J. Math. Kyoto Univ. (JMKYAZ)
37-2 (1997) 317-326

Absolute continuity of similar translations

Dedicated to Professor Shinzo Watanabe on his sixtieth birthday

By

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§ 1. Introduction

Let $\mathbf{X}=\{X_k\}_k$ be an IID, let $\mathbf{Y}=\{Y_k\}_k$ be an independent random sequence defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, and assume that $\mathbf{X}$ and $\mathbf{Y}$ are independent. Denote by $\mu_{\mathbf{X}}$, $\mu_{\mathbf{Y}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ the probability measures on $\mathbf{R}^{\mathbf{N}}$ (the space of all real sequences), induced by $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{X}+\mathbf{Y}=\{X_k+Y_k\}_k$, respectively. Since $\mathbf{X}$ and $\mathbf{X}+\mathbf{Y}$ are independent random sequences, $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ are product measures :

$$\mu_{\mathbf{X}}=\prod_k \mu_{X_k} \quad \text{and} \quad \mu_{\mathbf{X}+\mathbf{Y}}=\prod_k \mu_{X_k+Y_k},$$

where μ_{X_k} and $\mu_{X_k+Y_k}$ are marginal distributions.

When μ_{X_1} is absolutely continuous with respect to the Lebesgue measure dx, define $f(x)=\frac{d\mu_{X_1}}{dx}(x)$. If f is an absolutely continuous function, f' denotes the derivative of f in the distribution sense, and if f' is an absolutely continuous function, f'' denotes the derivative of f' in the same sense. In these cases, define

$$I_1(\mathbf{X})=\int_{-\infty}^{+\infty}\frac{f'(x)^2}{f(x)}\,dx \quad \text{and} \quad I_2(\mathbf{X})=\int_{-\infty}^{+\infty}\frac{f''(x)^2}{f(x)}\,dx \quad \text{if } f>0 \text{ a.e..}$$

Sato and Watari [8, Theorem 1] proved the relation $I_1(\mathbf{X})\leq\frac{3}{2}\sqrt{I_2(\mathbf{X})}$, so that $I_2(\mathbf{X})<\infty$ implies $I_1(\mathbf{X})<\infty$.

Several authors have investigated the conditions for satisfying $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ (mutually absolutely continuous) in terms of the distribution of $\mathbf{Y}$, but necessary and sufficient conditions are not yet known in general (see Sato [7]). In the present paper we concentrate on the case in which $\mathbf{Y}$ is a similar random sequence, that is, $\mathbf{Y}=\mathbf{a}\boldsymbol{\Theta}=\{a_k\Theta_k\}_k$, where $\boldsymbol{\Theta}=\{\Theta_k\}_k$ are independent copies of a random variable Θ, and $\mathbf{a}=\{a_k\}_k$ is a real sequence. In the following with the exception of Section 2, we fix the above notation and assume $\mathrm{P}(\Theta\neq 0)>0$. The following results are known.

Communicated by Prof. S. Watanabe, August 1, 1996

Theorem A (Shepp [9]). *Assume* $\Theta\equiv 1$ *a. s.. Then we have*:
(1) $\mu_{X+a\Theta}\sim\mu_X$ *implies* $\sum_k a_k^2<\infty$.
(2) *Assume* $I_1(\mathbf{X})<\infty$. *Then* $\sum_k a_k^2<\infty$ *implies* $\mu_{X+a\Theta}\sim\mu_X$.
(3) *If* $\sum_k a_k^2<\infty$ *implies* $\mu_{X+a\Theta}\sim\mu_X$, *then* $I_1(\mathbf{X})<\infty$ *holds.*

Theorem B (Okazaki and Sato [6], Sato and Watari [8], and Okazaki [5]). *Let* $\mathbf{\Theta}=\{\Theta_k\}_k$ *be the Rademacher sequence, that is,* $\mathrm{P}(\Theta=1)=\mathrm{P}(\Theta=-1)=\frac{1}{2}$. *Then we have*:
(1) $\mu_{X+a\Theta}\sim\mu_X$ *implies* $\sum_k a_k^4<\infty$.
(2) *Assume* $I_2(\mathbf{X})<\infty$. *Then* $\sum_k a_k^4<\infty$ *implies* $\mu_{X+a\Theta}\sim\mu_X$.
(3) *If* $\sum_k a_k^4<\infty$ *implies* $\mu_{X+a\Theta}\sim\mu_X$, *then* $I_2(\mathbf{X})<\infty$ *holds.*

Theorem C (Kakutani [3]). *Let* $\mathbf{X}=\{X_k\}_k$ *be a standard Gaussian sequence and* Θ *be a standard Gaussian random variable. Then* $\mu_{X+a\Theta}\sim\mu_X$ *holds if and only if* $\sum_k a_k^4<\infty$.

In this paper we first prove, without assumption of the similarity of $\mathbf{Y}$, a variation of Theorem 3 of Sato and Watari [8], and then prove the following theorems for similar $\mathbf{Y}=\mathbf{a\Theta}$. We begin with necessary conditions for the relation $\mu_{X+a\Theta}\sim\mu_X$.

Theorem 1. (1) $\mu_{X+a\Theta}\sim\mu_X$ *implies* $\sum_k a_k^4<\infty$.
(2) *If* $\liminf_{x\to\infty}|\mathrm{E}[\Theta : |\Theta|\le x]|>0$, *then* $\mu_{X+a\Theta}\sim\mu_X$ *implies* $\sum_k a_k^2<\infty$.
(3) *If* $\liminf_{x\to\infty}x^p\mathrm{P}(|\Theta|>x)>0$ *for some* $p>0$, *then* $\mu_{X+a\Theta}\sim\mu_X$ *implies* $\sum_k|a_k|^{2p}<\infty$.

The following corollary is an immediate consequence of Theorem 1 (2).

Corollary 1. *If* Θ *is integrable and* $\mathrm{E}[\Theta]\neq 0$, *then* $\mu_{X+a\Theta}\sim\mu_X$ *implies* $\sum_k a_k^2<\infty$.

Then sufficient conditions are:

Theorem 2. (1) *Assume* $I_1(\mathbf{X})<\infty$ *and* $\mathrm{E}[\Theta^2]<\infty$. *Then* $\sum_k a_k^2<\infty$ *implies* $\mu_{X+a\Theta}\sim\mu_X$.
(2) *Assume* $I_2(\mathbf{X})<\infty$, $\mathrm{E}[|\Theta|^p]<\infty$ *for some* $p\ge 2$, *and* $\mathrm{E}[\Theta]=0$. *Then* $\sum_k|a_k|^{p\wedge 4}<\infty$ *implies* $\mu_{X+a\Theta}\sim\mu_X$, *where* $p\wedge 4=\min(p, 4)$.

Combining Theorems 1 and 2, we obtain necessary and sufficient conditions for several cases, extending (1) and (2) of both Theorems A and B, and Theorem C.

Theorem 3. (1) *Assume* $I_1(\mathbf{X})<\infty$, $\mathrm{E}[\Theta^2]<\infty$, *and* $\mathrm{E}[\Theta]\neq 0$. *Then* $\mu_{\mathbf{X}+\mathbf{a}\Theta}\sim\mu_{\mathbf{X}}$ *holds if and only if* $\sum_k a_k^2<\infty$.

(2) *Assume* $I_2(\mathbf{X})<\infty$, $\mathrm{E}[\Theta^4]<\infty$, *and* $\mathrm{E}[\Theta]=0$. *Then* $\mu_{\mathbf{X}+\mathbf{a}\Theta}\sim\mu_{\mathbf{X}}$ *holds if and only if* $\sum_k a_k^4<\infty$.

We now refine certain sufficient conditions. In the following Proposition 1, we weaken the assumption $\mathrm{E}[|\Theta|^p]<\infty$ to $\sup_{x>0}x^p\,\mathrm{P}(|\Theta|>x)<\infty$.

Proposition 1. *Assume* $I_2(\mathbf{X})<\infty$ *and* $\sup_{x\geq 0}x^p\ \mathrm{P}(|\Theta|>x)<\infty$ *for some* $p>0$. *If one of the following* (1)~(4) *holds, then* $\sum_k|a_k|^{p\wedge 4}<\infty$ *implies* $\mu_{\mathbf{X}+\mathbf{a}\Theta}\sim\mu_{\mathbf{X}}$:

(1) $0<p\leq 2$.

(2) $2<p<4$ *and* $\mathrm{E}[\Theta]=0$.

(3) $p=4$, $\mathrm{E}[\Theta]=0$, *and there exists* $\varepsilon>0$ *such that*

$$\sup_{|z|<\varepsilon}(\varepsilon-|z|)^2\int_{-\infty}^{+\infty}\frac{f''(x+z)^2}{f(x)}\,dx<\infty,\quad \textit{where}\quad f(x)=\frac{d\mu_{X_1}}{dx}(x).$$

(4) $p>4$ *and* $\mathrm{E}[\Theta]=0$.

On the other hand, we have the following.

Theorem 4. (1) *If* $\sum_k a_k^2<\infty$ *implies* $\mu_{\mathbf{X}+\mathbf{a}\Theta}\sim\mu_{\mathbf{X}}$, *then we have* $\limsup_{x\to\infty}|\mathrm{E}[\Theta : |\Theta|\leq x]|<\infty$.

(2) *If* $\sum_k a_k^4<\infty$ *implies* $\mu_{\mathbf{X}+\mathbf{a}\Theta}\sim\mu_{\mathbf{X}}$, *then we have* $\mathrm{E}[\Theta^2]<\infty$ *and* $\mathrm{E}[\Theta]=0$.

If $\Theta\geq 0$ a.s., then $\limsup_{x\to\infty}|\mathrm{E}[\Theta : |\Theta|\leq x]|<\infty$ in Theorem 4 (1) implies $\mathrm{E}[\Theta]<\infty$, so that we have the following.

Corollary 2. *Assume* $\Theta\geq 0$ *a. s.. If* $\sum_k a_k^2<\infty$ *implies* $\mu_{\mathbf{X}+\mathbf{a}\Theta}\sim\mu_{\mathbf{X}}$, *then we have* $\mathrm{E}[\Theta]<\infty$.

Example 1. Let $\mathbf{X}=\{X_k\}_k$ be an IID such that $I_1(\mathbf{X})<\infty$, and let $\mathbf{Y}=\{Y_k\}_k$ be an independent random sequence, independent of $\mathbf{X}$, such that each Y_k is exponentially distributed. Then the following (1)~(4) are equivalent:

(1) $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$.

(2) $\sum_k \mathrm{E}[Y_k]^2<\infty$.

(3) $\sum_k \mathrm{E}[Y_k^2]<\infty$.

(4) $\sum_k Y_k^2 < \infty$ a. s..

In fact, μ_Y is expressed as $\mu_Y = \mu_{a\Theta}$, where Θ is exponentially distributed, $E[\Theta]=1$, and $a_k = E[Y_k]$, $k \in \mathbf{N}$. Then by Theorem 3 (1), $\mu_{X+a\Theta} \sim \mu_X$ is equivalent to $\sum_k a_k^2 < \infty$. They are also equivalent to $\sum_k E[Y_k^2] < \infty$ because $E[Y_k^2] = a_k^2 E[\Theta^2] = 2a_k^2$. Moreover, we have by Kolmogorov's three series theorem, $\sum_k Y_k^2 < \infty$ a. s. if and only if $\sum_k a_k^2 < \infty$.

Example 2. Let $\mathbf{X} = \{X_k\}_k$ be an IID such that $I_2(\mathbf{X}) < \infty$, and let $\mathbf{Y} = \{Y_k\}_k$ be an independent random sequence, which is independent of $\mathbf{X}$, such that each Y_k is a symmetric α-stable random variable, where $0 < \alpha \leq 2$. Let

$$E[e^{itY_k}] = e^{-c_k |t|^\alpha}, \quad \text{where} \quad c_k \geq 0,\ k \in \mathbf{N}.$$

Then μ_Y is expressed as $\mu_Y = \mu_{a\Theta}$, where $E[e^{it\Theta}] = e^{-|t|^\alpha}$ and $a_k = c_k^{\frac{1}{\alpha}}$, $k \in \mathbf{N}$. In addition, we have by Blumenthal and Getoor [1, Theorem 2.1],

$$0 < \liminf_{x \to \infty} x^\alpha P(|\Theta| > x) \leq \sup_{x \geq 0} x^\alpha P(|\Theta| > x) < \infty.$$

Hence by Proposition 1, $\sum_k |a_k|^\alpha < \infty$ implies $\mu_{X+Y} \sim \mu_X$, and by Theorem 1 (3), $\mu_{X+Y} \sim \mu_X$ implies $\sum_k |a_k|^{2\alpha} < \infty$.

§ 2. General Case

In this section we do not assume that $\mathbf{Y}$ is similar. We first give preliminaries and then prove a variation of Theorem 3 of Sato and Watari [8]. A general characterization of $\mu_{X+Y} \sim \mu_X$ has been given by Kitada and Sato [4, Theorem 2] as follows.

Lemma 1 (Kitada and Sato [4]). *Assume* $\mu_{X_k+Y_k} \sim \mu_{X_k}$ *for every* $k \in \mathbf{N}$, *and define*

$$Z_k(x) = \frac{d\mu_{X_k+Y_k}}{d\mu_{X_k}}(x) - 1, \qquad k \in \mathbf{N}.$$

Then $\mu_{X+Y} \sim \mu_X$ *holds if and only if the following hold* :

$$\sum_k E[Z_k(X_k) : Z_k(X_k) \geq 1] < \infty,$$
$$\sum_k E[Z_k(X_k)^2 : |Z_k(X_k)| < 1] < \infty.$$

This is a necessary and sufficient condition, but $Z_k(x)$ depends on the distribution of X_1 and is not always easily estimated. Starting from Lemma 1, Hino [2, Theorem 1.8] proved certain conditions for the relation $\mu_{X+Y} \sim \mu_X$ as follows. His conditions are described in terms only of the distribution of $\mathbf{Y}$, but they are necessary *or* sufficient conditions.

Lemma 2 (Hino [2]). (1) *If $\mu_{X+Y}\sim\mu_X$, then we have for every $\varepsilon>0$,*

$$\sum_k P(|Y_k|>\varepsilon)^2+\sum_k E[Y_k : |Y_k|\leq\varepsilon]^2+\sum_k E[Y_k^2 : |Y_k|\leq\varepsilon]^2<\infty.$$

(2) *Assume $I_2(\mathbf{X})<\infty$. If there exists $\varepsilon>0$ such that*

$$\sum_k P(|Y_k|>\varepsilon)+\sum_k E[Y_k : |Y_k|\leq\varepsilon]^2+\sum_k E[Y_k^4 : |Y_k|\leq\varepsilon]<\infty,$$

then we have $\mu_{X+Y}\sim\mu_X$.

(3) *If there exists $\varepsilon>0$ such that*

$$\sup_{|z|<\varepsilon}(\varepsilon-|z|)^2\int_{-\infty}^{+\infty}\frac{f''(x+z)^2}{f(x)}dx<\infty$$

and

$$\sum_k P(|Y_k|>\varepsilon)+\sum_k E[Y_k : |Y_k|<\varepsilon]^2+\sum_k E[Y_k^2 : |Y_k|\leq\varepsilon]^2<\infty,$$

then we have $\mu_{X+Y}\sim\mu_X$.

Applying Lemma 2, we have the following theorem.

Theorem 5. *If $I_2(\mathbf{X})<\infty$, $E[Y_k]=0$, $\sup_k E[Y_k^2]<\infty$ and $\sum_k Y_k^4<\infty$ a. s., then we have $\mu_{X+Y}\sim\mu_X$.*

Proof. Since $\sum_k Y_k^4<\infty$ a. s., we have by Kolmogorov's three series theorem,

$$\sum_k P(|Y_k|>1)<\infty \quad\text{and}\quad \sum_k E[Y_k^4 : |Y_k|\leq 1]<\infty.$$

Since $E[Y_k]=0$, we have

$$E[Y_k : |Y_k|\leq 1]=-E[Y_k : |Y_k|>1],$$

and thus, by the Schwarz inequality,

$$\begin{aligned}\sum_k E[Y_k : |Y_k|\leq 1]^2&=\sum_k E[Y_k : |Y_k|>1]^2\leq\sum_k E[Y_k^2]\,P(|Y_k|>1)\\&\leq\Big(\sup_k E[Y_k^2]\Big)\sum_k P(|Y_k|>1)<\infty.\end{aligned}$$

Hence by Lemma 2 (2), we have $\mu_{X+Y}\sim\mu_X$.

Corollary 3. *$I_2(\mathbf{X})<\infty$, $\sum_k E[Y_k^2]<\infty$ and $E[Y_k]=0$ together imply $\mu_{X+Y}\sim\mu_X$.*

Sato and Watari [8, Theorem 3] proved that $\mu_{X+Y} \sim \mu_X$ holds if $I_2(\mathbf{X}) < \infty$, $\sum_k Y_k^4 < \infty$ a. s. and each Y_k is symmetric. We assume $\mathrm{E}[Y_k]=0$ and $\sup_k \mathrm{E}[Y_k^2] < \infty$ instead of assuming the symmetry of Y_k. Then the case $p=4$ of Theorem 2 (2) is a special case of Theorem 5. In fact, $\mathrm{E}[\Theta^4] < \infty$ and $\sum_k a_k^4 < \infty$ together imply $\sum_k \mathrm{E}[a_k^4 \Theta_k^4] < \infty$, so that we have $\sum_k a_k^4 \Theta_k^4 < \infty$ a. s. and $\sup_k \mathrm{E}[a_k^2 \Theta_k^2] < \infty$.

§ 3. Proofs

Proof of Theorem 1. (1) Since $\mathrm{P}(\Theta \neq 0) > 0$, there exists $K > 0$ such that

$$\mathrm{P}(0 < |\Theta| \leq K) > 0 \qquad \text{and} \qquad \mathrm{P}(|\Theta| \geq K) > 0.$$

Then by Lemma 2 (1), we have

$$\infty > \sum_{a_k \neq 0} \mathrm{P}\left(|\Theta| > \frac{1}{|a_k|}\right)^2 \geq \sum_{|a_k| K > 1} \mathrm{P}\left(|\Theta| > \frac{1}{|a_k|}\right)^2$$
$$\geq \sum_{|a_k| K > 1} \mathrm{P}(|\Theta| \geq K)^2,$$

so that $|a_k| K > 1$ holds for finitely many $k \in \mathbf{N}$. Hence there exists $k_0 \in \mathbf{N}$ such that

$$|a_k| K \leq 1 \quad \text{for} \quad k \geq k_0.$$

We therefore have, by Lemma 2 (1),

$$\infty > \sum_{k \geq k_0} \mathrm{E}[Y_k^2 : |Y_k| \leq 1]^2$$
$$= \sum_{\substack{k \geq k_0 \\ a_k \neq 0}} a_k^4 \, \mathrm{E}\left[\Theta_k^2 : |\Theta_k| \leq \frac{1}{a_k}\right]^2 \geq \sum_{k \geq k_0} a_k^4 \, \mathrm{E}[\Theta^2 : |\Theta| \leq K]^2,$$

so that $\sum_k a_k^4 < \infty$.

(2) Let $m = \frac{1}{2} \liminf_{x \to \infty} |\mathrm{E}[\Theta : |\Theta| \leq x]| > 0$. In the case that $|\Theta| \leq M$ a. s. for some $M > 0$, we have $m = \frac{1}{2} |\mathrm{E}[\Theta]|$, so that by Lemma 2 (1), we have

$$\infty > \sum_k \mathrm{E}[Y_k : |Y_k| \leq M]^2 = \sum_k \mathrm{E}[Y_k]^2 = \sum_k a_k^2 \, \mathrm{E}[\Theta]^2 = 4m^2 \sum_k a_k^2.$$

In the case that $\mathrm{P}(|\Theta| > x) > 0$ for all $x > 0$, there exists $L > 0$ such that

$$|\mathrm{E}[\Theta : |\Theta| \leq x]| \geq m \quad \text{for} \quad x \geq L.$$

By Lemma 2 (1), we have

$$\infty > \sum_k P(|Y_k|>1)^2 = \sum_{a_k \neq 0} P\left(|\Theta| > \frac{1}{|a_k|}\right)^2$$

$$\geq \sum_{|a_k|L>1} P\left(|\Theta| > \frac{1}{|a_k|}\right)^2 \geq \sum_{|a_k|L>1} P(|\Theta|>L)^2.$$

Then since $P(|\Theta|>L)>0$, $|a_k|L>1$ holds for finitely many $k \in \mathbf{N}$, so that there exists $n_0 \in \mathbf{N}$ such that $|a_k|L \leq 1$ for $k \geq n_0$. We thus have

$$\left| E\left[\Theta : |\Theta| \leq \frac{1}{|a_k|}\right] \right| \geq m \quad \text{for} \quad k \geq n_0 \quad \text{with} \quad a_k \neq 0.$$

Therefore by Lemma 2 (1), we have

$$\infty > \sum_k E[Y_k : |Y_k| \leq 1]^2 \geq \sum_{\substack{k \geq n_0 \\ a_k \neq 0}} a_k^2 \, E\left[\Theta_k : |\Theta_k| \leq \frac{1}{|a_k|}\right]^2$$

$$\geq m^2 \sum_{k \geq n_0} a_k^2.$$

(3) Let $r = \frac{1}{2} \liminf_{x \to \infty} x^p P(|\Theta|>x) > 0$. Then there exists $L>0$ such that

$$x^p P(|\Theta|>x) \geq r \quad \text{for} \quad x \geq L.$$

Since $P(|\Theta|>L)>0$, as in the proof of (1), we know there exists $n_0 \in \mathbf{N}$ such that

$$|a_k|L \leq 1 \quad \text{for} \quad k \geq n_0.$$

Hence by Lemma 2 (1), we have

$$\infty > \sum_{k \geq n_0} P(|Y_k|>1)^2 \geq \sum_{\substack{k \geq n_0 \\ a_k \neq 0}} P\left(|\Theta| > \frac{1}{|a_k|}\right)^2 \geq r^2 \sum_{k \geq n_0} |a_k|^{2p},$$

so that $\sum_k |a_k|^{2p} < \infty$.

Proof of Theorem 2. (1) Since $E[\sum_k a_k^2 \Theta_k^2] = E[\Theta^2] \sum_k a_k^2 < \infty$, we have $\sum_k a_k^2 \Theta_k^2 < \infty$ a. s., that is, $\mu_{\mathbf{a}\Theta}(l_2) = 1$, and since $\mathbf{X}$ and $\mathbf{a}\Theta$ are independent, we therefore have $\mu_{\mathbf{X}+\mathbf{a}\Theta} = \mu_{\mathbf{X}} * \mu_{\mathbf{a}\Theta}$. It follows that

$$\mu_{\mathbf{X}+\mathbf{a}\Theta}(A) = \int_{l_2} \mu_{\mathbf{X}+\mathbf{y}}(A) d\mu_{\mathbf{a}\Theta}(\mathbf{y})$$

for every Borel set A in $\mathbf{R}^{\mathbf{N}}$. On the other hand, since $I_1(\mathbf{X}) < \infty$, we have by Theorem A, $\mu_{\mathbf{X}+\mathbf{y}} \sim \mu_{\mathbf{X}}$ for every $\mathbf{y} \in l_2$. If $\mu_{\mathbf{X}+\mathbf{a}\Theta}(A) = 0$, then $\mu_{\mathbf{X}+\mathbf{y}}(A) = 0$ for some $\mathbf{y} \in l_2$, so that $\mu_{\mathbf{X}}(A) = 0$. Conversely, if $\mu_{\mathbf{X}}(A) = 0$, then $\mu_{\mathbf{X}+\mathbf{y}}(A) = 0$ for all $\mathbf{y} \in l_2$, so that $\mu_{\mathbf{X}+\mathbf{a}\Theta}(A) = 0$. We therefore have $\mu_{\mathbf{X}+\mathbf{a}\Theta} \sim \mu_{\mathbf{X}}$.

(2) Since $E[|\Theta|^p] < \infty$, we have $M = \sup_x |x|^p P(|\Theta|>x) < \infty$. Then for every $k \in \mathbf{N}$,

$$P(|Y_k|>1) = P\left(|\Theta| > \frac{1}{|a_k|}\right) \leq M|a_k|^p,$$

$$\mathrm{E}[Y_k : |Y_k| \leq 1]^2 = a_k^2 \mathrm{E}\left[\Theta : |\Theta| \leq \frac{1}{|a_k|}\right]^2 = a_k^2 \mathrm{E}\left[\Theta : |\Theta| > \frac{1}{|a_k|}\right]^2$$
$$\leq a_k^2 \mathrm{E}[\Theta^2] \mathrm{P}\left(|\Theta| > \frac{1}{|a_k|}\right)$$
$$\leq a_k^2 \mathrm{E}[\Theta^2] a_k^2 \mathrm{E}[\Theta^2] = a_k^4 \mathrm{E}[\Theta^2]^2$$

and $\mathrm{E}[Y_k^4 : |Y_k| \leq 1] \leq \mathrm{E}[|Y_k|^{p\wedge 4} : |Y_k| \leq 1]$

$$= |a_k|^{p\wedge 4} \mathrm{E}\left[|\Theta|^{p\wedge 4} : |\Theta| \leq \frac{1}{|a_k|}\right]$$
$$\leq |a_k|^{p\wedge 4} \mathrm{E}[|\Theta|^{p\wedge 4}].$$

Hence by Lemma 2 (2), $\sum_k |a_k|^{p\wedge 4} < \infty$ implies $\mu_{X+a\Theta} \sim \mu_X$.

Proof of Theorem 3. (1) By Corollary 1, $\mu_{X+a\Theta} \sim \mu_X$ implies $\sum_k a_k^2 < \infty$, and by Theorem 2 (1), $\sum_k a_k^2 < \infty$ implies $\mu_{X+a\Theta} \sim \mu_X$.
(2) By Theorem 1 (1) and Theorem 2 (2), we have $\mu_{X+a\Theta} \sim \mu_X$ if and only if $\sum_k a_k^4 < \infty$.

Proof of Proposition 1. Let $M = \sup_{x \geq 0} x^p \mathrm{P}(|\Theta| > x) < \infty$. Then we have for $0 < p < 4$,

$$\mathrm{P}(|Y_k| > 1) = \mathrm{P}\left(|\Theta| > \frac{1}{|a_k|}\right) \leq M |a_k|^p$$

and $\mathrm{E}[Y_k^4 : |Y_k| \leq 1] = a_k^4 \mathrm{E}\left[\Theta^4 : |\Theta| \leq \frac{1}{|a_k|}\right] = -a_k^4 \int_0^{\frac{1}{|a_k|}} x^4 d\mathrm{P}(|\Theta| > x)$

$$\leq 4a_k^4 \int_0^{\frac{1}{|a_k|}} x^3 \mathrm{P}(|\Theta| > x) dx \leq 4M a_k^4 \int_0^{\frac{1}{|a_k|}} x^{3-p} dx$$
$$= \frac{4M}{4-p} |a_k|^p.$$

Therefore in (1) and (2), it is sufficient by Lemma 2 (2) to prove that $\sum_k |a_k|^p < \infty$ implies $\sum_k \mathrm{E}[Y_k : |Y_k| \leq 1]^2 < \infty$.

Assume $\sum_k |a_k|^{p\wedge 4} < \infty$. Then there exists $k_0 \in \mathbf{N}$ such that

$$|a_k| \leq 1 \quad \text{for} \quad k \geq k_0.$$

We thus have for $k \geq k_0$,

$$|\mathrm{E}[Y_k : |Y_k| \leq 1]| \leq |a_k| \mathrm{E}\left[|\Theta| : |\Theta| \leq \frac{1}{|a_k|}\right]$$
$$\leq |a_k| + |a_k| \mathrm{E}\left[|\Theta| : 1 < |\Theta| \leq \frac{1}{|a_k|}\right]$$
$$\leq |a_k| + |a_k| \int_1^{\frac{1}{|a_k|}} \mathrm{P}(|\Theta| > x) dx$$
$$\leq |a_k| + M |a_k| \int_1^{\frac{1}{|a_k|}} x^{-p} dx.$$

(1) If $0<p<1$, then for $k \geq k_0$,

$$|\mathrm{E}[Y_k : |Y_k| \leq 1]| \leq |a_k| + M|a_k| \int_1^{\frac{1}{|a_k|}} x^{-p} dx$$
$$= \left(1 - \frac{M}{1-p}\right)|a_k| + \frac{M}{1-p}|a_k|^p.$$

Therefore $\Sigma_k |a_k|^p < \infty$ implies $\Sigma_k \mathrm{E}[Y_k : |Y_k| \leq 1]^2 < \infty$.

If $p=1$, then for $k \geq k_0$,

$$|\mathrm{E}[Y_k : |Y_k| \leq 1]| \leq |a_k| + M|a_k| \int_1^{\frac{1}{|a_k|}} \frac{1}{x} dx$$
$$= |a_k| (1 + M|\log|a_k||).$$

Therefore $\Sigma_k |a_k| < \infty$ implies $\Sigma_k \mathrm{E}[Y_k : |Y_k| \leq 1]^2 < \infty$.

If $1<p\leq 2$, then for $k \geq k_0$,

$$|\mathrm{E}[Y_k : |Y_k| \leq 1]| \leq |a_k| + M|a_k| \int_1^{\frac{1}{|a_k|}} x^{-p} dx \leq \frac{M+1}{p-1} |a_k|.$$

Therefore $\Sigma_k |a_k|^p < \infty$ implies $\Sigma_k \mathrm{E}[Y_k : |Y_k| \leq 1]^2 < \infty$.

(2) Since

$$\mathrm{E}\left[\Theta : |\Theta| \leq \frac{1}{|a_k|}\right] = -\mathrm{E}\left[\Theta : |\Theta| > \frac{1}{|a_k|}\right] \quad \text{and} \quad \mathrm{E}[\Theta^2] < \infty,$$

the following holds:

$$\mathrm{E}[Y_k : |Y_k| \leq 1]^2 = a_k^2 \mathrm{E}\left[\Theta : |\Theta| \leq \frac{1}{|a_k|}\right]^2 = a_k^2 \mathrm{E}\left[\Theta : |\Theta| > \frac{1}{|a_k|}\right]^2$$
$$\leq a_k^2 \mathrm{E}[\Theta^2] \mathrm{P}\left(|\Theta| > \frac{1}{|a_k|}\right)$$
$$\leq a_k^2 \mathrm{E}[\Theta^2] a_k^2 \mathrm{E}[\Theta^2] = a_k^4 \mathrm{E}[\Theta^2]^2,$$

so that $\Sigma_k |a_k|^p < \infty$ implies $\Sigma_k \mathrm{E}[Y_k : |Y_k| \leq 1]^2 < \infty$.

(3) We have

$$\mathrm{P}(|Y_k| > \varepsilon) = \mathrm{P}\left(|\Theta| > \frac{\varepsilon}{|a_k|}\right) \leq \frac{M}{\varepsilon^4} a_k^4,$$

and since

$$\mathrm{E}\left[\Theta : |\Theta| \leq \frac{\varepsilon}{|a_k|}\right] = -\mathrm{E}\left[\Theta : |\Theta| > \frac{\varepsilon}{|a_k|}\right] \quad \text{and} \quad \mathrm{E}[\Theta^2] < \infty,$$

if follows that

$$\mathrm{E}[Y_k : |Y_k| \leq \varepsilon]^2 = a_k^2 \mathrm{E}\left[\Theta : |\Theta| \leq \frac{\varepsilon}{|a_k|}\right]^2 = a_k^2 \mathrm{E}\left[\Theta : |\Theta| > \frac{\varepsilon}{|a_k|}\right]^2$$
$$\leq a_k^2 \mathrm{E}[\Theta^2] \mathrm{P}\left(|\Theta| > \frac{\varepsilon}{|a_k|}\right)$$
$$\leq a_k^2 \mathrm{E}[\Theta^2] \frac{a_k^2}{\varepsilon^2} \mathrm{E}[\Theta^2] = a_k^4 \frac{\mathrm{E}[\Theta^2]^2}{\varepsilon^2},$$

and $\mathrm{E}[Y_k^2 : |Y_k| \leq \varepsilon]^2 = a_k^4 \mathrm{E}\left[\Theta_k^2 : |\Theta_k| \leq \frac{\varepsilon}{|a_k|}\right]^2 \leq a_k^4 \mathrm{E}[\Theta^2]^2$.

Hence by Lemma 2 (3), $\sum_k a_k^4 < \infty$ implies $\mu_{X+a\Theta} \sim \mu_X$.
(4) If $p>4$, then we have $E[\Theta^4] < \infty$. This case is proved in Theorem 2 (2).

Proof of Theorem 4. (1) Assume $\limsup_{x\to\infty} |E[\Theta : |\Theta| \le x]| = \infty$. Then for $T(x) = E\left[\Theta : |\Theta| \le \frac{1}{x}\right]^2$, we have $\limsup_{x\to 0} T(x) = \infty$, so that there exists, by Shepp [9, Lemma 4], a sequence $\mathbf{a} = \{a_k\}_k$ such that

$$\sum_k a_k^2 < \infty \quad \text{and} \quad \sum_k E[Y_k : |Y_k| \le 1]^2 = \sum_{a_k \neq 0} a_k^2 \, E\left[\Theta : |\Theta| \le \frac{1}{|a_k|}\right]^2 = \infty.$$

Therefore by Lemma 2 (1), we have $\mu_{X+a\Theta} \not\sim \mu_X$.
(2) We first prove $E[\Theta^2] < \infty$. If $E[\Theta^2] = \infty$, it follows that $\lim_{x\to+0} E\left[\Theta : |\Theta| \le \frac{1}{\sqrt{x}}\right]^2 = \infty$. Hence by Shepp [9, Lemma 4], there exists a sequence $\mathbf{a} = \{a_k\}_k$ such that

$$\sum_k a_k^4 < \infty \quad \text{and} \quad \sum_k E[Y_k^2 : |Y_k| \le 1]^2 = \sum_{a_k \neq 0} a_k^4 \, E\left[\Theta_k^2 : |\Theta_k| \le \frac{1}{\sqrt{a_k^2}}\right]^2 = \infty.$$

Therefore by Lemma 2 (1), we have $\mu_{X+a\Theta} \not\sim \mu_X$.

Next we prove $E[\Theta] = 0$. If $E[\Theta] \neq 0$, then $\lim_{x\to\infty} |E[\Theta : |\Theta| \le x]| = |E[\Theta]| > 0$. Hence by Theorem 1 (2), $\mu_{X+a\Theta} \sim \mu_X$ implies $\sum_k a_k^2 < \infty$. Then $\sum_k a_k^4 < \infty$ implies $\mu_{X+a\Theta} \sim \mu_X$, and $\mu_{X+a\Theta} \sim \mu_X$ implies $\sum_k a_k^2 < \infty$, so that $\sum_k a_k^4 < \infty$ implies $\sum_k a_k^2 < \infty$. This is a contradiction.

It is therefore shown that $E[\Theta^2] < \infty$ and $E[\Theta] = 0$.

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References

[1] R. M. Blumenthal and R. K. Getoor, Some theorems on stable processes, Trans. Amer. Math. Soc., **95** (1960), 263-273.
[2] M. Hino, On equivalence of product measures by random translation, J. Math. Kyoto Univ., **34**-4 (1994), 755-765.
[3] S. Kakutani, On equivalence of infinite product measures, Ann. Math., **49** (1948), 214-224.
[4] K. Kitada and H. Sato, On the absolute continuity of infinite product measure and its convolution, Probab. Th. Rel. Fields, **81** (1989), 609-627.
[5] Y. Okazaki, On equivalence of product measure by symmetric random l_4-translation, J. Funct. Anal., **115** (1993), 100-103.
[6] Y. Okazaki and H. Sato, Distinguishing a random sequence from a random translate of itself, Ann. Probab., **22** (1994), 1092-1096.
[7] H. Sato, Infinite products and infinite sum, in Stochastic analysis on infinite dimensional spaces, 289-296, Longman Scientific & Technical, 1994.
[8] H. Sato and C. Watari, Some integral inequalities and absolutely continuity of a symmetric random translation, J. Funct. Anal., **114**-1 (1993), 257-266.
[9] L. A. Shepp, Distinguishing a sequence of random variables from a translate of itself, Ann. Math. Stat., **36** (1965), 1107-1112.