x^4-x^3y+x^2y^2) \\ c(\kappa_2) &= (1+y)(1+x-y)(1-y)(1-x+y) \\ &= 1-(x^2+xy-y^2)+(y^4+xy^3+x^2y^2) \end{aligned}$$

Consider a polynomial $x^3y-xy^3=xy(x+y)(x-y)$ in $H^8(C_3\times C_3)^{C_4}$. We can see immediately that x^3y-xy^3 is not gotten by the linear combination of $(x^2+xy-y^2)^2=x^4-x^3y-x^2y^2+xy^3+y^4$, $x^4-x^3y+x^2y^2$ and $y^4+xy^3+x^2y^2$. x^3y-xy^3 is an universal cycle in the Atiyah's spectral sequence $H^*(C_3\times C_3)\Rightarrow \widehat{R(C_3\times C_3)}$ because x, y are Chern classes. By the naturality of the spectral sequence, x^3y-xy^3 in $H^*(G)$ is also an universal cycle.

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