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Lévy flows on manifolds and Lévy processes on Lie groups

By

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1. Introduction

The main concern of this paper is to construct stochastic flows of diffeomorphisms of manifolds by solving stochastic differential equations (SDE's) driven by Lévy processes. In two earlier papers of Fujiwara-Kunita ([2]) and Fujiwara ([1]), existence and uniqueness of the solutions of such equations were established in the first place on $\mathbf{R}^d$ and in the second place when the manifold (M) was compact. Herein we will not restrict ourselves to compact manifolds and will aim to find some natural classes of SDE's whose solutions take values in the diffeomorphism group of the manifold. In fact we will aim to generalize the well known result for flows driven by Brownian motion wherein the solution consists of diffeomorphisms (almost surely) provided each of the vector fields driving the equation is deterministically complete and the Lie algebra which they generate is finite dimensional (see [6] Theorem 4.8.7).

We note that in [1] and [2] it was shown that the solutions of the stochastic differential equations described therein define Lévy processes (i.e. cádlág processes with independent increments) on G_+ and G_+^m respectively where G_+ (G_+^m) is the topological semigroup comprising continuous maps (C^m maps) from $\mathbf{R}^d$ or M into itself. Furthermore, it is shown in [5] that under some additional conditions, the solution in [2] defines a Lévy process on G^m where G^m is the topological group of C^m-diffeomorphisms of $\mathbf{R}^d$. However the latter argument cannot be applied in this case since it depends critically on the global properties of Euclidean space. Hence we develop a completely different method. A major difference between this paper and its predecessors is that we restrict our Lévy process driving the SDE to possess finitely many degrees of freedom so that in particular the Poisson random measure component of the process is itself defined on the finite dimensional manifold N. We construct two distinct classes of Lévy flows in this paper which are obtained as follows.

(i) $N = \mathbf{R}^d$ and the vector fields driving the SDE satisfy the condition on the Lie algebra described in the first paragraph above.

(ii) N is a finite dimensional Lie group and the vector fields driving the SDE belong to the Lie algebra of N.

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We note that class (i) seems to be the simplest class of Lévy flows which contains flows driven by a finite number of independent Brownian motions or Poisson processes. Class (ii) arises naturally through the representation of a Lévy process on a Lie group when the group acts as diffeomorphisms on the manifold. We obtain an explicit decomposition for such processes by utilizing Hunt's formula for the infinitesimal generator of Markov semigroups associated to the weakly continuous convolution semigroups of probability measures on N which describe the law of the process. This generalizes a result obtained by Holevo [3], in the case where N is a matrix Lie group.

The organization of this paper is as follows. We split the paper into two sections to describe each class of flows. In the first of these (Sect. 2), after describing some preliminaries in Sect. 2.1, we proceed to construct the class (i) flows in Sections 2.2 and 2.3. In fact Sect. 2.2 is devoted to solving the "canonical extension" [7] of our equation in local coordinates. We recall that this extension generalizes the role of the Stratonovich integral for Brownian motion in providing a form for the equation which is invariant under changes of local coordinates. In Sect. 2.3 we use the result of Sect. 2.2 to construct our required flow on a manifold. In Sect. 3 we construct the class (ii) flows by means of our representation for Lévy flows on Lie groups as described above.

Notation. We use Einstein summation convention throughout this paper. Diff(M) denotes the group of all C^∞-diffeomorphisms of smooth manifold M. A stochastic process with values in M is cádlág if it is right continuous and the left limits always exists. If S is a topological space, $\mathscr{B}(S)$ denotes the σ-algebra generated by the Borel sets in S.

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2. Stochastic differential equations driven by Lévy processes and Lévy flows on manifolds

2.1. Preliminaries. Let $X = ((X^1(t), \dots, X^n(t)), t \geq 0)$ be an $\mathbf{R}^n$-valued Lévy process defined on some probability space $(\Omega, \mathscr{F}, P)$ and having the following *Lévy-Itô decomposition,*

$$X^k(t) = b^k t + \sigma_l^k B^l(t) + \int_0^{t+} \int_{|x| \geq 1} x^k N(dudx) + \int_0^{t+} \int_{|x| < 1} x^k \tilde{N}(dudx) \tag{2.1}$$

for each $t \in \mathbf{R}^+$, $1 \leq k \leq n$. Here $b = (b^1, \dots, b^n) \in \mathbf{R}^n$, $\sigma = (\sigma_l^k)$ is a real $n \times m$ matrix with $m \leq n$, $B = (B^1, \dots, B^m)$ is an m-dimensional standard Brownian motion. N is a Poisson random measure (independent of B) on $\mathbf{R}^+ \times (\mathbf{R}^n - \{0\})$ with intensity measure $dtd\nu(x)$, where ν is a Lévy measure on $\mathbf{R}^n - \{0\}$ satisfying

$\int_{\mathbf{R}^n-\{0\}} |x|^2/(1+|x|^2)d\nu(x) < \infty$ and $x = (x^1,\ldots,x^n) \in \mathbf{R}^n - \{0\}$. Further, $\tilde{N}$ is the compensator defined by $\tilde{N}(dtdx) = N(dtdx) - dtd\nu(x)$.

Let M be a finite dimensional connected paracompact smooth manifold. Let $\Phi = \{\Phi_{s,t}; 0 \le s \le t < \infty\}$ be a family of measurable maps from $M \times \Omega \to M$ and define for each $\omega \in \Omega$, $\Phi_{s,t}^{\omega}: M \to M$ by $\Phi_{s,t}^{\omega} = \Phi_{s,t}(\cdot, \omega)$. We say that Φ is a *(forward) Lévy flow of diffeomorphisms* of M if the following conditions are satisfied

(i) $\Phi_{s,t}^{\omega} \in \mathrm{Diff}(M)$ for all $t \ge s$ and a.a. $\omega \in \Omega$.

(ii) $\Phi_{t,u}^{\omega} \circ \Phi_{s,t}^{\omega} = \Phi_{s,u}^{\omega}$ for all $s \le t \le u$ and $\Phi_{s,s}^{\omega}(p) = p$ for all $0 \le s < \infty$, $p \in M$, for a.a. $\omega \in \Omega$.

(iii) The map $t \to \Phi_{s,t}^{\omega}$ from $[s, \infty)$ into $\mathrm{Diff}(M)$ is cádlág for any $s \ge 0$ and a.a. $\omega \in \Omega$.

(iv) For each positive integer N, $0 \le t_1 < t_2 < \cdots < t_{N+1} < \infty$, the random variables $\Phi_{t_i,t_{i+1}}$ for $1 \le i \le N$ are independent.

Our aim in this section is to construct such Lévy flows described by stochastic differential equations driven by the Lévy process (2.1).

To this end let $(Y_1,\ldots,Y_n)$ be complete smooth vector fields on M and let $\mathscr{L}$ be the Lie algebra which they generate. We will assume throughout that $\mathscr{L}$ has finite dimension, so that in particular every member of $\mathscr{L}$ is itself a complete smooth vector field. Let $X_{\mathscr{L}}(t)$ be the "vector-field valued Lévy process" defined by

$$X_{\mathscr{L}}(t) = X^k(t)Y_k\,, \tag{2.2}$$

so that $X_{\mathscr{L}}(t)$ has the Lévy-Itô decomposition

$$X_{\mathscr{L}}(t) = b^kY_kt + \sigma_l^kY_kB^l(t) + \int_0^{t+}\int_{|x|\ge 1} x^kY_kN(dudx) + \int_0^{t+}\int_{|x|<1} x^kY_k\tilde{N}(dudx)\,. \tag{2.3}$$

To simplify the notation, we will in the sequel write $Z_0 = b^kY_k$ and $Z_l = \sigma_l^kY_k$ for $1 \le l \le m$.

To construct Lévy flows we might consider trying to solve SDE's of the form

$$d\Phi_{s,t} = dX_{\mathscr{L}}(t)(\Phi_{s,t-})\,, \tag{2.4}$$

(where we have slightly adapted the notation of [2]). When we try to use (2.4) directly to construct $\Phi_{s,t}$ through its paths we find that it has the disadvantage of failing to be invariant under changes of co-ordinates. To overcome this difficulty, we replace (2.4) by its canonical extension in the sense of [7] (see also the discussion in Section 5 of [1]) which is the equation

$$\Phi_{s,t}(p) = p + \int_s^t Z_0(\Phi_{s,u-}(p))du + \int_s^t Z_l(\Phi_{s,u-}(p)) \circ dB^l(u)$$
$$+ \int_s^{t+}\int_{|x|\ge 1} x^kY_k(\Phi_{s,u-}(p))N(dudx)$$

$$+\int_s^{t+}\int_{|x|<1} x^k Y_k(\Phi_{s,u-}(p))\tilde{N}(dudx)$$

$$+\sum_{s<u\le t}\{\xi(\pi(u))(\Phi_{s,u-}(p))-\Phi_{s,u-}(p)-\pi^k(u)Y_k(\Phi_{s,u-}(p))\}\,, \tag{2.5}$$

where the integrals by Brownian motions are Stratonovich integrals, $\xi(x)$ denotes the diffeomorphism $\mathrm{Exp}(x^i Y_i)$ and $\pi(t)$ is the point process $\Delta X(t)$. A little algebraic manipulation shows that (2.5) can be written as

$$\Phi_{s,t}(p)=p+\int_s^t Z_0(\Phi_{s,u-}(p))\,du+\int_s^t Z_l(\Phi_{s,u-}(p))\circ dB^l(u)$$

$$+\int_s^{t+}\int_{|x|\ge 1}(\xi(x)(\Phi_{s,u-}(p))-\Phi_{s,u-}(p))N(dudx)$$

$$+\int_s^{t+}\int_{|x|<1}(\xi(x)(\Phi_{s,u-}(p))-\Phi_{s,u-}(p))\tilde{N}(dudx)$$

$$+\int_s^t\int_{|x|<1}(\xi(x)(\Phi_{s,u-}(p))-\Phi_{s,u-}(p)-x^kY_k(\Phi_{s,u-}(p)))\nu(dx)du\,. \tag{2.6}$$

A more precise interpretation of (2.6) is that for each $f\in C^\infty(M)$, $p\in M$ we have

$$f(\Phi_{s,t}(p))=f(p)+\int_s^t Z_0f(\Phi_{s,u-}(p))\,du+\int_s^t Z_lf(\Phi_{s,u-}(p))\circ dB^l(u)$$

$$+\int_s^{t+}\int_{|x|\ge 1}(f(\xi(x)\circ\Phi_{s,u-}(p))-f(\Phi_{s,u-}(p)))N(dudx)$$

$$+\int_s^{t+}\int_{|x|<1}(f(\xi(x)\circ\Phi_{s,u-}(p))-f(\Phi_{s,u-}(p)))\tilde{N}(dudx)$$

$$+\int_s^{t+}\int_{|x|<1}(f(\xi(x)\circ\Phi_{s,u-}(p))-f(\Phi_{s,u-}(p))-x^kY_kf(\Phi_{s,u-}(p)))\nu(dx)du\,. \tag{2.7}$$

We will call the above equation (2.6) or (2.7) a *stochastic differential equation driven by* $X_{\mathscr{L}}(t)$. In the next subsection we will solve it in the case where M is a Euclidean space and in Section 2.3 in the case of a more general manifold.

2.2. The canonical extension in $\mathbf{R}^d$. In this subsection we take $M=\mathbf{R}^d$. We write, for $p\in\mathbf{R}^d$, $Y_k(p)=a_k^i(p)\partial_i$, $1\le k\le n$ and denote $\gamma_0^i(p)=b^ka_k^i(p)$, $\gamma_l^i(p)=\sigma_l^ka_k^i(p)$ for $1\le l\le m$, $1\le i\le d$. Let $(\xi_t(x), t\in\mathbf{R})$ denote the one parameter subgroup of $\mathrm{Diff}(\mathbf{R}^d)$ given by $\mathrm{Exp}(tx^kY_k)$ and define a smooth function $\xi(x)$: $\mathbf{R}\times\mathbf{R}^d\to\mathbf{R}^d$ by

$$\xi(x)(t,p)=\xi_t(x)(p)\,, \tag{2.8}$$

where $p = (p^1, \ldots, p^d) \in \mathbf{R}^d$. We note that $\xi(x)$ is the unique solution of the system of differential equations given by

$$\begin{cases} \dfrac{d}{dt}\xi(x)^i(t, p) = x^k a_k^i(\xi(x)(t, p)), \\ \xi(x)^i(0, p) = p^i, \end{cases} \tag{2.9}$$

for $1 \leq i \leq d$. Hence for $0 \leq t \leq 1$, we have

$$\xi(x)^i(t, p) = p^i + \int_0^t x^k a_k^i(\xi(x)(s, p))ds. \tag{2.10}$$

We write $\xi(x)(p) = \xi(x)(1, p)$. We define an L^2-martingale $Y_t(q) = (Y_t^1(q), \ldots, Y_t^d(q))$; $t \in \mathbf{R}^+$ for $q \in \mathbf{R}^d$ by

$$Y_t^j(q) = \gamma_l^j(q)B^l(t) + \int_0^{t+} \int_{|x|<1} (\xi(x)^j(q) - q^j)\tilde{N}(dudx) \tag{2.11}$$

and a process of bounded variation $C_t(q) = (C_t^1(q), \ldots, C_t^d(q))$; $t \in \mathbf{R}^+$ by

$$\begin{aligned} C_t^j(q) = &\int_0^{t+} \int_{|x|\geq 1} (\xi(x)^j(q) - q^j)N(dudx) + \Bigg\{\gamma_0^j(q) + \frac{1}{2}\sum_{l=1}^m (\partial_i \gamma_l^j)(q)\gamma_l^i(q) \\ &+ \int_{|x|<1} [\xi(x)^j(q) - q^j - x^k a_k^j(q)]\nu(dx)\Bigg\}t, \end{aligned} \tag{2.12}$$

for $1 \leq j \leq d$. Then (2.6) takes the form

$$\Phi_{s,t}(p) = p + \int_s^{t+} dY_u(\Phi_{s,u-}) + \int_s^{t+} dC_u(\Phi_{s,u-}). \tag{2.13}$$

We will show that there exists a unique solution to (2.13) by demonstrating that the conditions of Theorem 2.1 of [2] are satisfied.

We note first of all that, using the notation of [2] p. 82, we have for $1 \leq i, j \leq d$, $p, q \in \mathbf{R}^d$, $\langle Y_t^i(p), Y_t^j(q)\rangle = tA^{ij}(p, q)$ where

$$A^{ij}(p, q) = \sum_l \gamma_l^i(p)\gamma_l^j(q) + \int_{|x|<1} (\xi(x)^i(p) - p^i)(\xi(x)^j(q) - q^j)\nu(dx). \tag{2.14}$$

We write $\tau(p, q) = (\tau^{ij}(p, q))$ to denote the $(d \times d)$ matrix given by $\gamma(p)\gamma(q)^T$. We also write $\zeta(x)(p) = \xi(x)(p) - p$ and define $d(p)$ via (2.12) by

$$td(p) = C_t(p) - \int_0^{t+} \int_{|x|\geq 1} \zeta(x)(p)N(dudx). \tag{2.15}$$

It follows then from [2], that (2.13) has a unique solution provided the following conditions hold

(i) $\tau(p, q)$ is bi-Lipschitz continuous.

(ii) $d(p)$ is Lipschitz continuous.

(iii) $\int_{|x|<1} |\zeta(x)(p) - \zeta(x)(q)|^2\nu(dx) \leq L|p - q|^2$ for some $L > 0$ and for all $p, q \in \mathbf{R}^d$.

To establish (i) to (iii) we make the following assumptions.

Condition (A). a_k^i, $\partial_j(a_k^i)$ and $\partial_l\partial_j(a_k^i)$ are bounded functions on $\mathbf{R}^d$ for $1 \le i, j, l \le d$, $1 \le k \le n$.

We can now verify (i) immediately by a standard use of the mean value theorem. To establish (ii) and (iii) we need Lemmata 2.1 and 2.2 below. First we introduce one further piece of notation and write for $0 \le t \le 1$, $p \in \mathbf{R}^d$, $x \in \mathbf{R}^n - \{0\}$

$$\zeta(x)(t, p) = \xi(x)(t, p) - p\,. \tag{2.16}$$

Lemma 2.1. *There exist constants C_1, $C_2 > 0$ such that*

$$\max_{1 \le i \le d} \left| \frac{\partial \zeta(x)^i}{\partial p^j}(p) \right| \le C_1 |x| e^{C_2 |x|} \tag{2.17}$$

for all $1 \le j \le d$, $p \in \mathbf{R}^d$, $x \in \mathbf{R}^n - \{0\}$.

Proof. By (2.10) we obtain

$$\begin{aligned} \frac{\partial}{\partial p^j} \zeta(x)^i(t, p) &= \int_0^t x^k \frac{\partial a_k^i}{\partial \xi^m}(\xi(x)(s, p)) \frac{\partial \xi(x)^m}{\partial p^j}(s, p) ds \\ &= \int_0^t x^k \frac{\partial a_k^i}{\partial \xi^j}(\xi(x)(s, p)) ds + \int_0^t x^k \frac{\partial a_k^i}{\partial \xi^m}(\xi(x)(s, p)) \frac{\partial \zeta(x)^m}{\partial p^j}(s, p) ds\,. \end{aligned}$$

Write $\gamma_j(t, x, p) = \max_{1 \le i \le d} \left| \frac{\partial \zeta^i(x)}{\partial p^j}(t, p) \right|$. Then by repeated use of the Schwarz inequality and condition (A), we may assert the existence of C_1 and C_2 such that

$$\gamma_j(t, x, p) \le |x| C_1 t + |x| C_2 \int_0^t \gamma_j(s, x, p) ds\,.$$

The required result then follows from Gronwall's inequality upon putting $t = 1$.

For each $p \in \mathbf{R}^d$, $x \in \mathbf{R}^n - \{0\}$, $t \in [0, 1]$ define

$$\phi(x)^i(t, p) = \xi(x)^i(t, p) - p^i - t x^k a_k^i(p) \tag{2.18}$$

and write $\phi(x)^i(p) = \phi(x)^i(1, p)$ for $1 \le i \le d$.

Lemma 2.2. *There exist constants D_1, $D_2 > 0$ such that*

$$\max_{1 \le i \le d} \left| \frac{\partial \phi(x)^i}{\partial p^j}(p) \right| \le D_1 |x|^2 e^{D_2 |x|} \tag{2.19}$$

for all $1 \le j \le d$, $p \in \mathbf{R}^d$, $x \in \mathbf{R}^n - \{0\}$.

Proof. Arguing as in Lemma 2.1, we obtain

$$\begin{aligned} \frac{\partial \phi(x)^i}{\partial p^j}(t, p) &= \int_0^t x^k \frac{\partial a_k^i}{\partial \xi^m}(\xi(x)(s, p)) \frac{\partial \xi(x)^m}{\partial p^j}(s, p) ds - t x^k \frac{\partial a_k^i}{\partial p^j}(p) \\ &= I_1(t) + I_2(t)\,, \end{aligned}$$

where

$$I_1(t) = \int_0^t x^k \frac{\partial a_k^i}{\partial \xi^m}(\xi(x)(s, p)) \frac{\partial \phi(x)^m}{\partial p^j}(s, p) ds ,$$

$$I_2(t) = \int_0^t x^k \frac{\partial a_k^i}{\partial \xi^m}(\xi(x)(s, p)) \left(\delta_j^m + s x^l \frac{\partial a_l^m}{\partial p^j}(p) \right) ds - t x^k \frac{\partial a_k^i}{\partial p^j}(p)$$

$$= \int_0^t x^k \left(\frac{\partial a_k^i}{\partial \xi^j}(\xi(x)(s, p)) - \frac{\partial a_k^i}{\partial p^j}(p) \right) ds + \int_0^t s x^l x^k \frac{\partial a_k^i}{\partial \xi^m}(\xi(x)(s, p)) \frac{\partial a_l^m}{\partial p^j}(p) ds ,$$

for $1 \le i,\ j \le d$.

Let $F_j^i(s, x, p) = x^k \left(\frac{\partial a_k^i}{\partial \xi^j}(\xi(x)(s, p)) - \frac{\partial a_k^i}{\partial p^j}(p) \right)$. Then by the Schwarz inequality, the mean value theorem and Condition (A), we find

$$|F_j^i(s, x, p)|^2 \le |x|^2 \sum_{k=1}^n \left(\frac{\partial a_k^i}{\partial \xi^j}(\xi(x)(s, p)) - \frac{\partial a_k^i}{\partial p^j}(p) \right)^2$$

$$\le |x|^2 \alpha_j^i \left(\sum_{l=1}^d |\xi(x)^l(s, p) - p^l| \right)^2 ,$$

where

$$\alpha_j^i = n \max_{\substack{1 \le l \le d \\ 1 \le k \le n}} \sup_{p \in \mathbf{R}^d} \left| \frac{\partial^2 a_k^i}{\partial p^l \partial p^j}(p) \right|^2 .$$

By (2.10) and the Schwarz inequality again, we obtain

$$|F_j^i(s, x, p)|^2 \le \alpha_j^i |x|^4 \left(\sum_{l=1}^d \left(\sum_{k=1}^n \left(\int_0^s a_k^l(\xi(x)(\tau, p)) d\tau \right)^2 \right)^{1/2} \right)^2 .$$

Hence by Condition (A), there exists a constant E_1 such that

$$|F_j^i(s, x, p)| \le E_1 s |x|^2 .$$

Returning to I_2 and making further use of the Schwarz inequality and Condition (A) we see there exists a constant D_1 such that

$$|I_2(t)| \le D_1 t^2 |x|^2 .$$

Similarly we find

$$|I_1(t)| \le D_2 |x| \int_0^t \max_{1 \le i \le d} \left| \frac{\partial \phi(x)^i}{\partial p^j}(s, p) \right| ds$$

and the result now follows by Gronwall's inequality as in Lemma 2.1.

Theorem 2.3. *Under Condition* (A), *there exists a unique solution to* (2.13).

Proof. We verify (iii). By the mean value theorem and Lemma 2.1, we find that

$$\int_{|x|<1} |\zeta(x)(p) - \zeta(x)(q)|^2 \nu(dx) \le d|p-q|^2 \int_{|x|<1} \max_{1\le i,j\le d} \sup_{p\in \mathbf{R}^d} \left|\frac{\partial \zeta(x)^i}{\partial p^j}(p)\right|^2 \nu(dx)$$
$$\le \tilde{C}|p-q|^2 \int_{|x|<1} e^{2C_2|x|}|x|^2 \nu(dx)$$
$$\le L|p-q|^2 ,$$

since ν is a Lévy measure. (ii) is verified similarly by use of Lemma 2.2.

2.3. Construction of flows on manifolds. We now return to the general case of a manifold M and show that under the conditions of Section 2.1, we can construct a Lévy flow of diffeomorphisms satisfying (2.6) (or alternatively the equivalent form (2.7)). We note that in the case where $\nu = 0$, the problem was solved in [5], Theorem 4.8.7 (p. 194–5).

Instead of the stochastic differential equation (2.7) on the manifold M, we shall consider the stochastic differential equation on a certain Lie transformation group of M. It is known (Palais [8]) that associated to $\mathscr{L}$ there exists a Lie group G with properties (i)–(iii) below.

(i) G is a Lie transformation group of M, i.e., there exists a C^∞-map $\psi: G \times M \to M$ such that $\psi(e, \cdot) =$ identity and $\psi(\tau\sigma, \cdot) = \psi(\sigma, \psi(\tau))$, where e is the identity of G.

(ii) The map $\tau \to \psi(\tau)$ is an isomorphism from G into Diff(M).

(iii) Let g be the Lie algebra of G (left invariant vector fields on G). For any X of $\mathscr{L}$ there exists $\hat{X}$ of g such that

$$\hat{X}(f \circ \psi_p)(\tau) = Xf(\psi(\tau, p)) \tag{2.20}$$

holds for all f of $C^\infty(M)$. Here $f \circ \psi_p$ is a C^∞-function on G such that $f \circ \psi_p(\tau) = f \circ \psi(\tau, p)$.

Now let $\hat{Y}_1, \ldots, \hat{Y}_n$ be elements of g determined by $Y_1, \ldots, Y_n$ through (2.20), respectively. Define the g-valued process

$$X_g(t) = \hat{Z}_0 t + \hat{Z}_l B^l(t) + \int_0^{t+} \int_{|x|\ge 1} x^k \hat{Y}_k N(dudx) + \int_0^{t+} \int_{|x|<1} x^k \hat{Y}_k \tilde{N}(dudx) , \tag{2.21}$$

where $\hat{Z}_0 = b^k \hat{Y}_k$ and $\hat{Z}_l = \sigma_l^k \hat{Y}_k$. We consider a stochastic differential equation on G driven by $X_g(t)$:

$$\begin{aligned} f(\gamma(t)) = f(e) &+ \int_s^t \hat{Z}_0 f(\gamma(u-))du + \int_s^t \hat{Z}_l f(\gamma(u-)) \circ dB^l(u) \\ &+ \int_s^{t+} \int_{|x|\ge 1} (f(\gamma(u-)\hat{\xi}(x)) - f(\gamma(u-)))N(dudx) \\ &+ \int_s^{t+} \int_{|x|<1} (f(\gamma(u-)\hat{\xi}(x)) - f(\gamma(u-)))\tilde{N}(dudx) \\ &+ \int_s^{t+} \int_{|x|<1} (f(\gamma(u-)\hat{\xi}(x)) - f(\gamma(u-)) - x^k \hat{Y}_k f(\gamma(u-)))\nu(dx)du , \end{aligned} \tag{2.22}$$

where $\hat{\xi}(x) = \exp(x^k \hat{Y}_k)$ and $\exp(tx^k X_k)$ is the one parameter subgroup of G generated by $x^k X_k \in g$.

If $\gamma_s(t)$ is a solution of the above equation, then $\psi(\gamma_s(t), p) \equiv \Phi_{s,t}(p)$ satisfies (2.7).

Theorem 2.4. *Given the initial data $s = 0$, the stochastic differential equation* (2.22) *has a unique global solution $\gamma(t)$, $0 \le t < \infty$. Further the solution $\gamma(t)$ is measurable with respect to the σ-field $\sigma(X_g(s); 0 \le s \le t)$ for any $t > 0$.*

Proof. Before constructing the solution, we make a preliminary observation on $X_g(t)$. We will rewrite it using a basis of g. Let $\{L_1, \ldots, L_n, L_{n+1}, \ldots, L_d\}$ be such a basis. We define $\hat{Y}_k = 0$ for $n < k \le d$ so that there exists a $(d \times d)$ orthogonal matrix C such that

$$\hat{Y}_k = C_k^j L_j \qquad \text{for } 1 \le k \le d\,.$$

Let $\tilde{x} = (x^1, \ldots, x^n, 0, \ldots, 0)$ (d-vector) and define $\hat{x} = \tilde{x} C^T$ so that $\exp(\tilde{x}^k \hat{Y}_k) = \exp(\hat{x}^j L_j)$ and $\hat{x}$ is a canonical co-ordinate for G. Now define $\tilde{b} \in \mathbf{R}^d$ and $\tilde{\sigma} \in M_d(\mathbf{R})$ by $\tilde{b} = (b_1, \ldots, b_n, 0, \ldots, 0)$ (d-vector) and

$$\tilde{\sigma}_l^k = \begin{matrix} \sigma_l^k & \text{if } 1 \le k \le n,\ 1 \le l \le m \\ 0 & \text{if } n < k \le d,\ m < l \le d \end{matrix}$$

and let $\hat{\sigma} = \tilde{\sigma} C^T$ and $\hat{b} = \tilde{b} C^T$, then clearly (2.21) is invariant under the transformations $\tilde{x} \to \hat{x}$, $\tilde{b} \to \hat{b}$, $\tilde{\sigma} \to \hat{\sigma}$ and $\tilde{Y} \to L$.

In the sequel, to save on notation, we will assume (without loss of generality) that $\hat{Y}_1, \ldots, \hat{Y}_n$ extend to a basis of g so that we may take x (which we identify with $\tilde{x}$) as a canonical co-ordinate for G. Write $X_g(t) = X_1^\varepsilon(t) + X_2^\varepsilon(t)$ for $0 \le t < \infty$ where

$$X_1^\varepsilon(t) = \hat{Z}_0 t + \hat{Z}_l B^l(t) + \int_0^{t+} \int_{|x| < \varepsilon/2} x^k \hat{Y}_k \tilde{N}(dudx)\,,$$

$$X_2^\varepsilon(t) = \int_0^{t+} \int_{\varepsilon/2 \le |x|} x^k \hat{Y}_k N(dudx)\,.$$

We first consider the equation driven by $X_1^\varepsilon(t)$. We denote the solution of this equation starting at $s = 0$ (when it exists) by $\gamma^\varepsilon(t)$. To show that $\gamma^\varepsilon(t)$ exists, we work in a canonical co-ordinate neighborhood of $e \in G$ given by $U_\varepsilon = \{g \in G; |x(g)| < \varepsilon\}$. Denote by ϕ_ε the homeomorphism between U_ε and an open set in $\mathbf{R}^d$ which we denote by $\tilde{U}_\varepsilon$. We can extend the functions $a_j^i(x)$ from $\tilde{U}_\varepsilon$ onto the whole of $\mathbf{R}^d$ so that Condition (A) of Section 2.2 is satisfied. We hence obtain the solution of (2.13) on $\mathbf{R}^d$. Let $\Phi_{s,t}^\varepsilon$ be the solution starting from 0 at time s. Set

$$\sigma_1 = \inf\{t > 0; \Phi_{0,t}^\varepsilon \notin \tilde{U}_{\varepsilon/2}\},\ \ldots,\ \sigma_n = \inf\{t > \sigma_{n-1}; \Phi_{\sigma_{n-1},t}^\varepsilon \notin \tilde{U}_{\varepsilon/2}\}\,.$$

Then, the sequence of random variables $\sigma_1, \sigma_2 - \sigma_1, \ldots, \sigma_n - \sigma_{n-1}, \ldots$ are independent and identically distributed, whose expectations are positive. Then by the law of the large numbers, $\lim_{n\to\infty}\sigma_n = \infty$ a.s.. (See Theorem 4.8.7 of [5]). Note that $\Phi^\varepsilon_{\sigma_{n-1},t} \in \tilde{U}_\varepsilon$ for $\sigma_{n-1} \le t \le \sigma_n$. Then we can define $\gamma^\varepsilon(t)$ as follows. For $0 \le t \le \sigma_1$ define $\gamma^\varepsilon(t) = \phi_\varepsilon^{-1}(\Phi^\varepsilon_{0,t})$. For $\sigma_1 < t \le \sigma_2$, define $\gamma^\varepsilon(t) = \gamma^\varepsilon(\sigma_1)\phi_\varepsilon^{-1}(\Phi^\varepsilon_{\sigma_1,t})$ and so on inductively: for $\sigma_n < t \le \sigma_{n+1}$ $\gamma^\varepsilon(t) = \gamma^\varepsilon(\sigma_n)\phi_\varepsilon^{-1}(\Phi^\varepsilon_{\sigma_n,t})$. Then $\gamma^\varepsilon(t)$ is defined on G for all $t \ge 0$. For $s \le t$, we write $\gamma^\varepsilon(s,t) = \gamma^\varepsilon(s)^{-1}\gamma^\varepsilon(t)$.

Now the solution $\gamma(t)$ driven by $X_g(t)$ is constructed as follows. For $t \in [0,\infty)$ such that $\Delta X^2_\varepsilon(t) \ne 0$, we set $\pi_\varepsilon(t) = \Delta X^\varepsilon_2(t)$. We denote by $\{\beta_1, \beta_2, \dots\}$ the domain of $\pi_\varepsilon(t)$. For $0 \le t < \beta_1$, $\gamma(t) = \gamma^\varepsilon(t)$. For $t = \beta_1$, $\gamma(t) = \gamma(\beta_1 -)\hat{\xi}(\pi_\varepsilon(\beta_1))$. For $\beta_1 < t < \beta_2$, $\gamma(t) = \gamma(\beta_1)\gamma^\varepsilon(\beta_1, t)$. For $t = \beta_2$, $\gamma(t) = \gamma(\beta_2 -)\hat{\xi}(\pi_\varepsilon(\beta_2))$ and we continue this inductively. The explosion time of $\gamma(t)$ is infinite a.s. by a similar reasoning as the above.

Next we shall prove the uniqueness of the solution. Suppose that $\tilde{\gamma}(t)$ be any solution driven by $X_g(t)$ with the initial data $s = 0$ and $p = e$. For $\varepsilon > 0$ given above, we define $\tilde{\gamma}^\varepsilon(t)$ as follows. For $0 \le t < \beta_1$, $\tilde{\gamma}^\varepsilon(t) = \tilde{\gamma}(t)$. For $t = \beta_1$, $\tilde{\gamma}^\varepsilon(t) = \tilde{\gamma}^\varepsilon(\beta_1)\hat{\xi}(\pi_\varepsilon(\beta_1))^{-1}$. For $\beta_1 < t < \beta_2$, $\tilde{\gamma}^\varepsilon(t) = \tilde{\gamma}^\varepsilon(\beta_1)\tilde{\gamma}(\beta_1)^{-1}\tilde{\gamma}(t)$. For $t = \beta_2$, $\tilde{\gamma}^\varepsilon(t) = \tilde{\gamma}^\varepsilon(\beta_2)\hat{\xi}(\pi_\varepsilon(\beta_2))^{-1}$ and we continue thus inductively. Then $\tilde{\gamma}^\varepsilon(t)$ is a solution of the equation driven by $X^\varepsilon_1(t)$. Set $\tilde{\Phi}^\varepsilon_t = \phi_\varepsilon(\tilde{\gamma}^\varepsilon(t))$ for $t < \tilde{\sigma}_1 \equiv \inf\{t > 0;\ \tilde{\gamma}^\varepsilon(t) \notin U_{\varepsilon/2}\}$. Then $\tilde{\Phi}^\varepsilon_t$ satisfies an equation of the form (2.13). Since it has a unique solution, $\tilde{\Phi}^\varepsilon_t = \Phi^\varepsilon_{0,t}$ holds for $t < \tilde{\sigma}_1$, or equivalently, $\gamma^\varepsilon(t) = \tilde{\gamma}^\varepsilon(t)$ holds for $t < \tilde{\sigma}_1 = \sigma_1$. We can prove inductively that $\gamma^\varepsilon(t) = \tilde{\gamma}^\varepsilon(t)$ holds for $\sigma_n \le t < \sigma_{n+1}$ for all n. Then the equality holds for all t. Recall how $\gamma(t)$ and $\tilde{\gamma}(t)$ are constructed from $\gamma^\varepsilon(t)$ and $\tilde{\gamma}^\varepsilon(t)$, respectively. Then $\gamma(t) = \tilde{\gamma}(t)$ holds for all t.

Finally in view of the method of constructing the solution, it is obvious that $\gamma(t)$ is measurable with respect to the σ-field $\sigma(X_g(s); 0 \le s \le t)$.

Let ϕ_t, $t \in \mathbf{R}^+$ be a stochastic process continuous in probability with values in G. It is called a *Lévy process on the Lie group G* if it satisfies (a), (b) below.

(a) ϕ_t is cádlág and $\phi_0 = e$ a.s.

(b) For any $0 = t_0 < t_1 < \cdots < t_m$, $\{\phi^{-1}_{t_{k-1}}\phi_{t_k};\ k = 1, \dots, m\}$ are independent.

Corollary to Theorem 2.4. *The solution* $\gamma(t)$ *of equation* (2.22) *is a Lévy process on the Lie group G.*

Proof. It is sufficient to prove that $\gamma(t)$ has independent increments (property (b)). Let p be any element of G and let L_p be the left translation defined by $L_p f(\sigma) = f(p\sigma)$. Apply L_p to equation (2.22) by setting $s = 0$. Since $\hat{Z}_j$ are left invariant vector fields, we have $L_p\hat{Z}_j f = \hat{Z}_j L_p f$. Therefore dividing the integrals over the interval $[0, t]$ into the integrals over two intervals $[0, s]$ and $[s, t]$, we obtain

$$f(p\gamma(t)) = f(p\gamma(s)) + \int_s^t \hat{Z}_0 f(p\gamma(u-))du + \int_s^t \hat{Z}_l f(p\gamma(u-)) \circ dB^l(u)$$

$$+ \int_s^{t+} \int_{|x|\ge 1} (f(p\gamma(u-)\hat{\xi}(x)) - f(p\gamma(u-)))N(dudx)$$

$$+\int_s^{t+}\int_{|x|<1}(f(p\gamma(u-)\hat{\xi}(x)) - f(p\gamma(u-)))\tilde{N}(dudx)$$
$$+\int_s^{t+}\int_{|x|<1}(f(p\gamma(u-)\hat{\xi}(x)) - f(p\gamma(u-)) - x^k\hat{Y}_k f(p\gamma(u-)))\nu(dx)du\,. \tag{2.23}$$

The above formula is valid even if we set $p = \gamma(s)^{-1}$, since $\gamma(s)$ is independent of the future σ-field $\mathscr{F}_{s,t} = \sigma(X_g(u) - X_g(s); s \le u \le t)$. This means that $\gamma(s)^{-1}\gamma(t)$, $t \in [s, T]$ is a solution of equation (2.22) (starting from e at time s). Therefore it is measurable with respect to the above σ-field $\mathscr{F}_{s,t}$. Since $X_g(t)$ is a Lévy process on the Lie algebra g, $\mathscr{F}_{0,s}$ and $\mathscr{F}_{s,t}$ are independent for any $0 < s < t$. Therefore $\gamma(s)^{-1}\gamma(t)$ is independent of $\mathscr{F}_{0,s}$. Consequently $\gamma(t)$ has independent increments.

Again by use of the technique of Theorem 4.8.7 of [6], we see that our required flow on M is given by

$$\Phi_{s,t}(p) = \psi(\gamma(s)^{-1}\gamma(t), p)$$

for $0 \le s$, $t \le T$, $p \in M$. Indeed it is a Lévy flow of diffeomorphisms by the above Corollary and it satisfies (2.9). The uniqueness of the solution of equation (2.7) with fixed p can be verified by the similar argument as in the proof of Theorem 2.4. Hence we have the following:

Theorem 2.5. *Assuming that* $\dim(\mathscr{L}) < \infty$, *there exists a unique Lévy flow of diffeomorphisms of M which satisfies the SDE* (2.7).

3. Representations and constructions of Lévy processes on Lie groups

3.1 Preliminaries. Let G be a connected Lie group of dimension d with identity e and let g be the Lie algebra of left invariant vector fields on G. A basis $\{X_1, \dots, X_d\}$ of g will be fixed through this section. Let C be the set of all bounded continuous functions f on G such that $\lim_{p\to\infty} f(p)$ exists, where ∞ is the infinity (one point compactification of G). It is a Banach space by the supremum norm. Let C_2 be the set of all functions f of C such that $X_i f$, $X_i X_j f$, $i, j = 1, \dots, d$ exist and belong to C. In the following we will choose functions $x^1, \dots, x^d$ of C_2 which satisfy

$$x^i(e) = 0\,, \qquad X_j x^i(e) = \delta^i_j\,,$$

and a function h of C_2 which is strictly positive on $G \cup \{\infty\} - \{e\}$ and behaves near e like $\sum_i (x^i)^2$. The existence of such functions is shown in [4].

Let $\{\nu_t, 0 < t < \infty\}$ be a family of probability measures on G with the semi-group property $\nu_{s+t} = \nu_s * \nu_t$, s, $t > 0$. Here the convolution of two measures ν_s and ν_t is defined by

$$\nu_s * \nu_t(E) = \int_G \nu_s(d\sigma)\nu_t(\sigma^{-1}E)\,, \qquad \text{for } E \in \mathscr{B}(G)\,.$$

We assume that $\lim_{t \downarrow 0} \nu_t(E) = 1$ holds for any neighborhood E of e. Then the semigroup of measures $\{\nu_t; 0 < t < \infty\}$ gives a strongly continuous semigroup $\{S_t; 0 < t < \infty\}$ of linear operators on C by setting

$$S_t f(\tau) = \int_G f(\tau\sigma)\nu_t(d\sigma). \tag{3.1}$$

Hunt [4] has shown that the infinitesimal generator of $\{S_t; 0 < t < \infty\}$ is defined at least on C_2 and it is represented by

$$\begin{aligned} Af(\tau) &= a^i X_i f(\tau) + a^{ij} X_i X_j f(\tau) \\ &\quad + \int_{G-\{e\}} \{f(\tau\sigma) - f(\tau) - X_i f(\tau) x^i(\sigma)\}\mu(d\sigma). \end{aligned} \tag{3.2}$$

Here a^i are real constants, (a^{ij}) is a symmetric nonnegative definite matrix and μ is a positive measure on $G - \{e\}$ for which the integral $\int h(\tau)\mu(d\tau)$ is finite. The matrix (a^{ij}) and the measure μ are uniquely determined from the convolution semigroup, but the constants a^i may depend on the choice of the functions $x^1, \ldots, x^d$.

Now, on a certain probability space $(\Omega, \mathscr{F}, P)$ we can define a Lévy process ϕ_t on the Lie group G such that the law of $\phi_s^{-1}\phi_t$ coincides with ν_{t-s} for any $t > s \geq 0$. It is called a *Lévy process associated with the semigroup of measures* $\{\nu_t; 0 < t < \infty\}$. We will show in Sect. 3.2 that any Lévy process on a Lie group G can be represented as a solution of a stochastic differential equation driven by a Brownian motion on the Lie algebra g and a Poisson random measure on $G - \{e\}$. The equation is a certain generalization of the equation driven by a g-valued Lévy process which is discussed in the previous section. Then the result will be applied to show that a Lévy flow on a manifold can be represented as a solution of a stochastic differential equation driven by a Brownian motion on a Euclidean space and a Poisson random measure on a finite dimensional subgroup of Diff(M), provided that the Lévy flow takes values in a finite dimensional subgroup. In Sec. 3.4 we will prove that any Lévy process on a Lie group can be constructed by solving a stochastic differential equation driven by a g-valued brownian motion B and a Poisson random measure on $G - \{e\}$.

3.2. Representation of Lévy processes. Let $\{\phi_t; 0 \leq t < \infty\}$ be a Lévy process on a connected Lie group of dimension d with infinitesimal generator A represented by (3.2). Set

$$\mathscr{F}_t = \sigma(\phi_s; s \leq t). \tag{3.3}$$

Theorem 3.1. *There exists an $\mathscr{F}_t$-adapted time homogeneous Brownian motion $B(t) = (B^1(t), \ldots, B^d(t))$ with mean 0 and $\operatorname{cov}(B^i(t), B^j(t)) = 2ta^{ij}$, and an $\mathscr{F}_t$-adapted time homogeneous Poisson random measure $N((s, t] \times E)$ on $\mathbf{R}^+ \times (G - \{e\})$ with the intensity measure $dtd\mu(\sigma)$ independent of $B(t)$, for which ϕ_t satisfies the following stochastic differential equation for any $f \in C_2$:*

$$f(\phi_t) = f(e) + \int_0^t X_i f(\phi_{u-}) \circ dB^i(u) + a^i \int_0^t X_i f(\phi_{u-}) du$$
$$+ \int_0^{t+} \int_{G-\{e\}} (f(\phi_{u-}\sigma) - f(\phi_{u-}))\tilde{N}(dud\sigma)$$
$$+ \int_0^{t+} \int_{G-\{e\}} (f(\phi_{u-}\sigma) - f(\phi_{u-}) - X_i f(\phi_{u-}) x^i(\sigma))\mu(d\sigma)du\,, \qquad (3.4)$$

where $\tilde{N}((s, t] \times E) = N((s, t] \times E) - (t - s)\mu(E)$ *for* $E \in \mathscr{B}(G - \{e\})$.

Further the pair of the Brownian motion $B(t)$ *and the Poisson random measure* N *satisfying* (3.4) *is uniquely determind by* ϕ_t. *It satisfies*

$$\mathscr{F}_t = \sigma(B(s), N((s, t] \times E);\ 0 \le s \le t, E \in \mathscr{B}(G - \{e\}))\,. \qquad (3.5)$$

Remark. It is sometimes convenient to rewrite equation (3.4) in the following way

$$f(\phi_t) = f(e) + \int_0^t X_i f(\phi_{u-}) \circ dB^i(u) + \hat{a}^i \int_0^t X_i f(\phi_{u-}) du$$
$$+ \int_0^{t+} \int_U (f(\phi_{u-}\sigma) - f(\phi_{u-}))\tilde{N}(dud\sigma)$$
$$+ \int_0^{t+} \int_{U^c} (f(\phi_{u-}\sigma) - f(\phi_{u-}))N(dud\sigma)$$
$$+ \int_0^{t+} \int_U (f(\phi_{u-}\sigma) - f(\phi_{u-}) - X_i f(\phi_{u-}) x^i(\sigma))\mu(d\sigma)du\,, \qquad (3.6)$$

where U is a neighborhood of e with compact closure and $\hat{a}^i = a^i - \int_{U^c} x^i(\sigma)\mu(d\sigma)$. Note that the above is valid for any twice continuously differentiable function f which is not necessarily bounded together with its derivatives.

For the proof of the theorem, we shall introduce several martingales and discuss their properties. We set $\phi_{s,t}(p) = p\phi_s^{-1}\phi_t$ for $0 \le s < t < \infty$ and $p \in G$. Then if $s \le t < u$,

$$E[f(\phi_{s,u}(p))|\mathscr{F}_t] = S_{u-t} f(\phi_{s,t}(p))$$

holds. Therefore $\{\phi_{s,t}(p);\ s \le t < \infty\}$ is a Markov process with semigroup $\{S_t;\ 0 < t < \infty\}$ starting from p at time s. For fixed $f \in C_2$, $s \ge 0$, $p \in G$, define

$$M_{s,t} f(p) = f(\phi_{s,t}(p)) - f(p) - \int_s^t Af(\phi_{s,u}(p))du\,, \qquad t \ge s\,. \qquad (3.7)$$

It is an L^2-martingale for any $f \in C_2$, $s \ge 0$ and $p \in G$. Now consider the product of two such L^2-martingales $M_{s,t} f(p)$ and $M_{s,t} g(q)$ where $g \in C_2$ and $q \in G$. There exists a unique continuous $(\mathscr{F}_t)$-measurable process of bounded variation $\{A_s(t);$

$s \leq t < \infty\}$ such that $M_{s,t}f(p)M_{s,t}g(q) - A_s(t)$ is a martingale in view of Meyer's decomposition. We denote $A_s(t)$ by $\langle M_{s,t}f(p), M_{s,t}g(q)\rangle$ and call it the bracket process of $M_{s,t}f(p)$ and $M_{s,t}g(q)$. It coincides with the joint quadratic variation $[M_{s,t}f(p), M_{s,t}g(q)]$ of two martingales $M_{s,t}f(p)$ and $M_{s,t}g(q)$ if these are continuous processes. If these are not continuous processes, $[M_{s,t}f(p), M_{s,t}q(q)] - \langle M_{s,t}f(p), M_{s,t}g(q)\rangle$ is a nontrivial martingale of bounded variation.

Lemma 3.2. *For any $f, g \in C_2$ and $p, q \in G$,*

$$B(f, g)(p, q) \equiv 2a^{ij}X_i f(p)X_j g(q) + \int_{G-\{e\}} (f(p\sigma) - f(p))(g(q\sigma) - g(q))\mu(d\sigma) \tag{3.8}$$

is well defined. It gives a symmetric and nonnegative definite bilinear form on C_2. Further,

$$\langle M_{s,t}f(p), M_{s,t}g(q)\rangle = \int_s^t B(f, g)(\phi_{s,u}(p), \phi_{s,u}(q))du \tag{3.9}$$

holds for all $f, g \in C_2$, $0 \leq s \leq t < \infty$ and $p, q \in G$.

Proof. A direct computation yields

$$A(f^2)(p) - 2f(p)Af(p) = B(f, f)(p, p) < \infty .$$

Indeed, the inequality $(f(\sigma p) - f(p))^2 \leq ch(\sigma)$ holds with some positive constant c by [4, p. 272]. Therefore $\int (f(\sigma p) - f(p))^2\mu(d\sigma) < \infty$ holds. Then $B(f, g)$ is also well defined for all $f, g \in C_2$ by the polarization identity.

We shall first give the proof of (3.9) in case $f = g$ and $p = q = e$. Set $M_{s,t}f(e) = M_t$ and $\phi_{s,t}(e) = \phi_t$. Since $f(\phi_t) = f(e) + M_t + \int_s^t Af(\phi_u)du$ is a semi-martingale, we have by Itô's formula

$$f(\phi_t)^2 - f(e)^2 = 2\int_s^{t+} f(\phi_{u-})dM_u + 2\int_s^t f(\phi_u)Af(\phi_u)du + [M_t, M_t] .$$

Therefore,

$$\begin{aligned} M_{s,t}(f^2)(e) &= f(\phi_t)^2 - f(e)^2 - \int_s^t Af^2(\phi_u)du \\ &= 2\int_s^{t+} f(\phi_{u-})dM_u + [M_t, M_t] - \langle M_t, M_t\rangle \\ &\quad + \left\{\langle M_t, M_t\rangle + 2\int_s^t f(\phi_u)Af(\phi_u)du - \int_s^t A(f^2)(\phi_u)du\right\}. \end{aligned}$$

Note that $M_{s,t}(f^2)(e)$, $\int_s^t f(\phi_{u-})dM_u$ and $[M, M]_t - \langle M, M\rangle_t$ are martingales and the last term $\{\cdots\}$ is a continuous process of bounded variation. Hence the latter is identically 0. Therefore we have

$$\langle M_t, M_t\rangle = \int_s^t (A(f^2) - 2fAf)(\phi_u)du = \int_s^t B(f, f)(\phi_u, \phi_u)du\,.$$

This proves (3.9) in the case $f = g$ and $p = q = e$.

Note the polarization identity

$$\begin{aligned}&\langle M_{s,t}f(e), M_{s,t}g(e)\rangle\\ &\quad = \tfrac{1}{4}(\langle M_{s,t}(f+g)(e), M_{s,t}(f+g)(e)\rangle - \langle M_{s,t}(f-g)(e), M_{s,t}(f-g)(e)\rangle).\end{aligned}$$

Then we get (3.9) in the case $f \neq g$ and $p = q = e$. Finally let L_p be the left translation defined by $L_pf(\sigma) = f(p\sigma)$. Since $L_pAf = AL_pf$ is satisfied we have $M_{s,t}f(p) = M_{s,t}L_pf(e)$. We have further $B(L_pf, L_qg)(\sigma, \tau) = B(f, g)(p\sigma, q\tau)$ since $L_pXf(\sigma) = XL_pf(\sigma)$ is satisfied for any left invariant vector field X. Consequently,

$$\begin{aligned}\langle M_{s,t}f(p), M_{s,t}g(q)\rangle &= \langle M_{s,t}L_pf(e), M_{s,t}L_qg(e)\rangle\\ &= \int_s^t B(L_pf, L_qg)(\phi_{s,u}(e), \phi_{s,u}(e))du\\ &= \int_s^t B(f, g)(\phi_{s,u}(p), \phi_{s,u}(q))du\,.\end{aligned}$$

For a partition $\delta = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ of $[0, \infty)$, we define a process $\{Y_t^\delta f(p);\ 0 \leq t < \infty\}$ by

$$Y_t^\delta f(p) = \sum_k M_{t\wedge t_k, t\wedge t_{k+1}} f(p)\,. \tag{3.10}$$

Since $M_{s,t}f(p)$ is an L^2-martingale, $Y_t^\delta f(p)$ is also an L^2-martingale. Set $|\delta| = \max_k(t_{k+1} - t_k)$. We can prove similarly as in [2, Lemma 3.2] the following lemma.

Lemma 3.3. *$Y_tf(p) \equiv \lim_{|\delta|-0} Y_t^\delta f(p)$ exists in L^2-sense for any $f \in C_2$, $p \in G$ and $t > 0$. It is a Lévy process with values in $\mathbf{R}^1$, i.e., $Y_{t_{k+1}}f(p) - Y_{t_k}f(p)$, $k = 0, 1, \ldots$ are independent for any $t_0 < t_1 < \cdots < t_n$. The bracket process of $Y_tf(p)$ and $Y_tg(q)$ is given by*

$$\langle Y_tf(p), Y_tg(q)\rangle = tB(f, g)(p, q)\,. \tag{3.11}$$

Furthermore, $M_{s,t}f(p)$ is represented by the nonlinear integral of $\phi_{s,u}$ by $Y_uf(p)$, i.e.,

$$M_{s,t}f(p) = \int_s^t dY_uf(\phi_{s,u-}(p))\,. \tag{3.12}$$

Now let $Y_tf = Y_t^cf + Y_t^df$ be the unique decomposition such that $Y_t^cf(p)$ is a continuous L^2-martingale and $Y_t^df(p)$ is a discontinuous L^2-martingale. We claim:

Lemma 3.4. (i) *Define*

$$B^i(t) = Y_t^c(x^i)(e), \qquad i = 1, \dots, d. \tag{3.13}$$

Then $(B^1(t), \dots, B^d(t))$ *is a d-dimensional Brownian motion whose covariance is given by* $\mathrm{Cov}(B^i(t), B^j(t)) = 2ta^{ij}$. *Further,* $Y_t^c f(p)$ *is represented by*

$$Y_t^c f(p) = X_i f(p) B^i(t) \tag{3.14}$$

for any $f \in C_2$ *and* $p \in G$.

(ii) *Define*

$$N((s, t] \times E) = \#\{u \in (s, t]; \phi_{u-}^{-1}\phi_u \in E\}. \tag{3.15}$$

Then it is a Poisson random measure on $\mathbf{R}^+ \times (G - \{e\})$ *with intensity measure* $dtd\mu$. *Further,* $Y_t^d f(p)$ *is represented by*

$$Y_t^d f(p) = \int_0^{t+} \int_{G-\{e\}} (f(p\sigma) - f(p))\tilde{N}(dud\sigma), \tag{3.16}$$

where $\tilde{N}((s, t] \times E) = N((s, t] \times E) - (t - s)\mu(E)$.

Proof. We shall first consider (ii). Set $Z_E(t) = N((0, t] \times E)$. It is a time homogeneous process with values in $\{0, 1, 2, \dots\}$ increasing with jump 1 only. Further $Z_E(t_i) - Z_E(t_{i-1})$, $i = 1, \dots, n - 1$ are independent for any $t_0 < t_1 < \cdots < t_n$, since ϕ_t has independent increments in the Lie group G. Hence it is a Poisson process with parameter $t\hat{\mu}(E) \equiv E[Z_E(t)]$. This shows that N is a Poisson random measure with intensity measure $dtd\hat{\mu}$. We shall prove $\hat{\mu} = \mu$ later.

Now consider the L^2-martingale $M_{s,t}f(p)$. It is decomposed as $M_{s,t}f(p) = M_{s,t}^c f(p) + M_{s,t}^d f(p)$, where $M_{s,t}^c f(p)$, $s \le t < \infty$ is a continuous L^2-martingale and $M_{s,t}^d f(p)$, $s \le t < \infty$ is a discontinuous L^2-martingale. The latter is represented by

$$M_{s,t}^d f(p) = \int_s^{t+} \int_{G-\{e\}} (f(\phi_{s,u-}(p)\sigma) - f(\phi_{s,u-}(p)))\hat{N}(dud\sigma),$$

where $\hat{N}((s, t] \times E) = N((s, t] \times E) - (t - s)\hat{\mu}(E)$. For a partition $\delta = \{0 = t_0 < \cdots < t_n < \cdots\}$ of $[0, \infty)$, we set $Y_t^{d,\delta}f(p) = \sum_k M_{t_k \wedge t, t_{k+1} \wedge t}^d f(p)$ as before. Then it converges in L^2 to $Y_t^d f(p)$ as $|\delta| \to 0$. The limit is represented by

$$Y_t^d f(p) = \int_0^{t+} \int_{G-\{e\}} (f(p\sigma) - f(p))\hat{N}(dud\sigma). \tag{3.17}$$

We shall prove $\hat{N} = \tilde{N}$ or equivalently $\hat{\mu} = \mu$. Note that

$$\langle Y_t f(p), Y_t g(q) \rangle = \langle Y_t^c f(p), Y_t^c g(q) \rangle + \langle Y_t^d f(p), Y_t^d g(q) \rangle \tag{3.18}$$

holds since $Y_t^c f(p)$ and $Y_t^d f(p)$ are orthogonal. The last term is given by

$$\langle Y_t^d f(p), Y_t^d g(q) \rangle = \int_{G-\{e\}} (f(p\sigma) - f(p))(g(q\sigma) - g(q))\hat{\mu}(d\sigma), \tag{3.19}$$

in view of (3.17). Therefore we have from (3.8), (3.11), (3.18) and (3.19)

$$\langle Y_t^c f(p), Y_t^c g(q)\rangle - 2ta^{ij}X_i f(p)X_j g(q)$$
$$= t\int_{G-\{e\}} (f(p\sigma) - f(p))(g(q\sigma) - g(q))(\mu - \hat{\mu})(d\sigma)\,. \tag{3.20}$$

Set $B_t^c(f, g)(p, q) = \langle Y_t^c f(p), Y_t^c g(q)\rangle$. We can show similarly as in [1] that $B_t^c(f, g)$ is a symmetric bilinear form on C_2 and satisfies the derivation property:

$$B_t^c(f_1 f_2, g) = f_1 B_t^c(f_2, g) + f_2 B_t^c(f_1, g)\,.$$

The same derivation property is valid for the bilinear form $2ta^{ij}X_i f(p)X_j g(q)$. On the other hand, the derivation property is not valid for the right hand side of (3.20) unless $\mu - \hat{\mu} = 0$. We have thus proved the assertion (ii) of the lemma and the formula (3.16).

We shall next consider (i). We have shown above the equality

$$\langle Y_t^c f(p), Y_t^c g(q)\rangle = 2ta^{ij}X_i f(p)X_j g(q)\,. \tag{3.21}$$

In particular, the continuous martingales $B^i(t)$, $i = 1, \dots, d$ satisfies $\langle B^i(t), B^j(t)\rangle = 2ta^{ij}$. Therefore $B(t)$ is an d-dimensional Brownian motion with covariance $2t(a^{ij})$ by Lévy's characterization of a Brownian motion. Define the right hand side of (3.14) by $\tilde{Y}_t^c f(p)$. By (3.21), we have the following equalities.

$$\langle Y_t^c f(p), Y_t^c f(p)\rangle = 2ta^{ij}X_i f(p)X_j f(p)\,,$$
$$\langle Y_t^c f(p), \tilde{Y}_t^c f(p)\rangle = X_j f(p)\langle Y_t^c f(p), B^j(t)\rangle = 2ta^{ij}X_i f(p)X_j f(p)\,,$$
$$\langle \tilde{Y}_t^c f(p), \tilde{Y}_t^c f(p)\rangle = X_i f(p)X_j f(p)\langle B^i(t), B^j(t)\rangle = 2ta^{ij}X_i f(p)X_j f(p)\,.$$

These three equalities imply $\langle Y_t^c f(p) - \tilde{Y}_t^c f(p), Y_t^c f(p) - \tilde{Y}_t^c f(p)\rangle = 0$. Therefore we have $Y_t^c f(p) = \tilde{Y}_t^c f(p)$ for any p.

Remark. Equality (3.14) tells us that $B^i(t)$ defined by (3.13) does not depend on the choice of the function x^i. Indeed, let $\tilde{x}^i$ be another function of C_2 satisfying $\tilde{x}^i(e) = 0$ and $X_j \tilde{x}^i(e) = \delta_j^i$. Then $Y_t^c(\tilde{x}^i)(e) = B^i(t)$ holds by (3.14).

Proof of Theorem 3.1. We have defined Brownian motion $(B^1(t), \dots, B^d(t))$ by (3.13) and Poisson random measure $N((s, t] \times E)$ by (3.15). These are $(\mathscr{F}_t)$-measurable, obviously. Further by Lemma 3.4, $Y_t f(p)$ is represented by

$$Y_t f(p) = X_i f(p)B^i(t) + \int_0^{t+}\int_{G-\{e\}} (f(p\sigma) - f(p))\tilde{N}(dud\sigma)\,.$$

Then from Lemma 3.3, we have

$$\begin{aligned} M_{0,t}f(e) &= \int_0^t X_i f(\phi_{u-})dB^i(u) + \int_0^{t+}\int_{G-\{e\}} (f(\phi_{u-}\sigma) - f(\phi_{u-}))\tilde{N}(dud\sigma) \\ &= \int_0^t X_i f(\phi_{u-}) \circ dB^i(u) + \int_0^{t+}\int_{G-\{e\}} (f(\phi_{u-}\sigma) - f(\phi_{u-}))\tilde{N}(dud\sigma) \\ &\quad - a^{ij}\int_0^t X_i X_j f(\phi_{u-})du\,. \end{aligned}$$

We have further $\int_0^t Af(\phi_u)du = \int_0^t Af(\phi_{u-})du$ since the discontinuous times of ϕ_t are at most countable. Then by (3.7), the above is equivalent to (3.4).

Conversely suppose that a Lévy process ϕ_t is represented by (3.4) with a Brownian motion $B'(t)$ and a Poisson random measure N' which are mutually independent. Then $N = N'$ holds because of (3.17). Further, $Y_t^c f(p) = X_i f(p)B^i(t) = X_i f(p)B'^i(t)$ holds because of the definition of $Y_t^c f(p)$. Therefore $B(t) = B'(t)$ holds.

The last statement of the theorem will be proved at the end of this section.

Example. Suppose that G is a matrix group of dimension n and let g be its Lie algebra. We can identify the basis $\{X_1, \ldots, X_n\}$ of g as matrices. Let $f(\sigma) = (f_j^i(\sigma))$ be a smooth function with values in matrices such that $f_j^i(\sigma) = \sigma_j^i$, where σ_j^i is the (i, j)-element of the matrix σ. Then ϕ_t is represented as a solution of the stochastic differential equation:

$$\begin{aligned}\phi_t = I &+ \int_0^t \phi_{u-}X_i \circ dB^i(u) + \int_0^t \phi_{u-}\hat{a}^i X_i du \\ &+ \int_0^{t+}\int_U \phi_{u-}(\sigma - I)\tilde{N}(dud\sigma) + \int_0^{t+}\int_{U^c} \phi_{u-}(\sigma - I)N(dud\sigma) \\ &+ \int_0^{t+}\int_U \phi_{u-}(\sigma - I - X_i x^i(\sigma))du\mu(d\sigma)\,, \end{aligned} \tag{3.22}$$

where $(x^1, \ldots, x^n)$ is the canonical cordinate in U. A representation similar to the above is obtained by Holevo [3].

3.3. Representation of Lévy flows. Let M be a connected orientable paracompact manifold of dimension n. Suppose that we are given a Lévy flow $\{\Phi_{s,t}; 0 \le s < \infty\}$ of diffeomorphisms of M. We will show that it is obtained by solving a stochastic differential equation of jump type, assuming that $\Phi_{s,t}$ takes values in a finite dimensional transformation group acting on M.

We assume that there exists a Lie group G with properties (i)–(ii) mentioned in Section 2.3 and the following
(iv) The Lévy flow $\Phi_{s,t}$ takes values in the subgroup $\psi(G) = \{\psi(g, \cdot); g \in G\}$ of Diff(M).

Let ϕ_t be a right continuous stochastic process on G such that $\psi(\phi_t, p) = \Phi_{0,t}(p)$. Then $\psi(\phi_s^{-1}\phi_t, p) = \Phi_{0,t}\Phi_{0,s}^{-1}(p) = \Phi_{s,t}(p)$ holds. Therefore ϕ_t is a Lévy process on G and hence the assertions of Theorem 3.1 are valid.

Corollary to Theorem 3.1. *The Lévy flow $\Phi_{s,t}$ is obtained as a solution of the stochastic differential equation*

$$f(\Phi_{s,t}(p)) = f(p) + \int_s^t X_i f(\Phi_{s,u-}(p)) \circ dB^i(u) + a^i \int_s^t X_i f(\Phi_{s,u-}(p))du$$
$$+ \int_s^{t+} \int_{G-\{e\}} (f(\sigma \circ \Phi_{s,u-}(p)) - f(\Phi_{s,u-}(p)))\hat{N}(dud\sigma)$$
$$+ \int_s^t \int_{G-\{e\}} (f(\sigma \circ \Phi_{s,u-}(p)) - f(\Phi_{s,u-}(p)) - x^i(\sigma)X_i f(\Phi_{s,u-}(p)))\mu(d\sigma)du .$$

We remark that Fujiwara-Kunita [2] proved that a Lévy flow defined on G_+^m is represented by a solution of a stochastic differential equation similar to the above, and Fujiwara [1] proved the similar fact for a Lévy flow defined on G_+, in the case where M is a Euclidean space and is a compact manifold, respectively.

3.4. Construction of Lévy processes. Suppose that we are given a time homogeneous d-dimensional Brownian motion $B(t) = (B^1(t), \dots, B^d(t))$ with mean 0 and $\mathrm{cov}(B^i(t), B^j(t)) = 2ta^{ij}$, and a time homogeneous Poisson random measure N on $\mathbf{R}^+ \times (G - \{e\})$ with the intensity measure $dtd\mu(\sigma)$ which are mutually independent. Consider the stochastic differential equation (3.6) on G. It is a certain generalization of stochastic differential equation considered in Section 2. Indeed, the stochastic differential equation (2.22) can be written as (3.6). We will show that equation (3.6) has also a unique solution.

Theorem 3.5. *The stochastic differential equation* (3.6) *has a unique solution. The solution is a Lévy process on the Lie group G. Further the solution is measurable with respect to the σ-fields*:

$$\tilde{\mathscr{F}}_t = \sigma(B(s), N((s, t] \times E); 0 \le s \le t, E \in \mathscr{B}(G - \{e\})) . \tag{3.23}$$

Proof. Let $x = (x^1, \dots, x^d)$ be a canonical coordinate in a coordinate neighborhood U of e. Then every σ of U is represented by $\sigma = \exp(x^i X_i)$, where $x^i = x^i(\sigma)$. In the sequel we set $\xi(x) = \exp(x^i X_i)$. Set $U_\varepsilon = \{\sigma \in G; |x(\sigma)| < \varepsilon\}$. Define a Poisson random measure $\hat{N}$ and its intensity measure ν on $\mathbf{R}^+ \times \{x \in \mathbf{R}^d; |x| < \varepsilon/2\}$ by

$$\hat{N}((s, t] \times E) = N((s, t] \times \{\xi(x); x \in E\}) , \qquad \nu(E) = \mu(\{\xi(x); x \in E\}) .$$

Then equation (3.6) is written as

$$f(\phi_t) = f(e) + \int_0^t X_i f(\phi_{u-}) \circ dB^i(u) + \hat{a}^i \int_0^t X_i f(\phi_{u-})du$$
$$+ \int_0^{t+} \int_{|x|<\varepsilon/2} (f(\phi_{u-}\xi(x)) - f(\phi_{u-}))(\hat{N}(dudx) - du\nu(dx))$$
$$+ \int_0^{t+} \int_{|x|<\varepsilon/2} (f(\phi_{u-}\xi(x)) - f(\phi_{u-}) - X_i f(\phi_{u-})x_i)\nu(dx)du ,$$
$$+ \int_0^{t+} \int_{U_{\varepsilon/2}^c} (f(\phi_{u-}\sigma) - f(\phi_{u-}))N(dud\sigma) , \tag{3.24}$$

We first consider the equation in the case where the last term of the above is identically 0, i.e., $N(t, U_{\varepsilon/2}^c) = 0$. The equation is of the same form as equation (2.22). Hence it has a unique solution by Theorem 2.4. We denote it by $\phi^\varepsilon(t)$. Let $\pi_\varepsilon(t)$ be a Poisson point process on G associated with the Poisson random measure $N_\varepsilon(t, E) = N(t, U_\varepsilon^c \cap E)$ and let $\{\beta_1, \beta_2, \dots\}$ be its domain. We define $\phi(t)$ as in the proof of Theorem 2.4 replacing $\gamma^\varepsilon(t)$ by the above $\phi^\varepsilon(t)$. Then this $\phi(t)$ is a solution of equation (3.6). Further in view of the method of the construction of the solution, $\phi(t)$ is measurable with respect to the driving processes B and N. Therefore $\phi(t)$ is measurable with respect to (3.23).

We can show similarly as in the proofs of Theorem 2.4 and its Corollary that the solution is unique and it is in fact a Lévy process.

Proof of Theorem 3.1, *continued.* We shall prove (3.5). Let $\mathscr{F}_t$ and $\tilde{\mathscr{F}}_t$ be σ-fields defined by (3.3) and (3.23), respectively. Then the inclusion property $\tilde{\mathscr{F}}_t \subset \mathscr{F}_t$ is obvious since both $B(t)$ and N are $\mathscr{F}_t$-measurable. Conversely ϕ_t is measurable with respect to $\tilde{\mathscr{F}}_t$ as is shown in Theorem 3.5. Therefore we get the equality (3.5).

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Added in proof: After submitting this paper, the attention of the authors was drawn to the following work: *A. Estrade, Exponentielle stochastique et intégrale multiplicative discontinues, Ann. Inst. Henri Poincaré, Probabilités et Statistiques, 28 (1992), 107–129.* Results similar to our Theorem 2.3 are obtained in the paper.