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A norm inequality for Itô processes

By

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Introduction

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ such that $A\in\mathscr{F}_0$ whenever $A\in\mathscr{F}$ and $P(A)=0$. The adapted real Brownian motion $B=(B_t)_{t\geq 0}$ starts at 0 and the process $(B_t-B_s)_{t\geq s}$ is independent of $\mathscr{F}_s$ for all $s\geq 0$.

Let φ and ϕ be real predictable processes such that

$$P\Big(\int_0^t (|\varphi_s|^2+|\phi_s|)ds<\infty \text{ for all } t>0\Big)=1.$$

Also, let ζ and ξ be $\mathbf{R}^\nu$-valued predictable processes, where ν is a positive integer. We assume the same constraint for ζ and ξ as above. The Itô processes X and Y are defined by

$$X_t=X_0+\int_0^t \varphi_s\, dB_s+\int_0^t \phi_s\, ds,$$
$$Y_t=Y_0+\int_0^t \zeta_s\, dB_s+\int_0^t \xi_s\, ds.$$

We assume that X and Y are continuous.

For $1<p<\infty$ we set $p^{**}=\max\{2p, p/(p-1)\}$ and $|||X|||_p=\sup \|X_\tau\|_p$ where the supremum is taken over all bounded stopping times τ.

Definition 1. We define that Y *is strongly differentially subordinate to* X if $|Y_0|\leq|X_0|$, $|\zeta|\leq|\varphi|$ and $|\xi|\leq|\phi|$.

The following inequality is due to Burkholder [2].

Theorem 2. *If* $X\geq 0$, $\phi\geq 0$, *and* Y *is strongly differentially subordinate to* X, *then*

$$|||Y|||_p\leq(p^{**}-1)\,|||X|||_p.$$

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A norm Inequality

Let $(\Omega, \mathscr{F}, P)$ and $(\mathscr{F}_t)_{t\geq 0}$ be as in the introduction. The adapted square integrable real martingale M starts at 0 and, for all $s\geq 0$, the process $(M_t-M_s)_{t\geq s}$ is independent of $\mathscr{F}_s$. Let $\langle M\rangle$ be the quadratic variational process of M. The adapted integrable increasing process A starts at 0. We asume that M and A are continuous. Thus $\langle M\rangle$ is also continuous.

We follow [3] for notions of stochastic processes. Thus increasing in the above means non-decreasing. Terms, positive, negative and decreasing, will be used similarly. Also one may see [3] for the basic facts about stochastic integrals.

Consider real predictable processes φ and ϕ such that

$$\int_0^t |\varphi_s|^2 d\langle M\rangle_s < \infty \text{ and } \int_0^t |\phi_s|\, dA_s < \infty, \text{ for all } t>0.$$

Let $\mathbf{H}$ be a Hilbert space over $\mathbf{R}$. For $x, y\in\mathbf{H}$ we denote by $x\cdot y$ the inner product of x and y and put $|x|^2=x\cdot x$. The $\mathbf{H}$-valued predictable processes ζ and ξ have the same growth condition as above. The Itô processes X and Y are difined by

$$X_t=X_0+\int_0^t \varphi_s\, dM_s+\int_0^t \phi_s\, dA_s,$$
$$Y_t=Y_0+\int_0^t \zeta_s\, dM_s+\int_0^t \xi_s\, dA_s.$$

We assume that X and Y have continuous paths.

Let $0\leq\alpha\leq 1$ and $1<p<\infty$. Put $r=r(\alpha, p)=\max\{(\alpha+1)p, p/(p-1)\}$. Observe that $p^{**}=r(1, p)$. Define $|||X|||_p$ as in the introduction.

Definition 3. We define that *Y is α-subordinate to X* if $|Y_0|\leq|X_0|$, $|\zeta|\leq|\varphi|$ and $|\xi|\leq\alpha|\phi|$.

Theorem 4. *If $X\geq 0$, $\phi\geq 0$, and Y is α-subordinate to X, then*

$$|||Y|||_p\leq(r-1)\,|||X|||_p$$

and the constant $r-1$ is best possible.

Proof of the Inequality. In order to make the key points of the proof clear we defer some technical details to the following section. Thus we use some unproved claims and lemmas in this proof.

We may asume $|||X|||_p<\infty$.

Claim 5. *We may further assume that $X>0$ and $|Y|>0$.*

Put $S=\{(x, y): 0<x<\infty$ and $y\in\mathbf{H}$ with $|y|>0\}$. Define two functions U and V on S by

$$U(x, y)=(|y|-(r-1)x)(x+|y|)^{p-1},$$
$$V(x, y)=|y|^{p}-(r-1)^{p}x^{p}.$$

Observe that U is smooth and (X, Y) has value in S.

Claim 6. *It suffices to prove* $\|Y_\tau\|_p\leq(r-1)\|X_\tau\|_p$ *whenever* τ *is a bounded stopping time for which there is a number* $n\geq 0$ *such that* $\int_0^\tau|\varphi_s|^2 d\langle M\rangle_s\leq n$ *and* $X_t+|Y_t|+|U_x(X_t, Y_t)|+|U_y(X_t, Y_t)|\leq n$ *for* $0\leq t\leq\tau$.

Let τ and n be as in Claim 6. Observe that the random variables $U(X_\tau, Y_\tau)$ and $V(X_\tau, Y_\tau)$ are integrable. The inequality in Claim 6 becomes $\mathbf{E}V(X_\tau, Y_\tau)\leq 0$, which follows from the inequality $\mathbf{E}U(X_\tau, Y_\tau)\leq 0$ and

Lemma 7. *There is a constant* $c>0$ *such that* $V\leq cU$ *on* S.

Observe that $r\geq 2$, hence $U(x, y)\leq 0$ if $|y|\leq x$. Since $X>0$ and Y is α-subordinate to X we have $|Y_0|\leq X_0$, hence $\mathbf{E}U(X_0, Y_0)\leq 0$. Thus it is enough to show

$$\mathbf{E}U(X_\tau, Y_\tau)\leq\mathbf{E}U(X_0, Y_0).$$

Since τ is bounded, Itô's formula gives

$$\begin{aligned}U(X_\tau, Y_\tau)=U(X_0, Y_0)&+\int_0^\tau(U_x(X_s, Y_s)\varphi_s+U_y(X_s, Y_s)\cdot\zeta_s)dM_s\\&+\int_0^\tau(U_x(X_s, Y_s)\phi_s+U_y(X_s, Y_s)\cdot\xi_s)dA_s\\&+\frac{1}{2}\int_0^\tau(U_{xx}(X_s, Y_s)|\varphi_s|^2+2U_{xy}(X_s, Y_s)\cdot\varphi_s\zeta_s\\&+U_{yy}(X_s, Y_s)\zeta_s\cdot\zeta_s)d\langle M\rangle_s.\end{aligned}$$

Here $U_{yy}(X_s, Y_s)$ can be regarded as a linear transformation from $\mathbf{H}$ to $\mathbf{H}$.

For differentiation of vector functions one may see [4].

We like to finish the proof by showing that the above three integrals have negative expectations.

The first integral has zero expectation because τ is bounded and the process

$$t\rightarrow\int_0^{\tau\wedge t}(U_x(X_s, Y_s)\varphi_s+U_y(X_s, Y_s)\cdot\zeta_s)dM_s$$

is a martingale. This follows from

$$\mathbf{E}\int_0^\tau \left| U_x(X_s, Y_s)\varphi_s + U_y(X_s, Y_s)\cdot\zeta_s \right|^2 d\langle M\rangle_s$$
$$\leq n^2\mathbf{E}\int_0^\tau |\varphi_s|^2 d\langle M\rangle_s \leq n^3 < \infty$$

where we used the assumption that Y is α-subordinate to X, that is, $|\zeta_s| \leq |\varphi_s|$ and the assumptions about τ.

The rest two integrals have negative integrands; thus they have negative expectations because the processes A and $\langle M\rangle$ are increasing. For this we need

Lemma 8. *(a)* $U_x(x, y) + \alpha |U_y(x, y)| \leq 0$ *for all* $(x, y) \in S$.
(b) $U_{xx}(x, y)|h|^2 + 2U_{xy}(x, y)\cdot hk + U_{yy}(x, y)k\cdot k \leq 0$ *whenever* $(x, y) \in S$, $h \in \mathbf{R}$, $k \in \mathbf{H}$ *and* $|h| \geq |k|$.

Because $\phi \geq 0$ and Y is α-subordinate to X, using the Cauchy-Schwarz inequality and (a) of Lemma 8, we have

$$U_x(X_s, Y_s)\phi_s + U_y(X_s, Y_s)\cdot\xi_s \leq U_x(X_s, Y_s)\phi_s + |U_y(X_s, Y_s)|\,|\xi_s|$$
$$\leq (U_x(X_s, Y_s) + \alpha|U_y(X_s, Y_s)|)\phi_s \leq 0.$$

Similarly, the integrand of the third integral is negative because of (b) of Lemma 8, Claim 5 and the assumption that Y is α-subordinate to X.

This proves the inequality in Theorem 4 under the assumption of Claim 5, Claim 6, Lemma 7 and Lemma 8. We will elaborate on these claims and lemmas in the following section. In the last section we construct examples which show that the constant $r-1$ is best possible.

Proof of Claims and Lemmas

Proof of Claim 5. Let X and Y be Itô processes such that $X \geq 0$, $\phi \geq 0$, and Y is α-subordinate to X. For each $\varepsilon > 0$, consider new Itô processes $X + \varepsilon$ and (Y, ε), where (Y, ε) has value in the standard product Hilbert space $\mathbf{H}\times\mathbf{R}$. Observe that these new processes satisfy the extra assumption in Claim 5 as well as the assumptions of Theorem 4. Assuming the inequality in Theorem 4 for these new processes, we get

$$|||Y|||_p \leq |||(Y, \varepsilon)|||_p \leq (r-1)|||X+\varepsilon|||_p \leq (r-1)|||X|||_p + \varepsilon(r-1),$$

which, as $\varepsilon \to 0$, gives the inequlity in Theorem 4. This proves Claim 5.

Proof of Claim 6. Let X and Y be Itô processes such that $X \geq 0$, $\phi \geq 0$, and Y is α-subordinate to X. And let τ be a bounded stopping time. Consider sequences of stopping times $(\mu_n)_{n\geq 0}$ and $(\sigma_n)_{n\geq 0}$ given by

$$\mu_n=\inf\{t>0: X_t+|Y_t|+|U_x(X_t, Y_t)|+|U_y(X_t, Y_t)|>n\}$$

and

$$\sigma_n=\inf\left\{t>0: \int_0^t |\varphi_s|^2 d\langle M\rangle_s>n\right\}.$$

Notice that μ_n increase to infinity as n increase to infinity because the process in the definition of μ_n is finite. Concerning σ_n, we have assumed that

$$\int_0^t |\varphi_s|^2 d\langle M\rangle_s<\infty \text{ for all } t>0,$$

hence σ_n increase to infinity as n increase to infinity.

Putting $\tau_n=\tau\wedge\mu_n\wedge\sigma_n$, we have $X_t+|Y_t|+|U_x(X_t, Y_t)|$ $+|U_y(X_t, Y_t)|\leq n$ for $0\leq t\leq\tau_n$ and

$$\int_0^{\tau_n} |\varphi_s|^2 d\langle M\rangle_s\leq n$$

because the process in the definition of μ_n and the process $\langle M\rangle$ are continuous. Thus, assuming the inequality in Claim 6 for τ_n, we have

$$\| Y_{\tau_n}\|_p\leq(r-1)\| X_{\tau_n}\|_p\leq(r-1)|||X|||_p.$$

Since Y is continuous and $\tau_n\to\tau$ we have $Y_{\tau_n}\to Y_\tau$, hence

$$\| Y_\tau\|_p\leq\liminf_{n\to\infty}\| Y_{\tau_n}\|_p\leq(r-1)|||X|||_p$$

by Fatou's lemma. Now taking supremum over all bounded stopping times τ, we have $|||Y|||_p\leq(r-1)|||X|||_p$. This proves Claim 6.

Proof of Lemma 7. Put $c=p(1-1/r)^{p-1}$. We want to prove $V(x, y)-cU(x, y)=|y|^p-(r-1)^p x^p-c(|y|-(r-1)x)(x+|y|)^{p-1}$ ≤ 0 for all $(x, y)\in S$. By the homogeneity we may consider only those $(x, y)\in S$ with $x+|y|=1$. Thus, with

$$F(x)=(1-x)^p-(r-1)^p x^p-c(1-rx),$$

We need to show that $F(x)\leq 0$ if $0<x<1$.

Observe that F is continuous on $[0, 1]$ and smooth on the open interval $(0, 1)$. Thus, for $0<x<1$, we have

$$F'(x)=-p((1-x)^{p-1}+(r-1)^p x^{p-1})+rc,$$
$$F''(x)=p(p-1)((1-x)^{p-2}-(r-1)^p x^{p-2}).$$

Notice that $0<1/r<1$. One can check $F(1/r)=F'(1/r)=0$.

We divide the rest of the proof into three cases.

In case $p=2$ we have $F''=2[1-(r-1)^2]\leq 0$ on $(0, 1)$ because $r\geq 2$. Hence F has the maximum over $[0, 1]$ at $t=1/r$, which impies that $F\leq 0$ on $[0, 1]$.

Now let $1<p<2$. From the formula of F'' we see that $F''(x)<0$ if and only if $1-x>(r-1)^{p/(p-2)}x$, or $x<x^*$ where $1/x^*=1+(r-1)^{p/(p-2)}$. Here $0<1/r<x^*$. Thus, $F\leq 0$ on $[0,x^*]$ for the same reason as the previous case. On the interval $[x^*, 1]$ the function F is convex. Hence it suffices to check $F(1)\leq 0$. For this observe that $\log x$ is concave. Thus

$$\log 1\geq(p-1)\log(p-1)+(2-p)\log p, \text{ or } (p-1)^{p-1}\leq p^{p-2}.$$

Hence

$$r^{p-1}\geq\left(\frac{p}{p-1}\right)^{p-1}=p\frac{p^{p-2}}{(p-1)^{p-1}}\geq p$$

and

$$\begin{aligned} F(1)&=-(r-1)^p-c(1-r)\\ &=-(r-1)^p+p(r-1)\left(1-\frac{1}{r}\right)^{p-1}\\ &=\frac{(r-1)^p}{r^{p-1}}(p-r^{p-1})\leq 0. \end{aligned}$$

The case $p>2$ is proved similarly. This time one needs to check $F(0)\leq 0$ for which the inequality $(p-1)^{p-1}\geq p^{p-2}$ is necessary.

Basic facts about convex functions can be found in [5].

Proof of Lemma 8.

Proof of (*a*). From the definition of U we get

$$\begin{cases} U_x(x, y)=((p-r)(x+|y|)-r(p-1)x)(x+|y|)^{p-2}\\ U_y(x, y)=(p(x+|y|)-r(p-1)x)(x+|y|)^{p-2}\dfrac{y}{|y|}. \end{cases}$$

By the homogeneity the inequality in (a) of Lemma 8 is reduced to the inequality that $L\leq 0$ on $(0, 1)$, where

$$L(x)=(p-r)-r(p-1)x+\alpha\,|p-r(p-1)x|.$$

Observe that L is convex on $[0, 1]$. Hence it is enough to check $L(0)\leq 0$ and $L(1)\leq 0$ for which one just needs to keep in mind that $(\alpha+1)p\leq r$, $p/(p-1)\leq r$ and that $0\leq\alpha\leq 1$.

Proof of (*b*). Let $x, h\in\mathbf{R}$, $y, k\in\mathbf{H}$, $(x, y)\in S$ and $|k|\leq|h|$. Put $I=\{t\in\mathbf{R}: x+th>0 \text{ and } |y+tk|>0\}$ and define a function G on I by

$$G(t)=U(x+th, y+tk).$$

Observe that I is an open set, $0\in I$ and that $G(t)$ is smooth at $t=0$. By the chain rule one has

$$G''(0)=U_{xx}(x, y)h^2+2U_{xy}(x, y)\cdot hk+U_{yy}(x, y)k\cdot k.$$

Hence the proof is complete if we can check $G''(t)\leq 0$ for all $t\in I$. If no confusoin arises, we will not write the arguement t. On I define more functions K, Q and R by $K=K(t)=x+th$, $Q=|y+tk|$ and $R=K+Q$. Then, differentiation gives $QQ'=k\cdot(y+tk)$ and $QQ''=|k|^2-(Q')^2$, hence, by the Cauchy-Schwarz inequality, we have $Q|Q'|=|QQ'|\leq|k||y+tk|=|k|Q$. Thus, $|Q'|\leq|k|$ and $R''=Q''\geq 0$.

Writing $G=R^p-rKR^{p-1}$, we compute

$$\begin{aligned} G' &=pR'R^{p-1}-rhR^{p-1}-r(p-1)KR'R^{p-2}, \\ G'' &=pR''R^{p-1}+p(p-1)(R')^2R^{p-2}-2r(p-1)hR'R^{p-2} \\ &\qquad -r(p-1)KR''R^{p-2}-r(p-1)(p-2)K(R')^2R^{p-3}. \end{aligned}$$

Thus, putting $1/H=(p-1)R^{p-3}$, noting $-rKR''R=-rR''R^2+rRQR''$, and inserting terms $rR(R')^2-rR(R')^2$, we have

$$\begin{aligned} HG'' &=\left(\frac{p}{p-1}-r\right)R''R^2+rR(QR''-2hR'+(R')^2)+(pR-rR-r(p-2)K)(R')^2 \\ &\leq rR(|k|^2-|h|^2)+((p-r)Q+(p-r(p-1))K)(R')^2\leq 0 \end{aligned}$$

because $R'=h+Q'$, $|k|\leq|h|$, $(\alpha+1)p\leq r$ and $p/(p-1)\leq r$. This proves $G''\leq 0$ on I and Lemma 8 has been all proved.

About the Best Constant

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ such that $A\in\mathscr{F}_0$ whenever $A\in\mathscr{F}$ and $P(A)=0$. The adapted real Brownian motion $(B_t)_{t\geq 0}$ starts at 0 and, for all $s\geq 0$, the process $(B_t-B_s)_{t\geq s}$ is independent of $\mathscr{F}_s$.

Let $0\leq\alpha\leq 1$, $1<p<\infty$ and $r=r(\alpha, p)=\max\{(\alpha+1)p, p/(p-1)\}$.

The constant $r-1$ is best possible in the sense that if $0<\beta<r-1$, then there are random variables X_0, Y_0 and real predictable processes φ, ψ, ζ and ξ such that the Itô processes X and Y defined by

$$\begin{aligned} X_t &=X_0+\int_0^t\varphi_s\,dB_s+\int_0^t\psi_s\,ds, \\ Y_t &=Y_0+\int_0^t\zeta_s\,dB_s+\int_0^t\xi_s\,ds, \end{aligned}$$

satisfy the conditions $X\geq 0$, $\psi\geq 0$, and that Y is α-subordinate to X but the opposite inequality

$$|||Y|||_p>\beta|||X|||_p$$

holds.

We need the following series test from Calculus:

Gauss Test. *If* $a_n>0$ *and* $a_{n+1}/a_n=1-\lambda/n+O(1/n^2)$ *as* $n\to\infty$, *then* $\sum_{n=0}^{\infty} a_n<\infty$ *if and only if* $\lambda>1$.

Let $0<\beta<r-1$. We cosider two cases $1<p\leq(\alpha+2)/(\alpha+1)$ and $(\alpha+2)/(\alpha+1)<p<\infty$ separately.

Case 1. $1<p\leq(\alpha+2)/(\alpha+1)$.

In this case $r=p/(p-1)$ and $p=r/(r-1)$. First define sequences $(x_n)_{n\geq1}$ and $(b_n)_{n\geq1}$ by $rx_n=n+1$ and $2b_n=2n+1$. Also, define a sequence of stopping times $(\sigma_n)_{n\geq0}$: put $\sigma_0=0$ and, for $n>0$, let

$$\sigma_n=\inf\{s>\sigma_{n-1}: B_s-B_{\sigma_{n-1}}\notin(x_n-b_n,\ 1)\}.$$

Observe that σ_n is finite almost surely and, with $p_n=P(B_{\sigma_n}-B_{\sigma_{n-1}}=x_n-b_n)$, we have $P(B_{\sigma_n}-B_{\sigma_{n-1}}=1)=1-p_n$ and

$$p_n=\frac{1}{b_n-x_n+1}=\frac{2p}{2n+p+2}\geq\frac{p}{n+2p}.$$

Thus, as $n\to\infty$, we have

$$p_n=\frac{p}{n}+O\left(\frac{1}{n^2}\right) \text{ and } \frac{p_{n+1}}{p_n}=1-\frac{1}{n}+O\left(\frac{1}{n^2}\right).$$

We also need the following

Proposition 9. (a) *The series* $\sum_{n=1}^{\infty}\left(|x_n|^p p_n\prod_{k=1}^{n-1}(1-p_k)\right)$ *diverges.*

(b) *The sequence* $\left((n+3/2)^p\prod_{k=1}^{n}(1-p_k)\right)_{n\geq1}$ *is bounded.*

Proof of (a). Let a_n be the n-th term of the above series. Then, (a) of Proposition 9 follows from the Gauss test and

$$\begin{aligned}\frac{a_{n+1}}{a_n}&=\left(1+\frac{1}{n+1}\right)^p\frac{p_{n+1}}{p_n}(1-p_n)\\&=\left(1+\frac{p}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{p}{n}+O\left(\frac{1}{n^2}\right)\right)\\&=1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\end{aligned}$$

Proof of (b). From

$$\prod_{k=1}^{n}(1-p_k)\leq\exp\Big(-\sum_{k=1}^{n}p_k\Big)\leq\exp\Big(-\sum_{k=1}^{n}\frac{p}{k+2p}\Big)\leq\exp\Big(-p\int_{1+2p}^{n+2p+1}\frac{dx}{x}\Big)$$

we have

$$\Big(n+\frac{3}{2}\Big)^p\prod_{k=1}^{n}(1-p_k)\leq\Big(n+\frac{3}{2}\Big)^p\Big(\frac{n+2p+1}{1+2p}\Big)^{-p}\leq(1+2p)^p.$$

Now we go back to the contruction in the first case. Let a_n be as in the proof of (a) of Proposition 9. Since $0<\beta<r-1$ Proposition 9 enables us to choose a large positive integer N so that

$$\sum_{n=1}^{N}a_n>\Big(\frac{\beta}{r-1}\Big)^p\Big(\sum_{n=1}^{N}a_n+\Big(N+\frac{3}{2}\Big)^p\prod_{n=1}^{N}(1-p_n)\Big).$$

Putting $X_0=3/2$ and $Y_0=1/2$, define Itô processes X and Y by their predictable integrands φ, ψ, ζ and ξ: for $\sigma_{n-1}<s\leq\sigma_n$, let

$$(\varphi_s,\ \zeta_s)=\begin{cases}(1,\ -2Y_{\sigma_{n-1}})1_{\{2|Y_{\sigma_{n-1}}|=1\}} & \text{if } 0<n\leq N,\\ (0,\ 0) & \text{if } n>N\end{cases}$$

and $\psi=\xi=0$.

Then, one can check that $X\geq0$, $\psi\geq0$, Y is α-subordinate to X, and that

$$(X_{\sigma_N},\ Y_{\sigma_N})=\begin{cases}x_n(1,\ (-1)^{n+1}(r-1)) & \text{with probability } p_n\prod_{k=1}^{n-1}(1-p_k)\\ & \qquad\text{for } 1\leq n\leq N,\\ \Big(N+\frac{3}{2},\ \frac{(-1)^N}{2}\Big) & \text{with probability } \prod_{n=1}^{N}(1-p_n),\end{cases}$$

thus

$$\|X_{\sigma_N}\|_p^p=\sum_{n=1}^{N}a_n+\Big(N+\frac{3}{2}\Big)^p\prod_{n=1}^{N}(1-p_n),$$

$$\|Y_{\sigma_N}\|_p^p\geq(r-1)^p\sum_{n=1}^{N}a_n.$$

Hence we have

$$\|Y_{\sigma_N}\|_p>\beta\|X_{\sigma_N}\|_p.$$

Almost surely σ_N is finite, hence $X_t\to X_{\sigma_N}$ and $Y_t\to Y_{\sigma_N}$ as $t\to\infty$. Also, $|\varphi|=|\zeta|\leq1$; thus both X and Y are martingales. As a matter of fact, X and Y are uniformly bounded by $N+3/2$. By Doob's optional sampling theorem, for any bounded stopping time τ, we have $\|X_\tau\|_p\leq\|X_{\sigma_N}\|_p$.

Hence $|||X|||_p \le \|X_{\sigma_N}\|_p$. Also, Lebesgue's dominated convergence theorem gives

$$\|Y_{\sigma_N}\|_p = \lim_{n\to\infty} \|Y_n\|_p \le |||Y|||_p.$$

Thus we have the inequality $|||Y|||_p > \beta |||X|||_p$ and this completes the sharpness in Case 1.

Case 2. $(\alpha+2)/(\alpha+1) < p < \infty$.

In this case $r = (\alpha+1)p$. Choose a small $\delta > 0$ so that $\delta(\alpha+1)(p-1) < 2$. Define a sequence $(x_n)_{n\ge 1}$ by $rx_n = n\delta(\alpha+1)+2$. Notice that $\delta < 2$ and $\delta < x_n$ for all $n \ge 1$. Also, define a sequence of stopping times $(\sigma_n)_{n\ge 0}$: put $\sigma_0 = 0$, $\sigma_1 = \inf\{s>0 : B_s \notin (-1, 1)\}$ and, for $n \ge 1$, let $\sigma_{2n} = 1 + \sigma_{2n-1}$ and

$$\sigma_{2n+1} = \inf\{s > \sigma_{2n} : B_s - B_{\sigma_{2n}} \notin (-\delta, x_n - \delta)\}.$$

With $p_n = P(B_{\sigma_{2n+1}} - B_{\sigma_{2n}} = x_n - \delta)$ we have $P(B_{\sigma_{2n+1}} - B_{\sigma_{2n}} = -\delta) = 1 - p_n$ and as $n \to \infty$

$$p_n = \frac{\delta}{x_n} = \frac{\delta r}{n\delta(\alpha+1)+2} = \frac{p}{n} + O\left(\frac{1}{n^2}\right) \text{ and } \frac{p_{n+1}}{p_n} = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Putting

$$a_n = |x_n|^p \frac{p_n}{2} \prod_{k=1}^{n-1} (1-p_k)$$

we have

Proposition 10. *The series $\sum_{n=1}^{\infty} a_n$ diverges to infinity.*

Proof. The proposition follows from the Gauss test and the computation

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{x_{n+1}}{x_n}\right)^p \frac{p_{n+1}}{p_n}(1-p_n) \\ &= \left(1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)^p \left(1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{p}{n}+O\left(\frac{1}{n^2}\right)\right) \\ &= 1-\frac{1}{n}+O\left(\frac{1}{n^2}\right) \qquad \text{as } n\to\infty.\end{aligned}$$

Recall that $0 < \beta < r-1$. Hence by Proposition 10 we can find N such that

$$\sum_{n=1}^{N} a_n > \left(\frac{\beta}{r-1}\right)^p \left(\sum_{n=1}^{N} a_n + 2^{p-1}\right).$$

We define Itô processes X and Y. Put $X_0=Y_0=1$. For $0<s\leq\sigma_1$ let $\phi_s=\xi_s=0$ and $\varphi_s=-\zeta_s=1$. For $1\leq n\leq N$, define

$$\begin{cases}\varphi_s=\zeta_s=0 \text{ and } \alpha\phi_s=\xi_s=\alpha\delta 1_{\{X_{\sigma_{2n-1}}=0\}} & \text{if } \sigma_{2n-1}<s\leq\sigma_{2n},\\ \varphi_s=-\zeta_s=1_{\{X_{\sigma_{2n}}=\delta\}} \text{ and } \phi_s=\xi_s=0 & \text{if } \sigma_{2n}<s\leq\sigma_{2n+1}.\end{cases}$$

Finally, for $s>\sigma_{2N+1}$ put $\varphi_s=\zeta_s=\phi_s=\xi_s=0$.

One can check that

$$(X_{\sigma_{2N+1}}, Y_{\sigma_{2N+1}})=\begin{cases}(2,0) & \text{with probability } \dfrac{1}{2},\\ x_n(1, r-1) & \text{with probability } \dfrac{p_n}{2}\prod_{k=1}^{n-1}(1-p_k)\\ & \qquad\text{for } 1\leq n\leq N,\\ (0, rx_n) & \text{with probability } \dfrac{1}{2}\prod_{n=1}^{N}(1-p_n).\end{cases}$$

Thus

$$\| X_{\sigma_{2N+1}}\|_p^p=\sum_{n=1}^N a_n+2^{p-1} \text{ and } \| Y_{\sigma_{2N+1}}\|_p^p\geq(r-1)^p\sum_{n=1}^N a_n$$

from which we have

$$\| Y_{\sigma_{2N+1}}\|_p>\beta\| X_{\sigma_{2N+1}}\|_p.$$

Observe that almost surely, σ_{2N+1} is finite, hence $X_t\to X_{\sigma_{2N+1}}$ and $Y_t\to Y_{\sigma_{2N+1}}$ as $t\to\infty$. Also, $|\varphi|=|\zeta|\leq 1$, $0\leq\phi\leq\delta$ and $0\leq\xi\leq\alpha\delta$. Thus, X and Y are submartingales: they are uniformy bounded by $rx_N=N\delta(\alpha+1)+2$. Clearly, $X\geq 0$, $\phi\geq 0$, and Y is α-subordinate to X. Besides, by Doob's optional sampling theorem, for any bounded stopping time τ, we have $\| X_\tau\|_p\leq\| X_{\sigma_{2N+1}}\|_p$; hence $|||X|||_p\leq\| X_{\sigma_N}\|_p$. Again, Lebesgue's dominated convergence theorem gives

$$\| Y_{\sigma_{2N+1}}\|_p=\lim_{n\to\infty}\| Y_n\|_p\leq|||Y|||_p.$$

Thus we have the inequality $|||Y|||_p>\beta|||X|||_p$.

This finishes the proof that $r-1$ is best possible.

Remark. The idea of considering stopping times σ_n in the above two constructions is due to Burkholder [1].

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References

[1] D. L. Burkholder, Sharp probability bounds for Itô processes, Statistics and Probability : A Raghu Raj Bahadur Festschrift, J. K. Ghosh, S. K. Mitra, K. R. Parathasarathy, and B. L. S. Prakasa Rao, editors. Published by Wiley Eastern Limited (1993), 135–145.

[2] D. L. Burkholder, Strong differential subordination and stochastic integration, Ann. Probab., **22** (1994), 995–1025.

[3] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam, 1981.

[4] S. Lang, Analysis I, Addison-Wesley, Reading, Mass., 1968.

[5] A. W. Roberts and D. E. Varberg, Convex functions, Academic Press, New York and London, 1973.