J. Math. Kyoto Univ. (JMKYAZ)
37-2 (1997) 261-316

Relations between unitary representations of diffeomorphism groups and those of the infinite symmetric group or of related permutation groups

By

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Introduction

In this paper, we study interrelations between unitary representations of two kinds of groups. The one is the group $G=\mathrm{Diff}_0(M)$ of diffeomorphisms with compact supports on a manifold of class $C^{(n)}$, $1 \leq n \leq \infty$, and the others are certain permutation groups S contained in $\tilde{\mathfrak{S}}_\infty$ of all permutations on the set $\mathbf{N}$ of natural numbers. In certain typical cases, the latter are equal to the infinite symmetric group $\mathfrak{S}_\infty$ of all finite permutations or its standard subgroups.

Let us expain in more detail. The representations treated here are principally infinite tensor products of natural representations $T_j^{s_j}$, $s_j \in \mathbf{R}$, of G on L^2-spaces $\mathscr{H}_j = L^2(X_j, \mathscr{B}_j, \mu_j)$, with $X_j = M$, $\mathscr{B}_j = \mathscr{B}(M)$ the σ-algebra of Borel subsets of M, and μ_j locally finite measures on M which are locally equivalent to Lebesques measures with respect to local coordinates. (The set of all such measures on M is denoted by $\mathscr{LFM}(M)$. Here $T_j^{s_j}$ is given as

$$T_j^{s_j}(g)f(p) = \left(\frac{d\mu_j(g^{-1}p)}{d\mu_j(p)}\right)^{\frac{1}{2}+is_j} f(g^{-1}p) \quad (g \in G,\ f \in \mathscr{H}_j,\ p \in M,\ i=\sqrt{-1}).$$

To have an infinite tensor product, we should fix a reference vector $\chi = (\chi_j)_{j \in \mathbf{N}}$ consisting of unit vectors $\chi_j \in \mathscr{H}_j$. As we see in § 1, it is enough for us to treat the cases where χ_j's are of the form $\chi_j = \| \chi_{E_j} \|_{\mathscr{H}_j}^{-1} \chi_{E_j}$ with χ_{E_j} the characteristic function of $E_j \in \mathscr{B}(M)$. Put $\mu = (\mu_j)_{j \in \mathbf{N}}$, $E = \prod_{j \in \mathbf{N}} E_j$ and $X = \prod_{j \in \mathbf{N}} X_j$, and assume two conditions (MU1) and (MU2) on (μ, E). Then we see as in § 2 that the infinite tensor product space $\otimes_{j \in \mathbf{N}}^{\chi} \mathscr{H}_j$ with respect to $\chi = (\chi_j)_{j \in \mathbf{N}}$ can be realized as an L^2-space for a measure given as infinite direct product of measures μ_j with respect to E_j's constructed as in [4, § 1] or in [5]. This realization of tensor product representations by means of

Received June 28, 1996

product measures gives, together with the actions of the infinite symmetric group $\mathfrak{S}_\infty$ on the infinite tensor product space, a background of our method of constructing irreducible unitary representations (=IURs) of the diffeomorphism group G in [4].

On the other hand, we ask if a permutation $\sigma\in\tilde{\mathfrak{S}}_\infty$ can work on the infinite tensor product space $\otimes_{j\in\mathbf{N}}^{\chi}\mathscr{H}_j$ by permuting the components as an intertwining operator of the tensor product representation $T\equiv T_{\mu,E,s}=\otimes_{j\in\mathbf{N}}^{\chi}T_j^{s_j}$. More neatly, put $s=(s_j)_{j\in\mathbf{N}}$ and $\tilde{\mathfrak{S}}_\infty(s)=\{\sigma\in\tilde{\mathfrak{S}}_\infty\,;\,s_{\sigma(j)}=s_j(j\in\mathbf{N})\}$. Then we ask, for an element $\sigma\in\tilde{\mathfrak{S}}_\infty(s)$, if the following formula defines a bounded operator $R(\sigma)$ on $\otimes_{j\in\mathbf{N}}^{\chi}\mathscr{H}_j$: for a decomposable element $f=\otimes_{j\in\mathbf{N}}f_j$, with $f_j\in\mathscr{H}_j$, $f_j=\chi_j(j\gg\mathbf{N})$, put $R(\sigma)f=\otimes_{j\in\mathbf{N}}h_j$ with

$$h_j(x_j)=\left(\frac{d\mu_{\sigma^{-1}(j)}(x_j)}{d\mu_j(x_j)}\right)^{\frac{1}{2}+is_j}f_{\sigma^{-1}(j)}(x_j)\quad(x_j\in X_j).$$

If $R(\sigma)$ is well defined, it gives an intertwining operator for T, that is, $R(\sigma)\in T(G)'$. The set of all such σ's is denoted by $\mathfrak{S}_{\mu,E,s}$.

The structure of this important subgroup of the permutation group $\tilde{\mathfrak{S}}_\infty$ is studied in §4. The group $\mathfrak{S}_{\mu,E,s}$ contains $\mathfrak{S}_\infty(s)$ and is properly contained in $\tilde{\mathfrak{S}}_\infty(s)$. In an interesting case where the μ-unital subset $E\subset X$ is μ-cofinal with another one $E'=\prod_{j\in\mathbf{N}}E'_j$ such that each E'_j's are mutually disjoint, the group $\mathfrak{S}_{\mu,E,s}$ is exactly equal to the subgroup $\mathfrak{S}_\infty(s)$ of the infinite symmetry group $\mathfrak{S}_\infty$.

In the above case, we have Theorem 5.1, one of our main results in this paper, which says that, under the infinite tensor product representation T, the diffeomorphism group G and the standard subgroup $\mathfrak{S}_\infty(s)$ of the symmetric group $\mathfrak{S}_\infty$ form a so-called dual pair, that is,

$$T(G)'=R(\mathfrak{S}_\infty(s))'',\quad T(G)''=R(\mathfrak{S}_\infty(s))'.$$

This case corresponds to the case of our previous work [4]. Further, the above result on dual pair expains well the meaning of our method of constructing IURs of G employed in [4] and [5], in connection to an irreducible decomposition of T through the action R of the so-called symmetry group $\mathfrak{S}_\infty(s)$ of T (cf. §5.1).

Another interesting case is also studied in §§5.7–5.8 and we get Theorem 5.9, where the μ-unital subset E is assumed to satisfy a weaker disjointness condition (wDIS) (see §5.7). These results on dual pairs for $G=\mathrm{Diff}_0(M)$ and certain permutation groups are, in a sense, analogous to Weyl's reciprocity law between k-times tensor product of the natural representation of $GL_n(\mathbf{C})$ and the k-th symmetry group $\mathfrak{S}_k$.

Now take a measure $\omega\in\mathscr{LFM}(M)$ and consider the subgroup $\mathrm{Diff}_0(M\,;\,\omega)$ of $\mathrm{Diff}_0(M)$ consisting of g which preserve the measure ω. Then, in the case where ω has densities of class $C^{(n)}$ with respect to local coordinates, the group $\mathrm{Diff}_0(M;\omega)$ is sufficiently big and contains many elements as is seen in Theorem 6.5. In such a case, similar results as for the whole group $\mathrm{Diff}_0(M)$ can be given. Some of them are given as Theorems 6.7 and 6.9.

We omit historical comments here, but cite simply [9] and [10], along with the classical work [18], among the studies on irreducible unitary representations of diffeomorphism groups.

Let us now explain the organization of this paper.

In §1, we discuss infinite tensor product of Hilbert spaces, and especially pay attention on a normalization of reference vectors in case of L^2-spaces.

In §2, first we discuss a product measure $\nu_{\mu,E}$ on $X=\prod_{j\in\mathbf{N}}X_j$, $X_j=M$, of measures μ_j on $X_j(j\in\mathbf{N})$ with respect to a subset $E=\prod_{j\in\mathbf{N}}E_j\subset X$ satisfying the condition (MU1) (such an E is called μ-unital). Next we discuss a realization of infinite tensor product of $\mathscr{H}_j=L^2(X_j, \mathscr{B}_j, \mu_j)$ with a reference vector $\chi=(\chi_j)_{j\in\mathbf{N}}$ of the form $\chi_j=\|\chi_{E_j}\|^{-1}_{\mathscr{H}_j}\chi_{E_j}$ as L^2-space $L^2(X, \mathscr{M}(\mu, E), \nu_{\mu,E})$ for the product measure. Then we introduce infinite tensor product $\otimes^{\chi}_{j\in\mathbf{N}}T_j$ of representations T_j on $\mathscr{H}_j$ of a group of measurable transformations on M.

In §3, we concentrate ourselves to the case of diffeomorphism group $G=\mathrm{Diff}_0(M)$. At this stage, to get an infinite tensor product $T=\otimes^{\chi}_{j\in\mathbf{N}}T_j^{s_j}$ of representations, we should ask that (μ, E) satisfies one more condition (MU2) in §3.2. We study the G-quasi-invariance of the product measure $\nu_{\mu,E}$. Then we see that, to have such a quasi-invariance in a general setting, it is necessary to choose an appropriate μ-unital subset E', μ-cofinal with E, and to restrict the σ-ring of measurable subsets to much smaller one, and thus we come to a product measure $\nu_{0,\mu,E'}$ to replace $\nu_{\mu,E}$ (for details, see §§ 3.2–3.3).

In §4, we study the subgroup $\mathfrak{S}_{\mu,E,s}$ of $\tilde{\mathfrak{S}}_\infty$ consisting of elements σ which give canonically intertwining operators for the representation T of G. We give some general properties, some interesting examples and propose open problems.

In §5, we establish a dual pair relation between the diffeomorphism group G and a permutaion group $\mathfrak{S}_\infty(s)\subset\mathfrak{S}_\infty$ through representaions T and R, in case where all E_j's are mutually disjoint or in case where $E=\prod_{j\in\mathbf{N}}E_j$ satisfies a weaker disjointness condition (wDIS).

In §6, we study the group of measure preserving diffeomorphisms $\mathrm{Diff}_0(M;\omega)$ and its representations. We obtain some results parallel to the case of the whole group G.

At last, in Appendix, we give, only for completeness, proofs of several facts in the case of finite tensor products of natural representations of G on $L^2(M)$'s.

A part of the results in this paper has been reported in [3] and [6].

Acknowledgements. The authors express their hearty thanks to Prof. K. Nishiyama, Prof. H. Yamashita, Prof. A. Hora and Mr. T. Yoshinaga for their kind discussions and suggestions.

§ 1. Infinite tensor products of L^2-spaces

1.1. Infinite tensor product of Hilbert spaces. Let $(\mathscr{H}_i)_{i\in\mathbf{N}}$ be a countable system of separable Hilbert spaces. We define an infinite tensor product of these Hilbert spaces according to von Neumann [12], and understand it through the interpretation by Guichardet [1]. For this, we take a unit vector $\phi_i\in\mathscr{H}_i$ for each $i\in\mathbf{N}$. Then we form the infinite tensor product $\mathscr{H}^\phi=\otimes_{i\in\mathbf{N}}^{\phi}\mathscr{H}_i$ of Hibert spaces $\mathscr{H}_i$ with a reference vector $\phi=(\phi_i)_{i\in\mathbf{N}}$ as the limit of the inductive system of Hilbert spaces:

$$\otimes_{i=1}^{n}\mathscr{H}_i\ni w\mapsto w\otimes\phi_{n+1}\in\otimes_{i=1}^{n+1}\mathscr{H}_i \qquad (n=1, 2, \ldots).$$

A complete orthonormal base (=CONB), called *standard* with respect to ϕ, is defined as follows. Take a CONB $\phi_{i,j}(j\in\mathbf{N})$ containing ϕ_i as $\phi_{i,1}=\phi_i$, for each $\mathscr{H}_i$, and then form a set of vectors $\otimes_{i\in\mathbf{N}}\phi_{i,j_i}$, where the sequences $(j_i)_{i\in\mathbf{N}}$ of natural numbers run over such ones that $j_i=1$ for almost all (or except a finite number of) $i\in\mathbf{N}$.

A vector $u=\otimes_{i\in\mathbf{N}}u_i$ with $u_i\in\mathscr{H}_i$, for which $\Pi_{i\in\mathbf{N}}\|u_i\|$ is unconditionally convergent, belongs to the space $\otimes_{i\in}^{\phi}\mathscr{H}_i$ if and only if

$$\Sigma_{i\in\mathbf{N}}\mid 1-<u_i, \phi_i>_{\mathscr{H}_i}\mid<+\infty,$$

where $<.,.>_{\mathscr{H}_i}$ denotes the inner product in $\mathscr{H}_i$. The above relation is written as $u\sim\phi$, and this kind of vectors u are called *decomposable*. Note that a product $\Pi_{i\in\mathbf{N}}c_i$, $c_i\in\mathbf{C}$, is called *unconditionally convergent* if $c_i\neq 0(i\in\mathbf{N})$ and $\Sigma_{i\in\mathbf{N}}\mid c_i-1\mid<\infty$.

For two such decomposable vectors $u=\otimes_{i\in\mathbf{N}}u_i$ and $v=\otimes_{i\in\mathbf{N}}v_i$, their inner product is given by

$$<u, v>=\Pi_{i\in\mathbf{N}}<u_i, v_i>_{\mathscr{H}_i}.$$

Note that if $v_i=a_iu_i$ with $a_i\in\mathbf{C}$ for $i\in\mathbf{N}$, then $v\sim u$ means that product $a=\Pi_{i\in\mathbf{N}}a_i$ is unconditionally convergent and $\otimes_{i\in\mathbf{N}}v_i=a\cdot\otimes_{i\in\mathbf{N}}u_i$.

Note further that if $\varphi=(\varphi_i)_{i\in\mathbf{N}}$, $\varphi_i\in\mathscr{H}_i$, $\|\varphi_i\|=1$, satisfies $\varphi\sim\phi$, then the Hilbert spaces $\otimes_{i\in\mathbf{N}}^{\phi}\mathscr{H}_i$ and $\otimes_{i\in\mathbf{N}}^{\varphi}\mathscr{H}_i$ are naturally isomorphic, since, for a vector $u=\otimes_{i\in\mathbf{N}}u_i$, the relation $u\sim\phi$ is equivalent to $u\sim\varphi$. We denote by $I_{\phi,\varphi}$ the natural isomorphism from the former to the latter.

1.2. Case of L^2-spaces. Now let us discuss the case where each space $\mathscr{H}_i$ is an L^2-space. Let $(X_i, \mathscr{B}_i, \mu_i)$, $i\in\mathbf{N}$, be measure spaces, where $\mathscr{B}_i$

denotes a σ-ring of subsets of a space X_i on which a measure μ_i is defined. According to Halmos, $\mathscr{B}_i$ is a σ-ring if $A_k\in\mathscr{B}_i(k\in\mathbf{N})$, then $\cup_{k\in\mathbf{N}}A_k\in\mathscr{B}_i$, and if A, $B\in\mathscr{B}_i$, then $A\backslash B\in\mathscr{B}_i$. Put $\mathscr{H}_i=L^2(X_i, \mathscr{B}_i, \mu_i)$, the Hilbert space of L^2-functions on X_i. We assume in this paper that these L^2-spaces are all separable.

For the infinite tensor products of L^2-spaces, we have several natural isomorphisms in addition to $I_{\phi,\phi}$. Firstly put for $\phi_i\in\mathscr{H}_i=L^2(X_i, \mathscr{B}_i, \mu_i)$,

$$\xi_i(p)=\begin{cases}\phi_i(p)/\,|\,\phi_i(p)\,| & \text{for } p\in\operatorname{supp}'(\phi_i),\\ 1 & \text{for } p\notin\operatorname{supp}'(\phi_i),\end{cases}$$

where $\operatorname{supp}'(f)=\{p\in X_i \mid f(p)\neq 0\}$ for a measurable function f on X_i. Further put $\eta_i=|\,\phi_i\,|$ and

$$\eta_i'(p)=\begin{cases}\eta_i(p) & \text{for } p\in E_i,\\ 1 & \text{for } p\notin E_i,\end{cases}$$

with $E_i=\operatorname{supp}'(\phi_i)$. Denote by $\chi_i=\chi_{E_i}$ the indicator function of the set E_i. Then,

$$\phi_i=\xi_i\cdot\eta_i, \quad \eta_i=\eta_i'\cdot\chi_i \qquad (i\in\mathbf{N}).$$

Put $\xi=(\xi_i)_{i\in\mathbf{N}}$, $\eta=(\eta_i)_{i\in\mathbf{N}}$, and $\chi=(\chi_i)_{i\in\mathbf{N}}$, then we can write the above relations symbolically as $\phi=\xi\cdot\eta$, $\eta=\eta'\cdot\chi$.

Let M_{ξ_i} be the operator of multiplication by ξ_i on the space $\mathscr{H}_i=L^2(X_i, \mathscr{B}_i, \mu_i)$. Then it sends η_i to ϕ_i, and therefore we have a natural unitary operator $M^\xi:=\otimes_{i\in\mathbf{N}}M_{\xi_i}$ from the Hilbert space $\otimes^{\eta}_{i\in\mathbf{N}}\mathscr{H}_i$ onto $\otimes^{\phi}_{i\in\mathbf{N}}\mathscr{H}_i$, which sends a decomposable vector $\otimes_{i\in\mathbf{N}}u_i$ to $\otimes_{i\in\mathbf{N}}(\xi_i\cdot u_i)$.

To give another natural isomorphism between the tensored Hilbert spaces, let us define a new set of measures $(\mu_i')_{i\in\mathbf{N}}$ as

$$d\mu_i'(p)=\eta_i'(p)^2\cdot d\mu_i(p) \qquad (p\in X_i).$$

Then, the multiplication operator $M_{\eta_i'}$ sends the vector χ_i in $\mathscr{H}_i'=L^2(X_i, \mathscr{B}_i, \mu_i')$ to η_i in $\mathscr{H}_i=L^2(X_i, \mathscr{B}_i, \mu_i)$, and they give naturally a unitary operator $M^{\eta'}:=\otimes_{i\in\mathbf{N}}M_{\eta_i'}$ from $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i'$ onto $\otimes^{\eta}_{i\in\mathbf{N}}\mathscr{H}_i$.

Note that $\chi_i=\chi_{E_i}$ with $\mu_i'(E_i)=1$. Then, for the tensored Hilbert space $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i'$, we can give a realization of it as an L^2-space, with respect to an infinite product of measures μ_i' on $X_i(i\in\mathbf{N})$ which is defined with reference to the system of sets $(E_i)_{i\in\mathbf{N}}$. This is done in the next section.

1.3. Tensor products of linear operators. Let T_i be a bounded linear operator on $\mathscr{H}_i$ for each $i\in\mathbf{N}$. We ask under what condition a tensor product $\otimes_{i\in\mathbf{N}}T_i$ can be defined as a bounded linear operator on the tensored space $\otimes_{i\in\mathbf{N}}^{\phi}\mathscr{H}_i$, where $\phi=(\phi_i)_{i\in\mathbf{N}}$ is the reference vector. To give a useful sufficient condition, we introduce the following definition.

Definition 1.1. A sequence $(u_i)_{i\in\mathbf{N}}$ of vectors $u_i\in\mathscr{H}_i\,(i\in\mathbf{N})$ is said to be *cofinal* with the reference vector ϕ if the product $\Pi_{i\gg 0}\|u_i\|$ is unconditionally convergent and

$$\Sigma_{i\in\mathbf{N}}\,|\,1-\langle u_i,\,\phi_i\rangle_{\mathscr{H}_i}\,|<+\infty.$$

Lemma 1.1. *Let T_i be a bounded linear operator on $\mathscr{H}_i\,(i\in\mathbf{N})$. Assume that $\Pi_{i\gg 0}\|T_i\|$ is unconditionally convergent, and that $(T_i\phi_i)_{i\in\mathbf{N}}$ is cofinal with the reference vector ϕ. Then a tensor product $T=\otimes_{i\in\mathbf{N}}T_i$ is well defined on $\otimes_{i\in\mathbf{N}}^{\phi}\mathscr{H}_i$ in such a way that*

$$u=\otimes_{i\in\mathbf{N}}u_i\mapsto Tu=\otimes_{i\in\mathbf{N}}(T_iu_i)$$

for any decomposable vector $u=\otimes_{i\in\mathbf{N}}u_i$ in $\otimes_{i\in\mathbf{N}}^{\phi}\mathscr{H}_i$.

Proof. Let $T^{(n)}$ be a linear operator of the subspace $\otimes_{i=1}^{n}\mathscr{H}_i$ to the whole space $\otimes_{i\in\mathbf{N}}^{\phi}\mathscr{H}_i$ given as

$$\otimes_{i=1}^{n}\mathscr{H}_i\ni w\mapsto(\otimes_{i=1}^{n}T_i)w\otimes(\otimes_{i>n}(T_i\phi_i))\in\otimes_{i\in\mathbf{N}}^{\phi}\mathscr{H}_i.$$

Then, the system of $(T^{(n)})_{n\in\mathbf{N}}$ is consistent with the inductive system $(\otimes_{i=1}^{n}\mathscr{H}_i)_{n\in\mathbf{N}}$ and further

$$\|T^{(n)}\|\leq\Pi_{i\in\mathbf{N}}\|T_i\|<\infty.$$

This means that the inductive system $(T^{(n)})_{n\in\mathbf{N}}$ defines a linear operator T and $\|T\|\leq\Pi_{i\in\mathbf{N}}\|T_i\|$.

§ 2. Infinite products of measures and their L^2-spaces

2.1. Definition of infinite products. Let $(X_i,\,\mathscr{B}_i,\,\mu_i)$, $i\in\mathbf{N}$, be measure spaces as in § 1.2. To define an infinite product of measures $(\mu_i)_{i\in\mathbf{N}}$ on the product space $X=\Pi_{i\in\mathbf{N}}X_i$, we first fix a system $(E_i)_{i\in\mathbf{N}}$ of mesurable sets $E_i\in\mathscr{B}_i$, with reference to which the infinite product is defined. We put $\mu=(\mu_i)_{i\in\mathbf{N}}$ and introduce some definitions.

Definition 2.1. A direct product subset $E=\Pi_{i\in\mathbf{N}}E_i$ of X with $E_i\in\mathscr{B}_i$ is called *μ-unital* if the product $\Pi_{i\in\mathbf{N}}\mu_i(E_i)$ is unconditionally convergent.

According to the definition, $E=\Pi_{i\in\mathbf{N}}E_i$ is *μ-unital* if the following condition holds:

$$\text{(MU1)}\qquad\begin{cases}0<\mu_i(E_i)<+\infty, & (\forall i)\\ \Sigma_{i\in\mathbf{N}}\,|\,1-\mu_i(E_i)\,|<+\infty.\end{cases}$$

Definition 2.2. Two unital subsets $E=\prod_{i\in\mathbf{N}}E_i$ and $F=\prod_{i\in\mathbf{N}}F_i$ of X are said to be *μ-cofinal* (Notation: $E\overset{\mu}{\sim}F$) if $\sum_{i\in\mathbf{N}}\mu_i(E_i\ominus F_i)<+\infty$. They are *strongly cofinal* (Notation $E\approx F$) if $E_i=F_i$ for $i\gg 0$ (i. e., for sufficiently large i).

Let $\mathcal{M}_0(E)$ (resp. $\mathcal{M}(\mu, E)$) be the σ-ring of subsets of X which is generated by the family $\mathcal{E}_0(E)$ (resp. $\mathcal{E}(\mu, E)$) of unital subsets F such that $F\approx E$ (resp. $F\overset{\mu}{\sim}E$). We define a product measure $\nu_{0,\mu,E}$ (resp. $\nu_{\mu,E}$) by patching together the standard product measure $\prod_{i\in\mathbf{N}}(\mu_i|F_i)$ on each $F=\prod_{i\in\mathbf{N}}F_i$ in $\mathcal{E}_0(E)$ (resp. in $\mathcal{E}(\mu, E)$), where $\mu_i|F_i$ denotes the restriction of μ_i on F_i. Then we have

$$\mathcal{M}(\mu, E)=\cup_{F\overset{\mu}{\sim}E}\mathcal{M}_0(F), \qquad \nu_{0,\mu,E}=\nu_{\mu,E}|\mathcal{M}_0(E).$$

Furtheremore it may be considered that the mesure $\nu_{\mu,E}$ on $\mathcal{M}(\mu, E)$ is a kind of completion of the one $\nu_{0,\mu,E}$ on $\mathcal{M}_0(E)$. In fact, any $F=\prod_{i\in\mathbf{N}}F_i$ in $\mathcal{E}(\mu, E)$ can be approximated, with respect to $\nu_{\mu,E}$, by a series of elements in $\mathcal{M}_0(E)$ as shown below. Put, for $N\in\mathbf{N}$, $F^{(N)}=(\prod_{i=1}^N F_i)\times(\prod_{i>N}E_i)$, then $F^{(N)}\in\mathcal{M}_0(E)$, and $F\cap F^{(N)}=(\prod_{i=1}^N F_i)\times(\prod_{i>N}F_i\cap E_i)$, and therefore

$$\begin{aligned}\nu_{\mu,E}(F\ominus F^{(N)}) &\leq\Big(\prod_{i=1}^{N}\mu_i(F_i)\Big)\Big\{\prod_{i>N}\mu_i(F_i\cup E_i)-\prod_{i>N}\mu_i(F_i\cap E_i)\Big\}\\ &\to\Big(\prod_{i\in\mathbf{N}}\mu_i(F_i)\Big)\{1-1\}=0.\end{aligned}$$

Concerning the relationship between two direct product mesures such as $(\nu_{\mu,E}, \mathcal{M}(\mu, E))$, we have the following

Lemma 2.1. *Suppose that two unital subsets E and E′ are not μ-cofinal. Then, for any subset A in $\mathcal{M}(\mu, E)\cap\mathcal{M}(\mu, E')$,*

$$\nu_{\mu,E}(A)=\nu_{\mu,E'}(A)=0.$$

2.2. Relation to infinite tensor products of L^2-spaces. Let us consider an infinite tensor product of L^2-spaces $\mathcal{H}_i=L^2(X_i, \mathcal{B}_i, \mu_i)$ and study a relation to an infinite product of measures μ_i.

Take a unital subset $E=\prod_{i\in\mathbf{N}}E_i$ of $X=\prod_{i\in\mathbf{N}}X_i$. Then $\chi_i=\|\chi_{E_i}\|_{\mathcal{H}_i}^{-1}\chi_{E_i}$ is a unit vector of $\mathcal{H}_i$. So we get a tensor product $\otimes_{i\in\mathbf{N}}^{\chi}\mathcal{H}_i$ of Hilbert spaces, with reference vector $\chi=(\chi_i)_{i\in\mathbf{N}}$.

On the other hand, we have product measures $(X, \mathcal{M}_0(E), \nu_{0,\mu,E})$ and $(X, \mathcal{M}(\mu, E), \nu_{\mu,E})$ for $\mu=(\mu_i)_{i\in\mathbf{N}}$ with respect to the unital subset E. Hence we obtain two L^2-spaces $L^2(X, \mathcal{M}_0(E), \nu_{0,\mu,E})$ and $L^2(X, \mathcal{M}(\mu, E), \nu_{\mu,E})$, which are naturally isomorphic, because the latter measure is a 'completion' of the former one. However the expression $L^2(X, \mathcal{M}_0(E), \nu_{0,\mu,E})$ is the most intimately related one to the tensor product $\otimes_{i\in\mathbf{N}}^{\chi}\mathcal{H}_i=\otimes_{i\in\mathbf{N}}^{\chi}L^2(X_i, \mathcal{B}_i, \mu_i)$.

In fact, a natural isomorphism from the latter to the former can be given as follows. Arbitrary element of a standard basis of $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$ is of the form $f=(\otimes^{N}_{i=1}f_i)\otimes(\otimes_{i>N}\chi_i)$ with $f_i\in\mathscr{H}_i=L^2(X_i,\ \mathscr{B}_i,\ \mu_i)$. Since $\chi_i=\|\chi_{E_i}\|^{-1}_{\mathscr{H}_i}\chi_{E_i}$, and the product $\Pi_{i>N}\|\chi_{E_i}\|^{-1}_{\mathscr{H}_i}=c_N$ (put) is convergent, the vector f can be interpreted as a function on X according to the expression

$$(\Pi^{N}_{i=1}f_i(p_i))\times(c_N\cdot\Pi_{i>N}\chi_{E_i}(p_i))\quad\text{for } x=(p_i)_{i\in\mathbf{N}}\in X.$$

Denote this function on X by Uf, then Uf is clearly measurable with respect to $\mathscr{M}_0(E)$ and belongs to the L^2-space $L^2(X,\ \mathscr{M}_0(E),\ \nu_{0,\mu,E})$. Let $\mathscr{H}(E)$ denote the linear span of vectors in $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$ of the form $(\otimes^{N}_{i=1}f_i)\otimes(\otimes_{i>N}\chi_i)$ with $f_i\in\mathscr{H}_i$. Then this is a dense subspace containing the CONS and U is defined on it as a linear map.

Lemma 2.2. *The above map U on $\mathscr{H}(E)\subset\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$ is uniquely extended to a unitary operator and it gives a natural isomorphism between two Hilbert spaces as*

$$U:\ \otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i\longrightarrow L^2(X,\ \mathscr{M}_0(E),\ \nu_{0,\mu,E})\cong L^2(X,\ \mathscr{M}(E),\ \nu_{\mu,E}),$$

where $\mathscr{H}_i=L^2(X_i,\ \mathscr{B}_i,\ \mu_i)$ and $\chi_i=\|\chi_{E_i}\|^{-1}_{\mathscr{H}_i}\chi_{E_i}$.

Let us now consider a non-zero decomposable element of $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$ of the form $\otimes_{i\in\mathbf{N}}\chi_{F_i}$ with $F_i\in\mathscr{B}_i$. Then we see that $F=\Pi_{i\in\mathbf{N}}F_i$ should be a unital subset of X, and be μ-cofinal with E. In fact, to see $F\overset{\mu}{\sim}E$, we check the criterion for $\otimes_{i\in\mathbf{N}}\chi_{F_i}\in\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$, that is,

$$\Sigma_{i\in\mathbf{N}}\ |\ 1-\langle\chi_{F_i},\ \chi_i\rangle_{\mathscr{H}_i}|\ <\infty.$$

Since $\chi_i=\|\chi_{E_i}\|^{-1}_{\mathscr{H}_i}\chi_{E_i}$ and the product $\Pi_{i\in\mathbf{N}}\|\chi_{E_i}\|_{\mathscr{H}_i}$ is convergent, the above inequality is equivalent to

$$\Sigma_{i\in\mathbf{N}}\ |\ 1-\langle\chi_{F_i},\ \chi_{E_i}\rangle_{\mathscr{H}_i}|\ <\infty.$$

This in turn is equivalent to $\Sigma_{i\in\mathbf{N}}\mu_i(E_i\ominus F_i)<\infty$, or $F\overset{\mu}{\sim}E$, because F and E are both μ-unital.

Note that if we make correspond to $f=\otimes_{i\in\mathbf{N}}\chi_{F_i}\in\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$ a function on X as $f'(x)=\Pi_{i\in\mathbf{N}}\chi_{F_i}(p_i)$ for $x=(p_i)_{i\in\mathbf{N}}\in X$, then it is $\mathscr{M}(\mu,\ E)$-measurable but not necessarily $\mathscr{M}_0(E)$-measurable.

2.3. Group of measurable transformations. Let G be a topological group consisting of measurable transformations g, which act on each X_i in such a way that the transformed measure ${}^g\mu_i$ is equivalent to μ_i, where ${}^g\mu_i(A):=\mu_i(g^{-1}A)$ $(A\in\mathscr{B}_i)$. From now on we assume that all the measures appearing are σ-finite, so that there exists the Jacobian between mutually equivalent measures.

We have a unitary operator $T^{\alpha_i}_i(g)$ on $\mathscr{H}_i=L^2(X_i,\ \mathscr{B}_i,\ \mu_i)$ given as

$$T_i^{\alpha_i}(g)f(p)=\alpha_i(g,\, p)\sqrt{\frac{d\mu_i(g^{-1}p)}{d\mu_i(p)}}f(g^{-1}p) \quad (f\in\mathscr{H}_i,\ p\in X_i).$$

Here $\alpha_i(g, p)$ is a so-called 1-cocycle, i. e., a function on $G\times X_i$, measurable in $p\in X_i$ and satisfying $|\alpha_i(g, p)|=1$, and for $g_1, g_2\in G$,

$$\alpha_i(g_ig_2,\, p)=\alpha_i(g_1,\, p)\alpha_i(g_2,\, g_1^{-1}p) \quad \text{for a. a. } p\in X_i.$$

Then we have $T_i^{\alpha_i}(g_1g_2)=T_i^{\alpha_i}(g_1)T_i^{\alpha_i}(g_2)$ for $g_1, g_2\in G$. Let ξ be a measurable function on X_i such that $|\xi(p)|=1\ (p\in X_i)$, and M_ξ the multiplication operator on $\mathscr{H}_i$ as in §1.2, then the transformed operator $M_\xi T_i^{\alpha_i}(g)M_\xi^{-1}$ has a similar form as $T_i^{\alpha_i}(g)$ with a different but equivalent 1-cocycle.

To have a representation of G, we need the continuity $G\ni g\mapsto T_i^{\alpha_i}(g)$. For this we have the following necessary and sufficient condition.

Lemma 2. 3. *Let $\mathscr{A}$ be a subfamily of $\mathscr{B}_i$ consisting of elements with finite measures. Assume that every element in $\mathscr{B}_i$ with a finite measure can be approxinated by elements of the σ-ring generated by $\mathscr{A}$. Then the map $g\mapsto T_i^{\alpha_i}(g)$ is continuous if and only if the following conditions hold: for any fixed $A\in\mathscr{A}$, and as $g\to e$,*

(a) $$\int_M \left|\sqrt{\frac{d\mu_i(g^{-1}p)}{d\mu_i(p)}}\chi_A(g^{-1}p)-\chi_A(p)\right|^2 d\mu_i(p)\to 0,$$

(b) $$\int_A \left|\alpha_i(g,\, p)-1\right|^2 d\mu_i(p)\to 0.$$

Furthermore the condition (b) is equivalent to the continuity in probability of the map $g\to\alpha_i(g, \cdot)$ at $g=e$, that is,

(b′) *for a fixed finite measure $\omega\simeq\mu_i$ on M,*

$$\forall\varepsilon>0 \textit{ fixed}, \quad \omega(\{p\,;\,|\alpha_i(g,\, p)-1|>\varepsilon\})\to 0\ (g\to e).$$

Proof. We prove here only the necessity of the conditions (a) and (b). The continuity of the representation $T_i^{\alpha_i}$ is equivalent to $\Phi(g):=\|T_i^{\alpha_i}(g)\chi_A-\chi_A\|_{L^2(\mu_i)}\longrightarrow 0\ (g\to e)$ for any $A\in\mathscr{A}$. Taking into account $|\alpha_i|\equiv 1$ and $||a|-|b||\le|a-b|$, we obtain from $\Phi(g)\to 0$,

$$\Psi(g):=\int_M \left|\sqrt{\frac{d\mu_j(g^{-1}p)}{d\mu_j(p)}}\chi_A(g^{-1}p)-\chi_A(p)\right|^2 d\mu_i(p)\to 0\ (g\to e).$$

From these two formulas, we have

$$\int_A |\alpha_i(g, p)-1|^2 d\mu_i(p) = \|\alpha_i\chi_A - \chi_A\|^2 \leq$$
$$\leq 2\|T_i^{\alpha_i}(g)\chi_A - \chi_A\|^2 + 2\|T_i^{\alpha_i}(g)\chi_A - \alpha_i\chi_A\|^2 =$$
$$= 2\Phi(g)^2 + 2\Psi(g) \to 0 \quad (g \to e).$$

Let us now prove the equivalence of (b) and (b′). For (b)⇒(b′), it is enough to approximate the density function $\frac{d\omega}{d\mu_i}(p)$ on M by linear combinations of χ_A, $A \in \mathscr{A}$.

For (b′)⇒(b), put $A_{g,\varepsilon} := \{p \in M; |\alpha_i(g, p)-1| > \varepsilon\}$. Then

$$\int_A |\alpha_i(g, p)-1|^2 d\mu_i(p) \leq \varepsilon^2 \mu_i(A) + 4\mu_i^A(A_{g,\varepsilon}),$$

where μ_i^A denotes the restriction of μ_i onto A : $\mu_i^A(B) = \mu_i(B \cap A)$. Since μ_i^A is a finite measure and $\mu_i^A \leq \omega$, we obtain $\mu_i^A(A_{g,\varepsilon}) \to 0$ from $\omega(A_{g,\varepsilon}) \to 0$. This gives (b). Q. E. D.

Note that the condition (a) in Lemma 2. 3 means the continuity of the representation in case of the trivial 1-cocycle: $\alpha_i \equiv 1$.

We have also the next sufficient condition.

Lemma 2. 4. *Let $\mathscr{A} \subset \mathscr{B}_i$ be as in Lemma 2. 3. Then the map $g \mapsto T_i^{\alpha_i}(g)$ is continuous if the following three functions in g are continuous at $g=e$ for any fixed $A \in \mathscr{A}$*:

$$\mu_i(A \ominus gA), \quad \int_A \left| \sqrt{\frac{d\mu_i(g^{-1}p)}{d\mu_i(p)}} - 1 \right| d\mu_i(p), \quad \int_A |\alpha_i(g, p)-1| \, d\mu_i(p).$$

Proof. It is sufficient for us to prove the continuity at $g=e$ of the map

$$G \ni g \mapsto \langle T_i^{\alpha_i}(g)\chi_A, \chi_B \rangle (=: \Phi(g)),$$

for any $A, B \in \mathscr{A}$. Then, the difference $|\Phi(g) - \Phi(e)|$ is majorized by the sum

$$\mu_i(A \ominus gA) + \int_B \left| \sqrt{\frac{d\mu_i(g^{-1}p)}{d\mu_i(p)}} - 1 \right| d\mu_i(p) + \int_B |\alpha_i(g, p)-1| \, d\mu_i(p).$$

For a particular case, we have a simple criterion for continuity as follows.

Lemma 2. 5. *Assume that the indicator function χ_{E_i} of a set $E_i \in \mathscr{B}_i$, with $0 < \mu_i(E_i) < \infty$, is cyclic under G, that is, the set of vectors $\{T_i^{\alpha_i}(g)\chi_{E_i} \mid g \in G\}$ is total in $\mathscr{H}_i$. Then the map $g \mapsto T_i^{\alpha_i}(g)$ is continuous if the function*

$$<T_i^{\alpha_i}(g)\chi_{E_i}, \chi_{E_i}>_{\mathscr{H}_i}=\int_{E_i\cap gE_i}\alpha_i(g, p)\sqrt{\frac{d\mu_i(g^{-1}p)}{d\mu_i(p)}}d\mu_i(p)$$

is continuous in g.

Now let us examine a condition for the existence of the product $\otimes_{i\in\mathbf{N}}^{\chi}T_i^{\alpha_i}(g)$ as an operator on $\otimes_{i\in\mathbf{N}}^{\chi}\mathscr{H}_i$, where $\chi=(\chi_i)_{i\in\mathbf{N}}$, $\chi_i=\|\chi_{E_i}\|_{\mathscr{H}_i}^{-1}\chi_{E_i}$, be as in §2.2. By Lemma 1.1, a necessary and sufficient condition is given for the existence of the product as

$$\sum_{i\in N}|1-<T_i^{\alpha_i}(g)\chi_i, \chi_i>|<\infty. \tag{2.1}$$

The left hand side is majorized by a constant multiple of

$$\sum_{i\in\mathbf{N}}\int_{E_j}\left|1-\alpha_i(g, p)\sqrt{\frac{d\mu_i(g^{-1}p)}{d\mu_i(p)}}\chi_{E_i}(g^{-1}p)\right|d\mu_i(p)$$
$$\leq C\sum_{i\in\mathbf{N}}\|\chi_{E_i}-T_i^{\alpha_i}(g)\chi_{E_i}\|_{L^2(E_i)},$$

where $C>0$ is a constant and $\|f\|_{L^2(E_i)}=\|f|E_i\|_{L^2(X_i, \mu_i)}$.

According to the natural isomorphisms of Hilbert spaces, $\otimes_{i\in\mathbf{N}}^{\chi}\mathscr{H}_i\cong L^2(X, \mathscr{M}_0(E), \nu_{0, \mu, E})\cong L^2(X, \mathscr{M}(\mu, E), \nu_{\mu, E})$, we can give, from another point of view, a sufficient condition for that a transformation of each $g\in G$ can be given on the Hilbert space. This is nothing but the so-called absolute continuous action of G on the product measures.

Lemma 2.6. *Assume that the σ-ring $\mathscr{M}_0(E)$ (resp. $\mathscr{M}(\mu, E)$) is invariant under G, and that the measure ${}^g\nu$ transformed by $g\in G$ of $\nu=\nu_{0, \mu, E}$ (resp. $=\nu_{\mu, E}$) is absolutely continuous. Then we have a unitary operator on the corresponding L^2-space $\mathscr{H}$ as*

$$f\mapsto\alpha(g, x)\sqrt{\frac{d\nu(g^{-1}x)}{d\nu(x)}}f(g^{-1}x)=: T^{\alpha}(g)f(x)\quad(x\in X, f\in\mathscr{H}),$$

for $g\in G$, where $\alpha(g, x)$ is a 1-cocycle. The case where $\alpha(g, x)=\Pi_{i\in\mathbf{N}}\alpha_i(g, p_i)$ for $x=(p_i)_{i\in\mathbf{N}}$, which is assumed to exist ν-almost everywhere, corresponds to the case of tensor products.

To get a tensor product of representations $T_i^{\alpha_i}$ of G, we should have the continuity of the map $g\mapsto\otimes_{i\in\mathbf{N}}^{\chi}T_i^{\alpha_i}(g)$, which is not automatic from that of each map $g\mapsto T_i^{\alpha_i}(g)$ $(i\in\mathbf{N})$. From the view point of Lemma 2.6, the continuity needed is that of the map $G\ni g\mapsto T^{\alpha}(g)f\in\mathscr{H}$ for any fixed $f\in\mathscr{H}$.

These representations will be discussed more in detail for cetain choice of G, in the next section.

§ 3. Tensor products of natural representations of diffeomorphism groups

3.1. Diffeomorphism groups and their natural representations. Let M be a non-compact, σ-compact, connected differential manifold of class $C^{(n)}$, $1\leq n\leq\infty$, and $G=\mathrm{Diff}_0(M)$ the group of all diffeomorphisms g on M such that the support $\mathrm{supp}(g):=\mathrm{Cl}\{p\in M \mid gp\neq p\}$ is compact. We introduce a so-called $C^{(n)}$-topology in G and consider its unitary representations. By definition, a net g_β in G converges to $g\in G$ if $\mathrm{supp}(g_\beta)$ are contained in a fixed compact set and every differential of g_β converges to that of g.

Denote by $\mathscr{LFM}(M)$ the set of all locally finite (i. e., finite on every compact subset) measures on M which are equivalent locally to a Lebesque measure with respect to the local co-ordinates.

For $i\in\mathbf{N}$, put $X_i=M$ and take a measure $\mu_i\in\mathscr{LFM}(M)$ on it. Then we call a *natural representation* of G the representation of the form $T_i^{\alpha_i}$ on the L^2-space $\mathscr{H}_i=L^2(X_i,\ \mathscr{B}_i,\ \mu_i)$, where $\mathscr{B}_i$ is the σ-algebra of all Lebesgue measurable subsets of X_i. Let us study when and how a tensor product of these representations can be defined.

3.2. Reference vectors for tensor product representations. To form a tensor product Hilbert space of $\mathscr{H}_i(i\in\mathbf{N})$, we fix a reference vector $\phi=(\phi_i)_{i\in\mathbf{N}}$. Note that under the replacement in § 1.2 of ϕ_i by χ_i, and μ_i by μ_i', the representation $T_i^{\alpha_i}$ on $L^2(X_i,\ \mathscr{B}_i,\ \mu_i)$ is transformed to a similar one $T_i^{\alpha_i'}$ on $L^2(X_i,\ \mathscr{B}_i,\ \mu_i')$ with another 1-cocycle α_i'. Therefore, taking also into account the results in §§ 2.1–2.2, we may assume from the beginning that the reference vector is of the form $\chi=(\chi_i)_{i\in\mathbf{N}}$, $\chi_i=\|\chi_{E_i}\|_{\mathscr{H}_i}^{-1}\chi_{E_i}\in\mathscr{H}_i=L^2(X_i,\ \mathscr{B}_i,\ \mu_i)$, with a μ-unital product subset $E=\prod_{i\in\mathbf{N}}E_i$ of $X=\prod_{i\in\mathbf{N}}X_i$, where $\mu=(\mu_i)_{i\in\mathbf{N}}$ as before.

Let us consider which conditions should be put on the μ-unital set E to get a tensor product representation of G.

We introduce the following condition on (μ, E):

(MU2) for any compact subset K of M, $\sum_{i\in\mathbf{N}}\mu_i(K\cap E_i)<\infty$.

As in § 2, denote by $\mathscr{H}(E)$ the linear span of vectors of $\otimes_{i\in\mathbf{N}}^{\chi}\mathscr{H}_i$ of the form

$$(\otimes_{i\leq N}f_i)\otimes(\otimes_{i>N}\chi_i)=\mathrm{const.}\ (\otimes_{i\leq N}f_i)\otimes(\otimes_{i>N}\chi_{E_i})$$

for some $N>0$ with $f_i\in\mathscr{H}_i=L^2(X_i,\ \mathscr{B}_i,\ \mu_i)$. Then $\mathscr{H}(E)$ is a dense subspace containing a CONS which is standard with respect to $\chi=(\chi_i)_{i\in\mathbf{N}}$ or rather to $E=\prod_{i\in\mathbf{N}}E_i$. Let us define, for $g\in G$, a map from $\mathscr{H}(E)$ to $\otimes_{i\in\mathbf{N}}^{\chi}\mathscr{H}_i$ as

$$\otimes_{i\in\mathbf{N}}^{\chi}f_i\longmapsto\otimes_{i\in\mathbf{N}}(T_i^{\alpha_i}(g)f_i),$$

where $f_i \in \mathscr{H}_i$ and $f_i = \chi_i\ (i \gg 1)$. Suppose that $\alpha_i(g, p) = 1$ if $p \notin \mathrm{supp}(g)$. Then we see that, under the condition (MU2) above, the element in the right hand side is actually in $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$, and the above formula defines a unitary operator on $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$, denoted by $T^{\tilde{\alpha}}_E(g)$ with $\tilde{\alpha} = (\alpha_i)_{i\in\mathbf{N}}$. In fact, by Lemma 1.1, it is sufficient to check that

$$\sum_{i\in\mathbf{N}} \mid 1 - < T^{\alpha_i}_i(g)\chi_i, \chi_i > \mid = \sum_{i\in\mathbf{N}} \mid < \chi_i - T^{\alpha_i}_i(g)\chi_i, \chi_i > \mid < \infty.$$

This is equivalent to $\sum_{i\in\mathbf{N}} \mid < \chi_{E_i} - T^{\alpha_i}_i(g)\chi_{E_i}, \chi_{E_i} > \mid < \infty$. Put $K_g = \mathrm{supp}(g)$. Then, since $\alpha_i(g, p) = 1 (p \notin K_g)$ by assumption, each term is evaluated by

$$\begin{aligned}
&\int_{K_g} \left| \chi_{E_i}(p) - \alpha_i(g, p)\sqrt{\frac{d\mu_j(g^{-1}p)}{d\mu_j(p)}}\chi_{E_i}(g^{-1}p) \right| \cdot \chi_{E_i}(p)d\mu_i(p) \\
&\leq \mu_i(K_g \cap E_i) + \left\| \sqrt{\frac{d\mu_j(g^{-1}p)}{d\mu_j(p)}}\chi_{E_i}(g^{-1}p) \cdot \chi_{E_i}(p) \right\|_{L^2(K_g)} \cdot \| \chi_{E_i} \|_{L^2(K_g)} \\
&\leq \mu_i(K_g \cap E_i) + \mu_i(K_g \cap E_i \cap g^{-1}E_i)^{1/2}\mu_i(K_g \cap E_i)^{1/2} \leq 2\mu_i(K_g \cap E_i),
\end{aligned}$$

where $\| \cdot \|_{L^2(K_g)}$ denotes the norm in $L^2(K_g, \mu_i | K_g)$. Thus the condition (MU2) guarantees the convergence of the infinite sum in question.

Now we state the following

Theorem 3.1. *Assume that a μ-unital subset E satisfies, together with $\mu = (\mu_i)_{i\in\mathbf{N}}$, the conditions (MU1)–(MU2), and that $\alpha_i(g, p) = 1$ if $p \notin \mathrm{supp}(g)$. Then a system of unitary representations $(T^{\alpha_i}_i, \mathscr{H}_i)$, $\mathscr{H}_i = L^2(X_i, \mathscr{B}_i, \mu_i)$, gives naturally a tensor product representation $T^{\tilde{\alpha}}_E$, with $\tilde{\alpha} = (\alpha_i)_{i\in\mathbf{N}}$, on the tensored Hilbert space $\otimes^{\chi}_{i\in\mathbf{N}}\mathscr{H}_i$.*

By Lemma 1.1, we can prove $T^{\tilde{\alpha}}_E(g_1g_2) = T^{\tilde{\alpha}}_E(g_1)T^{\tilde{\alpha}}_E(g_2)\ (g_1, g_2 \in G)$. To prove the strong continuity of the map $G \ni g \mapsto T^{\tilde{\alpha}}_E(g)$, and also to write down more neatly the operator $T^{\tilde{\alpha}}_E(g)$, we apply the following

Lemma 3.2. *Under the conditions (MU1)–(MU2), there exists a μ-unital subset $E' = \prod_{i\in\mathbf{N}} E'_i$ which satisfies $E' \overset{\mu}{\sim} E$ and the condition*

(MU2str) *for any compact subset K of M, $K \cap E'_i = \phi\ (i \gg 0)$.*

Furthermore E' can be so chosen that each E'_i is a relatively compact, open subset, and moreorer is connected in case $\dim M \geq 2$.

Proof. Step 1. Since M is σ-compact, there exists an increasing sequence of relatively compact, open subset U_k, $k \in \mathbf{N}$, such that $\cup_{k\in\mathbf{N}} U_k = M$. Put $K_k = \mathrm{Cl}(U_k)$. Choose an increasing sequence of integers $N_k (k \in \mathbf{N})$ such that

$$E_i \backslash K_k \neq \phi \ (i > N_k) \ \text{ and } \ \sum_{i>N_k} \mu_i(E_i \cap K_k) < 2^{-k}.$$

and put $E'_i = E_i \backslash \cup_{k:i>N_k} K_k$. Note that $\{k : i > N_k\}$ is finite. We have, for any k, $E'_i \cap K_k = \phi$ if $i > N_k$, and further $\phi \neq E'_i \subset E_i$ and $E_i \backslash E'_i = \cup_{k:i>N_k}(E_i \cap K_k)$. So we have $E' \overset{\mu}{\sim} E$ because

$$\sum_{i \in \mathbf{N}} \mu_i(E_i \ominus E'_i) \leq \sum_{i \in \mathbf{N}} \sum_{k:i>N_k} \mu_i(E_i \cap K_k)$$

$$\leq \sum_{k \in \mathbf{N}} \sum_{i>N_k} \mu_i(E_i \cap K_k) \leq \sum_{k \in \mathbf{N}} 2^{-k} = 1.$$

STEP 2. To see that the additional conditions can be put on E', we renormalize E' above satisfying (MU2str). Assume, in case $\dim M \geq 2$, that E' is obtained using such a sequence U_k that $M \backslash K_k$ is always connected.

Put $K_k = \phi$ for $k = 0$, and for each $i \in \mathbf{N}$, let J_i be the maximum of $\{k \in \mathbf{N}\,;\, K_k \cap E'_i = \phi\}$. Put $D_i := M \backslash K_{J_i}$, then $D_i \supset E'_i$, and so there exists a relatively compact, open subset E''_i of D_i such that $\mu_i(E'_i \ominus E''_i) < 2^{-i}$. In case $\dim M \geq 2$, since D_i is connected, E''_i can be chosen as connected. Put $E'' = \prod_{i \in \mathbf{N}} E''_i$, then $E'' \overset{\mu}{\sim} E$, and E'' satisfies the additional conditions demanded. Q. E. D.

Let us now consider a new dense subspace $\mathscr{H}(E')$, in place of $\mathscr{H}(E)$, for $E' \overset{\mu}{\sim} E$ chosen above. Then we can define a unitary operator for any $g \in G$ on $\mathscr{H}(E')$ which corresponds to a tensor product of representations $(T_i^{\alpha_i}, \mathscr{H}_i)$ as follows. For a $g \in G$, put $K_g = \mathrm{supp}(g)$, then by Condition (MU2str) there exists an integer $N_g > 0$ such that $K_g \cap E'_i = \phi$ for $i > N_g$. Then, taking $N \geq N_g$, we have $gp = p$ for $p \in E'_i$ if $i > N$, and $T_i^{\alpha_i}(g)\chi_{E'_i} = \chi_{E'_i}$, and so the map

$$(3.1) \qquad (\otimes_{i \leq N} f_i) \otimes (\otimes_{i>N} \chi_{E'_i}) \longmapsto (\otimes_{i \leq N}(T_i^{\alpha_i}(g) f_i)) \otimes (\otimes_{i>N} \chi_{E'_i})$$

is well-defined on $\mathscr{H}(E')$ and is unitary, i. e., isometric onto. This unitary operator on $\mathscr{H}(E')$ can be extended uniquely to such a one on the whole space $\otimes_{i \in \mathbf{N}}^{\chi} \mathscr{H}_i$. Denote it by $T_{E'}^{\bar{\alpha}}(g)$, then we have naturally $T_{E'}^{\bar{\alpha}}(g_1 g_2) = T_{E'}^{\bar{\alpha}}(g_1) T_{E'}^{\bar{\alpha}}(g_2)$ for $g_1, g_2 \in G$. In this way we obtain the following theorem which is a version of Theorem 3. 1.

Theorem 3. 3. *Assume that a μ-unital subset E satisfies, together with μ, the conditions (MU1)–(MU2). Then there exists a μ-unital set $E' \overset{\mu}{\sim} E$ satisfying Condition (MU2str). Assume that $\alpha_i(g, p) = 1$ for $p \notin \mathrm{supp}(g)$. Then the formula (3. 1) above defines the tensor product $(T_{E'}^{\bar{\alpha}}, \otimes_{i \in \mathbf{N}}^{\chi} \mathscr{H}_i)$ of unitary representations $(T_i^{\alpha_i}, \mathscr{H}_i)$, $\mathscr{H}_i = L^2(X_i, \mathscr{B}_i, \mu_i)$.*

Proof. It rests only to prove the continuity of the map $G \ni g \mapsto T_{E'}^{\bar{\alpha}}(g) f$ for any fixed $f \in \otimes_{i \in \mathbf{N}}^{\chi} \mathscr{H}_i$. However, since the operators are all unitary, it is enough to check it for any f in the dense subspace $\mathscr{H}(E')$. In turn, this

f is of the form $(\otimes_{i\le N}f_i)\otimes(\otimes_{i>N}\chi_{E'_i})$. For a relatively compact open set V of G containing e, the set $U_V=\{g\in G\,;\,K_g\subset V\}$ is an open neighbourhood of e in G. On the other hand, we can choose an integer $N_V>0$ such that $V\cap E'_i=\phi$ for $i>N_V$. Then, for any $g\in U_V$, choosing an $N>N_V$, the vector $T^{\mathfrak{a}}_{E'}(g)f$ is expressed by the formula (3.1). This means that the continuity of the map $U_V\ni g\mapsto T^{\mathfrak{a}}_{E'}(g)f$ comes from that of the maps $g\mapsto T^{\alpha_i}_i(g)$ for $i\le N$.

3.3. Properties of product measures. In §1, with reference to a given μ-unital set $E=\prod_{i\in\mathbf{N}}E_i$, we defined two kinds of product measures $(X,\ \mathscr{M}_0(E),\ \nu_{0,\mu,E})$ and $(X,\ \mathscr{M}(\mu,\ E),\ \nu_{\mu,E})$ on $X=\prod_{i\in\mathbf{N}}X_i$. We discuss about the G-quasi-invariance of these measures. The first point is the invariance under G of the σ-ring $\mathscr{M}_0(E)$ or $\mathscr{M}(\mu,\ E)$, and the second point is the existence of Radon-Nikodym derivatives.

By an example given later, we know that the condition (MU2) on $(\mu,\ E)$ is not sufficient to guarantee these two points affirmatively. Our primary answer coming from the results in §3.2 is the following.

Lemma 3.4. *Assume that $(\mu,\ E)$ satisfies Conditions (MU1)-(MU2). Let $E'=\prod_{i\in\mathbf{N}}E'_i$ be a μ-unital set such that $E'\overset{\mu}{\sim}E$ and that Condition (MU2str) holds. Then, the σ-ring $\mathscr{M}_0(E')$ is invariant under G, and the product measure $\nu_{0,\mu,E'}$ is G-quasi-invariant with the Radon-Nikodym derivative given for $g\in G$ by*

$$\frac{d\nu_{0,\mu,E'}(g^{-1}x)}{d\nu_{0,\mu,E'}(x)}=\prod_{i\in\mathbf{N}}\frac{d\mu_i(g^{-1}p_i)}{d\mu_i(p_i)}\tag{3.2}$$

for $x=(p_i)_{i\in\mathbf{N}}$ on every set of $\mathscr{M}_0(E')$. Here the product is actually a finite product on each $F\in\mathscr{E}_0(E')$.

Proof. Note that any set $A\in\mathscr{M}_0(E')$ is covered by a countable infinite number of $E'^{(k)}\in\mathscr{E}_0(E')$: $A\subset\bigcup_{k\in\mathbf{N}}E'^{(k)}$. Then, we see that, to prove the assertion for A, it is sufficient to prove it for each $A\cap E'^{(k)}$ or rather for $E'^{(k)}$ itself. Since $E'^{(k)}\in\mathscr{E}_0(E')$, it has the form $E'^{(k)}=(\prod_{i\le N_k}E_i'^{(k)})\times(\prod_{i>N_k}E'_i)$ for some $N_k>0$. For a $g\in G$, put $K_g=\mathrm{supp}(g)$. Then, by (MU2str) for E', there exists an $N_g>0$ such that $K_g\cap E'_i=\phi$ for $i>N_g$. Hence, taking $N>\max(N_k,\ N_g)$, we have, for $i>N$, $gp_i=p_i(\forall p_i\in E'_i)$ and so, on the set $E'^{(k)}$,

$$\frac{d\nu_{0,\mu,E'}(g^{-1}x)}{d\nu_{0,\mu,E'}(x)}=\prod_{i\le N}\frac{d\mu_i(g^{-1}p_i)}{d\mu_i(p_i)}=\prod_{i\in\mathbf{N}}\frac{d\mu_i(g^{-1}p_i)}{d\mu_i(p_i)}\quad(x=(p_i)_{i\in\mathbf{N}}).$$

Thus we have the assertion. Q.E.D.

In general, assuming Condition (MU1) apriori, we do not have the G-invariance of the σ-ring $\mathscr{M}_0(E)$ or $\mathscr{M}(\mu,\ E)$ by the condition (MU2) only. To show this we give the following example.

Example 3.5. Let $M=\mathbf{R}$, and put $E_i \subset M$ as $E_i=(0, a_i)\cup(i+1, i+2)$ with $0<a_i<1$, $a_i \downarrow 0$. Measures μ_i are given as $d\mu_i(u)=\rho_i(u)du$ with positive functions ρ_i satisfying

$$\sum_{i\in\mathbf{N}}\int_0^{a_i}\rho_i(u)du<\infty,\ \rho_i(u)\equiv\frac{1}{a_i}\text{ on }[1, 2],\ \rho_i(u)\equiv 1\text{ on }[i+1, i+2].$$

Take an element $g\in G$ such that

$$g(u)=u+1\quad\text{on}\quad(0, 1),\quad\text{and}\quad g(u)=u\quad\text{for } u\gg 0.$$

As is easily seen, $E=\prod_{i\in\mathbf{N}}E_i$ is a μ-unital subset of $X=\prod_{i\in\mathbf{N}}X_i$ with $X_i=M$, and it satisfies Condition (MU2).

Consider the set $gE:=\prod_{i\in\mathbf{N}}gE_i$. Then, for $i\gg 0$, $gE_i=(1, 1+a_i)\cup(i+1, i+2)$ and so $\mu_i(gE_i)=2$ and even $\mu_i(gE_i\cap[1, 2])=1$. This means that the set gE is no longer μ-unital nor does it satisfy Condition (MU2) for a compact $K=[1, 2]$. Furthermore we see that the set gE can not be covered by a *countable* infinite number of μ-unital sets so that it does not belong even to $\mathscr{M}(\mu, E)\supset\mathscr{M}_0(E)$.

Thus, neither $\mathscr{M}_0(E)$ nor $\mathscr{M}(\mu, E)$ is G-invariant.

Furthermore, put $E_i'=(i+1, i+2)$ and $E'=\prod_{i\in\mathbf{N}}E_i'$. Then $E'\overset{\mu}{\sim}E$ and Condition (MU2str) holds for E'.

3.4. Another expression of tensor product representations. Assume Conditions (MU1)–(MU2) for (μ, E). Then by Lemma 3.2 there exists a μ-unital set $E'=\prod_{i\in\mathbf{N}}E_i'\overset{\mu}{\sim}E$ which satisfies the condition (MU2 str). As is proved in Lemma 3.4, the σ-ring $\mathscr{M}_0(E')$ is G-invariant and a product measure $\nu_{0,\mu,E'}$ is G-quasi-invariant.

We express, in another form by means of this measure, the tensor product representation $\otimes_{i\in\mathbf{N}}^{\chi}(T_i^{\alpha_i}, \mathscr{H}_i)$ of G defined with reference to (μ, E), where $\mathscr{H}_i=L^2(X_i, \mathscr{B}_i, \mu_i)$ and $\chi=(\chi_i)_{i\in\mathbf{N}}$ be as before.

Lemma 3.6. *A 1-cocycle $\alpha(g, x)$ on X can be defined by the product of 1-cocycles α_i $(i\in\mathbf{N})$, if $\alpha_i(g, p_i)=1$ for $p_i\notin\mathrm{supp}(g)$. The product converges $\nu_{0,\mu,E'}$-almost-everywhere on every subset in $\mathscr{M}_0(E')$ in such a way that the right hand side of the following formula is actually a finite product on each $F\in\mathscr{E}_0(E')$ for each fixed $g\in G$:*

$$\alpha(g, x):=\prod_{i\in\mathbf{N}}\alpha_i(g, p_i)\quad(g\in G,\ x=(p_i)_{i\in\mathbf{N}}).\tag{3.3}$$

Proof. Take a compact subset K of M which contains supp(g) in its interior. Then, by (MU2str) for E', $E_i'\cap K=\phi$ for sufficiently large i. For any fixed $F=\prod_{i\in\mathbf{N}}F_i\in\mathscr{E}_0(E')$, we have $F_i=E_i'$ for sufficiently large i, and accordingly $F_i\cap K=\phi$, and so, for an $x=(p_i)_{i\in\mathbf{N}}\in F$,

$$\alpha_i(g, p_i)=1\ (i\gg 0).$$

This means that the product in the lemma is actually a finite product on F.

Moreover every set in $\mathcal{M}_0(E')$ is covered by a countable number of sets in $\mathcal{E}_0(E')$. This proves completely the assertion of the lemma. Q. E. D.

Using the Radon-Nikodym derivative in (3. 2) and the 1-cocycle α in (3. 3), we can define a unitary representation $T_{E'}^{\alpha}$ of G on the Hilbert space $L^2(X, \mathcal{M}_0(E'), \nu_{0,\mu,E'})$ as

$$T_{E'}^{\alpha}(g)f(x)=\alpha(g, x)\sqrt{\frac{d\nu_{0,\mu,E'}(g^{-1}x)}{d\nu_{0,\mu,E'}(x)}}f(g^{-1}x), \tag{3. 4}$$

where $g\in G$, f is an $\mathcal{M}_0(E')$-measurable L^2-function, and $x\in\operatorname{supp}'(f):=\{x\in X\,;f(x)\neq 0\}$.

This gives another expression of the tensor product representation $T_E^{\tilde{\alpha}}$ in § 3. 2 as stated in the following theorem.

Theorem 3. 7. *Let $E=\prod_{i\in\mathbf{N}}E_i$ be a unital subset of $X=\prod_{i\in\mathbf{N}}X_i$, $X_i=M$, which satisfies the conditions (MU1)–(MU2). Let $\chi=(\chi_i)_{i\in\mathbf{N}}$, $\chi_i=\|\chi_{E_i}\|_{\mathscr{H}_i}^{-1}\chi_{E_i}$, and take a μ-unital $E'\overset{\mu}{\sim}E$ satisfying the condition (MU2str). Through the natural isomorphism of the tensor product Hilbert space $\otimes_{i\in\mathbf{N}}^{\chi}\mathscr{H}_i$, $\mathscr{H}_i=L^2(X_i, \mathscr{B}_i, \mu_i)$, with the L^2-space $L^2(X, \mathcal{M}(\mu, E), \nu_{\mu,E})\cong L^2(X, \mathcal{M}_0(E'), \nu_{0,\mu,E'})$, we have a unitary equivalence of representations $T_E^{\tilde{\alpha}}$ with $T_{E'}^{\alpha}$ of the group $G=\mathrm{Diff}_0(M)$.*

3. 5. Conditions for G-quasi-invariance of $\nu_{\mu,E}$ on $\mathcal{M}(\mu, E)$. As is shown by Example 3. 5, we need some condition to have the G-invariance of the σ-ring $\mathcal{M}(\mu, E)$ and also the G-quasi-invariance of the product measure $\nu_{\mu,E}$ on it. As a reasonable sufficient condition we propose the following one on $\mu=(\mu_i)_{i\in\mathbf{N}}$:

(MK) For any $g\in G$ and any compact $K\subset M$,

$$\sum_{i\in\mathbf{N}}\mu_i(A_i\cap K)<\infty \text{ for } A_i\in\mathscr{B}(M), \text{ implies } \sum_{i\in\mathbf{N}}\mu_i(g(A_i\cap K))<\infty,$$

where $\mathscr{B}(M)$ denotes the σ-algebra of all Borel subsets of M.

Under this condition we can prove the desired results as shown below. For $g\in G$ and $x=(p_i)_{i\in\mathbf{N}}\in X$, we put $gx=(gp_i)_{i\in\mathbf{N}}$, and so $gF=\prod_{i\in\mathbf{N}}gF_i$ for $F=\prod_{i\in\mathbf{N}}F_i$, $F_i\subset M$.

Lemma 3. 8. *Assume (MU1)–(MU2) for (μ, E), and (MK) for μ. Then the σ-ring $\mathcal{M}(\mu, E)$ is G-invariant.*

Proof. It is sufficient to prove that, for any $F=\prod_{i\in\mathbf{N}}F_i\in\mathcal{E}(\mu, E)$, we have $gF\in\mathcal{E}(\mu, E)$. Put $K_g=\operatorname{supp}(g)$. To prove that gF is again μ-unital, it is sufficient to remark that $gF_i=(F_i\setminus K_g)\sqcup g(F_i\cap K_g)$, and therefore

$$|1-\mu_i(gF_i)| \leq |1-\mu_i(F_i)| + \mu_i(F_i \cap K_g) + \mu_i(g(F_i \cap K_g)).$$

To prove $gF \overset{\mu}{\sim} E$, we note that $gF_i \ominus E_i \subset g(F_i \cap K_g) \cup (E_i \cap K_g) \cup (F_i \ominus E_i)$. Then this gives the assertion.

Lemma 3.9. *Assume the conditions (MU1)-(MU2) for (μ, E) and (MK) for μ. Then, for any μ-unital set $F = \prod_{i \in \mathbf{N}} F_i \overset{\mu}{\sim} E$, i.e., $F \in \mathscr{E}(\mu, E)$, the products*

$$\prod_{1 \leq i \leq N} \left(\frac{d\mu_i(gp_i)}{d\mu_i(p_i)} \right)^{1/p} \quad \textit{for } x = (p_i)_{i \in \mathbf{N}} \in F$$

converges on F in the space $L^p(F, \nu_{\mu, E}|F)$ as $N \to \infty$, for any p, $1 \leq p < \infty$.

Proof. We see in [19, Chap. 2] that the convergence for any p, $1 \leq p \leq \infty$ is equivalent to the one for some p. So we prove it here for $p=2$. Put for $N < N'$,

$$I_{N,N'} = \int_{\prod_{i \leq N'} F_i} \left| \prod_{i \leq N} \sqrt{\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}} - \prod_{i \leq N'} \sqrt{\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}} \right|^2 \prod_{i \leq N'} d\mu_i(p_i).$$

Then, we should prove $I_{N,N'} \to 0$ as $N < N' \to \infty$. On the other hand, $I_{N,N'} = \prod_{i \leq N} \mu_i(gF_i) \times J_{N,N'}$ with

$$J_{N,N'} = \prod_{N < i \leq N'} \mu_i(F_i) + \prod_{N < i \leq N'} \mu_i(gF_i) - 2 \prod_{N < i \leq N'} \int_{F_i} \sqrt{\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}}\, d\mu_i(p_i).$$

Since the products $\prod_{i \in \mathbf{N}} \mu_i(F_i)$ and $\prod_{i \in \mathbf{N}} \mu_i(gF_i)$ are both convergent, $I_{N,N'} \to 0$ is equivalent to

$$\sum_{i \in \mathbf{N}} \left| 1 - \int_{F_i} \sqrt{\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}}\, d\mu_i(p_i) \right| < \infty.$$

Moreover, since $\sum_{i \in \mathbf{N}} |\mu_i(F_i) - 1|$ and $\sum_{i \in \mathbf{N}} |\mu_i(gF_i) - 1|$ are convergent, this is equivalent to

$$\sum_{i \in \mathbf{N}} \int_{F_i} \left| 1 - \sqrt{\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}} \right|^2 d\mu_i(p_i) < \infty.$$

A sufficient condition for the above is given by

$$\sum_{i \in \mathbf{N}} \int_{F_i} \left| 1 - \frac{d\mu_i(gp_i)}{d\mu_i(p_i)} \right| d\mu_i(p_i) < \infty.$$

Now put

$$A_i=\left\{p_i\in X_i\,;\,\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}>1\right\},\ B_i=\left\{p_i\in X_i\,;\,\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}<1\right\}.$$

Then, $A_i\bigsqcup B_i\subset K_g$, and the above sum is evaluated as

$$=\sum_{i\in\mathbf{N}}\{\mu_i(g(F_i\cap A_i))-\mu_i(F_i\cap A_i)+\mu_i(F_i\cap B_i)-\mu_i(g(F_i\cap B_i))\}$$

$$\leq\sum_{i\in\mathbf{N}}\{\mu_i(g(F_i\cap K_g))+\mu_i(F_i\cap K_g)\}<\infty.$$

This proves our assertion. Q. E. D.

The p-th power of the limit as $N\to\infty$ of the products in Lemma 3.9 gives the same function for any p, and it gives the Jacobian of the product measures (cf. [19], or [7] for $p=2$).

Theorem 3.10. *Let E be a μ-unital subset of X, and assume the conditions (MU1)-(MU2) for (μ, E), and (MK) for μ. Then the σ-ring $\mathscr{M}(\mu, E)$ is G-invariant, and the product measure $\nu_{\mu,E}$ on it is G-quasi-invariant with the Jacobian given on each $F\in\mathscr{E}(\mu, E)$ as*

$$\frac{d\nu_{\mu,E}(gx)}{d\nu_{\mu,E}(x)}=\prod_{i\in\mathbf{N}}\frac{d\mu_i(gp_i)}{d\mu_i(p_i)}\quad for\ x=(p_i)_{i\in\mathbf{N}}\in F\subset X,$$

where the infinite product converges in $L^1(F, \nu_{\mu,E}|F)$.

In the above case, the tensor product representation $(T_E^a, \otimes_{i\in\mathbf{N}}^{\chi}\mathscr{H}_i)$ with $\mathscr{H}_i=L^2(X_i, \mathscr{B}_i, d\mu_i)$, is realized with a natural expression for representation operators on the L^2-space $L^2(X, \mathscr{M}(\mu, E), \nu_{\mu,E})$, at least when all the 1-cocycles α_i are trivial.

3.6. Space of ordered configurations and Condition (MU2). A series $x=(p_i)_{i\in\mathbf{N}}$ of mutually different points in M is called an *ordered configuration* if it has no convergent subsequence, i. e., the set of points $\{p_i\,;\,i\in\mathbf{N}\}$ has no accumulation points in M. The set of all ordered configurations of points in M is denoted by $\tilde{X}$. We say that a measure $(\nu, \mathscr{B})$ on the product space X is supported by $\tilde{X}$ if, for any $A\in\mathscr{B}$,

$$A\cap\tilde{X}\in\mathscr{B}\quad\text{and}\quad\nu(A)=\nu(A\cap\tilde{X}).$$

Then we have the following

Theorem 3.11. *Let $\mu=(\mu_i)_{i\in\mathbf{N}}$ be a system of measures on $X_i=M\ (i\in\mathbf{N})$ taken from $\mathscr{LFM}(M)$, and E a μ-unital subset of $X=\prod_{i\in\mathbf{N}}X_i$, so that (MU1) holds for (μ, E). Let $\nu_{0,\mu,E}$ and $\nu_{\mu,E}$ be the product measures on the σ-rings $\mathscr{M}_0(E)$ and $\mathscr{M}(\mu, E)$ respectively. Then they are supported by the space $\tilde{X}$ of* *ordered configurations if and only if the condition (MU2) holds for* (μ, E).

Proof. For $\nu=\nu_{0,\mu,E}$ (resp. $\nu_{\mu,E}$), it is supported by $\tilde{X}$ if and only if for any $F=\prod_{i\in\mathbf{N}}F_i$ in $\mathscr{E}_0(E)$ (resp. $\mathscr{E}(\mu, E)$), we have $F\cap\tilde{X}\in\mathscr{M}_0(E)$ (resp. $\mathscr{M}(\mu, E)$) and $\nu(F\cap\tilde{X})=\nu(F)$.

Now take an increasing sequence K_k, $k\in\mathbf{N}$, of compact subsets of M such that $\bigcup_{k\in N}K_k=M$. Then the intersection $F\cap\tilde{X}$ is expressed as follows. Put $F_{n,k}=(\prod_{i\leq n}F_i)\times(\prod_{i>n}(F_i\backslash K_k))$, then $\bigcup_{n\in\mathbf{N}}F_{n,k}$ is the set of points in F which has no accumulation points in $K_k\subset M$ and so

$$F\cap\tilde{X}=\bigcap_{k\in\mathbf{N}}\bigcup_{n\in\mathbf{N}}F_{n,k}.$$

On the other hand, since $\nu(F)<\infty$, we have

$$\nu(F\cap\tilde{X})=\lim_{k\to\infty}\lim_{n\to\infty}\nu(F_{n,k})=\lim_{k\to\infty}\lim_{n\to\infty}\Big(\prod_{i\leq n}\mu_i(F_i)\Big)\cdot\Big(\prod_{i>n}\mu_i(F_i\backslash K_k)\Big).$$

This limit is positive if and only if

$$\sum_{i\in\mathbf{N}}\mu_i(F_i\cap K_k)<\infty \quad \text{for any } k\geq 1,$$

and this condition is equivalent to (MU2).

In this case the limit is equal to $\prod_{i\in\mathbf{N}}\mu_i(F_i)=\nu(F)$. Q. E. D.

For the space of (non-ordered) configurations of points and quasi-invariant measures on it, we cite here the works [16] and [17].

§ 4. Permutations as intertwining operators for an infinite tensor product representation of G

4.1. Algebra of intertwining operators. Let T be a unitary representation on a Hilbert space $H(T)$ of a certain group G. A weakly closed subalgebra

$$T(G)'=(T(G))'=\{L\in\mathscr{B}(H(T))\,;\,L\circ T(g)=T(g)\circ L\,(g\in G)\}$$

of the algebra $\mathscr{B}(H(T))$ of all bounded linear operators on $H(T)$ is called the algebra of intertwining operators for T, and is essential to analyze the structure of the representation T. In fact, we may say that it governs irreducible decompositions of T. Actually, in case the algebra of intertwining operators $T(G)'$ is of type I, a spatial irreducible decomposition of this algebra gives essentially an irreducible decomposition of the representation T of G.

Denote by $\mathscr{U}(H(T))$ the set of all unitary operators on $H(T)$. Then the algebra $T(G)'$ is generated weakly by the group $\mathscr{I}(T):=T(G)'\cap\mathscr{U}(H(T))$. Therefore we are interested in determining explicitly a certain subgroup of unitary intertwining operators which is weakly dense in the above group.

In our present case, we take as G the group $\mathrm{Diff}_0(M)$ and as T one of the tensor product representations T_E^a or their equivalents $T_{E'}^a$ in §3. Then there appear the infinite symmetric group $\mathfrak{S}_\infty$ or related groups of permutations which act as unitary intertwining operators on the space of infinite tensor product $H(T)\cong\otimes_{j\in\mathbf{N}}^{\chi}\mathscr{H}_j$, $\mathscr{H}_j=L^2(X_j, \mathscr{B}_j, \mu_j)$, through 'permutations' of components. The study of structure of these permutation groups is the subject of this section.

The denseness in the group $\mathscr{I}(T)$ of the set of intertwining operators corresponding to such a permutation group will be treated in the next section.

4.2. Representations of $G=\mathrm{Diff}_0(M)$. For the group G, we take one of the tensor products of its natural representaions. As in §3.2, we assume that the conditions (MU1)–(MU2) hold for (μ, E), and consider the tensor product T_E^a of $(T_j^{\alpha_j}, L^2(X_j, \mathscr{B}_j, \mu_j))$, $j\in\mathbf{N}$, with respect to a reference vector $\chi=(\chi_j)_{j\in\mathbf{N}}$. Here

$$(4.1)\qquad T_j^{\alpha_j}(g)h(p)=\alpha_j(g, p)\sqrt{\frac{d\mu_j(g^{-1}p)}{d\mu_j(p)}}\,h(g^{-1}p)\quad(p\in M,\ g\in G),$$

with a 1-cocycle α_j, $|\alpha_j(g, p)|=1$. For simplicity, we choose, in this and the next subsections, a realization of the tensor product by means of product measures, given in §3.4. So that $T=T_{E'}^a$ on the space $H(T)=L^2(X, \mathscr{M}_0(E'), \nu_{0,\mu,E'})$:

$$T_{E'}^a(g)f(x)=\alpha(g, x)\sqrt{\frac{d\nu_{0,\mu,E'}(g^{-1}x)}{d\nu_{0,\mu,E'}(x)}}\,f(g^{-1}x).$$

Here the 1-cocycle α is defined in terms of $\alpha_j\,(j\in\mathbf{N})$ by the product

$$\alpha(g, x)=\prod_{j\in\mathbf{N}}\alpha_j(g, x_j)\quad\text{for } x=(x_j)_{j\in\mathbf{N}}$$

which is essentially a finite product on every subset $F=\prod_{j\in\mathbf{N}}F_j$ in $\mathscr{E}_0(E')$, and so converges $\nu_{0,\mu,E'}$-almost everywhere on each $A\in\mathscr{M}_0(E')$ (cf. Theorem 3.7).

With a sufficient generality, we study from now on the case where each α_j is given as

$$(4.2)\qquad \alpha_j(g, x_j)=\frac{\gamma_j(g^{-1}x_j)}{\gamma_j(x_j)}\left(\frac{d\mu_j(g^{-1}x_j)}{d\mu_j(x_j)}\right)^{is_j}$$

with $i=\sqrt{-1}$, $s_j\in\mathbf{R}$, and $\gamma_j(x_j)$ a measurable function in $x_j\in X_j=M$ such that

$|\gamma_j|=1$. Put, for $F=\prod_{j\in\mathbf{N}}F_j\in\mathscr{E}_0(E')$,

$$\gamma(x)=\prod_{j\in\mathbf{N}}\gamma_j(x_j)\quad\text{for } x=(x_j)_{j\in\mathbf{N}}\in F,$$

which converges $\nu_{0,\mu,E'}$-almost everywhere as above. Then, we have the following expression of α:

$$\alpha(g,x)=\frac{\gamma(g^{-1}x)}{\gamma(x)}\prod_{j\in\mathbf{N}}\left(\frac{d\mu_j(g^{-1}x_j)}{d\mu_j(x_j)}\right)^{is_j}\quad\text{for } x\in F.$$

Here the first fractional part containing γ is a non-essential part.

Now choose all different values from $\{s_j\,;\,j\in\mathbf{N}\}$ and let them be $s(k)$, $k\in\mathbf{K}$, and put $\mathbf{N}(k)=\{j\in\mathbf{N}\,;\,s_j=s(k)\}$ for $k\in\mathbf{K}$. Then the above expression for α gives us

$$\alpha(g,x)=\frac{\gamma(g^{-1}x)}{\gamma(x)}\prod_{k\in\mathbf{K}}\left(\prod_{j\in\mathbf{N}(k)}\frac{d\mu_j(g^{-1}x_j)}{d\mu_j(x_j)}\right)^{is(k)}\quad\text{for } x\in F. \tag{4.3}$$

4.3. Infinite symmetric group $\mathfrak{S}_\infty$ and commuting relations. Let $\mathfrak{S}_\infty$ be the group of all finite permutations on the set $\mathbf{N}$ of natural numbers. For a permutation σ on $\mathbf{N}$, put $\operatorname{supp}(\sigma)=\{k\in\mathbf{N}\,;\,\sigma(k)\neq k\}$, then by definition σ is called *finite* if $\operatorname{supp}(\sigma)$ is finite. We define an action of $\mathfrak{S}_\infty$ or its subgroup S on the space $H(T)$ as follows. For $\sigma\in\mathfrak{S}_\infty$ and $x=(x_j)_{j\in\mathbf{N}}\in X$, put $x\sigma=(x_{\sigma(j)})_{j\in\mathbf{N}}$. Recalling the isomorphism $H(T)\cong\otimes^{\chi}_{j\in\mathbf{N}}\mathscr{H}_j$ in Theorem 3.3, we put for $\sigma\in S\subset\mathfrak{S}_\infty$

$$R(\sigma)f(x)=\beta(\sigma,x)\sqrt{\frac{d\nu_{0,\mu,E'}(x\sigma)}{d\nu_{0,\mu,E'}(x)}}f(x\sigma)\quad(x\sigma\in\operatorname{supp}'(f)), \tag{4.4}$$

where the existence of a 1-cocycle β for a subgroup $S\subset\mathfrak{S}_\infty$ is assumed:

$$\beta(\sigma_1\sigma_2,x)=\beta(\sigma_1,x)\beta(\sigma_2,x\sigma_1)\quad(\sigma_1,\sigma_2\in S,\ x\in X).$$

The commuting relation

$$R(\sigma)\circ T^{\alpha}_{E'}(g)=T^{\alpha}_{E'}(g)\circ R(\sigma) \tag{4.5}$$

is equivalent to the following relation between 1-cocycles: for any $A\in\mathscr{M}_0(E')$,

$$\beta(\sigma,x)\cdot\alpha(g,x\sigma)=\alpha(g,x)\cdot\beta(\sigma,g^{-1}x)\qquad\text{for }\nu_{0,\mu,E'}\text{-almost all } x\in A. \tag{4.6}$$

Assume that σ satisfies $s_{\sigma(j)}=s_j$ $(j\in\mathbf{N})$, or equivalently, $\sigma\mathbf{N}(k)=\mathbf{N}(k)$ $(k\in\mathbf{K})$. Then, putting

(4. 7) $$\beta(\sigma, x)=\frac{\gamma(x\sigma)}{\gamma(x)}\prod_{k\in\mathbf{N}}\Big(\prod_{j\in\mathbf{N}(k)}\frac{d\mu_{\sigma^{-1}(j)}(x_j)}{d\mu_j(x_j)}\Big)^{is(k)},$$

we get a 1-cocycle β for which the commuting relation (4. 5) holds.

The subgroup $\mathfrak{S}_\infty((s_j)_{j\in\mathbf{N}})$ of $\mathfrak{S}_\infty$ consisting of all σ such that $s_{\sigma(j)}=s_j$ $(\forall j\in\mathbf{N})$, is equal to the restricted direct product $\prod'_{k\in\mathbf{K}}\mathfrak{S}_{\mathbf{N}(k)}$, where, for a subset J of $\mathbf{N}$, $\mathfrak{S}_J=\{\sigma\in\mathfrak{S}_\infty\,;\,\mathrm{supp}(\sigma)\subset J\}$.

The above results give us the following

Lemma 4. 1. *The group of unitary intertwining operators $\mathcal{I}(T^{\alpha}_{E'})$ for the representation $T^{\alpha}_{E'}$ contains a subgroup $\{R(\sigma)\,;\,\sigma\in\mathfrak{S}_\infty(s)\}$, $s=(s_j)_{j\in\mathbf{N}}$, or in another expression, $T^{\alpha}_{E'}(G)'\supset R(\mathfrak{S}_\infty(s))''$.*

4. 4. More general permutations acting as intertwining operators. Hereafter it is convenient for us to take the realization of T originally given as infinite tensor product of $T_j^{\alpha_j}$'s : $T=T^{\underline{\alpha}}_{E}=\otimes_{j\in\mathbf{N}}T_j^{\alpha_j}$ on $H(T)=\otimes^{\chi}_{j\in\mathbf{N}}\mathcal{H}_j$ with $\mathcal{H}_j=L^2(X_j,\mathcal{B}_j,\mu_j)$. We may and do assume that $\gamma_j=1$ for $j\in\mathbf{N}$.

Now take a permutation σ on $\mathbf{N}:\sigma\in\tilde{\mathfrak{S}}_\infty$, and let us examine if one can define an intertwining operator $R(\sigma)$ on $H(T)$ through permutation of components by the formula equivalent to (4. 4). The commuting relation (4. 5) of $R(\sigma)$ with the representation $T^{\alpha}_{E'}$ gave us the expression (4. 7) of the 1-cocycle β. So, in particular, we have the invariance $s_{\sigma^{-1}(j)}=s_j(j\in\mathbf{N})$. We denote by $\tilde{\mathfrak{S}}_\infty(s)$, $s=(s_j)_{j\in\mathbf{N}}$, the subgroup of $\tilde{\mathfrak{S}}_\infty$ consisting of all such σ's.

Our problem here is to examine if a bounded intertwining operator can be given canonically through the following formula. Take a decomposable element $f=\otimes_{j\in\mathbf{N}}f_j$ such that, for $j\gg 0$, $f_j=\chi_j=\|\chi_{E_j}\|^{-1}_{\mathcal{H}_j}\chi_{E_j}$, in $H(T)=\otimes^{\chi}_{j\in\mathbf{N}}\mathcal{H}_j$ with $\chi=(\chi_j)_{j\in\mathbf{N}}$. Then it should be mapped to a decomposable $h=\otimes_{j\in\mathbf{N}}h_j$ with

$$h_j(x_j)=\Big(\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j)\Big)^{\frac{1}{2}+is_j}f_{\sigma^{-1}(j)}(x_j)$$

We discuss as in §1. 3. The decomposable element h belongs to $H(T)=\otimes^{\chi}_{j\in\mathbf{N}}\mathcal{H}_j$ if and only if $(h_j)_{j\in\mathbf{N}}$ is cofinal with the reference vector $\chi=(\chi_j)_{j\in\mathbf{N}}$, that is,

(4. 8) $$\sum_{j\in\mathbf{N}}|\,1-\langle h_j,\chi_j\rangle_{\mathcal{H}_j}|<\infty.$$

Since E is μ-unital, we have $\sum_{j\in\mathbf{N}}|\,1-\|\chi_{E_j}\|^2_{\mathcal{H}_j}|<\infty$, and so the above condition is equivalent to

(4. 9) $$\sum_{j\in\mathbf{N}}|\,1-\langle h'_j,\chi_{E_j}\rangle_{\mathcal{H}_j}|<\infty,$$

where, since $s_{\sigma^{-1}(j)}=s_j$,

$$h_j'(x_j)=\left(\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j)\right)^{\frac{1}{2}+is_j}\chi_{E_{\sigma^{-1}(j)}}(x_j).$$

Furthermore it is also equivalent to the following condition:

$$(4.10)\qquad \sum_{j\in\mathbf{N}} |\,\|\chi_{E_j}\|_{\mathscr{H}_j}^2-\langle h_j', \chi_{E_j}\rangle_{\mathscr{H}_j}| =\sum_{j\in\mathbf{N}} |\langle\chi_{E_j}-h_j', \chi_{E_j}\rangle_{\mathscr{H}_j}| <\infty,$$

Thus we have next

Lemma 4.2. *A permutation $\sigma\in\tilde{\mathfrak{S}}_\infty(s)$ gives a unitary operator $R(\sigma)$ on $H(T)$ by the formula stated above if and only if the condition (4.9) or the equivalent one (4.10) holds. This operator $R(\sigma)$ intertwines the infinite tensor product representation T of G.*

Denote by $\mathfrak{S}_{\mu,E,s}$ with $s=(s_j)_{j\in\mathbf{N}}$ the set of all elements $\sigma\in\tilde{\mathfrak{S}}_\infty(s)$ for which an operator $R(\sigma)$ is defined by the above formula. Then it contains $\mathfrak{S}_\infty(s)$ and forms a group as we can see without difficulty using the convergence $\sum_{j\in\mathbf{N}} |\,\|\chi_{E_j}\|_{\mathscr{H}_j}^2-\|h_j'\|_{\mathscr{H}_j}^2| <\infty$.

The important thing is that $R(\sigma)$ gives an intertwining operator for the representation T, that is,

$$R(\mathfrak{S}_{\mu,E,s})''\subset T(G)'.$$

So, we are interested in determining the structure of the group $\mathfrak{S}_{\mu,E,s}$ and also in the following problem.

Problem 4.3. *Does the equality hold in the above inclusion relation? In other words, is the algebra of all intertwining operators for the infinite tensor product representation T is generated (weakly) by $R(\mathfrak{S}_{\mu,E,s})=\{R(\sigma)\,;\,\sigma\in\mathfrak{S}_{\mu,E,s}\}$?*

Let us introduce in the group $\mathscr{U}(H(T))$ of all unitary operators on $H(T)$ the strong convergence topology, which is equivalent to the weak convergence topology. Introduce on it a compatible metric given by

$$d(U_1, U_2)=\sum_{i\in\mathbf{N}} 2^{-i}\,\|U_1h_i-U_2h_i\|+\sum_{i\in\mathbf{N}} 2^{-i}\,\|U_1^{-1}h_i-U_2^{-1}h_i\|$$

for $U_1, U_2\in\mathscr{U}(H(T))$, where $\{h_i\,;\,i\in\mathbf{N}\}$ is a fixed complete orthonormal system of $H(T)$. Then the group $\mathscr{U}(H(T))$ is a complete separable metric group, and accordingly so is the group $\mathscr{I}(T)=T(G)'\cap\mathscr{U}(H(T))$. Induce the topology and the metric onto the group $\mathfrak{S}_{\mu,E,s}$ of permutations through the representation R. We call this topology as s·w-topology and denote the metric again by d. Then there occurs a natural question:

Problem 4.4. *When the subgroup $\mathfrak{S}_\infty(s)$ is everywhere dense in $\mathfrak{S}_{\mu,E,s}$?*

4.5. Some generalities on the permutation group $\mathfrak{S}_{\mu,E,s}$. Note that any element $\sigma\in\tilde{\tilde{\mathfrak{S}}}_\infty$ is expressed uniquely as a (possibly infinite) product of mutually disjoint cyclic permutations as

$$\sigma=\prod_{k\in K}\sigma_k, \tag{4.11}$$

where σ_k 's are cyclic and supp(σ_k)'s are mutually disjoint. Denote by $\tilde{\tilde{\mathfrak{S}}}_{\infty,f}$ the subgroup of $\mathfrak{S}_\infty$ consisting of all σ with cyclic components σ_k from $\mathfrak{S}_\infty$.

Proposition 4.5. (i) *Let $E=\prod_{j\in\mathbf{N}}E_j$ and $F=\prod_{j\in\mathbf{N}}F_j$ be two μ-unital subset of X such that $E\overset{\mu}{\sim}F$. Then $\mathfrak{S}_{\mu,E,s}=\mathfrak{S}_{\mu,F,s}$.*

(ii) *For an element $\sigma\in\mathfrak{S}_{\mu,E,s}$, let (4.11) be its canonical decomposition into cyclic permutations. Then, for any subset $K'\subset K$, the product $\sigma_{K'}=\prod_{k\in K'}\sigma_k$ is again an element of $\mathfrak{S}_{\mu,E,s}$.*

(iii) *Any element of $\mathfrak{S}_{\mu,E,s}\cap\tilde{\tilde{\mathfrak{S}}}_{\infty,f}$ is a limit of a sequence in $\mathfrak{S}_\infty(s)\subset\mathfrak{S}_{\mu,E,s}$ in $s\cdot w$-topology.*

Proof. (i) We apply the criterion (4.9). Denote by $a_j(E)$ the inner product $<h_j', \chi_{E_j}>$ and by $a_j(F)$ the corresponding inner product for the unital subset F. Then we see from (4.9) that it is sufficient for us to prove

$$\sum_{j\in\mathbf{N}}|1-a_j(E)|<\infty \quad\Rightarrow\quad \sum_{j\in\mathbf{N}}|1-a_j(F)|<\infty.$$

From the equality

$$\chi_{E_{\tau(j)}}\chi_{E_j}+\chi_{F_{\tau(j)}}\chi_{F_j}=(\chi_{F_{\tau(j)}}-\chi_{E_{\tau(j)}})(\chi_{F_j}-\chi_{E_j})+\chi_{E_{\tau(j)}}\chi_{F_j}+\chi_{E_j}\chi_{F_{\tau(j)}},$$

we have

$$\{1-a_j(E)\}+\{1-a_j(F)\}=I_{1,j}+I_{2,j}+I_{3,j},$$

where, with $\tau=\sigma^{-1}$,

$$I_{1,j}=-\int_{X_j}\left(\frac{d\mu_{\tau(j)}}{d\mu_j}\right)^{\frac12+is_j}(\chi_{F_{\tau(j)}}-\chi_{E_{\tau(j)}})(\chi_{F_j}-\chi_{E_j})d\mu_j,$$

$$I_{2,j}=1-\int_{X_j}\left(\frac{d\mu_{\tau(j)}}{d\mu_j}\right)^{\frac12+is_j}\chi_{E_{\tau(j)}}\chi_{F_j}d\mu_j,$$

$$I_{3,j}=1-\int_{X_j}\left(\frac{d\mu_{\tau(j)}}{d\mu_j}\right)^{\frac12+is_j}\chi_{E_j}\chi_{F_{\tau(j)}}d\mu_j.$$

Applying Schwarz inequality for $I_{1,j}$, we have $|I_{1,j}|\le\mu_{\tau(j)}(E_{\tau(j)}\ominus F_{\tau(j)})^{\frac12}$ $\mu_j(E_j\ominus F_j)^{\frac12}$, whence $\sum_{j\in\mathbf{N}}|I_{1,j}|<\infty$.

Note that $\sum_{j\in\mathbf{N}}|1-<h_j', \chi_{E_j}>|<\infty$ and $\sum_{j\in\mathbf{N}}|1-<\chi_{F_j}, \chi_{E_j}>|<\infty$ imply $\sum_{j\in\mathbf{N}}|1-<h_j', \chi_{F_j}>|<\infty$. Then, applying the last inequality, we get $\sum_{j\in\mathbf{N}}|I_{2,j}|<\infty$.

Replace τ by τ^{-1} in the above discussion, then we have

$$\sum_{j\in\mathbf{N}} \left| 1-\int_{X_j}\left(\frac{d\mu_{\tau^{-1}(j)}}{d\mu_j}\right)^{\frac{1}{2}+is_j}\chi_{E_{\tau^{-1}(j)}}\chi_{F_j}d\mu_j \right| <\infty.$$

Replace indices j by $\tau(j)$ and take into account $s_{\tau(j)}=s_j$, Then the left hand side equals

$$\sum_{j\in\mathbf{N}} \left| 1-\int_{X_j}\left(\frac{d\mu_{\tau(j)}}{d\mu_j}\right)^{\frac{1}{2}-is_j}\chi_{E_j}\chi_{F_{\tau(j)}}d\mu_j \right| =\sum_{j\in\mathbf{N}} |I_{3,j}|.$$

Thus we get finally $\sum_{j\in\mathbf{N}} |1-a_j(F)| <\infty$.

(ii) For this, we apply the criterion (4.10). Then the assertion is clear.

(iii) It is enough to note that $<R(\sigma_{K'})f_1, f_2>\to<R(\sigma)f_1, f_2>$ as $K'\nearrow K$, $|K'|<\infty$, for any f_1, f_2 from the canonical dense subset of $H(T)$ used in §4.4. This is a consequence of the condition (4.9).

The proof of Proposition 4.5 is now complete. Q.E.D.

Proposition 4.6. *The set of unitary operators* $\{R(\sigma)\,;\sigma\in\mathfrak{S}_{\mu,E,s}\}$ *is closed in the group* $\mathcal{U}(H(T))$ *of all unitary operators on* $H(T)$ *in the strong operator topology.*

The group $\mathfrak{S}_{\mu,E,s}$ *is a complete separable metric group with the metric* d.

Proof. The second assertion follows from the first one because $\mathcal{U}(H(T))$ with d is a complete separable metric group. So we prove here the first assertion.

Assume that $R(\sigma_n)\to U$, $\sigma_n\in\mathfrak{S}_{\mu,E,s}$ in $\mathcal{U}(H(T))$. Then $R(\sigma_m\sigma_n^{-1})\to I$ $(m, n\to\infty)$. In particular, for the standard vector $v_0=\otimes_{j\in\mathbf{N}}\chi_j\in H(T)$ with $\chi_j=\|\chi_{E_j}\|_{\mathscr{H}_j}^{-1}\chi_{E_j}=\mu_j(E_j)^{-1/2}\chi_{E_j}$, we have $<R(\sigma_m\sigma_n^{-1})v_0, v_0>\to1$, that is,

$$\prod_{j\in\mathbf{N}}\int_M\sqrt{\frac{d\mu_{\sigma_n\sigma_m^{-1}(j)}}{d\mu_j}(p)}\,\chi_{\sigma_n\sigma_m^{-1}(j)}(p)\chi_j(p)d\mu_j(p)\to1.$$

Denote by $\alpha_j^{m,n}$ the j-th term in the product on the left hand side. Then $\sum_{j\in\mathbf{N}}\log\alpha_j^{m,n}\to0$ $(m, n\to\infty)$. Since $-\log(1-x)>2^{-1}x$ for sufficiently small $x>0$, we have $\sum_{j\in\mathbf{N}}|1-\alpha_j^{m,n}|\to0$ $(m, n\to\infty)$. Therefore, for any $\varepsilon>0$, there exists $N>0$ such that, for any m, $n>N$,

$$\sum_{j\in\mathbf{N}}\int_M\left|\sqrt{\frac{d\mu_{\sigma_m(j)}}{d\lambda}(p)}\,\chi_{\sigma_m(j)}(p)-\sqrt{\frac{d\mu_{\sigma_n(j)}}{d\lambda}(p)}\,\chi_{\sigma_n(j)}(p)\right|^2 d\lambda(p)\le\varepsilon, \tag{4.12}$$

where $\lambda\in\mathscr{L}\mathscr{F}\mathscr{M}(M)$ is a fixed standard measure.

Now fix j, and consider the set of natural numbers $\Sigma_j=\{\sigma_n(j)\,;n\in\mathbf{N}\}$. We assert that this is a finite set. In fact, assuming the contrary, we can take a series of integers $n_1, n_2, \cdots$, such that $\sigma_{n_k}(j)\to\infty$. Then, since

$$(\chi_{\sigma_m(j)}(p))^2\frac{d\mu_{\sigma_m(j)}}{d\lambda}(p)\to{}^{\exists}\rho(p) \quad \text{in } L^1(M;\lambda),$$

the following limit exists for any Borel subset $B\subset M$: for $\tau_k=\sigma_{n_k}$,

$$\lim_{k\to\infty}\frac{\mu_{\tau_k(j)}(E_{\tau_k(j)}\cap B)}{\mu_{\tau_k(j)}(E_{\tau_k(j)})}.$$

Put this limit as $\nu(B)$, then ν is a probability measure on M. On the other hand, we assumed the condition (MU2) on the μ-unital subset $E=\prod_{j\in\mathbf{N}}E_j$. Hence $\nu(K)=0$ for any compact suset K of M. Making $K\nearrow M$, we come to a contradiction $0=1$.

Since Σ_1 is finite as just proved, there exists at least one element $j_1\in\Sigma_1$ such that $S_1:=\{\sigma_n\,;\sigma_n(1)=j_1\}$ is infinite. Similarly, since $\Sigma_2':=\{\sigma(2)\,;\sigma\in S_1\}\subset\Sigma_2$ is finite, there exists a $j_2\in\Sigma_2'$ for which $S_2:=\{\sigma\in S_1\,;\sigma(2)=j_2\}$ is infinite. Successively, we define a series of integers $j_1, j_2, j_3, \cdots$, and a series $S_1\supset S_2\supset S_3\supset\cdots$, of infinite subsets of $\mathfrak{S}_{\mu,E,s}$ such that $\sigma(i)=j_i\,(i\leq k)$ for $\sigma\in S_k$.

Put $\sigma_0(i)=j_i\,(i\in\mathbf{N})$, then σ_0 is an injective transformation on $\mathbf{N}$, and $s_{\sigma_0(j)}=s_j\,(j\in\mathbf{N})$. In the evaluation (4.12), replace the infinite sum $\sum_{j\in\mathbf{N}}$ by a finite sum $\sum_{j=1}^k$ and take σ_m from S_k. Then, $\sigma_m(j)=\sigma_0(j)$ for $1\leq j\leq k$, and so we get

$$\sum_{j=1}^{k}\int_M\left|\sqrt{\frac{d\mu_{\sigma_0(j)}}{d\lambda}(p)}\,\chi_{\sigma_0(j)}(p)-\sqrt{\frac{d\mu_{\sigma_n(j)}}{d\lambda}(p)}\,\chi_{\sigma_n(j)}(p)\right|^2d\lambda(p)\leq\varepsilon.$$

Letting $k\to\infty$, we obtain

$$\text{(4.13)}\quad \sum_{j=1}^{\infty}\int_M\left|\sqrt{\frac{d\mu_{\sigma_0(j)}}{d\lambda}(p)}\,\chi_{\sigma_0(j)}(p)-\sqrt{\frac{d\mu_{\sigma_n(j)}}{d\lambda}(p)}\,\chi_{\sigma_n(j)}(p)\right|^2d\lambda(p)\leq\varepsilon.$$

From this we can get

$$\sum_{j=1}^{\infty}\int_M\left|\sqrt{\frac{d\mu_{\sigma_0(j)}}{d\lambda}(p)}\,\chi_{\sigma_0(j)}(p)-\sqrt{\frac{d\mu_j}{d\lambda}(p)}\,\chi\chi_j(p)\right|^2d\lambda(p)<\infty,$$

by applying (4.14) below, or $\sum_{j\in\mathbf{N}}I_j(\sigma)<\infty$ for any $\sigma\in\mathfrak{S}_{\mu,E,s}$ in the notation there.

Discussing as in §4.4, we see that the above evaluation guarantees that the following correspondense defines a bounded linear oparator on $H(T)=\otimes_{j\in\mathbf{N}}^{\chi}\mathscr{H}_j$:

$$R(\sigma_0):f=\otimes_{j\in\mathbf{N}}f_j\longrightarrow h=\otimes_{j\in\mathbf{N}}h_j$$

with

$$h_j(x_j)=\Bigl(\frac{d\mu_{\sigma_0(j)}}{d\mu_j}(x_j)\Bigr)^{\frac{1}{2}+is_j}f_{\sigma_0(j)}(x_j)$$

Furthermore, for an element of the form

$$f=f_1(x_1)f_2(x_2)\cdots f_k(x_k)\otimes(\otimes_{j=k+1}^{\infty}\chi_j(x_j)),\quad \|f_i\|=1,$$

we have, for $\sigma\in S_k$,

$$\|\{R(\sigma_0)-R(\sigma^{-1})\}f\|^2=2-2\prod_{j=k+1}^{\infty}r_j$$

with

$$r_j=\int_{X_j}\sqrt{\frac{d\mu_{\sigma_0(j)}}{d\mu_j}(x_j)}\sqrt{\frac{d\mu_{\sigma(j)}}{d\mu_j}(x_j)}\,\chi_{\sigma_0(j)}(x_j)\chi_{\sigma(j)}(x_j)d\mu_j(x_j).$$

On the other hand, we have, from (4.13), $\sum_{j=k+1}^{\infty}2(1-r_j)<\varepsilon$. Since $2x>-\log(1-x)$ for sufficiently small $x>0$, we have, for $\rho=2-2\prod_{j=k+1}^{\infty}r_j$,

$$\sum_{j=k+1}^{\infty}2(1-r_j)\geq-\sum_{j=k+1}^{\infty}\log r_j=-\log\Bigl(1-\frac{\rho}{2}\Bigr)\geq\frac{\rho}{2}.$$

Thus we get

$$\|\{R(\sigma_0)-R(\sigma^{-1})\}f\|^2\leq 2\varepsilon,$$

for $\sigma\in S_k$ with sufficiently large k. Hence $R(\sigma_0)=U^{-1}$.

Similar argument shows that there exists an injective transformation σ_0' on $\mathbf{N}$ such that $R(\sigma_0')=U$. Accordingly, $R(\sigma_0\sigma_0')=R(\sigma_0)R(\sigma_0')=I$, and also $R(\sigma_0'\sigma_0)=I$. Thus we see that $\sigma_0\sigma_0'=\sigma_0'\sigma_0=\mathrm{id}$, and so $\sigma_0'\in\tilde{\mathfrak{S}}_\infty$. The element σ_0' belongs to $\mathfrak{S}_{\mu,E,s}$ and $U=R(\sigma_0')$. This is what we want to prove. Q. E. D.

Denote by $\tilde{\mathfrak{S}}_{\infty,f}(s)$ the subgroup consisting of $\sigma\in\tilde{\mathfrak{S}}_{\infty,f}$ which satisfy $s_{\sigma(j)}=s_j\ (j\in\mathbf{N})$. Then, in general, we have

Proposition 4.7. *Assume $s=(s_j)_{j\in\mathbf{N}}$ be such that $\tilde{\mathfrak{S}}_\infty(s)$ is not completely contained in $\tilde{\mathfrak{S}}_{\infty,f}$. Then the group $\mathfrak{S}_{\mu,E,s}$ contains neither the whole of $\tilde{\mathfrak{S}}_{\infty,f}(s)$, nor of $\tilde{\mathfrak{S}}_\infty(s)\backslash\tilde{\mathfrak{S}}_{\infty,f}(s)$.*

Proof. Taking into account Proposition 4.5(i), we can assume from the beginning that the μ-unital subset $E=\prod_{j\in\mathbf{N}}E_j$ satisfies the condition (MU2str), that is, for any compact $K\subset M$, $K\cap E_j=\phi$ for $j\gg0$. Replacing E by its μ-cofinal one if necessary, we may also assume that each E_j is relatively compact.

For a subset $J\subset\mathbf{N}$, denote by $\mathfrak{S}_J$ (resp. $\tilde{\mathfrak{S}}_J$) the group of all finite permutations (resp. all permutations) on J, and consider it as a subgroup of $\mathfrak{S}_\infty$(resp. $\tilde{\mathfrak{S}}_\infty$). The assumption on s means that there exists an infinite subset $I\subset\mathbf{N}$ for which $s_i=s_{i'}$ for $i, i'\in I$. This means that $\tilde{\mathfrak{S}}_\infty(s)\supset\tilde{\mathfrak{S}}_I$. Then, there exists an infinite subset $J\subset I$ such that E_j, $j\in J$, are mutually disjoint. By the criterion (4.9), we see that

$$\mathfrak{S}_{\mu, E, s}\cap\tilde{\mathfrak{S}}_J=\mathfrak{S}_J(s):=\{\sigma\in\mathfrak{S}_J\,;\,s_{\sigma(j)}=s_j\,(j\in\mathbf{N})\}.$$

This proves the assertions of the proposition. Q. E. D.

Let us give another important consequence of the condition (4.9). This clarifies the situation for "$\sigma\in\mathfrak{S}_{\mu, E, s}$".

Proposition 4.8. *Assume that $\sigma\in\tilde{\mathfrak{S}}_\infty(s)$ belongs to $\mathfrak{S}_{\mu, E, s}$.*

(i) *$E\cap E\sigma^{-1}$ is μ-cofinal with E, and also is $\mu\sigma^{-1}$-cofinal with $E\sigma^{-1}$, where $E\sigma^{-1}=\prod_{j\in\mathbf{N}}E_{\sigma^{-1}(j)}$ and $\mu\sigma^{-1}=(\mu_{\sigma^{-1}(j)})_{j\in\mathbf{N}}$.*

(ii) *Let $F=\prod_{j\in\mathbf{N}}F_j$ with $F_j=E_j\cap E_{\sigma^{-1}(j)}$ if $E_j\cap E_{\sigma^{-1}(j)}\neq\phi$ (true except for a finite number of j's), and $F_j\subset X_j=M$, a relatively compact open set, otherwise. Then $F\overset{\mu}{\sim}E$ and $F\overset{\mu\sigma^{-1}}{\sim}E\sigma^{-1}$. Furthermore the following two product measures on F are mutually equivalent:*

$$\prod_{j\in\mathbf{N}}(\mu_j\,|\,F_j),\qquad \prod_{j\in\mathbf{N}}(\mu_{\sigma^{-1}(j)}\,|\,F_j).$$

Proof. We study the condition (4.9). Then, first we have

$$\|\,|h_j'|-\chi_{E_j}\|_{\mathscr{H}_j}^2\leq\|h_j'-\chi_{E_j}\|_{\mathscr{H}_j}^2\leq$$
$$\leq|1-\|h_j'\|_{\mathscr{H}_j}^2|+|1-\|\chi_{E_j}\|_{\mathscr{H}_j}^2|+2\,|1-\langle h_j', \chi_{E_j}\rangle_{\mathscr{H}_j}|,$$

and so

$$\sum_{j\in\mathbf{N}}\|\,|h_j'|-\chi_{E_j}\|_{\mathscr{H}_j}^2<\infty. \tag{4.14}$$

The j-th term is the integral

$$I_j(\sigma)=\int_{X_j}\Big|\Big(\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j)\Big)^{\frac{1}{2}}\chi_{E_{\sigma^{-1}(j)}}(x_j)-\chi_{E_j}(x_j)\Big|^2 d\mu_j(x_j).$$

Separate the integral on X_j into the sum of those on $E_j\backslash E_{\sigma^{-1}(j)}$, $E_{\sigma^{-1}(j)}\backslash E_j$, and on $E_j\cap E_{\sigma^{-1}(j)}$. Then,

$$I_j(\sigma)=\mu_j(E_j\backslash E_{\sigma^{-1}(j)})+\mu_{\sigma^{-1}(j)}(E_{\sigma^{-1}(j)}\backslash E_j)+$$
$$+\int_{E_j\cap E_{\sigma^{-1}(j)}}\Big|\Big(\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j)\Big)^{\frac{1}{2}}\chi_{E_{\sigma^{-1}(j)}}(x_j)-\chi_{E_j}(x_j)\Big|^2 d\mu_j(x_j).$$

Since $\sum_{j\in\mathbf{N}}I_j(\sigma)<\infty$, we get

(A) $\sum_{j\in\mathbf{N}}\mu_j(E_j\backslash E_{\sigma^{-1}(j)})<\infty$,

(B) $\quad \sum_{j\in\mathbf{N}}\mu_{\sigma^{-1}(j)}(E_{\sigma^{-1}(j)}\backslash E_j) < \infty,$

(C) $\quad \displaystyle\sum_{j\in\mathbf{N}}\int_{F_j}\left|\left(\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j)\right)^{\frac{1}{2}}-1\right|^2 d\mu_j(x_j) < \infty.$

The first inequality shows that $E\cap E\sigma^{-1}$ is $\mu\sigma^{-1}$-cofinal with $E\sigma^{-1}$. The third one is rewritten in a symmetric from as follows by means of a fixed measure $\lambda\in\mathscr{L}\mathscr{F}\mathscr{M}(M)$:

$$\sum_{j\in\mathbf{N}}\int_{F_j}\left|\left(\frac{d\mu_{\sigma^{-1}(j)}}{d\lambda}(x_j)\right)^{\frac{1}{2}}-\left(\frac{d\mu_j}{d\lambda}(x_j)\right)^{\frac{1}{2}}\right|^2 d\lambda(x_j)<\infty.$$

By Kakutani's theorem [7], this is a necessary and sufficient condition for that the two product measures on F in the proposition are mutually equivalent.

Thus the proposition is now completely proved. Q. E. D.

Proposition 4.9. *Assume that all s_j's are equal to zero: $s=(s_j)_{j\in\mathbf{N}}=(0)$. Then, for a $\sigma\in\tilde{\mathfrak{S}}_\infty=\tilde{\mathfrak{S}}_\infty(s)$, the conditions* (A), (B) *and* (C) *are necessary and sufficient for that σ belongs to $\mathfrak{S}_{\mu,E,s}$.*

A proof can be given by examining the proof of the preceeding proposition.

4.6. Examples. We give here several typical examples.

Example 4.10. Assume that all E_j's are mutually disjoint. Then, we see from the criterion (4.9) that $\mathfrak{S}_{\mu,E,s}=\mathfrak{S}_\infty(s)$. Furthermore, in this case, as will be seen in the next section, we have $R(\mathfrak{S}_\infty(s))''=T(G)'$, that is, any intertwining operator for T can be weakly approximated by linear combinations of $R(\sigma)$, $\sigma\in\mathfrak{S}_\infty(s)$. In another terminology, the group $G=\mathrm{Diff}_0(M)$ and a permutation group $\mathfrak{S}_\infty(s)$ form a dual pair.

Example 4.11. Assume that, for any $N>0$, there exists an integer $j_N>N$ such that $\cup_{j\le j_N}E_j$ and $\cup_{j>j_N}E_j$ are mutually disjoint. Then, again from (4.9), we see that any cyclic permutation in $\tilde{\mathfrak{S}}_\infty$ with infinite length cannot comes into $\mathfrak{S}_{\mu,E,s}$. This means that $\mathfrak{S}_{\mu,E,s}\subset\tilde{\mathfrak{S}}_{\infty,f}(s)$. Furthermore, the group $\mathfrak{S}_\infty(s)$ is dense in $\mathfrak{S}_{\mu,E,s}$ in s・w-topology. In this sense, as will be seen in §5, we can say that the groups G and $\mathfrak{S}_\infty(s)$ form a dual pair.

Example 4.12. Let us give an example of $\mathfrak{S}_{\mu,E,s}$ which contains a cyclic permutation with infinite length. This gives also an example for which the subgroup $\mathfrak{S}_\infty(s)=\mathfrak{S}_\infty\cap\mathfrak{S}_{\mu,E,s}$ is not dense in $\mathfrak{S}_{\mu,E,s}$ in s・w-topology.

Put $M=\mathbf{R}$ and consider $X=\prod_{j\in\mathbf{Z}}X_j$, $X_j=M$. Let σ_∞ be a cyclic permutation given by $\sigma_\infty(j)=j+1\,(j\in\mathbf{Z})$. For $j\in\mathbf{N}$, put $c_j=1+1/2+1/3+$

$\cdots+1/j$, and $c_0=0$. We define measures μ_j on X_j as follows. First take a positive function ρ on $[0, \infty)$ which satisfies

$$(1)\quad \int_0^\infty \rho(t)dt=1, \qquad (2)\quad \sum_{j\in\mathbf{N}}\int_0^{1/j}\rho(t)dt<\infty,$$

$$(3)\quad (\sqrt{\rho})'(t)\in L^2([0, \infty)\,;\lambda) \quad \text{with usual Lebesgue measure } \lambda.$$

As an example of such a function ρ, we have $\rho(t)=Ct^4(1+t^6)^{-1}$ with a normalization constant $C>0$. We put $d\mu_j(t)=\rho_j(t)\,dt$ for $j\geq 0$, and $d\mu_{-j}(t)=\rho_j(-t)\,dt$ for $j>0$ with

$$\rho_j(t)=\begin{cases}\rho(t-c_j) & (t\geq c_j)\\ \tau_j(t) & (t\leq c_j),\end{cases}$$

for $j\geq 0$, where $\tau_j(t)>0$ are locally summable functions, arbitrarily chosen.

Take $E_j\subset X_j$ as follows: $E_j=[c_j, \infty)$ for $j\geq 0$, and $E_{-j}=(-\infty, -c_j]$ for $j>0$. Then $\mu_j(E_j)=1$, and $E=\prod_{j\in\mathbf{Z}}E_j$ is a μ-unital subset of X which satisfies the condition (MU2str), i.e., for any compact $K\subset M$, $K\cap E_j=\phi(\,|\,j\,|\gg 0)$.

We put $s=(s_j)_{j\in\mathbf{Z}}=(0)$ with $s_j=0(\forall j)$.

Let us check the conditions (A), (B) and (C) in Proposition 4.9. For (A), $E_j\backslash E_{\sigma_\infty^{-1}(j)}=E_j\backslash E_{j-1}$ and is equal to ϕ for $j>0$. Moreover for $j\in\mathbf{N}$

$$\mu_{-j}(E_{-j}\backslash E_{-j-1})=\mu_j(E_j\backslash E_{j+1})=\int_{c_j}^{c_{j+1}}\rho(t-c_j)dt=\int_0^{1/(j+1)}\rho(t)dt.$$

Therefore we have

$$\sum_{j\in\mathbf{Z}}\mu_j(E_j\backslash E_{\sigma_\infty^{-1}(j)})=\sum_{j\geq 0}\mu_j(E_j\backslash E_{j+1})<\infty.$$

Similarly we see that the condition (B) holds.

Let us now prove the condition (C). Put $F_j=E_j\cap E_{\sigma_\infty^{-1}(j)}$, then $F_j=[c_j, \infty)$, $F_{-j}=(-\infty, -c_{j+1}]$ for $j>0$, and it is enough to prove

$$\sum_{j\in\mathbf{N}}\int_{F_j}|\sqrt{\rho_j(t)}-\sqrt{\rho_{j-1}(t)}|^2\,dt<\infty.$$

Then,

$$j\text{-th term}=\int_0^\infty\{\sqrt{\rho(t+1/j)}-\sqrt{\rho(t)}\}^2\,dt.$$

On the other hand,

$$\sqrt{\rho(t+a)}-\sqrt{\rho(t)}=a\cdot\int_0^1(\sqrt{\rho})'(t+sa)ds,$$

$$|\sqrt{\rho(t+a)}-\sqrt{\rho(t)}|^2\leq a^2\cdot\int_0^1|(\sqrt{\rho})'(t+sa)|^2\,ds.$$

Hence

$$\int_0^{\infty} | \sqrt{\rho(t+a)}-\sqrt{\rho(t)} |^2 \, dt \leq a^2 \cdot \int_0^{\infty} | (\sqrt{\rho})'(t) |^2 \, dt.$$

Put $a=1/j$, then we see that the sum over $j \in \mathbf{N}$ converges. This proves that the condition (C) holds, and so the cyclic permutation σ_∞ with infinite length does belong to the group $\mathfrak{S}_{\mu, E, s}$.

Finally we prove that $\mathfrak{S}_\infty(s)(=\mathfrak{S}_\infty$ here) is not dense in $\mathfrak{S}_{\mu, E, s}$ with respect to s·w-topology in this case. Consider the infinite cyclic permutation $\sigma_\infty \in \mathfrak{S}_{\mu, E, s}$. Take $\sigma \in \mathfrak{S}_\infty$ and put

$$I_k^\sigma := \int_{-\infty}^{\infty} \left| \sqrt{\frac{d\mu_{\sigma_\infty(k)}}{dt}(t)}\, \chi_{E_{\sigma_\infty(k)}}(t) - \sqrt{\frac{d\mu_{\sigma(k)}}{dt}(t)}\, \chi_{E_{\sigma(k)}}(t) \right|^2 dt$$

and $I(\sigma) := \sum_{k \in \mathbf{Z}} I_k^\sigma$. Assume that the element σ_∞ can be approximated by elements of $\mathfrak{S}_\infty$, then $I(\sigma)$ can become smaller and smaller without limit.

On the other hand, note that $\sigma_\infty(-1)=0$. Then, if $\sigma(-1)<0$, we have $I(\sigma) \geq I_{-1}^\sigma = 2$. If $\sigma(-1) \geq 0$, then there exists a $k \geq 0$ such that $\sigma(k)<0$. Since $\sigma_\infty(k)>0$, we have again $I(\sigma) \geq I_k^\sigma = 2$. Thus we come to a contradiction.

Remark 4.13. If the condition (C) is replaced by a stronger one

$$\text{(C}'\text{)} \qquad \sum_{j \in \mathbf{Z}} \int_{F_j} \left| 1 - \frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j) \right| d\mu_j(x_j) < \infty.$$

Then, we have no cyclic permutation $\sigma \in \tilde{\mathfrak{S}}_\infty$ with infinite length which satisfies the conditions (A), (B) and (C$'$).

In fact, let σ be an infinite cyclic permutation, and take a Borel measurable set $B \subset M$. Then for $\tau=\sigma^{-1}$

$$\begin{aligned} & | \mu_{\tau(j)}(B \cap E_{\tau(j)}) - \mu_j(B \cap E_j) | \leq \\ & \leq \mu_{\tau(j)}(E_{\tau(j)} \setminus E_j) + \mu_j(E_j \setminus E_{\tau(j)}) + | \mu_{\tau(j)}(B \cap E_j \cap E_{\tau(j)}) - \mu_j(B \cap E_j \cap E_{\tau(j)}) | \\ & \leq \mu_{\tau(j)}(E_{\tau(j)} \setminus E_j) + \mu_j(E_j \setminus E_{\tau(j)}) + \int_{E_j \cap E_{\tau(j)}} \left| 1 - \frac{d\mu_{\tau(j)}}{d\mu_j}(x_j) \right| d\mu_j(x_j). \end{aligned}$$

Therefore $\sum_{j \in \mathbf{Z}} | \mu_{\tau(j)}(B \cap E_{\tau(j)}) - \mu_j(B \cap E_j) | < \infty$.

Hence, there exists a limit $\lim_{k \to \infty} \mu_{\tau^k(j)}(B \cap E_{\tau^k(j)})$, where j is so chosen that $\tau(j) \neq j$. Denote this limit by $\omega(B)$, then, by a general theorem, ω is a measure on M. From the condition (MU2) for (μ, E), we see that $\omega(K)=0$ for any compact $K \subset M$, whence $\omega(M)=0$. But, this contradicts the fact that $\lim_{k \to \infty} \mu_{\tau^k(j)}(E_{\tau^k(j)})=1$.

4.7. Relation to quasi-invariance of the product measure $\nu_{\mu, E}$. When a permutation $\sigma \in \mathfrak{S}_\infty(s)$ is addmitted to have a unitary operator

$R(\sigma)$, or σ belongs to $\mathfrak{S}_{\mu,E,s}$, it acts on the product measures on $X=\prod_{j\in\mathbf{N}}X_j$, $X_j=M$, as shown in Proposition 4.8. Here we study from another point of view the quasi-invariance under σ of the product measure $\nu_{\mu,E}$ itself. In this study we find that for the vectors

$$\sqrt{\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j)}\cdot\chi_{E_{\sigma^{-1}(j)}}(x_j)\quad(j\in\mathbf{N}),$$

two multiplicative factors, the square-root of the density and a transformed function $\chi_{E_{\sigma^{-1}(j)}}$, cannot be separated in general, or they should be considered together, not separated, in connection with the tensored space $\otimes^{\chi}_{j\in\mathbf{N}}\mathscr{H}_j$. In other words, for a permutation σ, leaving the tensored space $\otimes^{\chi}_{j\in\mathbf{N}}\mathscr{H}_j$ invariant and leaving the measure $(\nu_{\mu,E},\ \mathscr{M}(\mu,E))$ quasi-invariant are different things.

The following example explains the situation.

Example 4.14. In Example 4.12, we have treated (μ, E) on $X=\prod_{j\in\mathbf{Z}}X_j$, $X_j=M=\mathbf{R}$. Assume in this example that $E\sigma=\prod_{j\in\mathbf{Z}}E_{\sigma(j)}$ is μ-cofinal with E, and accordingly that $\otimes_{j\in\mathbf{Z}}\chi_{E_{\sigma(j)}}\in\otimes^{\chi}_{j\in\mathbf{Z}}\mathscr{H}_j$, for an element $\sigma\in\mathfrak{S}_{\mu,E,s}$. The assumption means the following:

$$\textstyle\sum_{j\in\mathbf{Z}}|\,1-\mu_j(E_{\sigma(j)})\,|<\infty,\quad \sum_{j\in\mathbf{Z}}\mu_j(E_j\ominus E_{\sigma(j)})<\infty.$$

Now let us take as σ the infinite cyclic permutation $\sigma_\infty(j)=j+1\,(j\in\mathbf{Z})$. Then, for $j>0$, $E_{\sigma_\infty(-j)}\backslash E_{-j}=E_{-j+1}\backslash E_{-j}=(-c_j,\ -c_{j-1}]$, and

$$\mu_{-j}(E_{\sigma_\infty(-j)}\backslash E_{-j})=\int_{c_{j-1}}^{c_j}\tau_j(t)dt.$$

Since locally summable functions $\tau_j>0$ can be chosen arbitrarily, we take them in such a way that $\int_{c_{j-1}}^{c_j}\tau_j(t)dt=1\,(j\in\mathbf{N})$. Then we have $\sum_{j\in\mathbf{Z}}\mu_j(E_j\ominus E_{\sigma_\infty(j)})=\infty$, and so $E\sigma_\infty$ is not μ-cofinal with E. Accordingly the σ-ring $\mathscr{M}(\mu,E)$ is not stable under the action of σ_∞: $x=(x_j)_{j\in\mathbf{Z}}\mapsto x\sigma_\infty=(x_{\sigma_\infty(j)})_{j\in\mathbf{Z}}$ on X, whereas σ_∞ belongs to $\mathfrak{S}_{\mu,E,s}$ as shown in Example 4.12.

Here we give a good sufficient condition for the invariance of σ-ring $\mathscr{M}(\mu,E)$ and the quasi-invariance of the product measure $\nu_{\mu,E}$, under a permutation, as follows.

Proposition 4.15. *Let $\sigma\in\tilde{\mathfrak{S}}_\infty$. Assume, for (μ,E) on $X=\prod_{j\in\mathbf{N}}X_j$, $X_j=M$, that the next three conditions hold:*

$$(1)\quad \textstyle\sum_{j\in\mathbf{N}}\mu_j(E_j\ominus E_{\sigma(j)})<\infty,$$

$$(2)\quad \sum_{j\in\mathbf{N}}\int_{E_j}\left|\sqrt{\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}(x_j)}-1\right|^2 d\mu_j(x_j)<\infty,$$

$$(3)\quad \textstyle\sum_{j\in\mathbf{N}}\mu_j(N_j)<\infty \Rightarrow \sum_{j\in\mathbf{N}}\mu_j(N_{\sigma(j)})<\infty.$$

Then, (i) *for any μ-unital subset $F=\prod_{j\in\mathbf{N}}F_j$, μ-cofinal with $E: F\overset{\mu}{\sim}E$, $F\sigma$ is μ-unital and $F\sigma\overset{\mu}{\sim}E$. Accordingly the ring of measurable sets $\mathcal{M}(\mu, E)$ is stable under $S(\sigma): x\mapsto x\sigma^{-1}(x\in X)$.*

(ii) $\quad S(\sigma)\nu_{\mu,E}\simeq\nu_{\mu,E}$.

Proof. Firstly it follows from the condition (1) that $E\sigma$ is μ-unital, and $E\sigma\overset{\mu}{\sim}E$. Secondly, if $F\overset{\mu}{\sim}E$, then putting $N_j=E_j\ominus F_j$ in (3), we have $\sum_{j\in\mathbf{N}}\mu_j(E_{\sigma(j)}\ominus F_{\sigma(j)})<\infty$, i.e., $E\sigma\overset{\mu}{\sim}E\sigma$. Hence $F\sigma\overset{\mu}{\sim}E$.

Thus the transformation $S(\sigma)$ on X is $\mathcal{M}(\mu, E)$-measurable.

For the assertion (ii), the condition (2) means by Kakutani's theorem that $\nu_{\mu,E}|E\simeq S(\sigma)\nu_{\mu,E}|E$. So that it is enough for us to see that the condition (2) holds also for any $F\overset{\mu}{\sim}E$. To see this, we have

$$\int_{F_j}\left|\sqrt{\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}}-1\right|^2 d\mu_j\le\int_{E_j\cap F_j}\left|\sqrt{\frac{d\mu_{\sigma^{-1}(j)}}{d\mu_j}}-1\right|^2 d\mu_j+$$
$$+2(\mu_{\sigma^{-1}(j)}(F_j\backslash E_j)+\mu_j(F_j\backslash E_j)),$$

and so the condition (2) for F is obtained.

Example 4.16. Eventhough the situation where the above conditions (1)-(3) hold is rather general, we give such examples in the framework of Example 4.12. Assume that locally summable functions $(\tau_j)_{j\in\mathbf{N}}$ there satisfy

$$\sum_{j\ge0}\int_{-\infty}^{c_j}\tau_j(t)dt<\infty.$$

Then the conditions (1)-(3) hold for $\sigma=\sigma_\infty$, in this choice of $\mu=(\mu_j)_{j\in\mathbf{N}}$.

In fact, (1) and (2) are easy to prove, and (3) is essentially equivalent to the following: for $M_k\subset[0,\infty)$, $k\in\mathbf{N}$,

$$\sum_{k\in\mathbf{N}}\int_{M_k}\rho(t)dt<\infty \Longleftrightarrow \sum_{k\in\mathbf{N}}\int_{M_k}\rho\Big(t+\frac{1}{k}\Big)dt<\infty,\quad \text{with } \rho(t)=C\frac{t^4}{1+t^6}.$$

For $\Leftarrow$, we remark that there exists a constant $\gamma>0$ such that $\rho\left(t+\frac{1}{k}\right)\ge\gamma\rho(t)$ for $t>0$, $k\in\mathbf{N}$.

For $\Rightarrow$, we see that the essential part is to prove

$$J:=\sum_{k\in\mathbf{N}}\frac{1}{k}\int_{M_k}\frac{t^3}{1+t^6}dt<\infty.$$

By Hölder's inequality for $(p, q)=(4, 4/3)$,

$$J\leq\Big(\sum_{k\in\mathbf{N}}\frac{1}{k^4}\Big)^{1/4}\Big(\sum_{k\in\mathbf{N}}\Big(\int_{M_k}\frac{t^3}{1+t^6}dt\Big)^{4/3}\Big)^{3/4}.$$

Again by Hölder's inequality

$$\Big(\int_{M_k}\frac{t^3}{1+t^6}dt\Big)^{4/3}\leq a\cdot\int_{M_k}\rho(t)dt \quad (a>0,\ \text{constant}).$$

§ 5. Dual pairs between $\mathrm{Diff}_0(M)$ and certain permutation groups

5.1. Dual pairs. Let us first introduce the notion of a dual pair. Assume a unitary representation $(T, H(T))$ of a certain group G is given together with such a one R of another group U on the same Hilbert space $H(T)$. If there holds the relation $T(G)'=R(U)''$, then we call G and U form a *dual pair* (through T and R). Here $T(G)'$ denotes the commuting algebra for $T(G)$.

If the group U is compact, a dual pair gives a 1-1 correspondence $\pi\mapsto T_\pi$, from a subset of $\hat{U}$ into $\hat{G}$ by decomposing the representation $T\cdot R: G\times U\ni(g, u)\mapsto T(g)\circ R(u)$, of $G\times U$ into irreducibles: $T\cdot R\cong\sum_{\pi\in\hat{U}}\pi\times T_\pi$. Here $\hat{U}$ denotes the set of all equivalence classes of irreducible unitary representations of U, and T_π is realized naturally in the space $\mathrm{Hom}_U(H(\pi), H(T))$, with $H(\pi)$ the space for π. In this case, U is also called a *symmetry group* of T[14].

In our present case, we take as G the group $\mathrm{Diff}_0(M)$ and as T one of the tensor product representations $T_E^{\mathfrak{a}}$ or their equivalents $T_{E'}^{\alpha}$ in § 3. Then there appear the infinite symmetric group $\mathfrak{S}_\infty$ or related permutation groups as a symmetry group U. Here the group U is turned out to be non-compact, and accordingly the situation is not so simple as in the compact group case. However, we will show in another paper that, at least in case $U\subset\mathfrak{S}_\infty$, an IUR T_π can be constructed, for every IURs π of U, and that the representation T can be decomposed into these IURs T_π's.

We remark here that representations of the infinite symmetric group $\mathfrak{S}_\infty$ are studied from many different points of views, for instance, in [2], [8], [13] and [15].

5.2. Dual pair relations between G and subgroups of $\mathfrak{S}_\infty$. Here we assume $\dim M\geq 2$. Let us first treat a simple case where a certain disjointness condition on E_j's is assumed. The following theorem is one of our main results in this paper, and it explains well a background of our method of constructing IURs of G given in the previous paper [4].

Let $\mu=(\mu_j)_{j\in\mathbf{N}}$ be as in § 3.1 a system of measures on $X_j=M$ taken from $\mathscr{LFM}(M)$, and $E=\prod_{j\in\mathbf{N}}E_j$ be a μ-unital subset of $X=\prod_{j\in\mathbf{N}}X_j$ for which the conditions (MU1)-(MU2) hold. Consider the infinite tensor product $T_E^{\bar{\alpha}}=\otimes_{j\in\mathbf{N}}^{\chi}T_j^{\alpha_j}$ of representations $(T_j^{\alpha_j}, \mathscr{H}_j)$, $j\in\mathbf{N}$, with $\chi=(\chi_j)_{j\in\mathbf{N}}$, $\chi_j=\|\chi_{E_j}\|_{\mathscr{H}_j}^{-1}\chi_{E_j}$, $\mathscr{H}_j=L^2(X_j, \mathscr{B}_j, \mu_j)$, and α_j given in (4.2) with parameter s_j.

Theorem 5.1. *Let* $\dim M\geq 2$, *and assume that E is μ-cofinal with another μ-unital subset $F=\prod_{j\in\mathbf{N}}F_j$ for which $F_j\cap F_k=\phi$ for $j\neq k$. Then the group of permutations $\mathfrak{S}_{\mu,E,s}$ is equal to the subgroup $\mathfrak{S}_\infty(s)$ of the infinite symmetric group $\mathfrak{S}_\infty$, where $s=(s_j)_{j\in\mathbf{N}}$. The diffeomorphism group $G=\mathrm{Diff}_0(M)$ and $\mathfrak{S}_\infty(s)\subset\mathfrak{S}_\infty$ form a dual pair:*

$$T_E^{\bar{\alpha}}(G)''=R(\mathfrak{S}_\infty(s))', \quad T_E^{\bar{\alpha}}(G)'=R(\mathfrak{S}_\infty(s))''.$$

In particular, when all the parameters s_j, $j\in\mathbf{N}$, are equal to each other and all E_j's are mutually disjoint, the groups G and $\mathfrak{S}_\infty$ form a dual pair.

Put $S=\mathfrak{S}_\infty(s)$ and $\mathscr{C}=R(S)''=\{R(\sigma)\,;\,\sigma\in S\}''$, the weakly closed operator algebra generated by $R(\sigma)$'s. Then, $\mathscr{C}\subset T_E^{\bar{\alpha}}(G)'$, by Lemma 4.1. Therefore, to prove the theorem, it is enough to show the converse inclusion $\mathscr{C}\supset T_E^{\bar{\alpha}}(G)'$. To do so, we apply several lemmas given in the succeeding subsections.

5.3. A general lemma on a commuting operator. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{B}(\mathscr{H})$ the set of all bounded linear operators, and $\mathscr{C}\subset\mathscr{B}(\mathscr{H})$ a weakly closed subspace. Further let P_n, $n\in\mathbf{N}$, be a sequence of orthogonal projections on $\mathscr{H}$ approximating the identity operator I on $\mathscr{H}$ strongly.

Lemma 5.2. *Assume that an operator $A\in\mathscr{B}(\mathscr{H})$ satisfies the following condition:*

(P) *there exists a sequence of operators $A_n\in\mathscr{C}$, $n\in\mathbf{N}$, such that*

(a) $P_nAP_n=P_nA_nP_n\ (n\in\mathbf{N})$,

(b) $\|A_n\|\leq M_0\ (\forall n)$ *for some constant* $M_0>0$.

Then A belongs to $\mathscr{C}$: $A\in\mathscr{C}$.

Proof. Denote by $<.,.>$ the inner product on $\mathscr{H}$. Then, for any $\phi,\psi\in\mathscr{H}$,

$$\begin{aligned}|<A\phi,\psi>-<A_n\phi,\psi>|&\leq|<A\phi,(I-P_n)\psi>|+\\&+|<A(I-P_n)\phi,P_n\psi>|+|<A_n\phi,(I-P_n)\psi>|+\\&+|<A_n(I-P_n)\phi,P_n\psi>|\leq(\|A\|+M_0)\{\|\phi\|\ \|(I-P_n)\psi\|\\&+\|(I-P_n)\phi\|\ \|\psi\|\}\to 0\ (n\to\infty).\end{aligned}$$

5.4. Lemmas for finite tensor products of representations. To apply the above general lemma to our situation, we prepare the following 'so-called' dual pair relation for finite tensor products.

Take a finite number of representations $(T_j^{\alpha_j}, L^2(X_j, \mathscr{B}_j, \mu_j))$, $j\in J$, of $G=\mathrm{Diff}_0(M)$, where α_j is a 1-cocycle in (4.2) with parameter $s_j\in\mathbf{R}$, and J is a finite index set. Let $s(k)$, $k\in K_J$, be all the different numbers in s_j's, and put $J(k)=\{j\in J\,;\, s_j=s(k)\}$. Then the subgroup $\mathfrak{S}_J((s_j)_{j\in J})=\{\sigma\in\mathfrak{S}_J\,;\, s_{\sigma(j)}=s_j\ (j\in J)\}$ of $\mathfrak{S}_J$ is equal to $\prod_{k\in K_J}\mathfrak{S}_{J(k)}$. Consider the tensor product $T_J=\otimes_{j\in J}T_j^{\alpha_j}$ on the space $\mathscr{H}_J=\otimes_{j\in J}\mathscr{H}_j$ with $\mathscr{H}_j=L^2(X_j, \mathscr{B}_j, \mu_j)$. Then

$$T_J(g)h(x_J)=\frac{\gamma_J(g^{-1}x_J)}{\gamma_J(x_J)}\prod_{k\in K_J}\Big(\prod_{j\in J(k)}\frac{d\mu_j(g^{-1}x_j)}{d\mu_j(x_j)}\Big)^{\frac{1}{2}+is(k)}h(g^{-1}x_J),$$

for $h\in\mathscr{H}_J$ and $g\in G$, where $\gamma_J(x_J)=\prod_{j\in J}\gamma_j(x_j)$, $x_J=(x_j)_{j\in J}\in\prod_{j\in J}X_j$ with $X_j=M$, and $g^{-1}x_J=(g^{-1}x_j)_{j\in J}$. On the other hand, we can define an action of $\mathfrak{S}_J((s_j)_{j\in J})$ as follows:

$$R_J(\sigma)h(x_J)=\frac{\gamma_J(x_J\sigma)}{\gamma_J(x_J)}\prod_{k\in K_J}\Big(\prod_{j\in J(k)}\frac{d\mu_{\sigma^{-1}(j)}(x_j)}{d\mu_j(x_j)}\Big)^{\frac{1}{2}+is(k)}h(x_J\sigma),$$

where $x_J\sigma=(x_{\sigma(j)})_{j\in J}$.

The dual pair relation is claimed as in

Lemma 5.3. *The tensor product representation T_J of $G=\mathrm{Diff}_0(M)$ and the representation R_J of the subgroup $S=\mathfrak{S}_J((s_j)_{j\in J})=\prod_{k\in K_J}\mathfrak{S}_{J(k)}$ of $\mathfrak{S}_J$ form a dual pair, or*

$$T_J(G)''=R_J(S)',\qquad T_J(G)'=R_J(S)''.$$

For the sake of reference, we remark, at this point, about the relation between different finite tensor product representations. Denote by $|J|$ the number of elements in J.

Lemma 5.4. *Let J_1 and J_2 be two finite subsets of* $\mathbf{N}$. *Assume that $|J_1|\neq|J_2|$. Then, the tensor product representations T_{J_1} and T_{J_2} are mutually disjoint, or any intertwining operator between them is identically zero.*

For completeness, proofs of these lemmas are given in Appendix.

5.5. Fundamental Lemmas. The following observation is a key for our proof of dual pair. Let T_E^{α} on $\otimes_{j\in\mathbf{N}}^{\chi}\mathscr{H}_j$, $\mathscr{H}_j=L^2(X_j, \mathscr{B}_j, \mu_j)$, be the infinite tensor product representation of G in question. Let U be a connected open subset of M and put $V=M\backslash U$, and take a subspace $\mathscr{W}$ of the infinite tensor product space $\otimes_{j\in\mathbf{N}}^{\chi}\mathscr{H}_j$ given as follows: for a finite subset J of N and a series of vectors $f_j\in\mathscr{H}_j(j\notin J)$, $\mathscr{W}$ is expressed as

$$\mathscr{W}=(\otimes_{j\in J}\mathscr{H}_j(U))\otimes(\otimes_{j\notin J}f_j)\tag{5.1}$$
$$\text{with } \mathscr{H}_j(U)=L^2(U, \mathscr{B}_j|U, \mu_j|U)\hookrightarrow\mathscr{H}_j,$$

where $f_j \in \mathscr{H}_j(V)\,(j \notin J)$ are such that $\Pi_{j\notin J} \| f_j \|_{\mathscr{H}_j}$ is unconditionally convergent and $\sum_{j\notin J} | 1 - \langle f_j, \chi_j \rangle | < \infty$. Then we have the following simple lemma.

Lemma 5.5. *Take a subspace $\mathscr{W}$ of $\otimes_{j\in\mathbf{N}}^{\chi} \mathscr{H}_j$ of the form in (5.1), and let $P_{\mathscr{W}}$ be the orthogonal projection onto $\mathscr{W}$. Then, for any intertwining operator $A \in T_E^{\alpha}(G)'$, $P_{\mathscr{W}} A P_{\mathscr{W}} = P_{\mathscr{W}} A' P_{\mathscr{W}}$ with an $A' \in \mathscr{C} = R(\mathfrak{S}_{\mu, E, s})''$ such that $\| A' \| \le \| A \|$.*

Proof. Let $G(U) = \mathrm{Diff}_0(U) \subset G$. Then, for $g \in G(U)$, $T_j^{\alpha_j}(g) f_j = f_j$ for $j \notin J$, because $f_j = 0$ on U. Therefore the subspace $\mathscr{W}$ is invariant under $T_E^{\alpha}(G(U))$, and so

$$T_E^{\alpha}(g) P_{\mathscr{W}} = P_{\mathscr{W}} T_E^{\alpha}(g) = P_{\mathscr{W}} T_E^{\alpha}(g) P_{\mathscr{W}} \quad (g \in G(U)). \tag{5.2}$$

Using this, we get from $T_E^{\alpha}(g) A = A T_E^{\alpha}(g)$,

$$P_{\mathscr{W}} T_E^{\alpha}(g) P_{\mathscr{W}} \circ P_{\mathscr{W}} A P_{\mathscr{W}} = P_{\mathscr{W}} A P_{\mathscr{W}} \circ P_{\mathscr{W}} T_E^{\alpha}(g) P_{\mathscr{W}} \ (g \in G(U)). \tag{5.3}$$

On the other hand, the representation of $G(U)$ induced on $\mathscr{W}$ is isomorphic to the finite tensor product of $(T_j^{\alpha_j} | G(U), \mathscr{H}_j(U))$, $j \in J$, that is, $P_{\mathscr{W}} T_E^{\alpha}(g) P_{\mathscr{W}} | \mathscr{W}$ is equivalent to $T_J(g)$, $g \in G(U)$, in the notation in §5.4 (M and $G = G(M)$ are replaced by U and $G(U)$ here). Thus we can apply Lemma 5.3 and see that $P_{\mathscr{W}} A P_{\mathscr{W}}$ is a linear combination of $P_{\mathscr{W}} R_J(\sigma) P_{\mathscr{W}}$, $\sigma \in \mathfrak{S}_J(s_J)$, where $s_J = (s_j)_{j\in J}$. This means that $P_{\mathscr{W}} A P_{\mathscr{W}} = P_{\mathscr{W}} A' P_{\mathscr{W}}$ with $A' \in \langle R(\sigma)\,;\, \sigma \in \mathfrak{S}_J(s_J) \rangle$, the finite dimensional algebra generated by these $R(\sigma)$'s. Further we have $\| A' \| = \| P_{\mathscr{W}} A' P_{\mathscr{W}} \| = \| P_{\mathscr{W}} A P_{\mathscr{W}} \| \le \| A \|$.

Thus the proof of the lemma is now complete.

Applying this lemma, we obtain the following fundamental result.

Lemma 5.6. *Assume that there exists an increasing sequence U_n of open subsets of M such that*

(a) *$\cup_{n\in\mathbf{N}} U_n = M$ and each U_n is connected, and*

(b) *for each n, $E_j \subset U_n$ $(j \in J_n)$ and $E_j \cap U_n = \phi$ $(j \notin J_n)$ with a finite subset $J_n \subset \mathbf{N}$.*

Then, under representations T_E^{α} and R, the diffeomorphism group G and the permutation group $\mathfrak{S}_\infty(s) \subset \mathfrak{S}_\infty$ form a dual pair, and also the groups G and $\mathfrak{S}_{\mu, E, s}$ form a dual pair too:

$$T_E^{\alpha}(G)' = R(\mathfrak{S}_\infty(s))'' = R(\mathfrak{S}_{\mu, E, s})'', \quad T_E^{\alpha}(G)'' = R(\mathfrak{S}_\infty(s))' = R(\mathfrak{S}_{\mu, E, s})'.$$

Proof. First note that $\cup_{n\in\mathbf{N}} G(U_n) = G$, which comes from $U_n \nearrow M$. Fix n. Then each space $\mathscr{H}_j$ is decomposed into a direct sum as

$$\mathscr{H}_j = \mathscr{H}_j(U_n) \otimes \mathscr{H}_j(V_n) \quad \text{with } V_n = M \backslash U_n.$$

Put $\mathscr{W}_n=(\otimes_{j\in J_n}\mathscr{H}_j(U_n))\otimes(\otimes_{j\notin J_n}\chi_j)$ and $P_n=P_{\mathscr{W}_n}$. Then, $P_n\nearrow I$ the identity operator, or $\cup_{n\in\mathbf{N}}\mathscr{W}_n$ spans topologically the total space $\otimes^{\chi}_{j\in\mathbf{N}}\mathscr{H}_j$. Since $E_j\cap U_n=\phi\,(j\notin J_n)$ by assumption, we have $\chi_j\in\mathscr{H}_j(V_n)\,(j\notin J_n)$, and so we can apply Lemma 5.5 to $G(U_n)$ and $\mathscr{W}_n$. Hence, for any intertwining operator $A\in T^{a}_{E}(G)'$, we have $P_nAP_n=P_nA_nP_n$ with an $A_n\in\mathscr{C}=R(\mathfrak{S}_{\mu,E,s})''$ such that $\|A_n\|\le\|A\|$.

Thus we come to the situation where Lemma 5.2 can be applied and conclude that $A\in\mathscr{C}$ or $T^{a}_{E}(G)'\subset R(\mathfrak{S}_{\mu,E,s})''$. Since the converse inclusion is clear, the dual pair relation between G and $\mathfrak{S}_{\mu,E,s}$ is now established. Q. E. D.

Note that, in the case of Lemma 5.6, the subgroup $\mathfrak{S}_\infty(s)\subset\mathfrak{S}_\infty$ is everywhere dense in the permutation group $\mathfrak{S}_{\mu,E,s}$ in s·w-topology, and the latter is a subgroup of $\tilde{\mathfrak{S}}_{\infty,f}$.

5.6. Proof of Theorem 5.1. By assumption on (μ, E), we have a μ-unital subset $F=\prod_{j\in\mathbf{N}}F_j$ such that $F\overset{\mu}{\sim}E$ and F_j, $j\in\mathbf{N}$, are mutually disjoint. Here we normalize F to get a μ-unital subset, μ-cofinal with E, for which Lemma 5.6 is applicable. Cf. [4, §1.8] for another kind of normalization of E.

Lemma 5.7. (i) *There exists a μ-unital subset $E'=\prod_{j\in\mathbf{N}}E'_j$, μ-cofinal with E, such that the condition (MU2str) holds, and E'_j's are mutually disjoint, each relatively compact, open and with finite number of connected components.*

(ii) *There exists a μ-unital subset $E''=\prod_{j\in\mathbf{N}}E''_j\overset{\mu}{\sim}E$ for which E''_j's are relatively compact, not necessarily mutually disjoint but satisfy the following condition:*

(AB) *there exists an increasing sequence U_n, $n\in\mathbf{N}$, of connected, relatively compact, open subsets of M such that $\cup_{n\in\mathbf{N}}U_n=M$ and that, for each n, there exists a finite subset $J_n\subset\mathbf{N}$ for which $E''_j\subset U_n$ or $E''_j\cap U_n=\phi$ according as $j\in J_n$ or not.*

Proof. (i) Take an increasing sequence of connected, relatively compact, open subsets $W_n\,(n\in\mathbf{N})$ of M such that $\cup_{n\in\mathbf{N}}W_n=M$, and put $K_n=\mathrm{Cl}(W_n)$. Fix a small constant $\varepsilon>0$.

For each F_j, there exists a W_{m_j} such that $\mu_j(F_j\backslash W_{m_j})<\varepsilon/2^{j+1}$. Further, since E satisfies the condition (MU2), so does F, and so there exists an increasing sequence $N_n\in\mathbf{N}$ such that $\sum_{j>N_n}\mu_j(F_j\cap K_n)<\varepsilon/2^{n+1}$. Put $E^{(0)}_j=(F_j\cap W_{m_j})\backslash\cup_{n:j>N_n}K_n$, and $E^{(0)}=\prod_{j\in\mathbf{N}}E^{(0)}_j$. Then $E^{(0)}\overset{\mu}{\sim}E$, and $E^{(0)}$ satisfies the condition (MU2str): for any compact subset K of M, $K\cap E^{(0)}_j=\phi\,(j\gg0)$. The sets $E^{(0)}_j$ are mutually disjoint and relatively compact.

Hereafter, we consider only such open subsets U that their boundaries $\partial U=\mathrm{Cl}(U)\backslash U$ are null sets: $\mu_j(\partial U)=0$ for any μ_j. For each $E^{(0)}_j$, we take a relatively compact, open subset $E_j^{(1)}$ such that

$$\mu_j(E_j^{(1)} \ominus E_j^{(0)}) + \sum_{k \neq j} \mu_k(E_j^{(1)} \cap E_k^{(0)}) < \varepsilon/2^j. \tag{5.4}$$

Then, $E^{(1)} = \prod_{j \in \mathbf{N}} E_j^{(1)} \overset{\mu}{\sim} E^{(0)}$. Further, put inductively for $j = 1, 2, \cdots$,

$$E_j^{(2)} = E_j^{(1)} \setminus (\cup_{k<j} \mathrm{Cl}(E_k^{(2)})) = E_j^{(1)} \setminus \cup_{k<j} \mathrm{Cl}(E_k^{(1)}).$$

Then, $E_j^{(2)}$ are mutually disjoint, open subsets. Note that

$$\begin{aligned} E_j^{(2)} \ominus E_j^{(1)} &= E_j^{(1)} \cap (\cup_{k<j} \mathrm{Cl}(E_k^{(1)})) \equiv E_j^{(1)} \cap (\cup_{k<j} E_k^{(1)}) \\ &\subset (E_j^{(1)} \setminus E_j^{(0)}) \cup (E_j^{(0)} \cap \cup_{k<j} E_k^{(1)}) \quad \text{(modulo null sets)}, \end{aligned}$$

Then we get $\mu_j(E_j^{(2)} \ominus E_j^{(1)}) \leq \mu_j(E_j^{(1)} \ominus E_j^{(0)}) + \sum_{k<j} \mu_j(E_j^{(0)} \cap E_k^{(1)})$, and so

$$\sum_{j \in \mathbf{N}} \mu_j(E_j^{(2)} \ominus F_j^{(1)}) \leq \sum_{j \in \mathbf{N}} \mu_j(E_j^{(1)} \ominus E_j^{(0)}) + \sum_{k \in \mathbf{N}} \sum_{j=k+1}^{\infty} \mu_j(E_j^{(0)} \cap E_k^{(1)}) < \varepsilon. \tag{5.5}$$

This gives us $E^{(2)} = \prod_{j \in \mathbf{N}} E_j^{(2)} \overset{\mu}{\sim} E^{(1)}$. Finally, picking up finite number of connected components of each $E_j^{(2)}$ appropriately, we get E_j' and then $E' = \prod_{j \in \mathbf{N}} E_j'$ demanded in the assertion (i) in the lemma.

(ii) We start with E' in (i) but take a new increasing sequence $W_n \nearrow M\,(n \to \infty)$ of relatively compact, connected, open subsets. We note here the following elementary fact which will be repeatedly applied in the discussions below. Let $C \subset W_n$ be a closed subset of M and p a point outside of W_n, then a path connecting p with C meets necessarily with $W_n \setminus C$ before meeting C itself, and so we can connect p with W_n by a small open path without touching C inside of W_n.

We proceed inductively as follows. First consider E_1' and take a W_{m_1} containing its closure. Put $I_1 = \{i\,; E_i' \cap W_{m_1} \neq \phi\}$, $J_1 = \{i \in I_1\,; E_i' \subset W_{m_1}\}$. For $i \in J_1$, put $E_i'' = E_i'$. For j's in $I_1 \setminus J_1$, consider first the union $U_1' = W_{m_1} \cup (\cup_{i \in I_1} E_i')$. We can make it connected (since dim $M \geq 2$) by adding small open pathes appropriately to E_j', $j \in I_1 \setminus J_1$, to connect their connected components outside of W_{m_1} to W_{m_1}, so that we get E_j'', $j \in I_1 \setminus J_1$, and a connected open $U_1 = W_{m_1} \cup (\cup_{j \in I_1} E_j'')$ satisfying

$$\mu_j(E_j'' \ominus E_j') + \sum_{k \notin I_1} \mu_k(E_j'' \cap E_k') < \varepsilon/2^j \quad (j \in I_1 \setminus J_1). \tag{5.6}$$

Note that the above sum is actually a finite sum. Put $A_1 = \{k \notin I_1,\ E_k' \cap E_j'' \neq \phi\,(\exists j \in I_1 \setminus J_1)\}$ and

$$E_k'' = E_k' \setminus \cup_{j \in I_1 \setminus J_1} E_j'' \quad (k \in A_1). \tag{5.7}$$

Thus E_i'' are determined for $i \in B_1 := I_1 \cup A_1$. Note that $E_i'' \subset U_1$ for $i \in I_1$, and that $E_k''\,(k \in A_1)$ and $E_i'\,(i \notin B_1)$ are disjoint with U_1. Thus the first step of the induction is completed.

For the second step, we take a W_{m_2}, $m_2 > m_1$, which contains all of

$\mathrm{Cl}(E_i''),\ i\in B_1$. Then, $U_1\subset W_{m_2}$. Put $I_2=\{i\notin B_1\,;\,E_i'\cap W_{m_2}\neq\phi\}$ and $J_2=\{i\in I_2\,;\,E_i'\subset W_{m_2}\}$. For $i\in J_2$, put $E_i''=E_i'$. For j's in $I_2\backslash J_2$, consider first the union $U_2'=W_{m_2}\cup(\cup_{i\in I_2}E_i')$. We can make U_2' connected by adding small open pathes to E_j', $j\in I_2\backslash J_2$, to connect their connected components outside of W_{m_2}(finite number by (i)) to W_{m_2}, not touching $E_i''\subset W_{m_2}$, $i\in B_1$, already determined in the previous step (cf. the note at the beginning of the proof for (ii)), so that we get E_j'', $j\in I_2\backslash J_2$, and a connected open $U_2=W_{m_2}\cup(\cup_{j\in I_2}E_j'')$, such that

$$\mu_j(E_j''\ominus E_j')+\sum_{k\notin B_1\cup I_2}\mu_k(E_j''\cap E_k')<\varepsilon/2^j\ (j\in I_2\backslash J_2). \tag{5.6$'$}$$

Put $A_2=\{k\notin B_1\cup I_2\,;\,E_k'\cap E_j''\neq\phi\,(\exists j\in I_2\backslash J_2)\}$, and

$$E_k''=E_k'\backslash(\cup_{j\in I_2\backslash J_2}E_j'')\quad(k\in A_2). \tag{5.7$'$}$$

Thus E_i'' are determined for $i\in B_2:=B_1\cup I_2\cup A_2$. Note that $E_i''\subset U_2\,(i\in B_1\cup I_2)$, and that $E_i''(i\in A_2)$ and $E_i'(i\notin B_2)$ are disjoint with U_2.

For the third step, we take a W_{m_3}, $m_3>m_2$, containing all $\mathrm{Cl}(E_i'')$, $i\in B_2$. Put $I_3=\{i\notin B_2\,;\,E_i'\cap W_{m_3}\neq\phi\}$, and $J_3=\{i\in I_3\,;\,E_i'\subset W_{m_3}\}$. For $i\in J_3$, put $E_i''=E_i'$. For j's in $I_3\backslash J_3$, consider $U_3'=W_{m_3}\cup(\cup_{i\in I_3}E_i')$, and so on. We omit to state the n-th step since it is now clear.

Finally, thus obtained E_j'', $j\in\mathbf{N}$, give a μ-unital subset $E''\overset{\mu}{\sim}E'$, as is seen by an evaluation similar to (5.5), and satisfies the condition (AB).

The proof of the assertion (ii) of the lemma is now complete.

Proof of Theorem 5.1. Now let us return to the proof of Theorem 5.1. From $E''\overset{\mu}{\sim}E$, we see that the representations $T_{E''}^{\vec{\alpha}}$ and $T_E^{\vec{\alpha}}$ are unitary equivalent in a natural fashion, and that the permutation groups $\mathfrak{S}_{\mu,E'',s}$ and $\mathfrak{S}_{\mu,E,s}$ coincide with each other (cf. Proposition 4.5(i)). So the proof of the theorem is transferred from E to E''.

To establish the dual pair relation, we apply Lemma 5.6. The condition (AB) established in Lemma 5.7(ii) is nothing but the assumptions (a) and (b) in Lemma 5.6. So we can apply Lemma 5.6 and see that the groups G and $\mathfrak{S}_\infty(s)=\mathfrak{S}_{\mu,E,s}$ form a dual pair.

Thus the proof of Theorem 5.1 is now complete.

5.7. Cases of E satisfying a weaker disjointness condition. In our general situation where only the two conditions (MU1)–(MU2) are assumed for (μ, E), there gives rise to an interesting problem as

Problem 5.8. *Under the tensor product representation $T_E^{\vec{\alpha}}$, do the diffeomorphism group $G=\mathrm{Diff}_0(M)$ and the permutation group $\mathfrak{S}_{\mu,E,s}\subset\tilde{\mathfrak{S}}_\infty(s)$ form a dual pair?*

At this stage, we have no definite answer. However, if we assume a certain weak disjointness condition on E, the answer is yes and we have even more as is given in Theorem 5.9 below.

Let us introduce the following disjointness condition on $E=\prod_{j\in\mathbf{N}}E_j$:

(wDIS) there exists an increasing sequence I_n of finite subsets of $\mathbf{N}$ such that $I_n\nearrow\mathbf{N}$ and, for any $n\in\mathbf{N}$, $\cup_{j\in I_n}E_j$ and $\cup_{k\notin I_n}E_k$ are mutually disjoint.

Then we have another main result in this paper as in

Theorem 5.9. *Let $T_E^{\vec{a}}$ be the infinite tensor product representation of G determined from (μ, E) as in the preceeding theorem.*

Let $\dim M\geq 2$. *Assume that the μ-unital subset E is μ-cofinal to another μ-unital subset F which satisfies the condition (wDIS).*

(i) *The group $\mathfrak{S}_{\mu,E,s}$ is contained in $\tilde{\mathfrak{S}}_{\infty,f}$. The subgroup $\mathfrak{S}_\infty\cap\mathfrak{S}_{\mu,E,s}=\mathfrak{S}_\infty(s)$ of $\mathfrak{S}_\infty$ is everywhere dense in $\mathfrak{S}_{\mu,E,s}$ with respect to $s\cdot w$-topolygy.*

(ii) *The groups G and $\mathfrak{S}_{\mu,E,s}$ form a dual pair: $R(\mathfrak{S}_{\mu,E,s})''=T_E^{\vec{a}}(G)'$. Furthermore the groups G and $\mathfrak{S}_\infty(s)\subset\mathfrak{S}_{\mu,E,s}$ form also a dual pair in the sense that $R(\mathfrak{S}_\infty(s))''=T_E^{\vec{a}}(G)'$.*

5.8. Proof of Theorem 5.9. As in the proof of Theorem 5.1, we can replace E by F. For the first assertion (i), it is enough to see that $\mathfrak{S}_{\mu,E,s}\subset\tilde{\mathfrak{S}}_{\infty,f}$, thanks to Proposition 4.5(iii). In turn, this inclusion relation is not difficult to prove under the condition (wDIS) on F.

To prove the second assertion (ii), we apply Lemmd 5.6. Therefore the main part of the proof is to discuss a normalization of the μ-unital subset E, that is, a replacement of E by another *good* μ-unital subset to which Lemma 5.6 is applicable.

Lemma 5.10. *Let (μ, E) be a pair satisfying the conditions (MU1)–(MU 2). Assume that the condition (wDIS) holds for E.*

(i) *There exists a μ-unital subset $E'=\prod_{j\in\mathbf{N}}E'_j$ such that $E'\overset{\mu}{\sim}E$ and E' satisfies the conditions (wDIS) and (MU2str), and that each E'_j is a relatively compact, open subset of $X_j=M$, with finite number of connected components.*

(ii) *In case* $\dim$ *$M\geq 2$, there exists a μ-unital subset $E''=\prod_{j\in\mathbf{N}}E''_j\overset{\mu}{\sim}E$ with relatively compact E''_j's for which the condition (AB) in Lemma 5.7 holds.*

Proof. (i) From the condition (wDIS), there exists an increasing sequence $I_m\subset\mathbf{N}$, $I_m\nearrow\mathbf{N}$, such that $\cup_{j\in L_m}E_j\subset M$ $(m\geq 1)$, with $L_m=I_m\backslash I_{m-1}$, $L_1=I_1$, are mutually disjoint. We construct $E^{(0)}=\prod_{j\in\mathbf{N}}E_j^{(0)}\overset{\mu}{\sim}E$, and then $E^{(1)}=\prod_{j\in\mathbf{N}}E_j^{(1)}\overset{\mu}{\sim}E^{(0)}$, just as in the beginning of the proof of Lemma 5.7, but starting with E here in place of F there. Then, $Q_m^{(0)}:=\cup_{j\in L_m}E_j^{(0)}$, $m\in\mathbf{N}$, are mutually disjoint, and $E_j^{(1)}$'s are relatively compact and open.

Put, for $j\in L_1$, $E_j^{(2)}=E_j^{(1)}$, and put, for $j\in L_m$, $m\geq 2$, inductively on m,

$$E_j^{(2)}=E_j^{(1)}\setminus\cup_{k=1}^{m-1}\cup_{i\in L_k}\mathrm{Cl}(E_i^{(2)})=E_j^{(1)}\setminus\cup_{k=1}^{m-1}\cup_{i\in L_k}\mathrm{Cl}(E_i^{(1)}).$$

Then, $Q_m^{(2)}:=\cup_{i\in L_m}E_i^{(2)}$, $m\in\mathbf{N}$, are mutually disjoint, relatively compact and open. Similarly as in the proof of Lemma 5.7(i), we have an evaluation

$$\mu_j(E_j^{(2)}\ominus E_j^{(1)})\leq\mu_j(E_j^{(1)}\ominus E_j^{(0)})+\sum_{i\in\cup_{k=1}^{m-1}L_k}\mu_j(E_j^{(0)}\cap E_i^{(1)})\quad(j\in L_m).$$

Summing up this inequality, we get $\sum_{j\in\mathbf{N}}\mu_j(E_j^{(2)}\ominus E_j^{(1)})<\varepsilon$ as in (5.5), whence $E^{(2)}\overset{\mu}{\sim}E^{(1)}$. Picking up finite number of connected components from $E_j^{(2)}$ appropriately, we obtain E_j' and $E'=\prod_{j\in\mathbf{N}}E_j'\overset{\mu}{\sim}E^{(2)}\overset{\mu}{\sim}E$ in (i) of the lemma.

(ii) Just as in the proof of (ii) in Lemma 5.7, we start with E' given above, and with a new increasing sequence $W_n\nearrow M$ $(n\to\infty)$ having the same properties as there. However we discuss here not according to individual E_i', $i\in\mathbf{N}$, but according to families $\{E_i'\,;\,i\in L_m\}$, $m\in\mathbf{N}$.

Put $Q_m'=\cup_{i\in L_m}E_i'$, $m\in\mathbf{N}$. First consider Q_1', and take a W_{m_1} containing its closure. Put $I_1^w=\{m\,;\,Q_m'\cap W_{m_1}\neq\phi\}$, and $J_1^w=\{m\in I_1^w\,;\,Q_m'\subset W_{m_1}\}$. For $i\in L_m$ with $m\in J_1^w$, put $E_i''=E_i'$. For j's in L_m, $m\in I_1^w\setminus J_1^w$, consider first the union $U_1'=W_{m_1}\cup(\cup_{m'\in I_1^w}Q_{m'}')$. We make it connected by adding small open pathes appropriately to E_j', $j\in L_m$, $m\in I_1^w\setminus J_1^w$, to connect their connected components outside of W_{m_1} to W_{m_1}, so that we get E_j'', $j\in L_m$, $m\in I_1^w\setminus J_1^w$, and a connected open $U_1=W_{m_1}\cup(\cup_{m\in I_1^w}Q_m'')$ with $Q_m''=\cup_{j\in L_m}E_j''$ satisfying

$$(5.6'')\qquad\mu_j(E_j''\ominus E_j')+\sum_{k\notin L_m,\,m\in I_1^w}\mu_k(E_j''\cap E_k')<\varepsilon/2^j\quad(j\in L_m,\ m\in I_1^w\setminus J_1^w).$$

Put $A_1^w=\{a\notin I_1^w\,;\,Q_a'\cap(\cup_{m\in I_1^w\setminus J_1^w}Q_m'')\neq\phi\}$, and

$$(5.7'')\qquad E_k''=E_k'\setminus(\cup_{m\in I_1^w\setminus J_1^w}Q_m'')\quad(k\in L_a,\ a\in A_1^w).$$

Thus E_i'' are determined for $i\in L_m$, $m\in B_1^w:=I_1^w\cup A_1^w$. Note that $Q_m''\subset U_1$ for $m\in I_1^w$, and that $Q_m''(m\in A_1)$ and $Q_m'(m\notin B_1^w)$ are disjoint with U_1. Thus the first step of the induction is completed.

Now we state the n-th step. Assume that E_i'' have been determined for $i\in L_m$, $m\in B_{n-1}^w$, by the help of $W_{m_1}\subset W_{m_2}\subset\cdots\subset W_{m_{n-1}}$. Take a W_{m_n}, $m_n>m_{n-1}$, containing all $\mathrm{Cl}(E_i'')$, $i\in L_m$, $m\in B_{n-1}^w$, and put $I_n^w=\{m\notin B_{n-1}^w\,;\,Q_m'\cap W_{m_n}\neq\phi\}$, and $J_n^w=\{m\in I_n^w\,;\,Q_m'\subset W_{m_n}\}$. For $i\in L_m$ with $m\in J_n^w$, put $E_i''=E_i'$. For j's in L_m, $m\in I_n^w\setminus J_n^w$, consider first the union $U_n'=W_{m_n}\cup(\cup_{m'\in I_n^w}Q_{m'}')$. We make it connected by adding small open pathes appropriately to E_j', $j\in L_m$, $m\in I_n^w\setminus J_n^w$, to connect their connected components outside of W_{m_n} to W_{m_n} (not touching E_j'', $j\in L_m$, $m\in B_{n-1}^w$, already determined until the last step), so that we get E_j'', $j\in L_m$, $m\in I_n^w\setminus J_n^w$, and a connected open $U_n=W_{m_n}\cup(\cup_{m\in I_n^w}Q_m'')$

satisfying

$$(5.6''')\quad \mu_j(E_j''\ominus E_j')+\sum_{k\notin L_m,\ m\in B_{n-1}^w\cup I_n^w}\mu_k(E_j''\cap E_k')<\varepsilon/2^j\quad (j\in L_m,\ m\in I_n^w\setminus J_n^w).$$

Put $A_n^w=\{a\notin B_{n-1}^w\cup I_n^w\,;\,Q_a'\cap(\cup_{m\in I_n^w\setminus J_n^w}Q_m'')\neq\phi\}$, and

$$(5.7''')\qquad E_k''=E_k'\setminus(\cup_{m\in I_n^w\setminus J_n^w}Q_m'')\quad (k\in L_a,\ a\in A_n^w).$$

Then E_i'' are determined for $i\in L_m$, $m\in B_n^w:=B_{n-1}^w\cup I_n^w\cup A_n^w$. Note that $Q_m''\subset U_n$ for $m\in B_{n-1}^w\cup I_n^w$, and that $Q_m''(m\in A_n^w)$ and $Q_m'(m\notin B_n^w)$ are disjoint with U_n.

Repeating this process inductively on n, we obtain finally E_j'', $j\in\mathbf{N}$, which satisfy the condition (AB) in Lemma 5.7 for the sequence U_n, $n\in\mathbf{N}$, because $Q_m''\subset U_n$ or $Q_m''\cap U_n=\phi$ for any m, $n\in\mathbf{N}$.

This complete the proof of the lemma.

Proof of Theorem 5.9(ii). Since we have constructed $E''\overset{\mu}{\sim}E$ which satisfies the condition (AB) in Lemma 5.7, we are now ready to apply Lemma 5.6. Then the assertion (ii) of the theorem follows from this lemma.

The proof of Theorem 5.9 is now complete.

§ 6. Groups of volume-preserving diffeomorphisms

Assume that a connected $C^{(n)}$-manifold M, $n\geq 1$, is equipped with a measure $\omega\in\mathscr{LFM}(M)$. We consider here an important subgroup $G_\omega=\mathrm{Diff}_0(M\,;\omega)$ of $G=\mathrm{Diff}_0(M)$ consisting of volume-preserving $g\in G: d\omega(gp)=d\omega(p)$, $p\in M$.

6.1. The extension $\bar{G}_\omega$ of the group G_ω. Denote by $\mathscr{M}_\omega$ the group of all measurable transformations on M which are equal to the identity outside some compacts and preserve the volume ω, and also denote by $\bar{G}_\omega$ its subgroup consiting of all elements $g\in\mathscr{M}_\omega$ which can be approximated by nets g_ε, $\varepsilon>0$, in G_ω. Here, by definition, a net $g_\varepsilon\in G_\omega$, $\varepsilon>0$, approximates $g\in\mathscr{M}_\omega$ if (i) $\omega(\{p\in M;\ g_\varepsilon p\neq gp\})\to 0$ as $\varepsilon\downarrow 0$, and (ii) there exists a compact $K\subset M$ such that $g_\varepsilon=\mathrm{id}$ outside K.

Note that the fact that $\bar{G}_\omega$ becomes actually a group is seen from the following

Lemma 6.1. *Assume that two nets g_ε, h_ε, $\varepsilon>0$, in G_ω approximate g, $h\in\mathscr{M}_\omega$ respectively. Then the products $g_\varepsilon h_\varepsilon\in G_\omega$ approximates the product gh, and the inverse g_ε^{-1} approximates the inverse g^{-1}.*

In this section we investigate firstly what kind of transformations are contained in the extended group $\bar{G}_\omega$, and secondly whether or not the natural representations of the group G_ω or their tensor products can be extended to the group extension $\bar{G}_\omega$.

6.2. Rotation of a cubic body. In this and the succeeding two subsections, we treat the local case or the case where M is a connected open submanifold of $\mathbf{R}^d$ $(d \geq 2)$. Consider the measure

$$d\omega(x) = dx_1 dx_2 \cdots dx_d \qquad (x = (x_i)_{i=1}^d \in \mathbf{R}^d).$$

Let $D = J^d$ with closed interval J be a cubic body in $M \subset \mathbf{R}^d$. Intoduce coordinates for which $J = [-a, a]$, $a > 0$, so that the center of D is the origin O. We devide D into two pieces by $D_\pm = J_\pm \times J^{d-1}$ with $J_+ = [0, a]$, $J_- = [-a, 0]$. Denote by h_{D_+, D_-} (resp. g_D) the measurable transformation on M which exchanges D_+ and D_- (resp. rotates D arround the center O by the angle π) and equals to the identity outside of D. We prove here that the transformations h_{D_+, D_-} and g_D can be respectively approximated by a net $g_\varepsilon \in G_\omega$, $\varepsilon > 0$, as $\varepsilon \downarrow 0$, where the support of g_ε is contained in $D_\varepsilon = J_\varepsilon^d \subset M$ with $J_\varepsilon = [-a-\varepsilon, a+\varepsilon]$. To do so, it is enough to show it for the rotation g_D.

6.2.1. First we assume $d=2$ and follow the result of Neretin [11]. Introduce the polar coordinates (r, ϕ) for $(x_1, x_2) \in \mathbf{R}^2$. Take a smooth curve $r = \lambda(\phi)$ contained in the inside of $D_\varepsilon \backslash D$ such that $\lambda(\phi+\pi) = \lambda(\phi)$ $(\forall \phi)$, and also take a monotone smooth function $\tau(s)$ such that $\tau(s) = \pi$ for $s \leq 0$ and $\tau(s) = 0$ for $s \geq s_0 > 0$, with a sufficiently small number s_0 so that the curve $r = \sqrt{\lambda(\phi)^2 + s_0}$ is contained in D_ε. We define a transformation $g_\varepsilon \in \mathrm{Diff}_0(M, \omega)$ as follows. Let $g_\varepsilon(r, \phi) = (r_1, \phi_1)$ and, for $s = r^2 - \lambda(\phi)^2$,

$$(r_1, \phi_1) = \begin{cases} (r, \phi+\pi) & \text{for } s \leq 0, \\ (\sqrt{\lambda(\phi_1)^2 + s}, \ \phi + \tau(s)) & \text{for } 0 \leq s \leq s_0, \\ (r, \phi) & \text{for } s \geq s_0. \end{cases}$$

Note that, in the region between two curves $r = \lambda(\phi)$ and $r = \sqrt{\lambda(\phi)^2 + s_0}$, we have $2r\, dr\, d\phi = ds\, d\phi = 2r_1 dr_1 d\phi_1$.

6.2.2. Next we proceed to the general case. For $x = (x_1, x_2, x_3, \cdots, x_d) \in \mathbf{R}^d$, we put $\tilde{x} = (x_3, \cdots, x_d)$ and distinguish first two components (x_1, x_2), introducing for it the polar coordinates (r, ϕ) as in the case of $d=2$. Our transformation $g_\varepsilon \in \mathrm{Diff}_0(M, \omega)$, $\varepsilon > 0$, is given in the following form : for $x = (x_1, x_2, \cdots, x_d)$, use the coordinates $(r, \phi\,; \tilde{x})$, then

$$g_\varepsilon(r, \phi\,; \tilde{x}) = (r_1, \phi_1\,; \tilde{x})$$

with $(r_1, \phi_1) = g_{\varepsilon, \tilde{x}}(r, \phi)$, where $g_{\varepsilon, \tilde{x}}$ is a transformation depending on $\tilde{x} = (x_3, \cdots, x_d)$ with a similar form as g_ε in the case of $d=2$. To give $g_{\varepsilon, \tilde{x}}$, we introduce two monotone smooth functions $\xi(t)$, $\eta(t)$, $t \geq 0$, as

$$\xi(t) = 1 \ (0 \leq t \leq a), \qquad \xi(a + \varepsilon/3) = 0,$$
$$\eta(t) = 1 \ (0 \leq t \leq a + 2\varepsilon/3), \qquad \eta(a+\varepsilon) = 0,$$

and put $\xi(t)=\xi(-t)$, $\eta(t)=\eta(-t)$ for $t\leq 0$, and

$$\xi(\tilde{x})=\prod_{i=3}^{d}\xi(x_i), \qquad \eta(\tilde{x})=\prod_{i=3}^{d}\eta(x_i).$$

Define $(r_1, \phi_1)=g_{\varepsilon,\tilde{x}}(r, \phi)$ as follows: put first

$$\lambda(\phi\,;\tilde{x})=\frac{\lambda(\phi)}{\xi(\tilde{x})+(1-\xi(\tilde{x}))\lambda(\phi)},$$

and then, for $s=r^2-\lambda(\phi\,;\tilde{x})^2$,

$$(r_1, \phi_1)=\begin{cases}(r, \phi+\eta(\tilde{x})\pi) & \text{for } s\leq 0,\\ (\sqrt{\lambda(\phi_1\,;\tilde{x})^2+s}, \phi+\eta(\tilde{x})\tau(s)) & \text{for } 0\leq s\leq s_0,\\ (r, \phi) & \text{for } s\geq s_0.\end{cases}$$

Note that, for a fixed $\tilde{x}$, the curve $r=\lambda(\phi\,;\tilde{x})$, $0\leq\phi\leq 2\pi$, equals to a unit circle if $|x_i|\geq a+\varepsilon/3$ for some $i\geq 3$, and then the curve (r_1, ϕ_1), $0\leq\phi\leq 2\pi$, for a fixed parameter s, $0\leq s\leq s_0$, is a circle $r=\sqrt{1+s}$, and the rotation of angle $\eta(\tilde{x})\tau(s)$ on the circle is smoothly consistent at $s=s_0$ with the rotation of angle $\eta(\tilde{x})\pi$ of the unit disc $r\leq 1$ in the center, as it should be.

Note further that the transformation g_ε keeps the last $(d-2)$ components $\tilde{x}$ of x always invariant and, for each $\tilde{x}$, it equals a volume-preserving transformation on $(x_1, x_2)\in\mathbf{R}^2$ whose angle of rotation dicreases smoothly along with $\tilde{x}$. This implies in particular that the transformation g_ε on $M\subset\mathbf{R}^d$ preserves the volume element ω.

6.3. More general transformations in $\bar{G}_\omega$. We assume still $M\subset\mathbf{R}^d$. Let us divide M by the family of hyperplanes $x_i=n\delta\,(n\in\mathbf{Z})$ with sufficiently small $\delta>0$. Then, speaking about the cubic bodies cut off by these hyperplanes, we arrive at the following situation. Two cubic bodies are called *adjacent* to each other if they have in common one of their surfaces. Any two cubic bodies D_1 and D_2 inside of M can be connected by a chain of cubic bodies in M, $C_1=D_1$, C_2, $\cdots$, $C_n=D_2$, in such a way that C_i and C_{i+1} are adjacent for $1\leq i<n$. Then any permutation of C_i's can be given as a product of the transposition $h_{C_i, C_{i+1}}\in\bar{G}_\omega$ of C_i with C_{i+1} through the common surface, since this is well-known for the permutation group $\mathfrak{S}_n$. More in detail we have

Lemma 6.2. *For any two cubic bodies D_1, D_2 in M given as above, there exists a permutation h_{D_1, D_2} of D_1 and D_2 belonging to $\bar{G}_\omega$. Here, by definition, $h_{D_1, D_2}=\mathrm{id}$ outside of $D_1\cup D_2$. More exactly, take an arcwise-connected, relatively compact negighbourhood U of $D_1\cup D_2$, then there exists a net $g_\varepsilon\in G_\omega$, $\varepsilon>0$, and a permutation h_{D_1, D_2} such that g_ε approximates h_{D_1, D_2} and* $\mathrm{supp}(g_\varepsilon)\subset$ *U.*

Proof. According to the size of the narrowest neck of U, we devide M finer by hyperplanes $x_i=\delta':=\delta/N$ with sufficiently big N. Then new smaller cubic bodies $D_i'\subset D_i$ $(i=1, 2)$ can be connected by a chain of (new) cubic bodies contained in U. Then, the explicit form of measure-preserving transformations in §6.2 and the argument just above show that we can do everything inside of the open submanifold U. This means that a permutation h_{D_1, D_2} can be approximeted by a net $g_\varepsilon\in G_\omega$ with $\mathrm{supp}(g_\varepsilon)\subset U$. Take $K=\mathrm{Cl}(U)$, then $g_\varepsilon=\mathrm{id}$ outside K, and so we see that $h_{D_1, D_2}\in\bar{G}_\omega$.

6.4. Exchange of two equi-volume open sets. Let $M\subset\mathbf{R}^d$, $d\geq 2$. Take two relatively compact open sets O_1, O_2 with the same volume. Then we have

Proposition 6.3. *There exists in $\bar{G}_\omega$ a measurable transformation h_{O_1, O_2} which maps O_1 onto O_2, O_2 onto O_1 (modulo null sets), and equals to the identity outside of $O_1\cup O_2$. More exactly, for an arcwise-connected relatively compact open set U containing* $\mathrm{Cl}(O_1\cup O_2)$, *there exists a net g_ε, $\varepsilon>0$, in $G_\omega(U):=\{g\in G_\omega;\ \mathrm{supp}(g)\subset U\}$, which approximates h_{O_1, O_2}. We can choose h_{O_1, O_2} in such a way that, for certain open subsets V_j of O_j with $\omega(V_j)=\omega(O_j)$, it maps V_1 onto V_2, V_2 onto V_1, homeomorphically on each connected components.*

Proof. Step 1. Let $\gamma>0$. Then there exists a sufficiently fine decomposition of M by hypersurfaces $x_i=n\delta$ $(n\in\mathbf{Z})$ such that, for $j=1, 2$, let $\mathscr{C}_j$ be the set of cubic bodies for this decomposition contained in O_j, then the union $F_j=\bigcup_{D\in\mathscr{C}_j}D$ approximates O_j as $\omega(O_j\backslash F_j)<\gamma$. Let n_j be the number of elements in $\mathscr{C}_j$. Assume that $n_1\leq n_2$. Discurding (n_2-n_1) elements in $\mathscr{C}_2$, we get $\mathscr{C}_2'$. Put $\mathscr{C}_1'=\mathscr{C}_1$. Then $\omega(O_j\backslash F_j')<\gamma$ for $F_j'=\bigcup_{D\in\mathscr{C}_j'}D$. Make pairs (D_1, D_2), $D_1\in\mathscr{C}_1'$, $D_2\in\mathscr{C}_2'$, bijectively, and take $h_{D_1, D_2}\in\bar{G}_\omega$ in Lemma 6.2, then the product $h_{F_1', F_2'}$ of h_{D_1, D_2} over these pairs, maps F_1' onto F_2', F_2' onto F_1', and equal to the identity outside of $F_1'\cup F_2'$, and so it approximates in a sense the desired transformation h_{O_1, O_2}.

Note that $h_{F_1', F_2'}$ maps V_1' onto V_2' homeomorphically on each connected components, where $V_j'=\bigcup_{D\in C_j'}\mathrm{Int}(D)$, $\mathrm{Int}(D)=$ the interior of D, and that $\omega(F_j'\backslash V_j')=0$.

Step 2. Now we construct a net in $\bar{G}_\omega$ of certain $h_{F_1', F_2'}$'s which 'converges' to an h_{O_1, O_2}. Then we guarantee, by Lemma 6.4 given below, the exixtence of a net g_ε, $\varepsilon>0$, in the group G_ω, converging to h_{O_1, O_2}.

Take $\varepsilon=\varepsilon_k=2^{-k}$, $k\geq 1$, and put $\gamma=\varepsilon$. We discuss by induction on k and give a convergent series in $\bar{G}_\omega$. For $k=1$, we follow the process descrived in Step 1 and put

$$O_j^{(1)}=O_j, \quad F_j^{(1)}=F_j', \quad V_j^{(1)}=V_j' \ (j=1,\ 2), \quad h_{F_1^{(1)},\, F_2^{(1)}}.$$

For the next step $k=2$, we take $\varepsilon=\varepsilon_2$, $\gamma=\varepsilon$, and $O_j^{(2)}=O_j^{(1)}\backslash F_j^{(1)}$ for O_j in the discussion in Step 1. Then, we obtain $F_j^{(2)}\subset O_j^{(2)}$, $V_j^{(2)}\subset F_j^{(2)} (j=1,\ 2)$, and $h_{F_1^{(2)},\, F_2^{(2)}}$.

In general, for the k-th step, we take $O_j^{(k)}=O_j^{(k-1)}\backslash F_j^{(k-1)}$ for O_j in the discussion in Step 1. Then we get $F_j^{(k)}\subset O_j^{(k)}$, $V_j^{(k)}\subset F_j^{(k)}$ and $h_{F_1^{(k)},\, F_2^{(k)}}$.

Let us now put

$$h_n=h_{F_1^{(k)},\, F_2^{(k)}} \text{ on } F_1^{(k)}\cup F_2^{(k)} \text{ for } 1\leq k\leq n, \quad =\mathrm{id} \text{ elsewhere},$$
$$h=h_{F_1^{(k)},\, F_2^{(k)}} \text{ on } F_1^{(k)}\cup F_2^{(k)} \text{ for } 1\leq k<\infty, \quad =\mathrm{id} \text{ elsewhere},$$

Put $W_j=\cup_{k=1}^{\infty}F_j^{(k)}$, $V_j=\cup_{k=1}^{\infty}V_j^{(k)}$. Then, $h_n\in\bar{G}_\omega$ approximates the tansformation $h\in\mathcal{M}_\omega$ which is equal to the identity outside of $O_1\cup O_2$, and maps $W_1\subset O_1$ onto $W_2\subset O_2$, W_2 onto W_1. Note that $\omega(O_j\backslash V_j)=0$ and that h exchanges $V_1\subset W_1$ and $V_2\subset W_2$ homeomorphically on each connected components. We take this h as the transformation $h_{O_1,\, O_2}$ desired.

Step 3. The 'convergence' of h_n in $\bar{G}_\omega$ to h shows us the existence of a net g_ε, $\varepsilon>0$, in G_ω converging to h, by the help of the lemma below. To apply this lemma, we take $K=K_0=\mathrm{Cl}(U)$.

For the assertion on the existence of a convergent net in $G_\omega(U)$, we apply Lemmas 6.2 and 6.4. Q. E. D.

Lemma 6.4. *Assume that an element $g\in\mathcal{M}_\omega$ is approximated by a net g_ε, $\varepsilon>0$, in $\bar{G}_\omega$ in such a way that* (a) *for a compact K*, $\mathrm{supp}(g_\varepsilon)\subset K$, (b) $\omega(p\in M\,;\, g_\varepsilon p\neq gp\})\to 0$ *as $\varepsilon\downarrow 0$. Then g belongs to the extended group $\bar{G}_\omega$ if there exists another compact K_0 such that each $g_\varepsilon\in\bar{G}_\omega$ is approximated by a net $g_{\varepsilon\delta}$, $\delta>0$, $\delta\downarrow 0$, in G_ω such that* $\mathrm{supp}(g_{\varepsilon\delta})\subset K_0$.

We omit the proof of this lemma.

6.5. The group extension $\bar{G}_\omega$ in the general case. Let us treat now the general case. Let M be a connected $C^{(n)}$-manifold, $1\leq n\leq\infty$, and ω a measure on M taken from $\mathcal{LFM}(M)$. For each local chart $(U,\ \phi)$, in the co-ordinates $\phi(p)=x=(x_i)_{i=1}^d$, $d=\dim M$, we have $d\omega(p)=\rho(x)dx_1dx_2\cdots dx_d$ with a locally integrable, positive density ρ. Assume the following condition holds:

(Den) the density ρ is of class $C^{(n)}$ in every local chart.

Then we can transfer the results in §6.4 for local case to this general case, as shown below.

Define new local co-ordinates $(y_i)_{i=1}^d$ as $y_1=\int^{x_1}\rho(t_1,\ x_2,\ x_3,\ \cdots,\ x_d)dt_1$, and $y_i=x_i$ for $i>1$, then we have a standard expression of ω as

$$d\omega(p)=dy_1dy_2\cdots dy_d.$$

A local chart (U, ϕ) is called *admissible* if the measure ω is expressed in the standard form. For such a chart (U, ϕ), consider U as an open subset of $\mathbf{R}^d$ through $\phi: U \hookrightarrow \mathbf{R}^d$, and apply for U the results in §6.4. Put

$$\bar{G}_\omega(U) = \{g \in \bar{G}_\omega\,;\ \mathrm{supp}(g) \subset U\}.$$

Theorem 6.5. *Assume that a measure ω on M satisfies the condition (Den). Let O_1 and O_2 be two relatively compact, open subsets of M with the same volume. Then there exists an element $h_{O_1, O_2} \in \bar{G}_\omega$ which maps O_1 onto O_2, O_2 onto O_1(modulo null sets), and equals to the identity outside of $O_1 \cup O_2$.*

Furthermore, let U be an arcwise-connected open subset of M containing $\mathrm{Cl}(O_1 \cup O_2)$. *Then there exists a net g_ε, $\varepsilon > 0$, in $G_\omega(U)$, converging to h_{O_1, O_2}. Moreover h_{O_1, O_2} can be so chosen that there exist open subsets $V_j \subset O_j$ such that $\omega(O_j \setminus V_j) = 0$ and it maps V_1 onto V_2, V_2 onto V_1, homeomorphically on each connected component.*

Proof. Devide O_j into small open subsets $O_{j,m}$, $1 \leq m \leq N$, (up to subsets of smaller dimensions) in such a way that each $O_{j,m}$ is contained in an admissible chart $U_{j,m}$, and that $\omega(O_{1,m}) = \omega(O_{2,m})$ and $\omega(O_j \setminus \cup_{m=1}^N O_{j,m}) = 0$.

Fix an m. Then there exists a chain of admissible charts $W_1 = U_{1,m}$, W_2, $\cdots$, $W_n = U_{2,m}$, such that $W_i \subset U$, $W_i \cap W_{i+1} \neq 0 (1 \leq i < n)$. Devide again the pair $O_{1,m}$, $O_{2,m}$ into pairs of equi-volume open subsets, sufficiently small compairing to the sizes of W_i, $W_i \cap W_{i+1}$. Take one of these pairs and let it be O_1', O_2'. Choose a chain of open subsets $V_1 = O_1'$, $V_2 \subset W_1 \cap W_2$, $V_3 \subset W_2 \cap W_3$, $\cdots$, $V_n = O_2'$ all with the same volume. Then, by Proposition 6.3, V_i and V_{i+1} are exchanged by an $h_{i,i+1} = h_{V_i, V_{i+1}} \in \bar{G}_\omega(W_i)$. Therefore V_1 and V_n are exchanged by an appropriate product of transposition $h_{i,i+1}$, just as in the symmetric group $\mathfrak{S}_n$. Thus we see that $O_{1,m}$ and $O_{2,m}$ are exchanged by an element in $\bar{G}_\omega$, and finally so are the original O_1 and O_2.

Remark 6.6. Take an arbitrary $\omega_0 \in \mathscr{LFM}(M)$, or a locally finite measure on M which is locally equivalent to Lebesgues measures. Then there exists, for any $\varepsilon > 0$, a measure $\omega \in \mathscr{LFM}(M)$ satsifying the condition (Den) and $|\omega - \omega_0|(M) < \varepsilon$.

6.6. Representations of $G_\omega = \mathrm{Diff}_0(M, \omega)$ and of $\bar{G}_\omega$. On the Hilbert space $\mathscr{H}_0 = L^2(M, \omega)$, we have a natural representation T_0 of $G_\omega = \mathrm{Diff}_0(M, \omega)$ in the form

$$T_0(g)f(x) = f(g^{-1}x) \quad (g \in G_\omega,\ x \in M,\ f \in \mathscr{H}_0).$$

This representation can be extended by continuity to a representation of the group extension $\bar{G}_\omega$ consisting of measurable transformations approximated by a net in G_ω. In fact, suppose that g_ε, $\varepsilon > 0$, in G_ω converges to $g\in\bar{G}_\omega$ as $\varepsilon\downarrow 0$, then it induces a strong convergence of operators $T_0(g_\varepsilon)$, and the limiting operator can be attributed to g and gives $T_0(g)$, which is expressed by the same formula as above. Assume the condition (Den) for ω, then the group $\bar{G}_\omega$ contains a transformation which exchanges two equi-volume open subsets O_1 and O_2 in a compact, and equals to the identity outside of them. Even if there exist several such transformations, we denote any of them simply by h_{O_1, O_2}.

The irreducibility of the representation T_0 of the group G_ω is equivalent to the irreducibility under the bigger group $\bar{G}_\omega$, and the latter, even it is rather clear, is proved here in the simplest way.

Theorem 6.7. *Let* $\dim M\geq 2$, *and assume, for a measure* $\omega\in\mathcal{LFM}(M)$, *that the condition (Den) holds.*

(i) *In case* $\omega(M)=+\infty$, *the natural representation* T_0 *of the group* $G_\omega=\mathrm{Diff}_0(M,\omega)$ *is irreducible, and so is its extension to the group* $\bar{G}_\omega$.

(ii) *In case* $\omega(M)<+\infty$, *the 1-dimensional subspace consisting of constant functions on* M *is* G_ω*-invariant, and its orthogonal complement in* $\mathcal{H}_0$ *is irreducible under* G_ω. *The same is true also for* $\bar{G}_\omega$.

Proof. Enough to prove the assertions for the extended group $\bar{G}_\omega$. Let A be an intertwining operator of T_0. Take an open subset U with finite volume and its indicator function $f=\chi_U\in\mathcal{H}_0$. Put $\phi=Af$. Take any two relatively compact, open subsets O_1 and O_2 with the same volume, both contained in U or in $M\backslash U$, and take $h=h_{O_1, O_2}$, then $T_0(h)f=f$ and so we have $T_0(h)\phi=\phi$, that is,

$$\phi(h_{O_1, O_2}x)=\phi(x) \quad \text{for almost all } x\in M.$$

Since the pair O_1, O_2 are arbitrary both inside or outside of U, the function ϕ should be constant separately inside or outside of U. Therefore we have $\phi\equiv A(\chi_U)=c_U\chi_U+d_U\chi_M$ with constants c_U, $d_U\in\mathbf{C}$. In case $\omega(M)=+\infty$, the constant function χ_M does not belong to $\mathcal{H}_0$, and so $d_U=0$ or $A(\chi_U)=c_U\chi_U$. In case $\omega(M)<+\infty$, we have, in particular, $A(\chi_M)=a\cdot\chi_M$ with a constant $a\in\mathbf{C}$.

The formula $A(\chi_U)=c_U\chi_U+d_U\chi_M$ can be extended to a measurable subset U with finite volume such that $\omega(\partial U)=0$, where $\partial U:=\mathrm{Cl}(U)\backslash\mathrm{Int}(U)$ is the boundary of U. Now, for such a subset U with $\omega(M\backslash U)>0$, devide it into two disjoint, non-null, such subsets as $U=U_1\sqcup U_2$. Then, we see easily that $c_U=c_{U_1}=c_{U_2}$. Therefore $c_U=c_{U'}$ for any two open sets U and U' with finite volumes and $\neq M$. This means that $c_U=c$, a constant.

In case $\omega(M)=+\infty$, we have $Af_1=cf_1$ for any $f_1\in\mathcal{H}_0$ and so the representation T_0 is irreducible.

In case $\omega(M)<+\infty$, the 1-dimensional subspace $\mathcal{H}_{00}=\mathbf{C}\chi_M$ is invariant under $\bar{G}_\omega$ and so is its orthogonal complement $\mathscr{H}_{01}=(\mathscr{H}_{00})^\perp$. Put $\phi_U=\chi_U-(\omega(U)/\omega(M))\chi_M$. Then, $\phi_U\in\mathscr{H}_{01}$ and so $A(\phi_U)\in\mathscr{H}_{01}$. Since $c_U=c$, we get from this that $A(\phi_U)=c\cdot\phi_U$ and so $A\,|\,\mathscr{H}_{01}=c\cdot I_{\mathscr{H}_{01}}$. This means that the representation $T_0\,|\,\mathscr{H}_{01}$ is irreducible.

Note 6.8. For the bigger group $G=\mathrm{Diff}_0(M)$, its natural representations T_s, $s\in\mathbf{R}$, on $\mathscr{H}_0$ are always irreducible even when $\omega(M)<+\infty$. (The explicit form of T_s is given at the beginning of Appendix below, and its restriction for G_ω is nothing but T_0.) In fact, the subspace $\mathscr{H}_{00}$ consisting of constant functions on M is G_ω-invariant, but not G-invariant.

The same kind of arguments as in the above proof of Theorem 6.7 can be used for the irreducible decompositions of finite tensor products of the natural representations, and similar results are obtained for the small subgroup $G_\omega=\mathrm{Diff}_0(M,\omega)$ as those for the whole group $G=\mathrm{Diff}_0(M)$. Let $T_0^{(k)}=\otimes_{i=1}^k T_i$ with $T_i=T_0$ be the k-th tensor product of T_0 on the space $\mathscr{H}_0^{(k)}=\otimes_{i=1}^k\mathscr{H}_i$, $\mathscr{H}_i=\mathscr{H}_0$. The symmetric group $\mathfrak{S}_k$ acts on $\mathscr{H}_0^{(k)}$ naturally as permutations of the components of decomposable vectors.

Theorem 6.9. *The k-th tensor product $T_0^{(k)}$ of the natural representation T_0 of the group G_ω can be extended to the bigger group $\bar{G}_\omega$ by continuity. Assume* $\dim M\geq 2$, *and $\omega(M)=+\infty$. Then, on the representation space $\mathscr{H}_0^{(k)}$, the groups G_ω and $\mathfrak{S}_k$ form a dual pair, and so does the groups $\bar{G}_\omega$ and $\mathfrak{S}_k$.*

The assertion for G_ω and that for the extended group $\bar{G}_\omega$ are mutually equivalent. The proof is quite similar as for the group G itself and is based on the irreducibility of natural representation given in Theorem 6.7.

6.7. Infinite tensor products and dual pairs of $G_\omega\times\mathfrak{S}_\infty$. Let $\mu=(\mu_i)_{i\in\mathbf{N}}$ with $\mu_i=\omega$ on $X_i=M$, and also let $E=\prod_{i\in\mathbf{N}}E_i$, $E_i\subset X_i$, be a μ-unital subset of $X=\prod_{i\in\mathbf{N}}X_i$. Assume the condition (MU2) for (μ, E) : for any compact subset K of M, $\sum_{i\in\mathbf{N}}\omega(K\cap E_i)<\infty$. Then the infinite tensor product T_E of natural representation T_0 of G_ω is given as in Theorem 3.1. Moreover the σ-ring $\mathscr{M}(\mu, E)$ and the product measure $\nu_{\mu,E}$ are G_ω-invariant by the discussions in §3.5. So an explicit form of T_E is given as

$$T_E(g)f(x)=f(g^{-1}x)\quad(g\in G_\omega,\ x\in X,\ f\in L^2(X,\mathscr{M}(\mu,E),\nu_{\mu,E})).$$

Note that any element g in the extended group $\bar{G}_\omega$ has compact support, then we see easily that the above formula can be extended to give an infinite tensor product of the natural representation of the group $\bar{G}_\omega$. This representation is an extension by continuity from G_ω to $\bar{G}_\omega$, as shown by the following lemma, and is denoted again by the same symbol T_E.

Lemma 6.10. *Assume that a net g_ε, $\varepsilon>0$, in G_ω converges to an element g*

$\in\bar{G}_\omega$. *Then the net of operators* $T_E(g_\varepsilon)$ *converges strongly to* $T_E(g)$.

We make the infinite symmetric group $\mathfrak{S}_\infty$ act on the space $L^2(X, \mathscr{M}(\mu, E), \nu_{\mu,E})$ as

$$R(\sigma)f(x)=f(x\sigma) \quad (\sigma\in\mathfrak{S}_\infty,\ f\in L^2(X, \mathscr{M}(\mu, E), \nu_{\mu,E})),$$

where $x\sigma=(x_{\sigma(i)})_{i\in\mathbf{N}}$ for $x=(x_i)_{i\in\mathbf{N}}\in X$.

Similarly as for the infinite tensor product representations $T_E^{\bar{\alpha}}$ or $T_{E'}^{\alpha}$ for $G=\mathrm{Diff}_0(M)$ in §5, we have a dual pair relation for the group G_ω of measure preserving diffeomorphisms and the symmetric group $\mathfrak{S}_\infty$ as given in

Theorem 6.11. *Let M be a connected $C^{(n)}$-manifold, $n\geq 1$, with* $\dim M\geq 2$, *and ω be a measure on M locally finite, locally equivalent to Lebesgue measures, with $C^{(n)}$-class densities, and with $\omega(M)=+\infty$. Put $\mu=(\mu_i)_{i\in\mathbf{N}}$, $\mu_i=\omega$, and take a μ-unital subset $E=\prod_{i\in\mathbf{N}}E_i$. Assume that E_i's are mutually disjoint. Then, on the Hilbert space $L^2(X, \mathscr{M}(\mu, E), \nu_{\mu,E})$, the representation $T_E\cdot R$ of the product groups $G_\omega\times\mathfrak{S}_\infty$ gives a dual pair relation, and a similar fact holds also for $\bar{G}_\omega\times\mathfrak{S}_\infty$:*

$$T_E(G_\omega)'=R(\mathfrak{S}_\infty)'', \quad T_E(\bar{G}_\omega)'=R(\mathfrak{S}_\infty)''.$$

The proof is similar as for the case of $G\times\mathfrak{S}_\infty(s)$ in §§5.3–5.6, but for a special $s=(s_j)_{j\in\mathbf{N}}$ with all $s_j=0$ in Theorem 5.1. It is based on Theorem 6.9 for the k-th tensor product $T_0^{(k)}$ of T_0 and the symmetric group $\mathfrak{S}_k$. We omit the details here.

Remark 6.12. In the case where the disjointness condition (wDIS) on the μ-unital subset E is assumed, we can give a similar result as Theorem 5.9 in §5.7. Further, its proof is also similar.

Appendix. Finite tensor products of natural representations of a diffeomorphism group

Let M be a connected $C^{(n)}$-manifold with $n\geq 1$, and $G=\mathrm{Diff}_0(M)$ the group of diffeomorphisms on M with compact supports. Consider finite number of representations $(T_j^{\alpha_j}, \mathscr{H}_j)$, $\mathscr{H}_j=L^2(X_j, \mathscr{B}_j, \mu_j)$ with $X_j=M$, given by (4.1)–(4.2) in §4.2. Within the unitary equivalence, we may assume that $\gamma_j=1$ in (4.2) and all the μ_j's are equal to the same one $\omega\in\mathscr{LFM}(M)$. We can assume that ω has $C^{(n)}$-class density. Thus the representation $T_j^{\alpha_j}$ is determined uniquely by the parameter $s_j\in\mathbf{R}$ in the 1-cocycle α_j, and it is denoted also by T_{s_j}:

$$T_{s_j}(g)h(p)=\Big(\frac{d\omega(g^{-1}p)}{d\omega(p)}\Big)^{\frac{1}{2}+is_j}h(g^{-1}p).$$

At first we give the following simple lemma.

Lemma A. 1. *Any representation T_s, $s\in\mathbf{R}$, of G is irreducible. Two represbntations T_{s_1} and T_{s_2} are mutually equivalent if and only if $s_1=s_2$.*

Proof. For the irreducibility, it is enough for us to quote Note 6. 8. Let us prove the second assertion. Take a coordinate neighbourhood U of M. We may assume that the measure ω is given in this coordinates $p=(p_1, p_2, \cdots, p_d)$ as $d\omega(p)=dp_1dp_2\cdots dp_d$. Take a relatively compact, open subset U_0 of U such that $\mathrm{Cl}(U_0)\subset U$. Then, there exists an element $g\in G(U)=\mathrm{Diff}_0(U)\subset G$ such that $g^{-1}p=(\gamma p_1, \gamma p_2, \cdots, \gamma p_d)\equiv\gamma p$ for $p\in U_0$ with a positive constant $\gamma\neq 1$.

Decompose the representation spaces $\mathscr{H}_j=L^2(M, \omega)(j=1, 2)$ as $\mathscr{H}_j=L^2(U)\otimes L^2(M\backslash U)$. Then, the restrictions $T_{s_j}|G(U)$ are both irreducible on $L^2(U)$, by Note 6. 8, and trivial on $L^2(M\backslash U)$. Therefore an intertwining operator $A:\mathscr{H}_1\rightarrow\mathscr{H}_2$, leaves $L^2(U)$ and $L^2(M\backslash U)$ stable, and $A|L^2(U)$ is zero or invertible. Note that restrictions $T_{s_j}|G_\omega(U)$ to the subgroup $G_\omega(U)=G_\omega\cap G(U)$ are both identical on $L^2(U)$ and irreducible on its subspace $\{\chi_U\}^\perp$, and so A is a scalar multiplication operator on $L^2(U)$ and maps $L^2(M\backslash U)$ onto itself.

Thus, for any $h\in L^2(U)$, $Ah=ah$ with a constant $a\in\mathbf{C}$. So, taking the above element $g\in G(U)$, we have, for $h\in L^2(U_0)$,

$$T_{s_1}(g)h(p)=\gamma^{(\frac{1}{2}+is_1)d}h(\gamma p),\quad T_{s_2}(g)(Ah)(p)=\gamma^{(\frac{1}{2}+is_2)d}a\,h(\gamma p).$$

Since $AT_{s_1}(g)=T_{s_2}(g)A$, we have $a=0$ if $s_1\neq s_2$.

Lemma A. 2. *Let* $\dim M\geq 2$. *Take a finite number of representations T_{s_j}, $j\in J$, with the parameters $s_j\in\mathbf{R}$ of G, and the same number of $T_{s'_j}$, $j\in J$, with $s'_j\in\mathbf{R}$. Then an intertwining operator between the tensor products $\otimes_{j\in J}T_{s_j}$ and $\otimes_{j\in J}T_{s'_j}$ is a linear combination of the permutation operator $R(\sigma)$ with $\sigma\in\mathfrak{S}_J$ satisfying $s_{\sigma(j)}=s'_j\,(j\in J)$. Here $R(\sigma)(\otimes_{j\in J}f_j):=\otimes_{j\in J}f_{\sigma^{-1}(j)}$, for decomposable element $\otimes_{j\in J}f_j\in\otimes^{\chi}_{j\in J}\mathscr{H}_j$ with $f_j\in\mathscr{H}_j$.*

In particular, if the sets of parameters $\{s_j;\ j\in J\}$ and $\{s'_j;\ j\in J\}$ are different, these two tensor product representations are mutually disjoint.

Proof. For a subset V of M, put $\mathscr{H}_j(V)=L^2(V, \mathscr{B}_j|V, \mu_j|V)\subset\mathscr{H}_j$. Take connected open subsets U_j, $j\in J$, of M which are mutually disjoint, and put $U_\infty=M\backslash\cup_{j\in J}U_j$. Then we have an orthogonal decomposition of each $\mathscr{H}_j$ as

$$\text{(A. 1)}\qquad \mathscr{H}_j=\textstyle\sum^{\oplus}_{k\in J_\infty}\mathscr{H}_j(U_k),\quad \text{with } J_\infty=J\cup\{\infty\}.$$

Consider the subgroup $G'=\prod_{k\in J}G(U_k)$ of G and its representation under the tensor product $\otimes_{j\in J}T_{s_j}$ on the space $\mathscr{H}_J=\otimes_{j\in J}\mathscr{H}_j$. Then, inserting the decomposition (A. 1) in each $\mathscr{H}_j$, we get a decomposition of $\mathscr{H}_J$ as follows:

$$\mathscr{H}_J=\sum^{\otimes}_{Q\in(J_\infty)^J}\mathscr{H}[Q] \quad \text{with } \mathscr{H}[Q]=\otimes_{j\in J}\mathscr{H}_j(U_{q_j}),$$

where the sum runs over $Q=(q_j)_{j\in J}\in(J_\infty)^J$. As G'-modules, on each component $\mathscr{H}[Q]$, there acts a tensor product representation

$$\otimes_{j\in J_Q}(T_{s_j}|G(U_{q_j}),\ \mathscr{H}_j(U_{q_j})),$$

with $J_Q=\{j\in J;\ q_j\neq\infty\}$ on the factor $\otimes_{j\in J_Q}\mathscr{H}_j(U_{q_j})$, and the other factor $\otimes_{j\notin J_Q}\mathscr{H}_j(U_\infty)$ gives the multiplicity.

All the components which carry an irreducible tensor product of $G'=\prod_{j\in J}G(U_j)$, not containing the trivial representation of any of $G(U_j)$'s, are given as

$$\otimes_{j\in J}(T_{s_j}|G(U_{\sigma(j)}),\ \mathscr{H}_j(U_{\sigma(j)})),$$

where $\sigma\in\mathfrak{S}_J$. On any other component, some of $G(U_j)$'s acts trivially.

Take an intertwining operator A of $\otimes_{j\in J}T_{s_j}$ with $\otimes_{j\in J}T_{s'_j}$. Then the above fact means that A maps $\otimes_{j\in J}\mathscr{H}_j(U_j)$ onto some of $\otimes_{j\in J}\mathscr{H}_j(U_{\sigma(j)})$. If this is not zero, the representations of G' on these subspaces should be equivalent. By Lemma A. 1, applied to each of $G(U_j)$, we see that $s_{\sigma(j)}=s'_j$ $(j\in J)$ in that case. If this does not happen for any $\sigma\in\mathfrak{S}_J$, then A should be zero on $\otimes_{j\in J}\mathscr{H}_j(U_j)$.

Now let us study the way of changing when $\mathscr{U}=(U_j)_{j\in J}$ is replaced by another $\mathscr{U}'=(U'_j)_{j\in J}$. Put $\mathscr{H}(\mathscr{U})=\otimes_{j\in J}\mathscr{H}(U_j)$ anew and denote by $P_{\mathscr{U}}$ the orthogonal projection of $\mathscr{H}_J$ onto $\mathscr{H}(\mathscr{U})$. For a $\sigma\in\mathfrak{S}_J$, put $\mathscr{U}\sigma=(U_{\sigma(j)})_{j\in J}$, then $\mathscr{H}(\mathscr{U}\sigma)=R(\sigma)\mathscr{H}(\mathscr{U})$. Define $\mathfrak{S}_J((s_j)_{j\in J})=\{\sigma\in\mathfrak{S}_J;\ s_{\sigma(j)}=s_j\ (j\in J)\}$. Then the above argument shows that

$$A\circ P_{\mathscr{U}}=\sum_{\sigma\in\mathfrak{S}_J((s_j)_{j\in J})}a(\sigma,\mathscr{U})\cdot R(\sigma)\circ P_{\mathscr{U}}=\sum_{\sigma\in\mathfrak{S}_J((s_j)_{j\in J})}a(\sigma,\mathscr{U})\cdot P_{\mathscr{U}\sigma}\circ R(\sigma)\circ P_{\mathscr{U}},$$

where $a(\sigma,\mathscr{U})\in\mathbf{C}$ are constants.

Let us prove that these constants do not depend on $\mathscr{U}$. To do so, we introduce an equivalence relation in the set of all $\mathscr{U}=(U_j)_{j\in J}$ with mutually disjoint, connected open U_j's. Two elements $\mathscr{U}$ and $\mathscr{U}'=(U'_j)_{j\in J}$ are called *adjacent* to each other and denoted as $\mathscr{U}\approx\mathscr{U}'$ if $U_j\cap U'_j\neq\phi\,(j\in J)$. Further $\mathscr{U}$ and $\mathscr{U}'$ are called *equivalent* and denoted as $U\sim U'$, if there exists a finite number of elements $\mathscr{U}^{(1)},\mathscr{U}^{(2)},\cdots,\mathscr{U}^{(n)}$ such that $\mathscr{U}\approx\mathscr{U}^{(1)}$, $\mathscr{U}^{(k)}\approx\mathscr{U}^{(k+1)}(1\leq k<n)$, $\mathscr{U}^{(n)}\approx\mathscr{U}'$.

Assume now $\mathscr{U}\approx\mathscr{U}'$. For $j\in J$, take a connected component U''_j of $U_j\cap U'_j\neq\phi$, and put $\mathscr{U}''=(U''_j)_{j\in J}$. Then, $\mathscr{U}\approx\mathscr{U}''\approx\mathscr{U}'$ and $\mathscr{H}(\mathscr{U}\sigma)\cap\mathscr{H}(\mathscr{U}'\sigma)\supset\mathscr{H}(\mathscr{U}''\sigma)$ for any $\sigma\in\mathfrak{S}_J$. It can be seen from this that $a(\sigma,\mathscr{U})=a(\sigma,\mathscr{U}'')=$

$a(\sigma, \mathscr{U}')$ for $\sigma \in \mathfrak{S}_J((s_j)_{j\in J})$. Therefore we get $a(\sigma, \mathscr{U}) = a(\sigma, \mathscr{U}')$ if $\mathscr{U} \sim \mathscr{U}'$.

On the other hand, in case $d = \dim M \geq 2$, any two elements $\mathscr{U}$ and $\mathscr{U}'$ are mutually equivalent, as we can see without difficulty. This means that, on the dense subspace of $\mathscr{H}_J$ spanned by $\mathscr{H}(\mathscr{U})$'s, the operator A is expressed as $A = \sum_{\sigma \in \mathfrak{S}_J((s_j)_{j\in J})} a(\sigma) R(\sigma)$ with $a(\sigma) = a(\sigma, \mathscr{U})$. This expression holds also on the whole space $\mathscr{H}_J$.

Remark A. 3. In case $d=1$, take $M = \mathbf{R}$. Then $\mathscr{H}_J \cong L^2(\mathbf{R}^k, \Pi\lambda)$ with $k = |J|$ and λ a Lebesgue measure on $\mathbf{R}$. In this case, there exist $k!$ number of equivalence classes of $\mathscr{U}$'s. In fact, consider the order of elements in $\mathscr{U} = (U_j)_{j=1}^k$: for $x = (x_j)$, $x_j \in U_j$ $(1 \leq j \leq k)$,

$$x_{\tau(1)} < x_{\tau(2)} < \cdots < x_{\tau(k)},$$

with a certain $\tau \in \mathfrak{S}_k$. Then each τ represents an equivelence class of $\mathscr{U}$'s.

On the other hand, put $D = \{x = (x_j) \in \mathbf{R}^k ; x_1 < x_2 < \cdots < x_k\}$, and $x\tau = (x_{\tau(j)})_{j=1}^k$, $\tau \in \mathfrak{S}_k$. Denote by Q_τ the restriction on $D\tau \subset \mathbf{R}^k$ of function $f \in \mathscr{H}_J$:

$$Q_\tau f(x) = f(x) \ (x \in D_\tau); \quad = 0 \ (x \notin D\tau).$$

Then we have

Lemma A. 4. *Let $M = \mathbf{R}$. Then for the tensor product $\otimes_{j=1}^k T_{s_j}$ of representations $(T_{s_j}, \mathscr{H}_j)$, $\mathscr{H}_j = L^2(\mathbf{R}, \lambda)$ of $G = \mathrm{Diff}_0(\mathbf{R})$, the algebra of intertwining operators is generated by $\{R(\sigma); \sigma \in \mathfrak{S}_k((s_j)_{j=1}^k)\}$ and $\{Q_\tau; \tau \in \mathfrak{S}_k\}$. In case where all the s_j's are mutually equal, this algebra is isomorphic to $\mathfrak{gl}(k!, \mathbf{C})$ algebraically.*

It may be interesting to investigate the situation in case $k = |J| = \infty$. Returning to the general case, we remark here

Lemma A. 5. *Let J and J' be two finite sets of indices. Assume that $|J| \neq |J'|$. Then any two tensor product representations $\otimes_{j\in J} T_{s_j}$ and $\otimes_{k\in J'} T_{s'_k}$ of G are mutually disjoint.*

A proof can be given by a similar method as that in the proof of the above lemma.

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