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An initial-boundary value problem for the pseudo-hyperbolic equation of gravity-gyroscopic waves

By

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0. Introduction

The equation of gravity-gyroscopic waves (1.1) is a linear evolutionary partial differential equation of 4-th order and composite type. Similar equations yield both elliptic and hyperbolic characteristics and therefore they share properties of both elliptic and hyperbolic equations. Such equations are also called pseudo-hyperbolic. The equation (1.1) governs non-stationary internal waves in an ideal stratified and rotational inviscid incompressible fluid.

In [1]-[6] exact solutions of non-stationary boundary value problems for small oscillations of plates in an unbounded stratified rotational fluid were obtained. In doing so pressure or normal velocities were specified on both sides of the plate. This led to the first or second boundary value problem.

The problem of non-stationary internal waves in a two layer stratified fluid excited by small vibrations of a plate placed at the boundary of separation between layers was studied in [7]. Impulsively started vibrations of a sphere were discussed in [30], [31].

In [8]-[9] initial boundary value problems for small oscillations of plates in a bounded layer of a stratified fluid were considered.

Both classical and weak solvability of initial boundary value problems for the equation of gravity-gyroscopic waves in arbitrary simply connected regions was analysed in [5]. Problems in arbitrary multiply connected domains were studied in [33-36].

Solvability of the problem on non-stationary oscillations of an open arc in a stratified and rotational fluid was studied in [5], [16]-[18].

The problems of generation of stationary internal gravity waves by oscillations of a sphere and diffraction of internal waves from an oscillating cylinder were studied in [19], [20], [28], [29] (see also

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references in these articles). Diffraction problems with other geometries were studied in [21]–[27]. It is essential to note that diffraction problems for internal gravity waves lead to unusual boundary value problems for the hyperbolic equation, where propagation of singularities from the singular points on the boundary along the characteristics of the hyperbolic equation takes place.

In the present paper the explicit solution of the initial boundary value problem on vibrations of several double sided plates in a stratified and rotational fluid is obtained. In doing so dynamic pressure is specified on one side of each plate and normal velocities are specified on the other side. This is a mixed boundary condition.

Hence, the present paper is the attempt to consider excitation of nonsteady internal waves by vibrations of several bodies and to solve the pseudo-hyperbolic boundary value problem in a multiply connected domain with the mixed Dirichlet-Neumann boundary condition. All the previous papers mentioned above dealt with either Dirichlet or Neumann boundary condition.

The basic method for the analysis of the classical solvability of initial boundary value problems for the equation of gravity-gyroscopic waves is the potential theory, which has been constructed in [4], [5]. The potential theory for the pseudo-hyperbolic gravity-gyroscopic wave equation is similar to the potential theory for the parabolic equations. With the help of the potential theory, initial boundary value problems for the gravity-gyroscopic wave equation can be reduced to the time-dependent integral equations on a boundary of a region. The existence of solutions of the integral equations in the case of an arbitrary smooth boundary was studied in [5], and these solutions can be computed. Sometimes the solutions can be found in an explicit form.

The scheme of the present paper is as follows. The rigorous mathematical formulation of the initial mixed boundary value problem is given in Section 1 together with the uniqueness theorem. The reduction of the problem to the integral equations on the boundary by the method of dynamic potentials is presented in Section 2. The solution of the integral equations is found in Section 2 in an explicit form with an accuracy up to unknown functions depending on time only. These unknown functions are found from the linear algebraic systems of equations derived in Section 3. Thus, in Section 3 the construction of the explicit solution is completed and the theorem on the solvability of the problem is formulated.

1. Formulation of the problem

In Cartesian coordinates $x=(x_1, x_3)\in R^2$ let us consider an ideal fluid which is exponentially stratified along the Ox_3 axis and uniformly rotates around it. The dynamics of small two dimensional motions of such a fluid in the Boussinesq approximation are described by the equation of gravity-gyroscopic waves [5, 6, 10, 26, 32]:

$$\frac{\partial^2}{\partial t^2}\Delta\Phi+\omega_1^2\Phi_{x_1x_1}+\omega_2^2\Phi_{x_3x_3}=0, \quad \Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2} \tag{1.1}$$

where the Vasala-Brunt frequency ω_1 and twice the angular velocity of rotation ω_2 are constants and $\omega_1, \omega_2\geq 0$. The potential function $\Phi(t, x)$ was introduced in [10]. It is related to the dynamic pressure $p(t, x)$ and the velocity vector $\mathbf{v}=(v_1, v_3)$ by

$$\Phi=-\frac{1}{\rho_0}\frac{\partial p}{\partial t}, \quad \left(\frac{\partial^2}{\partial t^2}+\omega_1^2\right)v_3=\Phi_{x_3}, \quad \left(\frac{\partial^2}{\partial t^2}+\omega_2^2\right)v_1=\Phi_{x_1},$$

where ρ_0 is the average stationary density of the fluid.

We note that (1.1) is a fourth order equation of complex type. This equation yields both elliptic and hyperbolic characteristics. Such equations are called pseudo-hyperbolic and were not studied in classical mathematical physics.

We denote by Os the coordinate axis obtained from the Ox_1 axis by rotation through an angle θ around the origin.

Let two sets of plates (cuts) Γ^1 and Γ^2 be placed in the fluid along the Os axis. The first set Γ^1 involves $N_1\geq 0$ cuts and the second set Γ^2 involves $N_2\geq 0$ cuts:

$$\Gamma^1=\bigcup_{n=1}^{N_1}\Gamma_n^1, \quad \Gamma_n^1=\{x: \quad x_1=s\cos\theta,\ x_3=s\sin\theta,\ s\in(a_n^1, b_n^1)\}, \quad n=1, \dots, N_1;$$

$$\Gamma^2=\bigcup_{n=1}^{N_2}\Gamma_n^2, \quad \Gamma_n^2=\{x: \quad x_1=s\cos\theta,\ x_3=s\sin\theta,\ s\in(a_n^2, b_n^2)\}, \quad n=1, \dots, N_2.$$

The notations a_n^j and b_n^j $(n=1, \dots, N_j; j=1, 2)$ will be used for the points $(a_n^j\cos\theta, a_n^j\sin\theta)$ and $(b_n^j\cos\theta, b_n^j\sin\theta)$ in the plane (x_1, x_3) to make formulae shorter. The totality of cuts is denoted by Γ: $\Gamma=\Gamma^1\cup\Gamma^2$. Suppose that the closures of all cuts are disjoint.

Let η be a normal vector to Γ. The direction of η is chosen so that it will coincide with the direction of the Os axis if η is rotated clockwise through an angle of $\pi/2$.

We orient each cut Γ_n^j $(n=1, \dots, N_j; j=1, 2)$ by distinguishing between the sides $(\Gamma_n^j)^+$ and $(\Gamma_n^j)^-$, where $(\Gamma_n^j)^+$ is that side of the cut Γ_n^j which is on the left when the parameter s increases. The opposite side of Γ_n^j will be called $(\Gamma_n^j)^-$.

Similarly the side of the contour Γ^j, which is on the left when the parameter s increases, will be called $(\Gamma^j)^+$ and the opposite side will be called $(\Gamma^j)^-$ $(j=1, 2)$.

Definition. A function $W(t, x)$ belongs to the smoothness class G if $(k=0, 1, 2)$:

1) $D_t^k W \in C^0([0, \infty) \times \overline{R^2 \backslash \Gamma})$;

2) $D_t^k \nabla W \in C^0([0, \infty) \times \overline{R^2 \backslash \Gamma} \backslash X)$, where $D_t^k = \frac{\partial^k}{\partial t^k}$, X is the set of end-points of the plates, that is $X = \{a_n^j, b_n^j ; n=1, \dots, N_j, j=1, 2\}$;

3) in the vicinity of any point $d \in X$ we have

$$| D_t^k \nabla W | \leq A_k(t) |x-d|^{\delta}, \quad x \to d \tag{1.2}$$

for some $A_k(t) \in C^0[0, \infty)$ and $\delta > -1$.

Assuming that $N_1 + N_2 > 0$ we formulate the initial mixed boundary value problem $K(N_1, N_2)$.

Problem $K(N_1, N_2)$. To find a function $\Phi(t, x)$ of the class G which satisfies the equation (1.1) on $(0, \infty) \times (R^2 \backslash \Gamma)$ in a classical sense, the initial conditions $\Phi(0, x) = \Phi_t(0, x) = 0$, $x \in R^2 \backslash \Gamma$, the boundary conditions on Γ^1:

$$\Phi|_{x(s) \in (\Gamma^1)^+} = f_1(t, s), \tag{1.3a}$$
$$\mathbf{N}_{t,x} \Phi|_{x(s) \in (\Gamma^1)^-} = f_2(t, s), \tag{1.3b}$$

the boundary conditions on Γ^2:

$$\mathbf{N}_{t,x} \Phi|_{x(s) \in (\Gamma^2)^+} = f_2(t, s), \tag{1.3c}$$
$$\Phi|_{x(s) \in (\Gamma^2)^-} = f_1(t, s), \tag{1.3d}$$

and the regularity conditions at infinity

$$| D_t^k \Phi | \leq B_k(t), \quad k=0, 1, 2; \tag{1.4}$$
$$| D_t^k \nabla \Phi | \leq \bar{B}_k(t) |x|^{-1-\varepsilon}, \quad k=0, 1, 2; \tag{1.5}$$

where $B_k(t), \bar{B}_k(t) \in C^0[0, \infty)$, $\varepsilon > 0$ and $|x| = (x_1^2 + x_3^2)^{1/2} \to \infty$.

All conditions of the problem must be satisfied in a classical sense.

In the formulation of the problem the following operator on Γ was denoted by $\mathbf{N}_{t,x}$:

$$\begin{aligned}\mathbf{N}_{t,x} \Phi|_{x(s) \in \Gamma} &= \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial \eta} \Phi + \omega_1^2 \cos(\widehat{\eta x_1}) \Phi_{x_1} + \omega_2^2 \cos(\widehat{\eta x_3}) \Phi_{x_3} \\ &= \cos(\widehat{\eta x_1}) \left(\frac{\partial^2}{\partial t^2} + \omega_1^2 \right) \Phi_{x_1} + \cos(\widehat{\eta x_3}) \left(\frac{\partial^2}{\partial t^2} + \omega_2^2 \right) \Phi_{x_3},\end{aligned}$$

where $\cos(\widehat{\eta x_j})$ is the cosine of the angle between the vector η and the direction of the Ox_j axis $(j=1, 3)$.

From the definition of the class G it follows that

1) the boundary conditions (1.3a), (1.3d) must hold at the ends of the cuts Γ^1 and Γ^2;

2) the validity of the boundary conditions (1.3b), (1.3c) at the ends of the cuts Γ^1 and Γ^2 is not required.

The problem $K(N_1, N_2)$ describes non-stationary wave motions excited by small vibrations of the plates Γ starting at the moment $t=0$ (before this moment the system was at rest). In doing so pressure is specified at the side $(\Gamma^1)^+$ of the plates Γ^1 and at the side $(\Gamma^2)^-$ of the plates Γ^2 and this yields the first boundary condition.

Normal velocities are specified at the side $(\Gamma^2)^+$ of the plates Γ^2 and at the side $(\Gamma^1)^-$ of the plates Γ^1 and this produces the analog of the second boundary condition with time derivatives.

Let us note that the conditions (1.2) at the ends of the cuts and the regularity conditions at infinity (1.4), (1.5) ensure an absence of point sources at the ends of vibrating plates and at infinity.

It follows from [6, 10] that the statement holds:

Theorem 1. *There is not more than one solution of the problem $K(N_1, N_2)$.*

The proof of the theorem is based on the method of energy equalities for the equation (1.1).

2. Time-dependent integral equations on the boundary and their solution

Let

$$\begin{aligned} &f_1(t, s) \in C_0^2([0, \infty); C^{1,\lambda}(\bar{\Gamma})), \\ &f_2(t, s) \in C^0([0, \infty); C^\lambda(\bar{\Gamma})), \end{aligned} \tag{2.1}$$

where the Hölder index $\lambda \in (0, 1]$.

We denote by $C^k([0, \infty); \mathbf{B})$ the class of abstract functions $w(t)$ having k continuous derivatives with respect to t. For every t a function $w(t)$ belongs to the Banach space $\mathbf{B}$ in a spatial variable. We denote by $C_0^k([0, \infty); \mathbf{B})$ the class of abstract functions $w(t) \in C^k([0, \infty); \mathbf{B})$ which satisfy the initial conditions: $w(0) = \cdots = w^{(k-1)}(0) = 0$.

We denote by $\bar{\Gamma}$ the closure of Γ.

We can replace the boundary conditions (1.3a) on $(\Gamma^1)^+$ and on $(\Gamma^2)^-$ by the following equivalent conditions

$$\frac{\partial \Phi}{\partial s}\bigg|_{x(s) \in (\Gamma^1)^+} = \frac{\partial}{\partial s} f_1(t, s) = f_1'(t, s), \tag{2.2a}$$

$$\frac{\partial \Phi}{\partial s}\bigg|_{x(s) \in (\Gamma^2)^-} = \frac{\partial}{\partial s} f_1(t, s) = f_1'(t, s), \tag{2.2b}$$

$$\Phi(t, a_n^j)=f_1(t, a_n^j), \quad n=1, \dots, N_j, \quad j=1, 2. \tag{2.2c}$$

The conditions (2.2a), (2.2b) must hold at all points of $(\Gamma^1)^+$ and $(\Gamma^2)^-$ except their ends.

We will seek a solution of the problem $K(N_1, N_2)$ in the following form

$$\Phi(t, x)=V[\mu](t, x)+T[\nu](t, x)+c(t), \tag{2.3}$$

where $c(t)$ is an unknown function of time, so that $c(t)\in C_0^2[0, \infty)=\{c(t)\in C^2[0, \infty),\ c(0)=c_t(0)=0\}$ and $V[\mu](t, x)$, $T[\nu](t, x)$ are the dynamic potentials for the equation (1.1) which were studied in [4, 5, 12]. The potentials are defined by formulae

$$V[\mu](t, x)=\int_\Gamma \mu(t, s)\log|x-y(s)|ds$$
$$+\int_0^t\int_\Gamma \mu(t-\tau, s)\frac{1}{\tau}\left(1-\cos\left(\tau\frac{|x-y(s)|_0}{|x-y(s)|}\right)\right)dsd\tau,$$

$$T[\nu](t, x)=\int_\Gamma \nu(t, s)\phi(x, s)ds-\int_0^t\int_\Gamma \nu(t-\tau, s)U(\phi(x, s), \tau)dsd\tau,$$

where

$$y(s)=(s\cos\theta, s\sin\theta)\in\Gamma,$$
$$|x|=(x_1^2+x_3^2)^{1/2},$$
$$|x|_0=(\omega_1^2x_3^2+\omega_2^2x_1^2)^{1/2},$$
$$U(\phi(x, s), t)=\int_0^{\phi(x,s)}\Theta(\xi)\sin(t\Theta(\xi))d\xi,$$
$$\Theta(\xi)=(\omega_1^2\sin^2\xi+\omega_2^2\cos^2\xi)^{1/2}.$$

A function $\phi(x, s)$ is determined (up to indeterminacy $2\pi m$, $m=\pm1, \pm2, \dots$) by the formulae

$$\cos\phi(x, s)=\frac{x_1-s\cos\theta}{|x-y(s)|},$$
$$\sin\phi(x, s)=\frac{x_3-s\sin\theta}{|x-y(s)|}.$$

More precisely, we fix a point $x\notin\Gamma$ and choose an arbitrary fixed branch $\phi(x, s)$ of this function which varies continuously with s along each cut Γ_n^j, $(n=1, \dots, N_j,\ j=1, 2)$. Under this definition of $\phi(x, s)$, the potential $T[\nu](t, x)$ is a many-valued function. In order that the potential $T[\nu](t, x)$ be a single-valued function it is necessary to require the validity of the following (N_1+N_2) additional conditions for the function $\nu(t, s)$ (see [4]-[5], [12]-[13]):

$$\int_{\Gamma_n^j}\nu(t, s)ds=0, \quad t\geq0, \quad n=1, \dots, N_j, \quad j=1, 2. \tag{2.4}$$

The functions $\mu(t, s)$, $\nu(t, s)$ and $c(t)$ are unknown and must be found in the process of solving the problem.

We will seek functions $\mu(t, s)$, $\nu(t, s)$ in the following smoothness class:

$$\mu(t, s),\ \nu(t, s) \in C_0^2([0, \infty); C_{\kappa_0}^{\lambda_0}(\Gamma)),$$

where $\lambda_0 \in (0, 1]$, $\kappa_0 \in [0, 1)$. We denote by $C_{\kappa_0}^{\lambda_0}(\Gamma)$ a Banach space of functions $f(\xi)$ defined on Γ and such that

$$\left| \prod_{n=1}^{N_1} (\xi - a_n^1)(\xi - b_n^1) \prod_{n=1}^{N_2} (\xi - a_n^2)(\xi - b_n^2) \right|^{\kappa_0} f(\xi) \in C^{\lambda_0}(\bar{\Gamma}),$$

$$\| f(\xi) \|_{C_{\kappa_0}^{\lambda_0}(\Gamma)} = \left\| \left| \prod_{n=1}^{N_1} (\xi - a_n^1)(\xi - b_n^1) \prod_{n=1}^{N_2} (\xi - a_n^2)(\xi - b_n^2) \right|^{\kappa_0} f(\xi) \right\|_{C^{\lambda_0}(\bar{\Gamma})}.$$

If the conditions (2.4) hold and the functions $\mu(t, s)$, $\nu(t, s)$ belong to the required class of smoothness, then it can be verified directly using the properties of potentials from [4]-[5], that the function $\Phi(t, x)$ from (2.3) belongs to the class G and satisfies the equation (1.1) and the initial conditions of the problem $K(N_1, N_2)$.

In order for the function $\Phi(t, x)$ to satisfy the regularity conditions at infinity (1.4), (1.5) it is necessary to require the following additional condition

$$\int_\Gamma \mu(t, s) ds = 0, \quad t \geq 0. \tag{2.5}$$

We arrive at the theorem.

Theorem 2. *If $\mu(t, s)$, $\nu(t, s) \in C_0^2([0, \infty); C_{\kappa_0}^{\lambda_0}(\Gamma))$ where $\lambda_0 \in (0, 1]$, $\kappa_0 \in [0, 1)$ and the conditions (2.4), (2.5) hold, then the function (2.3) satisfies all conditions of the problem $K(N_1, N_2)$ except the boundary conditions.*

The theorem follows from [4, 5, 12], where dynamic potentials were studied. Besides, the theorem can be checked directly on the basis of the explicit formulae for the dynamic potentials introduced and discussed above.

By using the limiting formulae for the values of potentials on the boundary (see [4, 5, 11, 12]) and by satisfying the boundary conditions of the problem $K(N_1, N_2)$ on Γ (the conditions on $(\Gamma^1)^+$ and $(\Gamma^2)^-$ are taken in the form (2.2)) we obtain the following system of singular integral equations on Γ^1, Γ^2 for unknown functions $\mu(t, s)$, $\nu(t, s)$:

$$-\int_\Gamma \frac{\mu(t, \sigma)}{\sigma - s} d\sigma - \pi J_{\omega_1} * J_{\omega_2} * \nu(t, s) \Big|_{s \in \Gamma^1} = f_1'(t, s), \tag{2.6a}$$

$$-\int_{\Gamma}\frac{1}{\sigma-s}\frac{\partial^2}{\partial t^2}\nu(t,\sigma)d\sigma-\pi S_{\omega_1}*S_{\omega_2}*\frac{\partial^2}{\partial t^2}\mu(t,s)\Big|_{s\in\Gamma^1}=f_2(t,s), \tag{2.6b}$$

$$-\int_{\Gamma}\frac{1}{\sigma-s}\frac{\partial^2}{\partial t^2}\nu(t,\sigma)d\sigma+\pi S_{\omega_1}*S_{\omega_2}*\frac{\partial^2}{\partial t^2}\mu(t,s)\Big|_{s\in\Gamma^2}=f_2(t,s), \tag{2.7a}$$

$$-\int_{\Gamma}\frac{\mu(t,\sigma)}{\sigma-s}d\sigma+\pi J_{\omega_1}*J_{\omega_2}*\nu(t,s)\Big|_{s\in\Gamma^2}=f_1'(t,s), \tag{2.7b}$$

where the equations (2.6a), (2.6b) result from the boundary conditions (2.2a), (1.3b) and the equations (2.7a), (2.7b) result from the boundary conditions (1.3c), (2.2b).

We define the convolution operators $J_{\omega_j}*$ and $S_{\omega_j}*$ $(j=1,2)$ by

$$J_{\omega_j}*\Omega(t)=\Omega(t)-\omega_j\int_0^t J_1(\omega_j(t-\tau))\Omega(\tau)d\tau,$$

$$S_{\omega_j}*\Omega(t)=\Omega(t)-\omega_j\int_0^t S(\omega_j(t-\tau))\Omega(\tau)d\tau,$$

where $J_1(t)$ is a first-order Bessel function and

$$S(\omega_j t)=-\int_0^{\omega_j t}J_1(\sigma)\frac{d\sigma}{\sigma}.$$

It is essential to note that the operators $J_{\omega_j}*$ and $S_{\omega_j}*$ are self-inverse, that is $J_{\omega_j}*S_{\omega_j}*=S_{\omega_j}*J_{\omega_j}*=E$, $j=1,2$, where E is the identity operator.

We get the following assertion.

Theorem 3. *If the assumptions of the theorem 2 hold, $\mu(t,s)$, $\nu(t,s)$ satisfy the equations (2.6), (2.7) and the function (2.3) satisfies the conditions (2.2c), then the function (2.3) is a solution of the problem $K(N_1,N_2)$.*

Let us construct the solution of the system (2.6), (2.7).

By taking into account the assumptions introduced relative to $\mu(t,s)$, $\nu(t,s)$ and inverting the operators of convolution with respect to time we rewrite the equations (2.6) and (2.7) in the following form:

$$\frac{1}{\pi}\int_{\Gamma}\frac{\mu(t,\sigma)}{\sigma-s}d\sigma+\tilde{\nu}(t,s)\Big|_{s\in\Gamma^1}=-\frac{1}{\pi}f_1'(t,s), \tag{2.8a}$$

$$\mu(t,s)+\frac{1}{\pi}\int_{\Gamma}\frac{\tilde{\nu}(t,\sigma)}{\sigma-s}d\sigma\Big|_{s\in\Gamma^1}=-\frac{1}{\pi}\tilde{f}_2(t,s), \tag{2.8b}$$

$$\mu(t,s)-\frac{1}{\pi}\int_{\Gamma}\frac{\tilde{\nu}(t,\sigma)}{\sigma-s}d\sigma\Big|_{s\in\Gamma^2}=\frac{1}{\pi}\tilde{f}_2(t,s), \tag{2.8c}$$

$$-\frac{1}{\pi}\int_{\Gamma}\frac{\mu(t,\sigma)}{\sigma-s}d\sigma+\tilde{\nu}(t,s)\Big|_{s\in\Gamma^2}=\frac{1}{\pi}f_1'(t,s), \tag{2.8d}$$

where

$$\tilde{\nu}(t,\ s)=J_{\omega_1}*J_{\omega_2}*\nu(t,\ s), \tag{2.9a}$$

$$\tilde{f}_2(t,\ s)=J_{\omega_1}*J_{\omega_2}*\int_0^t(t-\tau)f_2(\tau,\ s)d\tau. \tag{2.9b}$$

In order for the function $\nu(t,\ s)$ to belong to the class $C_0^2([0,\ \infty);C_{\kappa_0}^{\lambda_0}(\Gamma))$, where $\lambda_0\in(0,\ 1]$, $\kappa_0\in[0,\ 1)$, we have to seek the function $\tilde{\nu}(t,\ s)$ from the same class.

We note that the equations (2.8) contain the time t as a parameter ($t\geq 0$).

Let us formulate the assertion.

Proposition 1. *A solution of the system (2.6), (2.7) is transformed by the substitution (2.9a) into a solution of the system (2.8); conversely, any solution of the latter system yields one of the former.*

We introduce the functions

$$\rho_+(t,\ s)=\mu(t,\ s)+\tilde{\nu}(t,\ s),$$
$$\rho_-(t,\ s)=\mu(t,\ s)-\tilde{\nu}(t,\ s).$$

By adding and subtracting relationships (2.8a) and (2.8b), we arrive at the following equations for the new unknown functions $\rho_+(t,\ s)$, $\rho_-(t,\ s)$:

$$\rho_+(t,\ s)+\frac{1}{\pi}\int_\Gamma\frac{\rho_+(t,\ \sigma)}{\sigma-s}d\sigma\Big|_{s\in\Gamma^1}=-\frac{1}{\pi}(f_1'(t,\ s)+\tilde{f}_2(t,\ s)), \tag{2.10a}$$

$$\rho_-(t,\ s)-\frac{1}{\pi}\int_\Gamma\frac{\rho_-(t,\ \sigma)}{\sigma-s}d\sigma\Big|_{s\in\Gamma^1}=\frac{1}{\pi}(f_1'(t,\ s)-\tilde{f}_2(t,\ s)). \tag{2.10b}$$

By adding and subtracting relationships (2.8c) and (2.8d), we get the following equations for $\rho_+(t,\ s)$ and $\rho_-(t,\ s)$ on Γ^2:

$$\rho_+(t,\ s)-\frac{1}{\pi}\int_\Gamma\frac{\rho_+(t,\ \sigma)}{\sigma-s}d\sigma\Big|_{s\in\Gamma^2}=\frac{1}{\pi}(f_1'(t,\ s)+\tilde{f}_2(t,\ s)), \tag{2.10c}$$

$$\rho_-(t,\ s)+\frac{1}{\pi}\int_\Gamma\frac{\rho_-(t,\ \sigma)}{\sigma-s}d\sigma\Big|_{s\in\Gamma^2}=-\frac{1}{\pi}(f_1'(t,\ s)-\tilde{f}_2(t,\ s)). \tag{2.10d}$$

Equations (2.10a), (2.10c) can be written in the form of a singular integral equation for the function $\rho_+(t,\ s)$. This new equation has to be valid on whole contour Γ and is

$$\rho_+(t,\ s)+\frac{R_+(s)}{\pi}\int_\Gamma\frac{\rho_+(t,\ \sigma)}{\sigma-s}d\sigma\Big|_{s\in\Gamma}=R_+(s)\varphi_+(t,\ s), \tag{2.11}$$

where

$$\varphi_+(t, s) = -\frac{1}{\pi}(f_1'(t, s) + \tilde{f}_2(t, s)), \quad s \in \Gamma, \tag{2.12a}$$

$$R_+(s) = \begin{cases} 1, & s \in \Gamma^1 \\ -1, & s \in \Gamma^2 \end{cases} \tag{2.12b}$$

In a similar manner by combining the equations (2.10b), (2.10d) we get a singular integral equation on the whole contour Γ for the function $\rho_-(t, s)$:

$$\rho_-(t, s) + \frac{R_-(s)}{\pi} \int_\Gamma \frac{\rho_-(t, \sigma)}{\sigma - s} d\sigma \Big|_{s \in \Gamma} = R_-(s) \varphi_-(t, s), \tag{2.13}$$

where

$$\varphi_-(t, s) = \frac{1}{\pi}(\tilde{f}_2(t, s) - f_1'(t, s)), \quad s \in \Gamma, \tag{2.14a}$$

$$R_-(s) = \begin{cases} -1, & s \in \Gamma^1 \\ 1, & s \in \Gamma^2 \end{cases} \tag{2.14b}$$

Thus the original system of singular integral equations with respect to the functions $\mu(t, s)$, $\nu(t, s)$ is reduced to the pair of independent singular integral equations (2.11), (2.13) for the new unknown functions $\rho_+(t, s)$ and $\rho_-(t, s)$ respectively. The equations (2.11), (2.13) must be solved for every $t \geq 0$. In order for the functions $\mu(t, s)$, $\nu(t, s)$ to belong to the class $C_0^2([0, \infty); C_{\kappa_0}^{\lambda_0}(\Gamma))$, $\lambda_0 \in (0, 1]$, $\kappa_0 \in [0, 1)$, the functions $\rho_+(t, s)$ and $\rho_-(t, s)$ have to belong to the same class.

We arrive at the following assertion.

Proposition 2. *A solution of the system (2.8) is transformed by the substitution*

$$\mu(t, s) = \frac{1}{2}(\rho_+(t, s) + \rho_-(t, s)),$$

$$\tilde{\nu}(t, s) = \frac{1}{2}(\rho_+(t, s) - \rho_-(t, s))$$

into a solution of the system (2.11), (2.13); conversely any solution of the latter system yields one of the former.

To solve equations (2.11), (2.13) we use the following Lemma.

Lemma 1. *Let L^+ and L^- be two sets of segments on a coordinate axis:*

$$L^+ = \bigcup_{n=1}^{N_-} (a_n^+, b_n^+), \quad L^- = \bigcup_{n=1}^{N_-} (a_n^-, b_n^-)$$

such that no two of the segments have any common points (including ends). Let us denote $L=L^{+}\cup L^{-}$ *and consider the singular integral equation*

$$h(s)+\frac{r(s)}{\pi}\int_L \frac{h(\sigma)}{\sigma-s}d\sigma=H_0(s)=H(s)r(s), \quad s\in L, \tag{2.15}$$

where

$$r(s)=\begin{cases}1, & s\in L^{+}\\ -1, & s\in L^{-}\end{cases}$$

and $H(s)$ *is an arbitrary Hölder function on the closed segments* L.

Then there exists a solution $h(s)$ *of the equation (2.15) such that* $h(s)\in C_{\kappa_0}^{\lambda_0}(\Gamma)$, $\lambda_0\in(0,1]$, $\kappa_0\in[0,1)$, *and the general form of this solution is*

$$h(s)=\frac{1}{2}H_0(s)-r(s)\left(\frac{1}{2\pi Q_0(s)}\int_L \frac{Q_0(\sigma)H_0(\sigma)}{\sigma-s}d\sigma-\frac{P_{N_{+}+N_{-}-1}(s)}{Q_0(s)}\right)$$

or

$$h(s)=\frac{1}{2}r(s)H(s)-\frac{1}{2\pi Q(s)}\int_L \frac{Q(\sigma)H(\sigma)}{\sigma-s}d\sigma+\frac{P_{N_{+}+N_{-}-1}(s)}{Q(s)}$$

where

$$\begin{aligned}Q_0(s)&=\prod_{n=1}^{N_{+}}|s-a_n^{+}|^{3/4}\cdot|s-b_n^{+}|^{1/4}\operatorname{sign}(s-a_n^{+})\\&\quad\times\prod_{n=1}^{N_{-}}|s-a_n^{-}|^{1/4}\cdot|s-b_n^{-}|^{3/4}\operatorname{sign}(s-b_n^{-})\\&=Q(s)r(s),\\ Q(s)&=\prod_{n=1}^{N_{+}}|s-a_n^{+}|^{3/4}\cdot|s-b_n^{+}|^{1/4}\operatorname{sign}(s-a_n^{+})\\&\quad\times\prod_{n=1}^{N_{-}}|s-a_n^{-}|^{1/4}\cdot|s-b_n^{-}|^{3/4}\operatorname{sign}(s-a_n^{-}),\end{aligned}$$

$P_{N_{+}+N_{-}-1}(s)$ *is an arbitrary polynomial of degree* $(N_{+}+N_{-}-1)$.

The validity of the Lemma follows from the results of the monographs [14], [15].

Let us return to the consideration of the equation (2.11). We set

$$\begin{aligned}&a_n^1=a_n^{+}, \quad b_n^1=b_n^{+}, \quad n=1,\dots,N_1=N_{+}, \quad \Gamma^1=L^{+};\\&a_n^2=a_n^{-}, \quad b_n^2=b_n^{-}, \quad n=1,\dots,N_2=N_{-}, \quad \Gamma^2=L^{-},\\&R_{+}(s)=r(s).\end{aligned}$$

We suppose that the time t is fixed and put $\rho_{+}(t,s)=h(s)$, $\varphi_{+}(t,s)=H(s)$. This change of notations transforms the equation (2.11) into the equation

(2.15). Hence according to Lemma 1 the solution of the equation (2.11) is

$$\rho_+(t,s)=\frac{1}{2}R_+(s)\varphi_+(t,s)-\frac{1}{2\pi Q_+(s)}\int_\Gamma\frac{Q_+(\sigma)\varphi_+(t,\sigma)}{\sigma-s}d\sigma+\frac{P^+_{N_1+N_2-1}(t,s)}{Q_+(s)}, \tag{2.16a}$$

where

$$Q_+(s)=\prod_{n=1}^{N_1}|s-a^1_n|^{3/4}\cdot|s-b^1_n|^{1/4}\operatorname{sign}(s-a^1_n)$$
$$\times\prod_{n=1}^{N_2}|s-a^2_n|^{1/4}\cdot|s-b^2_n|^{3/4}\operatorname{sign}(s-a^2_n) \tag{2.16b}$$

and

$$P^+_{N_1+N_2-1}(t,s)=\alpha^+_{N_1+N_2-1}(t)s^{N_1+N_2-1}+\cdots+\alpha^+_1(t)s+\alpha^+_0(t) \tag{2.16c}$$

is a polynomial of degree (N_1+N_2-1) in s whose coefficients are arbitrary functions of t of class $C^2_0[0,\infty)$. If the last requirement holds then proceeding from the explicit formula (2.16a) and properties of singular integrals presented in [14] one can show that

$$\rho_+(t,s)\in C^2_0([0,\infty);C^{\lambda_0}_{\kappa_0}(\Gamma)),\quad \lambda_0\in(0,1],\quad \kappa_0\in[0,1).$$

We solve the equation (2.13) in a similar way. The substitutions

$$a^1_n=a^-_n,\quad b^1_n=b^-_n,\quad n=1,\dots,N_1=N_-,\quad \Gamma^1=L^-;$$
$$a^2_n=a^+_n,\quad b^2_n=b^+_n,\quad n=1,\dots,N_2=N_+,\quad \Gamma^2=L^+,$$
$$R_-(s)=r(s),\quad \rho_-(t,s)=h(s),\quad \varphi_-(t,s)=H(s)$$

(for t fixed) transform the equation (2.13) into (2.15). By using Lemma 1 we obtain the solution of the equation (2.13)

$$\rho_-(t,s)=\frac{1}{2}R_-(s)\varphi_-(t,s)-\frac{1}{2\pi Q_-(s)}\int_\Gamma\frac{Q_-(\sigma)\varphi_-(t,\sigma)}{\sigma-s}d\sigma+\frac{P^-_{N_1+N_2-1}(t,s)}{Q_-(s)}, \tag{2.17a}$$

where

$$Q_-(s)=\prod_{n=1}^{N_1}|s-a^1_n|^{1/4}\cdot|s-b^1_n|^{3/4}\operatorname{sign}(s-a^1_n)$$
$$\times\prod_{n=1}^{N_2}|s-a^2_n|^{3/4}\cdot|s-b^2_n|^{1/4}\operatorname{sign}(s-a^2_n) \tag{2.17b}$$

and

$$P^-_{N_1+N_2-1}(t,s)=\alpha^-_{N_1+N_2-1}(t)s^{N_1+N_2-1}+\cdots+\alpha^-_1(t)s+\alpha^-_0(t) \tag{2.17c}$$

is a polynomial of degree (N_1+N_2-1) in s whose coefficients are arbitrary functions of t of class $C_0^2[0,\infty)$. If the last requirement holds then proceeding from the explicit formula (2.17a) and properties of singular integrals presented in [14] it follows that

$$\rho_-(t,s)\in C_0^2([0,\infty);C_{\kappa_0}^{\lambda_0}(\Gamma)),\quad \lambda_0\in(0,1],\quad \kappa_0\in[0,1).$$

Now the solution of the system (2.8) can be easily found, namely

$$\begin{aligned}\mu(t,s)&=\frac{1}{2}(\rho_+(t,s)+\rho_-(t,s))\\&=\frac{1}{2\pi}R_-(s)\tilde{f}_2(t,s)-\frac{1}{4\pi}\Big(\frac{1}{Q_+(s)}\int_\Gamma\frac{Q_+(\sigma)\varphi_+(t,\sigma)}{\sigma-s}d\sigma\\&\quad+\frac{1}{Q_-(s)}\int_\Gamma\frac{Q_-(\sigma)\varphi_-(t,\sigma)}{\sigma-s}d\sigma\Big)\\&\quad+\frac{1}{2}\Big(\frac{P_{N_1+N_2-1}^+(t,s)}{Q_+(s)}+\frac{P_{N_1+N_2-1}^-(t,s)}{Q_-(s)}\Big),\end{aligned}\tag{2.18}$$

$$\begin{aligned}\tilde{\nu}(t,s)&=\frac{1}{2}(\rho_+(t,s)-\rho_-(t,s))\\&=\frac{1}{2\pi}R_-(s)f_1'(t,s)-\frac{1}{4\pi}\Big(\frac{1}{Q_+(s)}\int_\Gamma\frac{Q_+(\sigma)\varphi_+(t,\sigma)}{\sigma-s}d\sigma\\&\quad-\frac{1}{Q_-(s)}\int_\Gamma\frac{Q_-(\sigma)\varphi_-(t,\sigma)}{\sigma-s}d\sigma\Big)\\&\quad+\frac{1}{2}\Big(\frac{P_{N_1+N_2-1}^+(t,s)}{Q_+(s)}-\frac{P_{N_1+N_2-1}^-(t,s)}{Q_-(s)}\Big),\end{aligned}\tag{2.19}$$

where $s\in\Gamma$, $t\geq 0$; the functions $f_1'(t,s)$, $\tilde{f}_2(t,s)$ are defined in (2.2a, b), (2.9b), the functions $\varphi_+(t,s)$, $\varphi_-(t,s)$, $R_-(s)$ are defined in (2.12a), (2.14a, b), the functions $\rho_+(t,s)$, $Q_+(s)$, $P_{N_1+N_2-1}^+(t,s)$ are defined in (2.16) and the functions $\rho_-(t,s)$, $Q_-(s)$, $P_{N_1+N_2-1}^-(t,s)$ are defined in (2.17).

We have proved the Lemma.

Lemma 2. *If $f_1(t,s)\in C_0^2([0,\infty);C^{1,\lambda}(\bar{\Gamma}))$; $f_2(t,s)\in C^0([0,\infty);C^\lambda(\bar{\Gamma}))$ and $\lambda\in(0,1]$ then the general solution from the class $\mu(t,s)$, $\tilde{\nu}(t,s)\in C_0^2([0,\infty);C_{\kappa_0}^{\lambda_0}(\Gamma))$, $\lambda_0\in(0,1]$, $\kappa_0\in[0,1)$ for the system of singular integral equations (2.8) is given by the formulae (2.18), (2.19) where the coefficients $\alpha_n^+(t)$, $\alpha_n^-(t)$ $(n=0,\ldots,N_1+N_2-1)$ of the polynomials $P_{N_1+N_2-1}^+(t,s)$ and $P_{N_1+N_2-1}^-(t,s)$ are arbitrary functions of t of class $C_0^2[0,\infty)$. Besides, for the general solution $\lambda_0=\min\{\lambda,1/4\}$, $\kappa_0=3/4$.*

The last statement of the Lemma follows from the explicit form of $\mu(t,s)$, $\tilde{\nu}(t,s)$ and properties of singular integrals from [14].

By using the formula (2.9a) and inverting convolution operators we obtain from (2.19) the expression for $\nu(t,s)$:

$$
\begin{aligned}
\nu(t, s)=&\frac{1}{2\pi}R_-(s)\bar{f}_1'(t, s)\\
&-\frac{1}{4\pi}\Big(\frac{1}{Q_+(s)}\int_\Gamma\frac{Q_+(\sigma)\bar{\varphi}_+(t, \sigma)}{\sigma-s}d\sigma\\
&-\frac{1}{Q_-(s)}\int_\Gamma\frac{Q_-(\sigma)\bar{\varphi}_-(t, \sigma)}{\sigma-s}d\sigma\Big)\\
&+\frac{1}{2}\Big(\frac{\bar{P}^+_{N_1+N_2-1}(t, s)}{Q_+(s)}-\frac{\bar{P}^-_{N_1+N_2-1}(t, s)}{Q_-(s)}\Big),
\end{aligned}
\tag{2.20}
$$

where

$$
\begin{aligned}
\bar{f}_1'(t, s)&=S_{\omega_1}*S_{\omega_2}*f_1'(t, s),\\
\bar{\varphi}_+(t, s)&=-\frac{1}{\pi}\Big(\int_0^t(t-\tau)f_2(\tau, s)d\tau+\bar{f}_1'(t, s)\Big),\\
\bar{\varphi}_-(t, s)&=\frac{1}{\pi}\Big(\int_0^t(t-\tau)f_2(\tau, s)d\tau-\bar{f}_1'(t, s)\Big),\\
\bar{P}^+_{N_1+N_2-1}(t, s)&=\bar{\alpha}^+_{N_1+N_2-1}(t)s^{N_1+N_2-1}+\cdots+\bar{\alpha}^+_1(t)s+\bar{\alpha}^+_0(t),\\
\bar{P}^-_{N_1+N_2-1}(t, s)&=\bar{\alpha}^-_{N_1+N_2-1}(t)s^{N_1+N_2-1}+\cdots+\bar{\alpha}^-_1(t)s+\bar{\alpha}^-_0(t),
\end{aligned}
$$

$$\bar{\alpha}^+_n(t)=S_{\omega_1}*S_{\omega_2}*\alpha^+_n(t), \quad n=0, \dots, N_1+N_2-1, \tag{2.21a}$$

$$\bar{\alpha}^-_n(t)=S_{\omega_1}*S_{\omega_2}*\alpha^-_n(t), \quad n=0, \dots, N_1+N_2-1, \tag{2.21b}$$

$\alpha^+_n(t)$ and $\alpha^-_n(t)$ are coefficients of the polynomials $P^+_{N_1+N_2-1}(t, s)$ and $P^-_{N_1+N_2-1}(t, s)$ from (2.16c), (2.17c); all other notations are the same as in (2.19).

It follows from (2.21) that if $\alpha^+_n(t)$, $\alpha^-_n(t)$ belong to $C^2_0[0, \infty)$ then $\bar{\alpha}^+_n(t)$, $\bar{\alpha}^-_n(t)$ belong to the same class ($n=0, \dots, N_1+N_2-1$). It now follows from (2.20) that $\nu(t, s)$ belongs to the required class of smoothness.

Thus the functions $\mu(t, s)$, $\nu(t, s)$ from (2.18) and (2.20) are solutions of the original system of singular integral equations (2.6), (2.7). The lemma holds.

Lemma 3. *If $f_1(t, s)\in C^2_0([0, \infty);C^{1,\lambda}(\bar{\Gamma}))$; $f_2(t, s)\in C^0([0, \infty);C^\lambda(\bar{\Gamma}))$ and $\lambda\in(0, 1]$ then the general solution from the class $\mu(t, s)$, $\nu(t, s)\in C^2_0([0, \infty);C^{\lambda_0}_{\kappa_0}(\Gamma))$, $\lambda_0\in(0, 1]$, $\kappa_0\in[0, 1)$ for the system of singular integral equations (2.6), (2.7) is given by the formulae (2.18), (2.20), where $\alpha^+_n(t)$, $\alpha^-_n(t)$ ($n=0, \dots, N_1+N_2-1$) are arbitrary functions of t of class $C^2_0[0, \infty)$. Moreover the indexes λ_0, κ_0 for the general solution are $\lambda_0=\min\{\lambda, 1/4\}$, $\kappa_0=3/4$.*

If we substitute the functions $\mu(t, s)$, $\nu(t, s)$ which were found in (2.3), then $\Phi(t, x)$ depends on $2(N_1+N_2)+1$ arbitrary functions of time $\alpha^+_0(t), \dots, \alpha^+_{N_1+N_2-1}(t)$; $\alpha^-_0(t), \dots, \alpha^-_{N_1+N_2-1}(t)$, $c(t)$. On the other hand the function $\Phi(t, x)$ must satisfy $2(N_1+N_2)+1$ additional conditions (2.2c), (2.4), (2.5).

Thus the functions $\alpha_n^+(t)$, $\alpha_n^-(t)$ $(n=0, \dots, N_1+N_2-1)$, which are coefficients of the polynomials $P^+_{N_1+N_2-1}(t, s)$, $P^-_{N_1+N_2-1}(t, s)$, and the function $c(t)$ from (2.3) have to be chosen to satisfy the conditions (2.2c), (2.4), (2.5).

3. The linear algebraic system of equations and the solution of the problem $K(N_1, N_2)$

Now we show how to satisfy the conditions (2.2c), (2.4), (2.5). We first consider the conditions (2.4). By inverting the convolution operator $J_{\omega_1} * J_{\omega_2} *$ we write conditions (2.4) in the form

$$\int_{\Gamma_n^j} \tilde{\nu}(t, s)ds=0, \quad t\geq 0, \quad n=1, \dots, N_j, \quad j=1, 2, \tag{3.1}$$

where $\tilde{\nu}(t, s)$ is defined in (2.9a), (2.19). By substituting here the expression for $\tilde{\nu}(t, s)$ from (2.19) we get (N_1+N_2) linear algebraic equations in unknowns $\alpha_n^+(t)$, $\alpha_n^-(t)$ $(n=0, \dots, N_1+N_2-1)$:

$$\sum_{m=0}^{N_1+N_2-1} (\Lambda_{nm}^{(j)+}\alpha_m^+(t)+\Lambda_{nm}^{(j)-}\alpha_m^-(t))=q_n^{(j)}(t), \tag{3.2a}$$

where $n=1, \dots, N_j$, $j=1, 2$ and

$$\Lambda_{nm}^{(j)\pm}=\pm\frac{1}{2}\int_{\Gamma_n^j}\frac{\sigma^m}{Q_\pm(\sigma)}d\sigma, \tag{3.2b}$$

$$\begin{aligned} q_n^{(j)}(t)=&\frac{1}{4\pi}\int_{\Gamma_n^j}\left(\frac{1}{Q_+(s)}\int_\Gamma\frac{Q_+(\sigma)\varphi_+(t, \sigma)}{\sigma-s}d\sigma\right.\\ &\left.-\frac{1}{Q_-(s)}\int_\Gamma\frac{Q_-(\sigma)\varphi_-(t, \sigma)}{\sigma-s}d\sigma\right)ds-\frac{1}{2\pi}\int_{\Gamma_n^j}R_-(s)f_1'(t, s)ds, \end{aligned} \tag{3.2c}$$

$m=0, \dots, N_1+N_2-1$, $\quad n=1, \dots, N_j$, $j=1, 2$.

Next we consider the conditions (2.2c). First we note that (up to an indeterminacy $2\pi m$, $m=\pm 1, \pm 2, \dots$)

$$\psi(a_n^j, s)=\begin{cases}\theta, & s<a_n^j\\ \pi+\theta, & s>a_n^j\end{cases}$$

consequently for all $n=1, \dots, N_j$, $j=1, 2$ the function $\psi(a_n^j, s)$ is constant on each segment of Γ. In view of (2.4) we have $T[\nu](t, a_n^j)=0$, $n=1, \dots, N_j$, $j=1, 2$. Hence conditions (2.2c) become

$$V[\mu](t, a_n^j)+c(t)=f_1(t, a_n^j), \quad n=1, \dots, N_j, \quad j=1, 2. \tag{3.3}$$

The potential $V[\mu](t, x)$ on the line Os (where $x=x(s)=(s\cos\theta, s\sin\theta)$) is

$$V[\mu](t, x(s))=\int_{\Gamma}\mu(t, \sigma)\log|s-\sigma|d\sigma+\int_0^t\int_{\Gamma}\mu(\tau, \sigma)d\sigma l(t-\tau)d\tau \\ =\int_{\Gamma}\mu(t, \sigma)\log|s-\sigma|d\sigma, \tag{3.4}$$

where

$$l(t)=\frac{1}{t}[1-\cos(t\sqrt{\omega_2^2\cos^2\theta+\omega_1^2\sin^2\theta})]$$

and the condition (2.5) is used.

Using (3.4) the conditions (3.3) can be transformed into

$$\int_{\Gamma}\mu(t, \sigma)\log|a_n^j-\sigma|d\sigma+c(t)=f_1(t, a_n^j), \quad n=1, \dots, N_j, \quad j=1, 2. \tag{3.5}$$

Substituting the expression for $\mu(t, s)$ from (2.18) into (3.5) we obtain (N_1+N_2) linear algebraic equations in unknowns $\alpha_0^+(t), \dots, \alpha_{N_1+N_2-1}^+(t)$; $\alpha_0^-(t), \dots, \alpha_{N_1+N_2-1}^-(t)$, $c(t)$:

$$\sum_{m=0}^{N_1+N_2-1}(Z_{nm}^{(j)+}\alpha_m^+(t)+Z_{nm}^{(j)-}\alpha_m^-(t))+c(t)=z_n^{(j)}(t), \tag{3.6a}$$

where $n=1, \dots, N_j$, $j=1, 2$ and

$$Z_{nm}^{(j)\pm}=\frac{1}{2}\int_{\Gamma}\frac{\sigma^m}{Q_{\pm}(\sigma)}\log|a_n^j-\sigma|d\sigma, \tag{3.6b}$$

$$z_n^{(j)}(t)=\frac{1}{4\pi}\int_{\Gamma}\left(\frac{1}{Q_+(s)}\int_{\Gamma}\frac{Q_+(\sigma)\varphi_+(t, \sigma)}{\sigma-s}d\sigma \right. \\ \left. +\frac{1}{Q_-(s)}\int_{\Gamma}\frac{Q_-(\sigma)\varphi_-(t, \sigma)}{\sigma-s}d\sigma\right)\log|a_n^j-s|ds \\ -\frac{1}{2\pi}\int_{\Gamma}R_-(s)\tilde{f}_2(t, s)\log|a_n^j-s|ds+f_1(t, a_n^j), \tag{3.6c}$$

$m=0, \dots, N_1+N_2-1$, $n=1, \dots, N_j$, $j=1, 2$.

Before substituting the expression for $\mu(t, \sigma)$ in (2.5) we compute some integrals which are easily derived with the help of the theory of complex analytic functions (see [14], [15]). Let σ be a real variable, then we put $(\beta\in(0, 1))$

$$\Omega_\beta(\sigma)=\prod_{n=1}^{N_1}|\sigma-a_n^1|^\beta|\sigma-b_n^1|^{1-\beta}\operatorname{sign}(\sigma-a_n^1) \\ \times\prod_{n=1}^{N_2}|\sigma-a_n^2|^{1-\beta}|\sigma-b_n^2|^\beta\operatorname{sign}(\sigma-a_n^2),$$

$$\hat{\Omega}_\beta(\sigma)=\prod_{n=1}^{N_1}(\sigma-a_n^1)^\beta(\sigma-b_n^1)^{1-\beta}\times\prod_{n=1}^{N_2}(\sigma-a_n^2)^{1-\beta}(\sigma-b_n^2)^\beta$$
$$=\begin{cases}\Omega_\beta(\sigma), & \sigma\notin\Gamma\\ -\exp(-i\pi\beta)\Omega_\beta(\sigma), & \sigma\in\Gamma^1\\ \exp(i\pi\beta)\Omega_\beta(\sigma), & \sigma\in\Gamma^2\end{cases}$$

Note that $\hat{\Omega}_\beta(\sigma)$ can be extended analytically from the real axis to the whole complex plane. Using analytical properties of the function $\hat{\Omega}_\beta(\sigma)$ and setting s a real variable we deduce

$$\int_\Gamma\frac{\sigma^m}{\Omega_\beta(\sigma)}\frac{d\sigma}{\sigma-s}=\begin{cases}-\dfrac{\pi s^m}{\Omega_\beta(s)\sin\pi\beta}, & s\notin\Gamma, \quad m=0,\ \dots,\ N_1+N_2-1,\\ \dfrac{\pi s^m\cos\pi\beta}{\Omega_\beta(s)\sin\pi\beta}, & s\in\Gamma^1, \quad m=0,\ \dots,\ N_1+N_2-1,\\ -\dfrac{\pi s^m\cos\pi\beta}{\Omega_\beta(s)\sin\pi\beta}, & s\in\Gamma^2, \quad m=0,\ \dots,\ N_1+N_2-1,\end{cases}$$

$$\int_\Gamma\frac{\sigma^m}{\Omega_\beta(\sigma)}d\sigma=\begin{cases}0, & m=0,\ \dots,\ N_1+N_2-2,\\ \dfrac{\pi}{\sin\pi\beta}, & m=N_1+N_2-1,\end{cases}$$

$$\int_\Gamma\frac{1}{\Omega_\beta(s)}\int_\Gamma\frac{h(\sigma)\Omega_\beta(\sigma)}{\sigma-s}d\sigma ds=\frac{\pi\cos\pi\beta}{\sin\pi\beta}\Big(\int_{\Gamma^2}h(\sigma)d\sigma-\int_{\Gamma^1}h(\sigma)d\sigma\Big),$$

where $h(\sigma)$ is a Hölder function on $\bar{\Gamma}$. By substituting the expression for $\mu(t, s)$ from (2.18) into (2.5) and by applying the formulae for integrals we reduce (2.5) to the equation

$$\alpha^+_{N_1+N_2-1}(t)+\alpha^-_{N_1+N_2-1}(t)=0. \tag{3.7}$$

It follows from (3.1) that

$$\int_\Gamma\tilde{\nu}(t,\ \sigma)d\sigma=0.$$

If we substitute $\tilde{\nu}(t,\ \sigma)$ from (2.19) and apply the above integral formulae, we obtain

$$\alpha^+_{N_1+N_2-1}(t)-\alpha^-_{N_1+N_2-1}(t)=0.$$

This, with (3.7) gives

$$\alpha^+_{N_1+N_2-1}(t)=0, \quad \alpha^-_{N_1+N_2-1}(t)=0,$$

but we will not use this fact below.

Thus the conditions (2.4), (2.2c), (2.5) for the functions $\nu(t,\ s)$, $\mu(t,\ s)$ are equivalent to the system of equations (3.2a), (3.6a), (3.7). The system consists of $2(N_1+N_2)+1$ linear algebraic equations for $2(N_1+N_2)+1$

unknown functions $\alpha_0^+(t), \ldots, \alpha_{N_1+N_2-1}^+(t)$; $\alpha_0^-(t), \ldots, \alpha_{N_1+N_2-1}^-(t)$, $c(t)$. The coefficients of the system do not depend on time t and t is included in the system as a parameter. The system (3.2a), (3.6a), (3.7) can be rewritten in the matrix form

$$M\alpha(t)=F(t), \tag{3.8a}$$

where

$$\alpha(t)=(\alpha_0^+(t), \ldots, \alpha_{N_1+N_2-1}^+(t);\ \alpha_0^-(t), \ldots, \alpha_{N_1+N_2-1}^-(t), c(t))^T, \tag{3.8b}$$

$$F(t)=(q^{(1)}(t), q^{(2)}(t), z^{(1)}(t), z^{(2)}(t), 0)^T, \tag{3.8c}$$

$$q^{(j)}(t)=(q_1^{(j)}(t), \ldots, q_{N_1+N_2}^{(j)}(t)), \quad j=1, 2,$$

$$z^{(j)}(t)=(z_1^{(j)}(t), \ldots, z_{N_1+N_2}^{(j)}(t)), \quad j=1, 2,$$

the functions $q_n^{(j)}(t)$, $z_n^{(j)}(t)$, $(n=1, \ldots, N_j, j=1, 2)$ are defined in (3.2c), (3.6c) and M is a square matrix of size $(2(N_1+N_2)+1)\times(2(N_1+N_2)+1)$, which consists of the coefficients of the equations (3.2a), (3.6a), (3.7). An exact expression for M can be easily written out by comparing (3.8a) with (3.2a), (3.6a), (3.7). We observe that matrix M does not depend on t.

Consider the following homogeneous system of linear algebraic equations

$$M\hat{\alpha}=0, \tag{3.9}$$

where $\hat{\alpha}=(\hat{\alpha}_0^+, \ldots, \hat{\alpha}_{N_1+N_2-1}^+;\ \hat{\alpha}_0^-, \ldots, \hat{\alpha}_{N_1+N_2-1}^-, \hat{c})^T$ is an unknown vector, which does not depend on any variables. Below we will use the scalar form of (3.9), namely

$$\sum_{m=0}^{N_1+N_2-1}(\Lambda_{nm}^{(j)+}\hat{\alpha}_m^+ + \Lambda_{nm}^{(j)-}\hat{\alpha}_m^-)=0, \tag{3.10a}$$

$$\sum_{m=0}^{N_1+N_2-1}(Z_{nm}^{(j)+}\hat{\alpha}_m^+ + Z_{nm}^{(j)-}\hat{\alpha}_m^-)+\hat{c}=0, \tag{3.10b}$$

$$\hat{\alpha}_{N_1+N_2-1}^+ + \hat{\alpha}_{N_1+N_2-1}^-=0, \tag{3.10c}$$

where $n=1, \ldots, N_j$, $j=1, 2$.

Let us prove that the matrix M is invertible. According to the Fredholm alternative, the matrix M is invertible if (3.10) has only the trivial solution. We will give a proof by a contradiction. Assume that $\hat{\alpha}$ is a non-trivial solution of the system (3.10), and this solution converts the equations (3.10) into identities.

We introduce the functions

$$\mu_0(s)=\frac{1}{2}\left(\frac{\hat{P}_{N_1+N_2-1}^+(s)}{Q_+(s)}+\frac{\hat{P}_{N_1+N_2-1}^-(s)}{Q_-(s)}\right),$$

$$\nu_0(s)=\frac{1}{2}\Big(\frac{\hat{P}^+_{N_1+N_2-1}(s)}{Q_+(s)}-\frac{\hat{P}^-_{N_1+N_2-1}(s)}{Q_-(s)}\Big),$$

$$\hat{P}^\pm_{N_1+N_2-1}(s)=\hat{a}^\pm_{N_1+N_2-1}s^{N_1+N_2-1}+\cdots+\hat{a}^\pm_1 s+\hat{a}^\pm_0,$$

where $Q_\pm(s)$ are defined in (2.16b), (2.17b).

By using formulae (3.2b), (3.6b), we write the identities (3.10) in terms of the functions $\mu_0(s)$, $\nu_0(s)$:

$$\int_{\Gamma^j_n}\nu_0(s)ds=0,\quad n=1,\dots,N_j,\quad j=1,2, \tag{3.11a}$$

$$\int_\Gamma \mu_0(s)\log|a^j_n-s|ds+\hat{c}=0,\quad n=1,\dots,N_j,\quad j=1,2, \tag{3.11b}$$

$$\int_\Gamma \mu_0(s)ds=0. \tag{3.11c}$$

We introduce the function $\Phi_0(x)=V_0[\mu_0](x)+T_0[\nu_0](x)+\hat{c}$, where

$$V_0[\mu_0](x)=\int_\Gamma \mu_0(s)\log|x-y(s)|ds$$

is a logarithmic harmonic potential and

$$T_0[\nu_0](x)=\int_\Gamma \nu_0(s)\phi(x,s)ds$$

is an angular harmonic potential studied in [13]. The kernel $\phi(x,s)$ of the angular potential was determined in Section 2. It follows from the identities (3.11a) that the function $\Phi_0(x)$ is a single-valued harmonic function.

By using properties of the angular harmonic potential from [13] and by taking into account the identities (3.11) one can show that the function $\Phi_0(x)$ satisfies all conditions of the following homogeneous mixed boundary value problem for the Laplace equation (we will call it problem L):

$$\Phi_0(x)\in C^0(\overline{R^2\backslash\Gamma}),\quad \nabla\Phi_0(x)\in C^0(\overline{R^2\backslash\Gamma}\backslash X),$$

$$\Delta\Phi_0(x)=0,\quad x\in R^2\backslash\bar{\Gamma},$$

$$\frac{\partial\Phi_0}{\partial s}\Big|_{x(s)\in(\Gamma^1)^+}=0,\quad \Phi_0(a^1_n)=0,\quad n=1,\dots,N_1, \tag{3.12a}$$

$$\frac{\partial\Phi_0}{\partial\eta}\Big|_{x(s)\in(\Gamma^1)^-}=0,\quad \frac{\partial\Phi_0}{\partial\eta}\Big|_{x(s)\in(\Gamma^2)^+}=0,$$

$$\frac{\partial\Phi_0}{\partial s}\Big|_{x(s)\in(\Gamma^2)^-}=0,\quad \Phi_0(a^2_n)=0,\quad n=1,\dots,N_2, \tag{3.12b}$$

$$|\nabla\Phi_0(x)|\le A|x-a^j_n|^{-3/4},\quad |x-a^j_n|\to 0,\quad n=1,\dots,N_j,\quad j=1,2,$$

$$|\nabla\Phi_0(x)|\le A|x-b^j_n|^{-3/4},\quad |x-b^j_n|\to 0,\quad n=1,\dots,N_j,\quad j=1,2,$$

$$|\Phi_0(x)|\le B,\quad |\nabla\Phi_0(x)|\le \bar{B}/|x|^2,\quad |x|\to\infty.$$

When verifying validity of the boundary conditions for $\Phi_0(x)$, the integral relationships listed above (see a derivation of (3.7)) can be applied.

Due to equivalence of the boundary conditions (3.12) to the following conditions

$$\Phi_0|_{x(s)\in(\Gamma^1)^+}=0, \quad \Phi_0|_{x(s)\in(\Gamma^2)^-}=0,$$

it can be shown with the help of the energy equality for the Laplace equation that the only solution of the problem L is $\Phi_0(x)\equiv 0$. Hence

$$\frac{\partial\Phi_0}{\partial\eta}\Big|_{x(s)\in\Gamma^+}-\frac{\partial\Phi_0}{\partial\eta}\Big|_{x(s)\in\Gamma^-}=2\mu_0(x)\equiv 0, \tag{3.13a}$$

$$\frac{\partial\Phi_0}{\partial s}\Big|_{x(s)\in\Gamma^+}-\frac{\partial\Phi_0}{\partial s}\Big|_{x(s)\in\Gamma^-}=-2\nu_0(x)\equiv 0, \tag{3.13b}$$

where the limiting formulae for derivatives of harmonic potentials from [13] were applied.

By adding and subtracting formulae (3.13a) and (3.13b) we obtain $\hat{P}^+_{N_1+N_2-1}(s)\equiv 0$ and therefore $\hat{\alpha}^+_0=\cdots=\hat{\alpha}^+_{N_1+N_2-1}=0$, $\hat{P}^-_{N_1+N_2-1}(s)\equiv 0$ and consequently $\hat{\alpha}^-_0=\cdots=\hat{\alpha}^-_{N_1+N_2-1}=0$. Now from (3.11b) we obtain $\hat{c}=0$.

Thus we get a contradiction to the assumption that $\hat{\alpha}$ is a non-trivial solution of the homogeneous system (3.10). Hence (3.10) has only the trivial solution, so the matrix M is invertible. This proves the following

Lemma 4. *There exists an inverse matrix M^{-1} for the matrix M of the system of linear algebraic equations (3.2a), (3.6a), (3.7) (or the system (3.8a) in a vector form).*

By inverting the matrix M we write the solution of the system (3.8a) in the form

$$\alpha(t)=M^{-1}F(t), \tag{3.14}$$

where M^{-1} is inverse to M, and the vectors $\alpha(t)$, $F(t)$ were defined in (3.8b, c).

It follows from (3.14) that the functions $\alpha^\pm_n(t)$ $(n=0, \dots, N_1+N_2-1)$ and $c(t)$ belong to the class $C^2_0[0, \infty)$. The functions $\bar{\alpha}^\pm_n(t)$ $(n=0, \dots, N_1+N_2-1)$ determined in (2.21) belong to $C^2_0[0, \infty)$ as well.

Thus we have found the functions $\mu(t, s)$, $\nu(t, s)$ and it follows from their expressions that $\mu(t, s)$, $\nu(t, s)\in C^2_0([0, \infty); C^{\lambda_0}_{\kappa_0}(\Gamma))$, $\lambda_0=\min\{1/4, \lambda\}$, $\kappa_0=3/4$, where λ is the Hölder index in the definition of the functions $f_1(t, s)$, $f_2(t, s)$, that is in (2.1). These functions $\mu(t, s)$, $\nu(t, s)$ satisfy all the conditions introduced for them in the beginning of Section 2.

From the properties of time-dependent dynamic potentials presented in [4], [5] it follows that the function $\Phi(t, x)$ from (2.3) belongs to the class G and satisfies all conditions of the problem $K(N_1, N_2)$.

In particular the function $\Phi(t, x)$ satisfies conditions (1.2) near the ends of Γ with the index $\delta=-3/4$, and the regularity conditions at infinity (1.4), (1.5) hold, where inequality (1.5) is valid with the index $\varepsilon=1$. This statement can be verified directly with the help of the explicit expression for $\Phi(t, x)$.

We have thus proved

Theorem 4. *Let* $f_1(t, s)\in C_0^2([0, \infty); C^{1,\lambda}(\bar{\Gamma}))$, $f_2(t, s)\in C^0([0, \infty); C^{\lambda}(\bar{\Gamma}))$, *where* $\lambda\in(0, 1]$, *then a solution of the problem* $K(N_1, N_2)$ *exists and is given by the formula (2.3) with the densities* $\mu(t, s)$, $\nu(t, s)$ *defined in (2.18), (2.20), where the functions* $\alpha_n^+(t)$, $\alpha_n^-(t)$ $(n=0, \dots, N_1+N_2-1)$, $c(t)$ *making up the vector* $\alpha(t)$ *from (3.8b) are given by the formulae (3.14) and functions* $\bar{\alpha}_n^+(t)$, $\bar{\alpha}_n^-(t)$ $(n=0, \dots, N_1+N_2-1)$ *are determined with the help of the functions* $\alpha_n^+(t)$, $\alpha_n^-(t)$ $(n=0, \dots, N_1+N_2-1)$ *in (2.21).*

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