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Absence of diffusion near the bottom of the spectrum for a random Schrödinger operator on $L^2(\boldsymbol{R}^3)$

By

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1. Introduction

Let $(\Omega, F, \boldsymbol{P})$ be a probability space whose precise definition will be given later. For each $\omega \in \Omega$, we consider Anderson type random Schrödinger operator on $\boldsymbol{L}^2(\boldsymbol{R}^3)$:

$$(1.1) \qquad \begin{cases} H_\omega = -\Delta + V_\omega(x), \\ V_\omega(x) = \sum_{i \in \boldsymbol{Z}^3} q_i(\omega) f(x-i) \end{cases}$$

where $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$. $\{q_i\}_{i \in \boldsymbol{Z}^3}$ satisty

(H.1) $\{q_i\}_{i \in \boldsymbol{Z}^3}$ are real-valued independent identically distributed random variables on $(\Omega, F, \boldsymbol{P})$ with uniform distribution on $[0, 1]$.

We suppose the following conditions:

(H.2) There exist two positive numbers η_0 and η_1 such that $\eta_0 \leq f(x) \leq \eta_1$ for $x \in [0, 1)^3$,

(H.3) $x \notin [0, 1)^3 \Rightarrow f(x) = 0$.

H_ω is considered to be the operator corresponding to the Hamiltonian of the electron in random metalic media. Let $\sigma(H_\omega)$ denote the spectrum of H_ω. Then the following is a known fact.

Proposition 1.1. (Kirsch and Martinelli).

$$\sigma(H_\omega) = [0, \infty) \ a.s.$$

For $E > 0$, we shall mean by g_E an arbitrary real-valued function which satisfies the following condition:

(A) $g_E \in C_0^\infty(\boldsymbol{R})$ and supp $g_E \subset (0, E)$,

where $C_0^\infty(O) = \{f \in C^\infty(O) \mid \text{supp} f \subset O\}$ for an open set $O \subset \boldsymbol{R}^n$.

In this paper we are interested in the following quantity:

$$(1.2) \qquad r_E^2(t) = \boldsymbol{E}\left[\int_{\boldsymbol{R}^3} |x|^2 |e^{-itH_\omega} g_E(H_\omega) \phi(x)|^2 dx\right]$$

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for $\psi \in L_2^2(\boldsymbol{R}^3) = \{f \in L^2(\boldsymbol{R}^3) \mid \langle x \rangle^2 f \in L^2(\boldsymbol{R}^3)\}$, where $\langle x \rangle = \sqrt{1+|x|^2}$ and $\boldsymbol{E}$ denotes the integration in ω with respect to the measure $\boldsymbol{P}$. $g_E(H_\omega)\psi$ is a wave function of a electron which is well localized in the sence of $L_2^2(\boldsymbol{R}^3)$ and has energy near the bottom of the spectrum. $r_E^2(t)$ represents the mean square distance from the origin of the time–evolution of the electron whose initial wave function is $g_E(H_\omega)\psi$.

When $V \equiv 0$ or V is periodic, $r_E^2(t)$ behaves asymptotically as

$$r_E^2(t) \sim Ct^2 \ (t \longrightarrow \infty).$$

But when V is random, we expect by physical consideration that $r_E^2(t)$ behaves asymptotically as

$$r_E^2(t) \sim Dt \ (t \longrightarrow \infty).$$

D is called the diffusion constant. In [6] J.M.Combes and P.D.Hislop proved Anderson localization, that is to say, there exists $E^* > 0$ such that in $[0, E^*]$ the spectrum is pure point and the corresponding eigenfunctions decay exponentially. Hence when E is sufficiently small, we expect that $D=0$. But this does not follow from Anderson localization (see e.g. [7]).

Our main theorem is the following.

Theorem 1.1. *We assume* $(H.1)$, $(H.2)$, $(H.3)$ *and* (A), *then there exists* $E^* > 0$ *such that if* $0 < E < E^*$, *then*

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T \frac{r_E^2(t)}{t} dt = 0.$$

By J. Fröhlich and T. Spencer [1], absence of diffusion was proved in the case of discrete random Schrödinger operators in multidimensions. In the continuous case F. Martinelli and H. Holden [5] studied random Schrödinger operator with potential

$$V_\omega(x) = \sum_{i \in \boldsymbol{Z}^3} q_i(\omega) X_{C_i}(x),$$

where

$$C_i = \left\{ x \in \boldsymbol{R}^3 \middle| -\frac{1}{2} < x_i \leq \frac{1}{2};\ i=1, 2, 3 \right\}$$

and $X_{C_i}(x)$ is the characteristic function of C_i.

Our proof relies heavily on [1] and [5].

Let $\Omega = \{\omega \colon \boldsymbol{Z}^3 \longrightarrow [0, 1]\}$ and F be the σ-algebra generated by of all cylinder sets of Ω. For a cylinder set $I = \{\omega | \omega(i_j) \in \Delta_j, i_j \in \boldsymbol{Z}^3,\ \Delta_j :$ Borel set of $\boldsymbol{R}, j=1, 2, \cdots, n\}$, we define

$$\text{(1.3)} \qquad \boldsymbol{P}(I) = \int_{\Delta_1} X_{[0,1]}(\lambda_1) d\lambda_1 \cdots \int_{\Delta_n} X_{[0,1]}(\lambda_n) d\lambda_n,$$

where $X_{[0,1]}(\lambda)$ is the characteristic function on interval $[0, 1]$. By E. Hopf's extension theorem, $\boldsymbol{P}$ is extended to a probability measure on (Ω, F). If we define $q_i(\omega)=\omega(i)$, the random variables $\{q_i\}_{i\in Z3}$ satisfy (H.1). We define the group of measure preserving ergodic transformations $T_i(i\in\boldsymbol{Z}^3)$ in Ω by

$$T_i\omega(j)=\omega(j-i),\ (j\in\boldsymbol{Z}^3)$$

for $\omega\in\Omega$. Then we have

$$H_{T_i\omega}=U_iH_\omega U_i^*\ (i\in\boldsymbol{Z}^3)$$

where U_i are the unitary operators in $L^2(\boldsymbol{R}^3)$ defined by

$$(U_if)(x)=f(x-i)\ \text{for}\ f\in L^2(\boldsymbol{R}^3),\ i\in\boldsymbol{R}^3.$$

For technical reasons, we shall rather work in the following extended probability space:

$$(\bar{\Omega}, \bar{F}, \bar{\boldsymbol{P}})=(\Omega, F, \boldsymbol{P})\times(\boldsymbol{R}^3/\boldsymbol{Z}^3, \boldsymbol{B}(\boldsymbol{R}^3/\boldsymbol{Z}^3), \mu)$$

where $\boldsymbol{B}(\boldsymbol{R}^3/\boldsymbol{Z}^3)$ is the topological Borel field and μ is the Lebesgue measure. $x\in\boldsymbol{R}^3$ can be written uniquely as follows:

$$x=\underline{x}+\dot{x},\ \underline{x}\in\boldsymbol{Z}^3,\ \dot{x}\in[0, 1)^3.$$

If we define the transformations $\bar{T}_x(x\in\boldsymbol{R}^3)$ on $\bar{\Omega}$ by

$$\bar{T}_x(\omega, k)=(T_{\underline{x+k}}\omega, (x+k)^{\cdot})$$

for $(\omega, k)\in\bar{\Omega}$ and $x\in\boldsymbol{R}^3$, we have the following proposition in [2].

Proposition 1.2 (Kirsch). (1) *$\{\bar{T}_x\}_{x\in\boldsymbol{R}^3}$ is a group of measure preserving ergodic transformations on $(\bar{\Omega}, \bar{F}, \bar{\boldsymbol{P}})$,*

(2) $H_{\bar{T}_x(\omega,k)}=U_xH_{(\omega,k)}U_x^*$ for $(\omega, k)\in\bar{\Omega}$ *and* $x\in\boldsymbol{R}^3$,
where $U_xf(\cdot)=f(\cdot-x)$ *and* $H_{\omega,k}=-\Delta+V_\omega(x-k)$.

We denote by $G_\omega(z; x, y)$ and $G_{(\omega,k)}(z; x, y)$ the Green functions of $H_\omega-z$ and $H_{(\omega,k)}-z$, respectively. It immediately follows that

$$G_{(\omega,k)}(z; x, y)=G_\omega(z; x-k, y-k) \tag{1.4}$$

and

$$G_{\bar{T}_t(\omega,k)}(z; x, y)=G_{(\omega,k)}(z; x-t, y-t). \tag{1.5}$$

The proof of Theorem 1.1 can be reduced to the following theorem as is shown in Section 2.

Theorem 1.2. *There exists $E^*>0$ such that*

$$\lim_{\varepsilon\downarrow 0}\varepsilon\int_{\boldsymbol{R}^3}(1+|x|)\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E'+i\varepsilon; x, 0)|^4]^{\frac{1}{4}}\,dx=0$$

uniformly in E' *on any compact set in* $(0, E^*]$, *where* $\bar{\boldsymbol{E}}$ *denotes the integration in* (ω, k) *with respect to* $\bar{\boldsymbol{P}}$.

In the proof of Theorem 1.2, the following theorem is essential.

Theorem 1.3. *For any* $p>0$, *there exist* $E^*>0$, $N^*\in\boldsymbol{N}$, $c_1>0$ *and* $K_p>0$ *such that if* $0<E\leq E^*$ *then*

$$\boldsymbol{P}\Big(|G_\omega(E+i\varepsilon; x, y)|\leq e^{m(E)(NL(E)^3-|x-y|)}max\Big\{1, \frac{1}{|x-y|}\Big\}$$
$$\text{for any } x\in\boldsymbol{R}^3 \text{ and any } y\in[0, 1)^3\Big)\leq 1-\frac{K_p}{N^p}$$

for any $N^*\leq N\in\boldsymbol{N}$ *uniformly in* $\varepsilon\neq 0$. *Here* $m(E)=c_1E^{\frac{1}{2}}$, $L(E)=\left[\frac{1}{E^{\frac{1}{2}}}\right]$ *where* [ ] *denotes the integer part.*

Theorem 1.3 is proved in Section 6.

2. Proof of Theorem 1.1

It is not difficult to check that the following proposition implies Theorem 1.1 (see e.g. [5, p. 203]).

Proposition 2.1. *There exists* $E^*>0$ *such that if* $0<E\leq E^*$ *then*

$$\lim_{T\to\infty}\frac{1}{T}\int_{\eta T}^{T}\frac{r_E^2(t)}{t}dt=0$$

for any $\eta\in(0, 1)$.

In this section we shall prove that this proposition, in turn, follows from Theorem 1.2.

Proof of Proposition 2.1 *assuming Theorem* 1.2. We denote by c constants independent of ω and ε, which may be different according to the situation. For $\varepsilon=\frac{1}{T}$, we have

$$\text{(2.1)}\qquad \frac{1}{T}\int_{\eta T}^{T}\frac{r_E^2(t)}{t}dt=\frac{1}{T}\int_{\eta T}^{T}e^{\varepsilon t}e^{-\varepsilon t}\frac{r_E^2(t)}{t}dt\leq\frac{e}{\eta}\varepsilon^2\int_0^\infty e^{-\varepsilon t}r_E^2(t)\,dt$$
$$=\frac{e}{2\pi\eta}\varepsilon^2\int_{-\infty}^{\infty}\boldsymbol{E}\Big[\int_{-\infty}^{\infty}|x|^2\Big|R_\omega\Big(E'+\frac{\varepsilon}{2}i\Big)g_E(H_\omega)\,\phi(x)\Big|^2dx\Big]dE'.$$

The last equality of (2.1) will be proved in Appendix 1. Let E^* be as in Theorem 1.2. Let $0<E<E^*$ and $\Psi_\omega=g_E(H_\omega)\,\phi$. We divide the last member of (2.1) in the three parts as follows:

$$\frac{e}{2\pi\eta}\varepsilon^2\int_{E^*}^{\infty}\boldsymbol{E}\Big[\int_{\boldsymbol{R}^3}|x|^2\Big|R_\omega\Big(E'+\frac{\varepsilon}{2}i\Big)\Psi_\omega(x)\Big|^2dx\Big]dE'$$

$$+\frac{e}{2\pi\eta}\varepsilon^2\int_{\bar{E}}^{E^*}\boldsymbol{E}\left[\int_{\boldsymbol{R}^3}|x|^2\left|R_\omega\left(E'+\frac{\varepsilon}{2}i\right)\Psi_\omega(x)\right|^2dx\right]dE'$$
$$+\frac{e}{2\pi\eta}\varepsilon^2\int_{-\infty}^{\bar{E}}\boldsymbol{E}\left[\int_{\boldsymbol{R}^3}|x|^2\left|R_\omega\left(E'+\frac{\varepsilon}{2}i\right)\Psi_\omega(x)\right|^2dx\right]dE'$$
$$=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where $\bar{E}$ is a positive number satisfying

$$\operatorname{supp} g_E \subset (\bar{E}, E).$$

To begin with, we shall estimate the terms I and III. If we set

$$f_{\varepsilon,E'}(x)=\frac{g_E(x)}{x-E'-i\varepsilon}\in C_0^\infty(\boldsymbol{R}),$$

we have

$$R_\omega(E'+i\varepsilon)\Psi_\omega(x)=f_{\varepsilon,E'}(H_\omega)\psi(x).$$

Then we get

$$\text{(2.2)}\qquad \int_{\boldsymbol{R}^3}|x|^2|R_\omega(E'+i\varepsilon)\Psi_\omega(x)|^2dx=\int_{\boldsymbol{R}^3}|x|^2|f_{\varepsilon,E'}(H_\omega)\psi(x)|^2dx$$
$$\leq\|f_{\varepsilon,E'}(H_\omega)\|^2_{L^2_2\to L^2_2}\|\psi\|^2_{L^2_2}$$

where $L^2_2=L^2_2(\boldsymbol{R}^3)$. For Banach spaces X and Y, we denote by $\|\cdot\|_{X\to Y}$ the operator norm of the bounded operator from X to Y. By Lemma A.1, we have uniformly for $E'\in(-\infty,\bar{E}]\cup[E^*,\infty)$

$$\text{(2.3)}\qquad \|f_{\varepsilon,E'}(H_\omega)\|^2_{L^2_2\to L^2_2}\leq\frac{c}{1+E'^2},$$

where constant c is independent of ω and ε. From (2.2) and (2.3) we obtain

$$\boldsymbol{E}\left[\int_{\boldsymbol{R}^3}|x|^2|R_\omega(E'+i\varepsilon)\Psi_\omega(x)|^2dx\right]\leq\frac{c}{1+E'^2}$$

for $E'\in(-\infty,\bar{E}]\cup[E^*,\infty)$ uniformly in $\varepsilon>0$. Then there exists a positive c such that

$$\int_{E^*}^{\infty}\boldsymbol{E}\left[\int_{\boldsymbol{R}^3}|x|^2|R_\omega(E'+i\varepsilon)\Psi_\omega(x)|^2dx\right]dE'\leq c$$

and

$$\int_{-\infty}^{\bar{E}}\boldsymbol{E}\left[\int_{\boldsymbol{R}^3}|x|^2|R_\omega(E'+i\varepsilon)\Psi_\omega(x)|^2dx\right]dE'\leq c$$

uniformly in $\varepsilon>0$. For this reason, I and III tend to 0 as $\varepsilon\to0$.

Next we shall estimate II. For $k\in[0,1)^3$, we have

$$\text{(2.4)}\qquad R_\omega(E'+i\varepsilon)\Psi_\omega(\cdot-k)=U_kR_\omega(E'+i\varepsilon)\Psi_\omega$$

$$=U_kR_\omega(E'+i\varepsilon)U_k^*U_kg_E(H_\omega)U_k^*U_k\phi$$
$$=R_{(\omega,k)}(E'+i\varepsilon)g_E(H_{(\omega,k)})U_k\phi,$$

where $R_{(\omega,k)}(E'+i\varepsilon)=(H_{(\omega,k)}-(E'+i\varepsilon))^{-1}$ by Proposition 1.2, (2) and $H_{(\omega,0)}=H_\omega$. Therefore if we put $\phi_{(\omega,k)}=U_k\phi$ and

$$\Psi_{(\omega,k)}=g_E(H_{(\omega,k)})\phi_{(\omega,k)},$$

then we obtain

$$\int_{R^3}|x|^2|R_\omega(E'+i\varepsilon)\Psi_\omega(x)|^2dx$$
$$=\int_{R^3}|x-k|^2|R_{(\omega,k)}(E'+i\varepsilon)g_E(H_{(\omega,k)})\phi_{(\omega,k)}(x)|^2dx$$
$$=\int_{R^3}|x-k|^2|R_{(\omega,k)}(E'+i\varepsilon)\Psi_{(\omega,k)}(x)|^2dx.$$

Integrating with respect to $\bar{\boldsymbol{P}}$, we get

$$\boldsymbol{E}\Big[\int_{R^3}|x|^2|R_\omega(E'+i\varepsilon)\Psi_\omega(x)|^2dx\Big] \tag{2.5}$$
$$=\bar{\boldsymbol{E}}\Big[\int_{R^3}|x|^2|R_\omega(E'+i\varepsilon)\Psi_\omega(x)|^2dx\Big]$$
$$=\bar{\boldsymbol{E}}\Big[\int_{R^3}|x-k|^2|R_{(\omega,k)}(E'+i\varepsilon)\Psi_{(\omega,k)}(x)|^2dx\Big].$$

Since $k\in[0,1)^3$, the last member of (2.5) is bounded by

$$\bar{\boldsymbol{E}}\Big[\int_{R^3}c(1+|x|^2)\Big|\int_{R^3}G_{(\omega,k)}(E'+i\varepsilon;x,y)\Psi_{(\omega,k)}(y)dy\Big|^2dx\Big] \tag{2.6}$$
$$\leq\bar{\boldsymbol{E}}\Big[\int_{R^3}c(1+|x|^2)\prod_{j=1,2}\int_{R^3}|G_{(\omega,k)}(E'+i\varepsilon;x,y_j)||\Psi_{(\omega,k)}(y_j)|dy_jdx\Big]$$
$$\leq\int_{R^3}c(1+|x|^2)\Big(\int_{R^3}\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E'+i\varepsilon;x,y)|^4]^{\frac{1}{4}}\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}(y)|^4]^{\frac{1}{4}}dy\Big)^2dx.$$

The last inequality is obtained by Fubini's theorem and by twice using the Schwarz inequality. Since $\bar{T}_y$ has the measure preserving property by Proposition 1.2, we obtain

$$\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E'+i\varepsilon;x,y)|^4]=\bar{\boldsymbol{E}}[|G_{\bar{T}_y(\omega,k)}(E'+i\varepsilon;x-y,0)|^4]$$
$$=\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E'+i\varepsilon;x-y,0)|^4]$$

Therefore the last member of (2.6) equals

$$\int_{R^3}(1+|x|^2)\Big(\int_{R^3}\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E'+i\varepsilon;x-y,0)|^4]^{\frac{1}{4}}\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}(y)|^4]^{\frac{1}{4}}dy\Big)^2dx. \tag{2.7}$$

Let

$$K(x)=\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E'+i\varepsilon;x,0)|^4]^{\frac{1}{4}}. \tag{2.8}$$

By taking $|x|^2$ into the integration with respect to y and using the inequality $|x|\leq|x-y|+|y|$, (2.7) is bounded by

$$
\begin{aligned}
(2.9)\quad & \int_{R^3}(K*\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}|^4]^{\frac{1}{4}})^2dx+2\int_{R^3}((|x|K)*\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}|^4]^{\frac{1}{4}})^2\,dx \\
&+2\int_{R^3}\Big(K*(|y|\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}|^4]^{\frac{1}{4}})\Big)^2\,dx \\
&\leq\Big(\int_{R^3}K(x)\,dx\Big)^2\int_{R^3}\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}(y)|^4]^{\frac{1}{2}}\,dy \\
&+2\Big(\int_{R^3}|x|K(x)\,dx\Big)^2\int_{R^3}\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}(y)|^4]^{\frac{1}{2}}\,dy \\
&+2\Big(\int_{R^3}K(x)\,dx\Big)^2\int_{R^3}\bar{\boldsymbol{E}}\big[|y\Psi_{(\omega,k)}(y)|^4\big]^{\frac{1}{2}}\,dy.
\end{aligned}
$$

We shall show

$$(2.10)\qquad \int_{R^3}\bar{\boldsymbol{E}}[|y\Psi_{(\omega,k)}(y)|^4]^{\frac{1}{2}}dy<\infty.$$

From Lemma A. 2 we have

$$|\langle y\rangle^2\Psi_{(\omega,k)}(y)|=|\langle y\rangle^2 g_E(H_{(\omega,k)})U_k\phi(y)|\leq\|U_k\phi\|_{L^2_2}\leq c$$

uniformly in $(\omega, k)\in\bar{\Omega}$. Therefore we get

$$(2.11)\qquad |\Psi_{(\omega,k)}(y)|\leq\frac{c}{1+y^2}$$

uniformly in $(\omega, k)\in\bar{\Omega}$. We have

$$
\begin{aligned}
&\int_{R^3}\bar{\boldsymbol{E}}[|y\Psi_{(\omega,k)}(y)|^4]^{\frac{1}{2}}\,dy \\
&\leq\Big(\int_{R^3}\Big(\frac{1}{1+y^2}\Big)^2dy\Big)^{\frac{1}{2}}\Big(\bar{\boldsymbol{E}}\Big[\int_{R^3}(1+y^2)^2|y|^4|\Psi_{(\omega,k)}(y)|^4dy\Big]\Big)^{\frac{1}{2}}
\end{aligned}
$$

and from (2.11) we get

$$
\begin{aligned}
&\int_{R^3}(1+y^2)^2|y|^4|\Psi_{(\omega,k)}(y)|^4dy \\
&\leq\int_{R^3}(1+y^2)^2|\Psi_{(\omega,k)}(y)|^2|y|^4\frac{c}{(1+|y|^2)^2}dy \\
&\leq c\|\Psi_{(\omega,k)}\|^2_{L^2_2}\leq\|g_E(H_{(\omega,k)})\|^2_{L^2_2\to L^2_2}\|U_k\phi\|^2_{L^2_2}\leq c
\end{aligned}
$$

uniformly in $(\omega, k)\in\bar{\Omega}$. The last inequality is obtained from Lemma A.1. Thus we have (2.10). In a similar fashion we can check

$$(2.12)\qquad \int_{R^3}\bar{\boldsymbol{E}}[|\Psi_{(\omega,k)}(y)|^4]^{\frac{1}{2}}\,dy<\infty.$$

By (2.5) - (2.10) and (2.12), to show that II tends to 0 as $\varepsilon\to 0$, we have only to prove

$$\varepsilon\int_{\boldsymbol{R}^3}(1+|x|)K(x)dx\to 0 \text{ as } \varepsilon\downarrow 0$$

uniformly in $E'\in[\bar{E}, E^*]$. In view of (2.8), this is nothing but the assertion of Theorem 1.2. Thus the proof of Proposition 2.1 is completed.

3. Proof of Theorem 1.2

In this section we shall give a proof of Theorem 1.2 by using Theorem 1.3 and Lemma A.4.

Proof of Theorem 1.2. To begin with we shall devide $\boldsymbol{R}^3$ as follows. Let

$$A_0=\{x\in\boldsymbol{R}^3||x|<1\},\ A_1=\{x\in\boldsymbol{R}^3|1\le|x|<R\} \tag{3.1}$$

and

$$A_j=\{x\in\boldsymbol{R}^3|2^{j-2}R\le|x|<2^{j-1}R\} \tag{3.2}$$

for $\boldsymbol{N}\ni j\ge 2$. For $E\ge\varepsilon>0$ we define

$$\begin{aligned} &V^{\varepsilon}_{N,E}\\ &=\left\{\omega\middle||G_\omega(E+i\varepsilon;x,y)|\le e^{m(E)(NL(E)^3-|x-y|)}\max\left\{1,\frac{1}{|x-y|}\right\}\right.\\ &\qquad\left.\text{for any } x\in\boldsymbol{R}^3 \text{ and any } y\in[0,1)^3\right\}. \end{aligned} \tag{3.3}$$

For $x\in A_0$, from Lemma A.4 we have

$$\begin{aligned} &\left(\int_{[0,1]^3}\boldsymbol{E}[|G_\omega(E+i\varepsilon;x+k,k)|^4]dk\right)^{\frac{1}{4}}\\ &\le\frac{e^{m(E)(N_0L(E)^3-|x|)}}{|x|}\boldsymbol{P}(V^{\varepsilon}_{N_0,E})^{\frac14}+\left(\frac{1}{|x|}+\frac{c}{\varepsilon}\right)\boldsymbol{P}(V^{\varepsilon c}_{N_0,E})^{\frac14} \end{aligned} \tag{3.4}$$

and for $x\in A_j\,(j\ge1)$, we have

$$\begin{aligned} &\left(\int_{[0,1]^3}\boldsymbol{E}[|G_\omega(E+i\varepsilon;x+k,k)|^4]dk\right)^{\frac{1}{4}}\\ &\le e^{m(E)(N_jL(E)^3-|x|)}\boldsymbol{P}(V^{\varepsilon}_{N_j,E})^{\frac14}+\left(\frac{1}{|x|}+\frac{c}{\varepsilon}\right)\boldsymbol{P}(V^{\varepsilon c}_{N_j,E})^{\frac14}, \end{aligned} \tag{3.5}$$

where N_j will be specified later and $V^{\varepsilon c}_{N_j,E}=\Omega\setminus V^{\varepsilon}_{N_j,E}$. Since we have by (1.4) and the definition of $\bar{\boldsymbol{E}}$

$$\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E+i\varepsilon;x,0)|^4]^{\frac14}=\left(\int_{[0,1]^3}\boldsymbol{E}[|G_\omega(E+i\varepsilon;x+k,k)|^4]dk\right)^{\frac14}, \tag{3.6}$$

we have by (3.4) and (3.5)

$$\begin{aligned} &\varepsilon\int_{\boldsymbol{R}^3}(1+|x|)\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E+i\varepsilon;x,0)|^4]^{\frac14}dx\\ &\le\varepsilon\int_{A_0}(1+|x|)\frac{e^{m(E)(N_0L(E)^3-|x|)}}{|x|}dx\boldsymbol{P}(V^{\varepsilon}_{N_0,E})^{\frac14} \end{aligned} \tag{3.7}$$

$$+\varepsilon\sum_{j=1}^{\infty}\int_{A_j}(1+|x|)e^{m(E)(N_jL(E)^3-|x|)}dx\boldsymbol{P}(V^{\varepsilon}_{N_j,E})^{\frac{1}{4}}$$

$$+\varepsilon\sum_{j=0}^{\infty}\int_{A_j}(1+|x|)\left(\frac{1}{|x|}+\frac{c}{\varepsilon}\right)dx\boldsymbol{P}(V^{\varepsilon c}_{N_j,E})^{\frac{1}{4}}.$$

By the definitions of A_j, we have

$$\varepsilon\int_{A_0}(1+|x|)\frac{e^{m(E)(N_0L(E)^3-|x|)}}{|x|}dx\boldsymbol{P}(V^{\varepsilon}_{N_0,E})^{\frac{1}{4}}\leq\varepsilon ce^{m(E)N_0L(E)^3}\tag{3.8}$$

and

$$\int_{A_j}(1+|x|)e^{m(E)(N_jL(E)^3-|x|)}dx\tag{3.9}$$

$$\leq e^{m(E)N_jL(E)^3}(1+2^{j-1}R)c(2^{j-1}R)^3\times\begin{cases}e^{-m(E)2^{j-2}R} & (j\geq 2)\\ e^{-m(E)} & (j=1)\end{cases}$$

$$\leq c(R2^{j-1})^4\times\begin{cases}e^{m(E)(N_jL(E)^3-R2^{j-2})} & (j\geq 2)\\ e^{m(E)(N_1L(E)^3-1)} & (j=1).\end{cases}$$

Since, from Theorem 1.3

$$\boldsymbol{P}(V^{\varepsilon c}_{N_j,E})\leq\frac{K_p}{N_j^p},$$

we have

$$\int_{A_j}(1+|x|)\left(\frac{1}{|x|}+\frac{c}{\varepsilon}\right)dx\boldsymbol{P}(V^{\varepsilon c}_{N_j,E})^{\frac{1}{4}}\tag{3.10}$$

$$\leq\left\{c|A_j|+c|A_j|2^{j-1}R\frac{1}{\varepsilon}\right\}\frac{K_p^{\frac{1}{4}}}{N_j^{\frac{p}{4}}}\leq c(R2^{j-1})^4\frac{1}{\varepsilon}\frac{K_p^{\frac{1}{4}}}{N_j^{\frac{p}{4}}}\text{ for } j\geq 1$$

and

$$\int_{A_0}(1+|x|)\left(\frac{1}{|x|}+\frac{c}{\varepsilon}\right)dx\boldsymbol{P}(V^{\varepsilon c}_{N_0,E})^{\frac{1}{4}}\leq\frac{c}{\varepsilon}\frac{K_p^{\frac{1}{4}}}{N_0^{\frac{p}{4}}}.\tag{3.11}$$

By (3.7)-(3.11), it follows that

$$\varepsilon\int_{\boldsymbol{R}^3}(1+|x|)\bar{\boldsymbol{E}}[|G_{(\omega,k)}(E+i\varepsilon;x,0)|^4]^{\frac{1}{4}}dx\tag{3.12}$$

$$\leq\varepsilon ce^{m(E)N_0L(E)^3}+\varepsilon cR^4e^{m(E)N_0L(E)^3-1}+\varepsilon c\sum_{j=2}^{\infty}(R2^{j-1})^4e^{m(E)(N_jL(E)^3-R2^{j-2})}$$

$$+c\frac{K_p^{\frac{1}{4}}}{N_0^{\frac{p}{4}}}+c\sum_{j=1}^{\infty}(R2^{j-1})^4\frac{K_p^{\frac{1}{4}}}{N_j^{\frac{p}{4}}}$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.$$

For $j \geq 0$ if we put

$$N_j = \left[\frac{R2^{j-2}}{2L(E)^3}\right], \tag{3.13}$$

then we have

$$N_j L(E)^3 - R2^{j-2} \leq -\frac{1}{4} R2^{j-1} \tag{3.14}$$

and

$$(R2^{j-1})^4 N_j^{-\frac{p}{4}} \leq (R2^{j-1})^4 \left(\frac{R2^{j-2}}{4L(E)^3}\right)^{-\frac{p}{4}} = (R2^{j-1})^{4-\frac{p}{4}} (8L(E)^3)^{\frac{p}{4}}. \tag{3.15}$$

If we take $p=17$, then it follows from (3.15) that

$$V \leq cK_p^{\frac{1}{4}} (8L(E)^3)^{\frac{p}{4}} R^{-\frac{1}{4}} \sum_{j=1}^{\infty} (2^{-\frac{1}{4}})^{j-1}. \tag{3.16}$$

For any $\varepsilon'>0$ if we take R sufficiently large, by (3.13) and (3.16) we have

$$\text{IV} < \frac{\varepsilon'}{5} \text{ and } \text{V} < \frac{\varepsilon'}{5}$$

independent of ε. Then if we take ε sufficiently small, it follows that

$$\text{I} < \frac{\varepsilon'}{5}, \ \text{II} < \frac{\varepsilon'}{5} \text{ and } \text{III} < \frac{\varepsilon'}{5}.$$

Therefore if ε is small enough, then we have

$$\varepsilon \int_{\boldsymbol{R}^3} (1+|x|) \bar{\boldsymbol{E}} [|G_{(\omega,k)}(E+i\varepsilon; x, 0)|^4]^{\frac{1}{4}} dx < \varepsilon'.$$

This estimate holds uniformly in $E \in [\bar{E}, E^*]$ because there exist two positive numbers c, c' such that for any $E \in [\bar{E}, E^*]$

$$0 < c < m(E) < c' \text{ and } 0 < c < L(E) < c'.$$

We have thus proved Theorem 1.2.

4. Singular sets

In this section we give the notion of singular sets and a theorem concerning them which will be used essentially in the proof of Theorem 1.3 in Section 6. We denote by E a small but arbitrarily fixed positive number in the sequel. We define the basic length scale:

$$L(E) = \left[\frac{1}{\sqrt{E}}\right]$$

where [] denotes the integer part. We choose E sufficiently small so that $L(E)\geq 1$ in the sequel. Let $\boldsymbol{Z}^3(E)=L(E)\boldsymbol{Z}^3$ and for $j\in\boldsymbol{Z}^3(E)$, $Q_E(j)=Q_E(0)+j$ where

$$Q_E(0)=\{x\in\boldsymbol{R}^3|0\leq x_i<L(E), j=1, 2, 3\}.$$

And we define the norm:

$$|j|_E=\max_{i=1,2,3}\frac{|j_i|}{L(E)}$$

for $j\in\boldsymbol{Z}^3(E)$.
We fix $\alpha>0$ and β satisfying

$$1<\alpha^2<\beta$$

and

$$\sqrt{2}<\beta<2$$

in the sequel.

Definition. A site $j\in\boldsymbol{Z}^3(E)$ is said to be singular if and only if

$$\lambda_1(H^N_{Q_E(j)}(\omega))\leq 2E.$$

Here $H^N_{Q_E(j)}(\omega)$ is $H_\omega|_{L^2(Q_E(j))}$ with Neumann boundary conditions and $\lambda_1(H^N_{Q_E(j)}(\omega))$ denotes the lowest eigenvalue of $H^N_{Q_E(j)}(\omega)$. We define a sequence of the singular sets inductively.

$S_0=\{j\in\boldsymbol{Z}^3(E)|\text{singular}\}$
$S_{i+1}=S_i\backslash S^g_i$
$S^g{}_i=\cup_x D^x_i$: maximal union of components D^x_i satisfying the following condition A (i)
Condition A (i):
(a) $D^x_i\subset S_i$
(b) $\operatorname{diam}_E D^x_i\leq d_i$
(c) $\operatorname{dist}_E(D^x_i,S_i\backslash D^x_i)\geq 2d_{i+1}$
(d) $\operatorname{dist}(\sigma(H^D_{Q_E(W(D^x_i,4d_i))}),E)\geq\exp(-d_i^{\frac{1}{2}})$

where

$$d_0=d_0(E)=L(E),\ d_i=d_0^{\alpha^i}$$

and

$$W(D, r)=\{j\ni\boldsymbol{Z}^3(E)|\operatorname{dist}_E(j, D)\leq r\},$$
$$Q_E(D)=\bigcup_{j\in D}Q_E(j)$$

for any $D \subset \boldsymbol{Z}^3(E)$. We denote by diam_E and dist_E the diameter and the distance measured by the norm $|\cdot|_E$. $H^D_{Q_E(W(D^X_i,4d_i))}$ is $H_\omega|_{L^2(Q_E(W(D^X_i,4d_i)))}$ with Dirichlet boundary conditions.

The main theorem of this section is the following.

Theorem 4.1. *For any $p>0$ there exists $E'>0$ such that if $0<E<E'$ then*

$$\boldsymbol{P}(i \in S^g_j) \leq d_j^{-p}$$

for any $i \in \boldsymbol{Z}^3(E)$ and any $j \geq 0$.

For proving this theorem we shall prepare some notations and some lemmas. We define the set of n-cubes $(n \geq 1)$:

$$\mathscr{C}_n = \{C_n | C_n = \{y \in \boldsymbol{R}^3 | \max_{i=1,2,3} |x_i - y_i| \leq 2^{n-1} L(E)\} \text{ for some } x \in 2^{n-1} \boldsymbol{Z}^3(E)\}$$

and the set of 0-cubes:

$$\mathscr{C}_0 = \boldsymbol{Z}^3(E).$$

Let $D \subset \boldsymbol{Z}^3(E)$ be finite set. We denote by $n_0(D)$ the smallest n_0 such that there exists an n_0-cube C_{n_0} including D and fix one $n_0(D)$-cube $C_{n_0(D)}$ including D. We define $\mathscr{C}_{n_0(D)}(D) = \{C_{n_0(D)}\}$ and for $n \leq n_0(D)$ let

$$V_n(D) = \min\{\#\mathscr{C}_n | \mathscr{C}_n \text{ is a family of n-cubes which cover } D\}$$

and $\mathscr{C}_n(D)$ be a family of n-cubes which attains this minimum. We shall fix one sequence of covers of D: $\{\mathscr{C}_n(D)\}_{n=1,2,\cdots,n_0(D)}$. We define

$$\mathscr{C}'_n(D) = \{C_n \in \mathscr{C}_n(D) | \mathrm{dist}_E(C_n, C'_n) \geq 2 d_0^\beta 2^{\beta n} \text{ for any } C'_n \in \mathscr{C}_n(D),\ C'_n \neq C_n\}$$

for $n_0(D) > n > 0$ and $\mathscr{C}'_n(D) = \emptyset$ for $n \geq n_0(D)$. We define

$$V(D) = \sum_{n=1}^{n_0(D)} \#\mathscr{C}_n(D),\ V'(D) = \sum_{n=1}^{\infty} \#\mathscr{C}'_n(D), \text{ and } \mathscr{C}_0(D) = \mathscr{C}'_0(D) = D.$$

We denote by $X_D(\omega)$ the characteristic function of the set:

(4.1) $\quad \Omega_D = \{\omega \in \Omega | \text{ there exists } k \text{ such that } D \text{ is a component of } S^g_k\}$.

Lemma 4.1. *Let D be a finite set of $\boldsymbol{Z}^3(E)$. For $n \geq 1$ let $j(n)$ be the smallest integer such that $d_{j(n)} \geq d_0 2^n$. For $C \in \mathscr{C}_n(D)$ we denote by $X_{n,C}$ $(n>0)$ the characteristic function of the set*:

$$\{\omega \in \Omega | \mathrm{dist}\ (\sigma(H^D_{Q_E(W(C \cap D, 4d_{j(n)}))}), E\Big) \leq \exp(-d_{j(n)}^{\frac{1}{2}})\}$$

and for $n=0$ let $X_{0,C}$ be the characteristic function of the set $\{\omega \in \Omega | C \in S_0\}$. Then

$$\boldsymbol{E}(X_D) \leq \boldsymbol{E}\Big(\prod_{n=0}^{\infty} \prod_{C \in \mathscr{C}'_n(D)} X_{n,C}\Big).$$

Here if $\mathscr{C}'_n(D)=\emptyset$, *then we set* $\prod_{C\in\mathscr{C}'_n(D)}X_{n,C}=1$.

Proof. For $\omega\in\Omega_D$ it is sufficient to prove that

$$\operatorname{dist}(\sigma(H^D_{Q_E(W(C\cap D,4d_{j(n)}))}),E)\leq\exp(-d_{j(n)}^{\frac{1}{2}})$$

for any $C\in\mathscr{C}'_n(D)$ $(n>0)$ and that $C\in S_0(\omega)$ for any $C\in D$ $(n=0)$.

Let $\omega\in\Omega_D$ and D be a component of S^g_k. First we consider the case $n=0$; $C\in D$ is contained in S_0 because D is a component of $S^g_k\subset S_0$. Next we consider the case $n>0$. We show that if $C\in\mathscr{C}'_n(D)$, then $C\cap D$ satisfies Condition $A(j(n))$ (a), (b) and (c). Noting the definition of $\mathscr{C}'_n(D)$ and $\alpha^2<\beta$ it follows that

$$\operatorname{diam}_E(D)\geq 2d_0^{\beta}2^{\beta n}>2d_0^{\alpha}2^{\alpha^2 n}=2(d_0^{\alpha}2^{\alpha n})^{\alpha}\geq 2d_{j(n)}^{\alpha}=2d_{j(n)+1}. \tag{4.2}$$

The last inequality follows from

$$d_0^{\alpha}2^{\alpha n}\geq d_{j(n)} \tag{4.3}$$

which can be easily seen by contradiction. Let $i(n)$ be the largest integer such that $d_{i(n)}\leq d_0^{\beta}2^{\beta n}$. Then we have $d_{i(n)}>d_0^{\alpha}2^{\alpha n}$ by contradiction and by $\alpha^2<\beta$. Because of this inequality and (4.3), we get $j(n)+1\leq i(n)$. By the definition of $i(n)$, (4.2) and Condition $A(k)$ (b), it follows that

$$d_k\geq\operatorname{diam}_E(D)\geq 2d_{i(n)}.$$

So we have $i(n)<k$. As a consequence we obtain

$$j(n)<i(n)<k.$$

From this inequality it follows that $D\in S^g_k\subset S_k\subset S_{j(n)}$. Therefore $C\cap D$ satisfies Condition $A(j(n))$ (a). Because of the definition of $j(n)$, $\operatorname{diam}_E C=2^n$ and $d_0=L(E)\geq 1$ it follows that

$$\operatorname{diam}_E(C\cap D)\leq 2^n\leq d_{j(n)},$$

which is Condition $A(j(n))$ (b) for $C\cap D$. We show Condition $A(j(n))$ (c) for $C\cap D$. It follows that

$$S_{j(n)}=S_k+\sum_{i=j(n)}^{k-1}S^g_i. \tag{4.4}$$

If D^x_i is a component of S^g_i for $i=j(n), j(n)+1,\cdots,k-1$ then we have that

$$\operatorname{dist}_E(D^x_i,C\cap D)\geq\operatorname{dist}_E(D^x_i,S_i\backslash D^x_i)\geq 2d_{i+1}\geq 2d_{j(n)+1} \tag{4.5}$$

by $C\cap D\subset D\subset S_k\subset S_i\backslash D^x_i$. Since D is a component of S^g_k, we have that

$$\operatorname{dist}_E(C\cap D,S_k\backslash D)\geq\operatorname{dist}_E(D,S_k\backslash D)\geq 2d_{k+1}>2d_{j(n)+1}.$$

Because of $C\in\mathscr{C}'_n(D)$ it follows that

$$\mathrm{dist}_E(C\cap D, D\setminus(C\cap D))\geq 2d_0^\beta 2^{\beta n}>2d_{j(n)+1}.$$

Hence we have

$$\mathrm{dist}_E(C\cap D, S_k\setminus(C\cap D))\geq 2d_{j(n)+1}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we conclude that

$$\mathrm{dist}_E(C\cap D, S_{j(n)}\setminus(C\cap D))\geq 2d_{j(n)+1},$$

which is Condition $A(j(n))$ (c) for $C\cap D$. Therefore if $C\cap D$ satisfies Condition $A(j(n))$ (d), then $C\cap D$ is a component of $S_{j(n)}^g$. But this contradicts the fact that D is a component of S_k^g because $j(n)<k$. As a consequence we get that

$$\mathrm{dist}(\sigma(H^D_{Q_E(W(C\cap D,4d_{j(n)}))}), E)\leq\exp(-d_{j(n)}^{\frac{1}{2}}).$$

We have thus proved Lemma 4.1.

In order to estimate $\boldsymbol{E}(X_D)$, we need the following two propositions.

Proposition 4.1 (Wegner estimate). *Let $Q(j)=Q(0)+j$, $j\in\boldsymbol{Z}^3$ and $Q(0)=[0,1)^3$. For a finite $J\subset\boldsymbol{Z}^3$, let*

$$\Lambda=\cup_{j\in J}Q(j),$$

and $\bar{J}\subset\boldsymbol{Z}^3$ be a finite subset such that

$$\bar{\Lambda}=\cup_{j\in\bar{J}}Q(j)$$

is the smallest cube containing Λ. Let

$$H^D_\Lambda(\omega)=-\Delta+V_\omega|_{L^2(\Lambda)}$$

with Dirichlet boundary conditions. Then we have

$$\boldsymbol{P}(\{\omega|\mathrm{dist}(\sigma(H^D_\Lambda(\omega)), E)\leq k\})\leq\frac{2c_0^{-1}}{3\pi^2}|\bar{\Lambda}|^2k(E-k+2\eta_0^{-1}\eta_1k)^{\frac{3}{2}}$$

for $k\geq 0$.

This proposition will be proved later. The following proposition has been proved by [4].

Proposition 4.2.

$$\boldsymbol{P}(\lambda_1(H^N_{Q_E(0)}(\omega))\leq E)\leq\exp(-cE^{-\frac{3}{2}}).$$

From these two propositions we can show the following lemma.

Lemma 4.2. *If $E>0$ is sufficiently small, then there exists $c>0$, $c'>0$ such that*

$$\boldsymbol{E}(X_D) \leq \exp(-cE^{-\frac{3}{2}} \# D - c'E^{-\frac{1}{4}} V'(D)).$$

Proof. For $n>0$ and for $C_1, C_2 \in \mathscr{C}'_n(D)$ $(C_1 \neq C_2)$,

$$\begin{aligned} &\mathrm{dist}_E(W(C_1 \cap D, 4d_{j(n)}), W(C_2 \cap D, 4d_{j(n)})) \\ &\geq 2d_0^{\beta} 2^{\beta n} - 8d_{j(n)} \geq 2d_0^{\beta} 2^{\beta n} - 2^3 d_0^{\alpha} 2^{\alpha n} \\ &= 2d_0^{\beta} 2^{\beta n}(1 - 2^2 d_0^{\alpha-\beta} 2^{(\alpha-\beta)n}) > 2d_0^{\beta} 2^{\beta n}(1 - 2^2 d_0^{\alpha-\beta}) \end{aligned}$$

by the definition of $\mathscr{C}'_n(D)$ and (4.3). If E is sufficiently small, then the last member of the above inequality is positive. Therefore we have

$$W(C_1 \cap D, 4d_{j(n)}) \cap W(C_2 \cap D, 4d_{j(n)}) = \emptyset.$$

Then $\{X_{n,C}\}_{C \in \mathscr{C}'_n(D)}$ are independent by (H.1). By the definition of $X_{0,C}$ and (H.1), it follows immediately that $\{X_{0,C}\}_{C \in D = \mathscr{C}'_0(D)}$ are independent. Hence by Lemma 4.1 and the independence of $X_{n,C}$, we have for $0<r<1$

$$\begin{aligned} (4.7) \qquad \boldsymbol{E}(X_D) &\leq \boldsymbol{E}\Big[\prod_{n=0}^{\infty} \prod_{C \in \mathscr{C}'_n(D)} X_{n,C}\Big] \\ &\leq \Big(\prod_{C \in \mathscr{C}'_0(D)} \boldsymbol{E}[X_{0,C}]\Big)^{1-r} \Big(\boldsymbol{E}\Big[\prod_{n\geq 1} \prod_{C \in \mathscr{C}'_n(D)} X_{n,C}\Big]\Big)^{r} \\ &\leq \cdots \leq \prod_{n=0}^{\infty} \prod_{C \in \mathscr{C}'_n(D)} (\boldsymbol{E}[X_{n,C}])^{r^n(1-r)}. \end{aligned}$$

For $n>0$, by the definition of $X_{n,C}$ it follows that

$$\boldsymbol{E}[X_{n,C}] = \boldsymbol{P}\{\omega | \mathrm{dist}(\sigma(H^D_{Q_E(W(C \cap D, 4d_{j(n)}))}), E) \leq \exp(-d_{j(n)}^{\frac{1}{2}})\}.$$

Since $Q_E(W(C \cap D, 4d_{j(n)}))$ is included in a cube with sides of $10d_{j(n)}d_0$ by $C \in \mathscr{C}'_n(D)$, $d_{j(n)} \geq 2^n d_0$ and $d_0 \geq 1$, we have by Proposition 4.1

$$(4.8) \qquad (\boldsymbol{E}[X_{n,C}])^{r^n(1-r)} \leq (cd_{j(n)}^6 d_0^6 \exp(-d_{j(n)}^{\frac{1}{2}}))^{r^n(1-r)}.$$

If E is sufficiently small, then it follows that

$$\begin{aligned} (4.9) \qquad &(d_{j(n)}^6 d_0^6 \exp(-d_{j(n)}^{\frac{1}{2}}))^{r^n(1-r)} \\ &\leq (\exp(-cd_{j(n)}^{\frac{1}{2}}))^{r^n(1-r)} \leq (\exp(-cd_0^{\frac{1}{2}} 2^{\frac{n}{2}}))^{r^n(1-r)}. \end{aligned}$$

Choose $1>r>\dfrac{1}{\sqrt{2}}$, then it follows that

(4.10) the last member of (4.9) $\leq \exp(-cE^{-\frac{1}{4}}(\sqrt{2}r)^n(1-r)) \leq \exp(-cE^{-\frac{1}{4}})$

uniformly in $n \geq 1$. Therefore by (4.8), (4.9) and (4.10) we have that

$$(4.11) \qquad (\boldsymbol{E}[X_{n,C}])^{r^n(1-r)} \leq \exp(-c'E^{-\frac{1}{4}}).$$

For $n=0$ and $C\in D$, it follows that

(4.12)
$$\begin{aligned}&\boldsymbol{E}[X_{0,C}]\\&=\boldsymbol{P}(C\in S_0)=\boldsymbol{P}(\lambda_1(H^N_{Q_E(C)}(\omega))\leq 2E)\\&=\boldsymbol{P}(\lambda_1(H^N_{Q_E(0)}(\omega))\leq 2E)\leq\exp(-cE^{-\frac{3}{2}})\end{aligned}$$

from Proposition 4.2. From (4.7), (4.11) and (4.12) we obtain that

$$\boldsymbol{E}[X_D]\leq\exp(-cE^{-\frac{3}{2}}\#D-c'E^{-\frac{1}{4}}V'(D)).$$

This completes the proof of Lemma 4.2.

In order to prove Theorem 4.1 we need the following lemmas.

Lemma 4.3.

$$V(D)\leq c(\log_2 E^{-1})^2\#D+c'V'(D).$$

This lemma will be proved later. The following lemma has been proved by [1].

Lemma 4.4. *For $V\in\boldsymbol{N}$, we have*

$$\#\{D\subset\boldsymbol{Z}^3(E)\,|\,V(D)=V \text{ and } 0\in D\}\leq\exp(10V).$$

Proof of Theorem 4.1. Because of the translation invariance by (H.1), we have

$$\boldsymbol{P}(i\in S^g_j)=\boldsymbol{P}(0\in S^g_j)$$

for $i\in\boldsymbol{Z}^3(E)$. We have

(4.13)
$$\boldsymbol{P}(0\in S^g_j)\leq\sum_{D\ni}\boldsymbol{P}(D \text{ is a component of } S^g_j)$$

Let $\boldsymbol{P}_{D,j}=\boldsymbol{P}$ (D is a component of S^g_j). If $\mathrm{diam}_E D>d_j$, it immediately follows that $\boldsymbol{P}_{D,j}=0$ because of Condition $A(j)$ (b). Therefore we shall estimate as follows:

(4.14)
$$\begin{aligned}&\boldsymbol{P}(0\in S^g_j)\\&\leq\sum_{V=1}^{\infty}\sum_{\substack{D\ni 0,V(D)=V\\ \mathrm{diam}_E D\leq d_{j-1}}}\boldsymbol{P}_{D,j}+\sum_{V=1}^{\infty}\sum_{\substack{D\ni 0,V(D)=V\\ d_{j-1}<\mathrm{diam}_E D\leq d_j}}\boldsymbol{P}_{D,j}=\mathrm{I}+\mathrm{II}.\end{aligned}$$

Let D be component of $S^g_j(\omega)$. As a first step, we consider the case where $\mathrm{diam}_E D\leq d_{j-1}$. Since $S_{j-1}=S^g_{j-1}+S_j$, it follows that

$$\mathrm{dist}_E(D,S_{j-1}\backslash D)=\min(\mathrm{dist}_E(D,S^g_{j-1}),\mathrm{dist}_E(D,S_j\backslash D))\geq 2d_j$$

by Conditions $A(j-1)$ (c) and $A(j)$ (c). Suppose that

$$\mathrm{dist}(\sigma(H^D_{Q_E(W(D,4d_{j-1}))}),E)\geq\exp(-d^{\frac{1}{2}}_{j-1}),$$

then D would satisfy Condition $A(j-1)$. But this contradicts the fact that D is a component of S_j^g. Therefore we have

$$\mathrm{dist}(\sigma(H^D_{Q_E(W(D,4d_{j-1}))}), E) \leq \exp(-d_{j-1}^{\frac{1}{2}}).$$

Consequently we can estimate $\boldsymbol{P}_{D,j}$ as follows. Let $X^1(\omega)$ be the characteristic function of the set:

$$\{\omega | \mathrm{dist}(\sigma(H^D_{Q_E(W(D,4d_{j-1}))}), E) \leq \exp(-d_{j-1}^{\frac{1}{2}})\}$$

and X_D is the characteristic function Ω_D. Since $Q_E(W(D, 4d_{j-1}))$ is included in a cube with side of $10d_{j-1}d_0$, by Proposition 4.1 we have

$$\boldsymbol{E}[X^1] c(d_{j-1}d_0)^6 \exp(-d_{j-1}^{\frac{1}{2}}).$$

Hence we have

$$\begin{aligned}(4.15)\qquad \boldsymbol{P}_{D,j} &\leq \boldsymbol{E}[X^1(\omega)X_D(\omega)] \leq (\boldsymbol{E}[X^1(\omega)])^{\frac{1}{2}}(\boldsymbol{E}[X_D(\omega)])^{\frac{1}{2}} \\ &\leq c d_{j-1}^3 d_0^3 \exp\left(-\frac{1}{2} d_{j-1}^{\frac{1}{2}}\right)(\boldsymbol{E}[X_D(\omega)])^{\frac{1}{2}}.\end{aligned}$$

On the other hand

$$(4.16)\qquad \mathbf{P}_{D,j} \leq \boldsymbol{E}(X_D(\omega)).$$

From Lemma 4.3, we have that

$$\begin{aligned} &cE^{-\frac{3}{2}} \# D + c'E^{-\frac{1}{4}} V'(D) \\ &\quad \geq (c(\log_2 E^{-1})^2 \# D + c'V'(D)) cE^{-\frac{1}{4}} \geq cE^{-\frac{1}{4}} V(D).\end{aligned}$$

Hence by Lemma 4.2, it follows that

$$(4.17)\qquad \boldsymbol{E}(X_D(\omega)) \leq \exp(-cE^{-\frac{1}{4}} V(D)).$$

From (4.15), (4.17) and 4.4, we have

$$\begin{aligned}(4.18)\qquad \mathrm{I} &\leq \sum_{V=1}^{\infty} \sum_{\substack{D\ni 0,\, V(D)=V \\ \mathrm{diam}_E D \leq d_{j-1}}} c d_{j-1}^3 d_0^3 \exp\left(-\frac{1}{2} d_{j-1}^{\frac{1}{2}}\right) \exp(-cE^{-\frac{1}{4}} V) \\ &\leq \sum_{V=1}^{\infty} c d_{j-1}^3 d_0^3 \exp\left(-\frac{1}{2} d_{j-1}^{\frac{1}{2}}\right) \exp((10 - cE^{-\frac{1}{4}}) V) \\ &\leq c d_j^{\frac{6}{\alpha}} \exp\left(-\frac{1}{2} d_j^{\frac{1}{2\alpha}}\right) \leq \frac{1}{2} d_j^{-p}\end{aligned}$$

provided $E>0$ is sufficiently small because $d_j \geq d_0$ and d_0 is large when E is small. It is easy to see that

(4.19) $$V(D) \geq n_0(D) \geq \log_2 \mathrm{diam}_E D.$$

Since we have (4.16) from the definition, it follows from (4.17), (4.19) and Lemma 4.4

(4.20) $$\begin{aligned} &\mathrm{II} \\ &\leq \sum_{V \geq \log_2 d_{j-1}} \exp((10 - cE^{-\frac{1}{4}})V) \leq c\exp((10 - cE^{-\frac{1}{4}})\log_2 d_{j-1}) \\ &\leq \frac{1}{2} d_j^{-p} \end{aligned}$$

provided $E>0$ is sufficiently small. From (4.16), (4.18) and (4.20), we have completed the proof of Theorem 4.1.

Proof of Lemma 4.3. Let $\gamma(x) = \left[\frac{1}{\beta}\left\{x - 1 - \log_2(4d_0^\beta + 3)\right\}\right]$. For $n \in \boldsymbol{N}$ such that $\gamma(n) > 0$, we claim

(4.21) $$V_n \leq \frac{1}{2} V_{\gamma(n)} + V'_{\gamma(n)}$$

where $V_m = V_m(D) = \# \mathscr{C}_m(D)$ and $V'_m = V'_m(D) = \# \mathscr{C}'_m(D)$. In fact, let $\mathscr{C}''_m = \mathscr{C}''_m(D) = \mathscr{C}_m(D) \mathscr{C}'_m(D)$. If $C \in \mathscr{C}''_{\gamma(n)}$, then there exists $C' \in \mathscr{C}''_{\gamma(n)}$ such that $\mathrm{dist}_E(C, C') < 2d_0^\beta 2^{\beta\gamma(n)}$ and $C \neq C'$. It follows that

$$\begin{aligned} &\mathrm{diam}_E(C \cup C') \\ &\quad < 2 \cdot 2^{\gamma(n)} + 2d_0^\beta 2^{\beta\gamma(n)} \\ &\quad < (2 + 2d_0^\beta) 2^{\beta\gamma(n)} = 2^{\beta\gamma(n) + \log_2(2 + 2d_0^\beta)} \end{aligned}$$

By the definition of $\gamma(n)$, we have $\beta\gamma(n) + \log_2(2 + 2d_0^\beta) + 1 \leq n$. Hence there exists an n-cube C'' such that $C \cup C' \subset C''$. And if C_1, C_2 and $C_3 \in \mathscr{C}''_{\gamma(n)}$ and $\mathrm{dist}_E(C_1, C_2) < 2d_0^\beta 2^{\beta\gamma(n)}$ and $\mathrm{dist}_E(C_1, C_3) < 2d_0^\beta 2^{\beta\gamma(n)}$, then it follows that

$$\begin{aligned} &\mathrm{diam}_E(C_1 \cup C_2 \cup C_3) \\ &\quad < 3 \cdot 2^{\gamma(n)} + 4d_0^\beta 2^{\beta\gamma(n)} \\ &\quad < (3 + 4d_0^\beta) 2^{\beta\gamma(n)} = 2^{\beta\gamma(n) + \log_2(3 + 4d_0^\beta)} \end{aligned}$$

and $\beta\gamma(n) + \log_2(3 + 4d_0^\beta) + 1 \leq n$. Therefore if $\{C_1, C_2, \cdots, C_l\} = \mathscr{C}''_{\gamma(n)}$, there exist at most $\left[\frac{l}{2}\right]$ pieces of n-cubes which cover $\{C_1, C_2, \cdots, C_l\}$. Hence we have that

$$V_n = \# \mathscr{C}_n \leq \frac{1}{2} \# \mathscr{C}''_{\gamma(n)} + \# \mathscr{C}'_{\gamma(n)} \leq \frac{1}{2} V_{\gamma(n)} + V'_{\gamma(n)}.$$

Thus (4.21) is proved. Let $\gamma^m(n) = \overbrace{\gamma(\gamma \cdots \gamma(\gamma(n)) \cdots)}^{m\text{times}}$. For n such that $\gamma(n) > 0$, $M(n)$ denotes the largest natural number such that $\gamma^{M(n)}(n) > 0$ and for n

such that $\gamma(n)\leq 0$, we define $M(n)=0$. If $\gamma(n)>0$, we can iterate (4.21) $M(n)$ times and we get

$$V_n\leq\left(\frac{1}{2}\right)^{M(n)}V_{\gamma^{M(n)}(n)}+\sum_{m=1}^{M(n)}\left(\frac{1}{2}\right)^{m-1}V'_{\gamma^m(n)}.$$

Hence we have

$$V_n\leq\left(\frac{1}{2}\right)^{M(n)}V_0+\sum_{m=1}^{M(n)}\left(\frac{1}{2}\right)^{m-1}V'_{\gamma^m(n)}. \tag{4.22}$$

If $\gamma(n)\leq 0$, this inequality (4.22) holds for $M(n)=0$ and for *the 2nd term of the right hand side of* (4.22) $=0$. Therefore we have

$$V=V(D)=\sum_{n=0}^{n_0(D)}V_n\leq\sum_{n=0}^{n_0(D)}\left(\frac{1}{2}\right)^{M(n)}V_0+\sum_{n=0}^{n_0(D)}\sum_{m=1}^{M(n)}\left(\frac{1}{2}\right)^{m-1}V'_{\gamma^m(n)}. \tag{4.23}$$

We put $d=1+\log_2(4d_0^\beta+3)$. Since $\frac{1}{\beta}(x-d)-1\leq\gamma(x)\leq\frac{1}{\beta}(x-d)$ and $\gamma(x)$ is a monotone increasing function, we have by induction

$$\left(\frac{1}{\beta}\right)^m x-d\sum_{j=1}^{m}\left(\frac{1}{\beta}\right)^j-\sum_{j=0}^{m-1}\left(\frac{1}{\beta}\right)^j\leq\gamma^m(x)\leq\left(\frac{1}{\beta}\right)^m x-d\sum_{j=1}^{m}\left(\frac{1}{\beta}\right)^j. \tag{4.24}$$

From (4.24) we have

$$\begin{aligned}\gamma^m(n)&>\left(\frac{1}{\beta}\right)^m n-d\sum_{j=1}^{m}\left(\frac{1}{\beta}\right)^j-\sum_{j=0}^{m-1}\left(\frac{1}{\beta}\right)^j\\&>\left(\frac{1}{\beta}\right)^m n-\frac{1}{\beta-1}(d+\beta).\end{aligned} \tag{4.25}$$

For $k\geq\frac{1}{\beta-1}(d+\beta)$, we denote by $l_0(k)$ the largest integer (≥ 0) such that

$$\left(\frac{1}{\beta}\right)^{l_0(k)}k-\frac{1}{\beta-1}(d+\beta)\geq 0. \tag{4.26}$$

Hence we have $\gamma^{l_0(k)}(k)>0$ and then $M(k)\geq l_0(k)$. Then we have

$$M(k)\geq\begin{cases}0 & \text{for } 0\leq k<\frac{1}{\beta-1}(d+\beta)\\ \left[\log_2\dfrac{k}{\frac{1}{\beta-1}(d+\beta)}(\log_2\beta)^{-1}\right] & \text{for } k\geq\frac{1}{\beta-1}(d+\beta).\end{cases} \tag{4.27}$$

From (4.27), we have

$$(4.28)\qquad \sum_{n=0}^{n_0(D)}\left(\frac{1}{2}\right)^{M(n)}$$
$$\leq\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{M(n)}$$
$$\leq\frac{1}{\beta-1}(d+\beta)+1+\sum_{k\geq\frac{1}{\beta-1}(d+\beta)}2^{-\log_2\frac{k}{\frac{1}{\beta-1}(d+\beta)}(\log_2\beta)^{-1}+1}.$$

We have

$$(4.29)\qquad 2^{-\log_2\frac{k}{\frac{1}{\beta-1}(d+\beta)}(\log_2\beta)^{-1}+1}=2\left(\frac{1}{\beta-1}(d+\beta)\right)^{(\log_2\beta)^{-1}}k^{-(\log_2\beta)^{-1}}.$$

Since $d=1+\log_2\left(4d_0^\beta+3\right)$, $d_0=\left[E^{-\frac{1}{2}}\right]$ and $\sqrt{2}<\beta<1$, we have

$$(4.30)\qquad \frac{1}{\beta-1}(d+\beta)\leq c\log_2 E^{-1}$$

and

$$(4.31)\qquad 2\left(\frac{1}{\beta-1}(d+\beta)\right)^{(\log_2\beta)^{-1}}\leq cd^2\leq c(\log_2 E^{-1})^2$$

for sufficiently small $E>0$. From (4.28) - (4.31) and $\log_2\beta<1$, we have

$$(4.32)\qquad \sum_{n=0}^{n_0(D)}\left(\frac{1}{2}\right)^{M(n)}\leq c\log_2 E^{-1}+c(\log_2 E^{-1})^2\leq c'(\log_2 E^{-1})^2$$

for sufficiently small $E>0$. For $m\in\boldsymbol{N}$ and $j\in\boldsymbol{Z}$, let $N_{m,j}=\{k\in\boldsymbol{Z}|\gamma^m(k)=j\}$. Let k_+ be the largest integer such that $\gamma^m(k_+)=j$ and k_- be the smallest integer such that $\gamma^m(k_-)=j$. By (4.24), we have

$$\left(\frac{1}{\beta}\right)^m k_+-d\sum_{j=1}^{m}\left(\frac{1}{\beta}\right)^j-\sum_{j=0}^{m-1}\left(\frac{1}{\beta}\right)^j\leq\gamma^m(k_+)$$
$$=j=\gamma^m(k_-)\leq\left(\frac{1}{\beta}\right)^m k_--d\sum_{j=1}^{m}\left(\frac{1}{\beta}\right)^j$$

and then

$$\left(\frac{1}{\beta}\right)^m(k_+-k_-)\leq\sum_{j=0}^{m-1}\left(\frac{1}{\beta}\right)^j=\frac{\beta}{\beta-1}\left(1-\left(\frac{1}{\beta}\right)^m\right).$$

Therefore it follows that

(4.33) $$\# N_{m,j}=k_{+}-k_{-}+1\leq\frac{1}{\beta-1}\beta^{m+1}.$$

We have

(4.34) $$\sum_{k=0}^{n_0(D)}\sum_{m=1}^{M(k)}\left(\frac{1}{2}\right)^{m-1}V'_{\gamma^{m}(k)}=\sum_{j=1}^{n_0(D)}\left(\sum_{k=0}^{n_0(D)}\sum_{m=1}^{M(k)}\left(\frac{1}{2}\right)^{m-1}\delta_{\gamma^{m}(k),j}\right)V'_j.$$

From (4.33) and $\beta<2$, we have

(4.35) $$\sum_{k=0}^{n_0(D)}\sum_{m=1}^{M(k)}\left(\frac{1}{2}\right)^{m-1}\delta_{\gamma^{m}(k),j}\leq\sum_{m,k=0}^{\infty}\left(\frac{1}{2}\right)^{m-1}\delta_{\gamma^{m}(k),j}=\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^{m-1}\# N_{m,j}\leq\frac{\beta^2}{\beta-1}\sum_{m=0}^{\infty}\left(\frac{\beta}{2}\right)^{m-1}=c'.$$

From (4.34) and (4.35), we have

(4.36) $$\sum_{k=0}^{n_0(D)}\sum_{m=1}^{M(k)}\left(\frac{1}{2}\right)^{m-1}V'_{\gamma^{m}(K)}\leq c'V'(D).$$

From (4.23), (4.32), and (4.36), we proved Lemma 4.3.

Proof of Proposition 4.1. *Let*

$$H_{\Lambda}^{D}(x_j; j\in J)=-\Delta+\sum_{j\in J}x_j f(\cdot-j)\Big|_{L^2(\Lambda)}$$

with Dirichlet boundary conditions. For $\lambda>0$, let

$$N(\lambda; x_j, j\in J)=\#\{\text{eigenvalues of } H_{\Lambda}^{D}(x_j; j\in J)\leq\lambda\}.$$

Since $\lambda-\lambda'\leq(\lambda-\lambda')\eta_0^{-1}f$ for $\lambda\geq\lambda'$ by (H.2), we have

(4.37) $$\begin{aligned}&N(\lambda; x_j, j\in J)\\&=N(\lambda'+\lambda-\lambda'; x_j, j\in J)\\&\leq N(\lambda'; x_j-\eta_0^{-1}(\lambda-\lambda'), j\in J).\end{aligned}$$

Let $X_{[0,1]}(x)$ be the characteristic function of the interval $[0, 1]$. From (H.1) and (4.37) we have

(4.38) $\boldsymbol{E}[N(\lambda; q_j(\omega), j\in J)-N(\lambda'; q_j(\omega), j\in J)]$

$$
\begin{aligned}
&=\int_{R^{\prime J}}(N(\lambda;\, x_j,\, j\in J)-N(\lambda';\, x_j,\, j\in J))\prod_{j\in J}X_{[0,1]}(x_j)\,dx_j\\
&\leq\int_{R^{\prime J}}(N(\lambda';\, x_j-\eta_0^{-1}(\lambda-\lambda'),\, j\in J)-N(\lambda';\, x_j,\, j\in J))\prod_{j\in J}X_{[0,1]}(x_j)\,dx_j\\
&=\int_{R^{\prime J}}N(\lambda';\, x_j,\, j\in J)\Big(\prod_{j\in J}X_{[0,1]}(x_j+\eta_0^{-1}(\lambda-\lambda'))-\prod_{j\in J}X_{[0,1]}(x_j)\Big)\prod_{j\in J}dx_j\\
&\leq N(\lambda';-\eta_0^{-1}(\lambda-\lambda')\, j\in J)\,2\eta_0^{-1}|\lambda-\lambda'||\Lambda|\\
&\leq N(\lambda';-\eta_0^{-1}(\lambda-\lambda')\, j\in \bar{J})\,2\eta_0^{-1}|\lambda-\lambda'||\bar{\Lambda}|
\end{aligned}
$$

where $|\Lambda|$ and $|\bar{\Lambda}|$ is the volume of Λ and $\bar{\Lambda}$ respectively. Let $l(\bar{\Lambda})=|\bar{\Lambda}|^{\frac{1}{3}}$. We have by (H.2)

$$
\begin{aligned}
(4.39)\qquad & N(\lambda';\,-\eta_0^{-1}(\lambda-\lambda'),\, j\in\bar{J})\\
&\leq N(\lambda'+\eta_0^{-1}\eta_1(\lambda-\lambda');\, 0,\, j\in\bar{J})\\
&=\#\left\{n\in \boldsymbol{N}^3\,\middle|\,\pi^2\sum_{j=1}^{3}\frac{n_j^2}{l(\bar{\Lambda})^2}<\lambda'+\eta_0^{-1}\eta_1(\lambda-\lambda')\right\}\\
&\leq\frac{1}{6}\pi\Big(\frac{l(\bar{\Lambda})^2}{\pi^2}(\lambda'+\eta_0^{-1}\eta_1(\lambda-\lambda'))\Big)^{\frac{3}{2}}\\
&=\frac{|\bar{\Lambda}|}{6\pi^2}(\lambda'+\eta_0^{-1}\eta_1(\lambda-\lambda'))^{\frac{3}{2}}.
\end{aligned}
$$

Let $\lambda=E+k$ and $\lambda'=E-k$. From (4.38) and (4.39), we have

$$
\begin{aligned}
(4.40)\qquad & \boldsymbol{P}(\mathrm{dist}(\sigma(H_\Lambda(\omega)),\, E)\leq k)\\
&\leq \boldsymbol{E}[N(E+k;\, q_j(\omega)\ \ j\in J)-N(E-k;\, q_j(\omega),\, j\in J)]\\
&\leq\frac{2c_0^{-1}}{3\pi^2}(E-k+2\eta_0^{-1}\eta_1 k)^{\frac{3}{2}}k.
\end{aligned}
$$

We have proved the proposition.

5. Sufficient condition of the exponential decay of the Green functions

Definition. For $A\subset \boldsymbol{Z}^3(E)$, let $A^c=\boldsymbol{Z}^3(E)\backslash A$,

$$\partial_{in}A=\{x\in A\,|\,\text{There exists } y\in A^C \text{ such that } \mathrm{dist}_E(x,\, y)=1\}$$

and

$$\partial_{out}A=\{x\in A^c\,|\,\text{There exists } y\in A \text{ such that } \mathrm{dist}_E(x,\, y)=1\}.$$

We define ∂A as follows:

$$\partial A=\partial_{in}A\cup\partial_{out}A.$$

$A\subset\boldsymbol{Z}^3(E)$ is said to be *k-admissible* if

$$\partial A \cap W(D_i^x, 4d_i) = \emptyset$$

for any component D_i^x of S_i^g and for $i=0, 1, \cdots, k$.
$A \subset \boldsymbol{Z}^3(E)$ is said to be *admissible* if A is *k-admissible* for all $k \geq 0$.

Let Λ, Λ_1 and $\Lambda_2 \subset \boldsymbol{R}^3$ be of the form $\cup_{j\in J} Q_E(j)$ for some $J \subset \boldsymbol{Z}^3(E)$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\Lambda_1 \cup \Lambda_2 = \Lambda$. Let $G_\Lambda(\omega, E+i\varepsilon; x, y)$ be the Green function of operator $H_{\Lambda,\omega} - (E+i\varepsilon) = H_\omega - (E+i\varepsilon)|_{L^2(\Lambda)}$ on $L^2(\Lambda)$ with Dirichlet boundary conditions and $G_{\Lambda_1|\Lambda_2}(\omega, E+i\varepsilon; x, y)$ be the Green function of operator $H_{\Lambda_1|\Lambda_2,\omega} - (E+i\varepsilon) = H_\omega - (E+i\varepsilon)|_{L^2(\Lambda_1)\oplus L^2(\Lambda_2)}$ on $L^2(\Lambda) \simeq L^2(\Lambda_1) \oplus L^2(\Lambda_2)$ with Dirichlet boundary conditions on $\partial\Lambda_1 \cup \partial\Lambda_2$. Let $\partial_{n_z} G_\Lambda(\omega, E+i\varepsilon; x, z)$, $x \in \Lambda$, $z \in \partial\Lambda$ be the outward normal derivative at z of $G_\Lambda(\omega, E+i\varepsilon; x, y)$. Then form Green's formula it follows that

$$\begin{aligned}(5.1)\qquad & G_\Lambda(\omega, E+i\varepsilon; x, y) \\ &= G_{\Lambda_1|\Lambda_2}(\omega, E+i\varepsilon; x, y) \\ &\quad - \int_{\partial\Lambda_1} \partial_{n_z} G_{\Lambda_1|\Lambda_2}(\omega, E+i\varepsilon; x, z) G_\Lambda(\omega, E+i\varepsilon; z, y) dz\end{aligned}$$

if $x \in \Lambda_1$, $y \in \Lambda_1 \cup \Lambda_2$ and $x \neq y$. We have

$$(5.2)\qquad G_{\Lambda_1|\Lambda_2}(\omega, E+i\varepsilon; x, y) = G_{\Lambda_j}(\omega, E+i\varepsilon; x, y)$$

if $x, y \in \Lambda_j$, $j=1, 2$ and

$$(5.3)\qquad G_{\Lambda_1|\Lambda_2}(\omega, E+i\varepsilon; x, y) = 0$$

if $x \in \Lambda_j$, $y \in \Lambda_k$, $j, k=1, 2$ and $j \neq k$.

The main theorem of this section is the following theorem.

Theorem 5.1. *For $x \in \boldsymbol{R}^3$ we denote by $j(x)$ the element of $\boldsymbol{Z}^3(E)$ which is uniquely determined by $x \in Q_E(j(x))$. There exists $E_1 > 0$ such that for $0 < E \leq E_1$ if $A \subset \boldsymbol{Z}^3(E)$ is k-admissible and $A \cap S_{k+1} = \emptyset$, then it follows that*

$$|G_{Q_E(A)}(E+i\varepsilon; x, y)| \leq \exp(-m(E)|x-y|)$$

provided $\mathrm{dist}_E(j(x), j(y)) \geq \frac{1}{5} d_{k+1}$ *and* $0 < \varepsilon \leq E$. *Here* $m(E) = c_1 E^{\frac{1}{2}}$ *and* c_1 *is independent of* A, k *and* E.

Proof. We denote by Θ_k the following assertion:
If A is $(k-1)$-*admissible* and $A \cap S_k = \emptyset$, then it follows that

$$|G_{Q_E(A)}(E+i\varepsilon; x, y)| \leq \exp(-m_k(E)|x-y|)$$

provided $\mathrm{dist}_E(j(x), j(y)) \geq \frac{1}{5} d_k$ and $0 < \varepsilon \leq E$.

Here $m_k(E) = \frac{1}{5} E^{\frac{1}{2}} \prod_{i=0}^{k-1} (1 - 77 d_i^{1-\alpha})$ and $m_0(E) = \frac{1}{5} E^{\frac{1}{2}}$.

By putting $c_1 = \frac{1}{5} \prod_{i=0}^{\infty} (1 - 77 d_i(E_1)^{1-\alpha})$, Theorem 5.1 follows from Θ_{k+1}.

We shall prove Θ_k for $k\geq 0$ by induction.

Step 1: *Proof of* Θ_0.

Since $A\cap S_0=\emptyset$, we have $\lambda_1(-\Delta^N_{Q_E(j)}+V_\omega)>2E$ for $j\in A$. Then we have

$$H^D_{Q_E(A)}(\omega)=-\Delta^D_{Q_E(A)}+V_\omega\geq-\Delta^N_{Q_E(A)}+V_\omega\geq\bigoplus_{j\in A}(-\Delta^N_{Q_E(j)}+V_\omega)>2E.$$

From Lemma $A.3$, the exists $E'>0$ such that if $0<E\leq E'$, then we have

$$|G_{Q_E(A)}(E+i\varepsilon;x,y)|\leq\exp\left(-\frac{1}{5}E^{\frac{1}{2}}|x-y|\right)$$

for any x and y such that $\mathrm{dist}_E(j(x),j(y))\geq\frac{1}{5}d_0$ and $0<\varepsilon\leq E$. This completes the proof of Θ_0.

Step2: *Proof of* Θ_{k+1} *under the assumption of* Θ_k.

Let A be *k-admissible* and $A\cap S_{k+1}=\phi$. If $A\cap S_k=\emptyset$, then Θ_{k+1} follows from Θ_k.

Hence we shall consider the case of $A\cap S_k\neq\emptyset$. In order to prove Θ_{k+1}, we distinguish the following two cases:

(i) $\frac{1}{5}d_{k+1}\leq\mathrm{diam}_E A\leq\frac{2}{3}d_{k+1}$.

(ii) $\frac{3}{2}d_{k+1}<\mathrm{diam}_E A$.

We first study the case (i).

Lemma 5.1. *Let* $R\subset\boldsymbol{Z}^3(E)$ *be a k-admissible set containing some* $D^x_k\in S^g_k$ *such that*

$$\frac{1}{5}d_{k+1}\leq\mathrm{diam}_E R\leq\frac{3}{2}d_{k+1}.$$

Then we have

$$|G_{Q_E(R)}(E+i\varepsilon;x,y)|\leq\exp\{-(m_k(E)-\mu_k(E))|x-y|\}$$

provided $\mathrm{dist}_E(j(x),j(y))\geq\frac{1}{5}d_{k+1}$. *Here* $\mu_k(E)=75m_k(E)d_k^{1-\alpha}$.

Proof. For simplicity we denote $D=D^x_k$. We fix $x\in Q_E(R)$ and $y\in Q_E(R)$ such that $\mathrm{dist}_E(j(x),j(y))\geq\frac{1}{5}d_k$. If $\mathrm{dist}_E(\{j(x),j(y)\},D)\geq 4d_k$, we put $D_1=D$ and $D_2=\{z\in R|\mathrm{dist}_E(z,D)\leq 3d_{k+1}\}$. If $\frac{3}{2}d_k\leq\mathrm{dist}_E(\{j(x),j(y)\},D)<4d_k$, we put $D_1=D$ and $D_2=\{z\in R|\mathrm{dist}_E(z,D)\leq\frac{1}{2}d_k\}$. If $\mathrm{dist}_E\{j(x),j(y)\},D)<\frac{3}{2}d_k$, we put $D_1=\{z\in R|\mathrm{dist}_E(z,D)\leq\frac{5}{2}d_k\}$ and $D_2=\{z\in R|\mathrm{dist}_E(z,D)\leq 3d_k\}$. Then for sufficientely small $E>0$, from Lemma B.1 it iollows that there exists a $(k-1)$*-admissible* set $B\subset\boldsymbol{Z}^3(E)$ such that

$$D \subset B \subset W(D, 4d_k),\ \mathrm{dist}_E(\partial B, \{j(x), j(y)\}\geq d_k$$

and

$$\mathrm{dist}_E(B^c, D) \leq 3d_k.$$

Let $Q=R\backslash B$, $\gamma=\partial Q_E(B)$ and $\bar{\gamma}=\partial Q_E(W(D, 4d_k))$.

Case (i, 1). Let x and y in $Q_E(Q)$.

From (5.1), we have

$$(5.4)\quad G_{Q_E(R)}(x, y)$$
$$=G_{Q_E(Q)|Q_E(B)}(x, z)-\int_\gamma \partial_{n_z} G_{Q_E(Q)|Q_E(B)}(x, z) G_{Q_E(R)}(z, y)dz$$
$$=G_{Q_E(Q)}(x, y)+\int_\gamma \partial_{n_z} G_{Q_E(R)}(x, z)\int_\gamma G_{Q_E(R)}(z, z')\partial_{n_{z'}} G_{Q_E(Q)}(z', y)dz'dz.$$

Since R and B is $(k-1)$*-admissible*, so is Q. Moreover by the assumption of R and Condition A(k) (c), $R\cap(S_k\backslash D)=\emptyset$. Then we have $Q\cap S_k=\emptyset$. Hence by applying Θ_k to Q, we get

$$(5.5)\qquad |G_{Q_E(Q)}(u, v)|\leq \exp\{-m_k(E)|u-v|\}$$

if $\mathrm{dist}_E(j(u), j(v))\geq \frac{1}{5}d_k$. Since $\mathrm{dist}_E(\partial B, \{j(x), j(y)\}\geq d_k$, $\mathrm{dist}_E(\{j(x), j(y)\}, j(z))-1>d_k-1\geq \frac{1}{5}d_k$ for $z\in\gamma$ for sufficiently small $E>0$. Therefore from Lemma A.5 and (5.5), we obtain for $z, z'\in\gamma$

$$(5.6)\qquad |\partial_{n_z} G_{Q_E(Q)}(x, z)|\leq c_3' \exp\{-m_k(E)|x-y|\}$$

and

$$(5.7)\qquad |\partial_{n_{z'}} G_{Q_E(Q)}(z', y)|\leq c_3' \exp\{-m_k(E)|z'-y|\}$$

since there exists a positive δ such that $m_k(E)<\delta$ uniformly in k and sufficiently small $E>0$. Next we shall estimate the term $G_{Q_E(R)}(z, z')$ in (5.4).

Lemma 5.2. *Let $u, w\in Q_E(B)$. Then we have*

$$|G_{Q_E(R)}(E+i\varepsilon; u, w)|\leq\frac{1}{|u-w|}+c_4' \exp(d_k^{\frac{1}{2}})$$

Here c_4' is independent of R, B, u, w, E and ε.

Proof of Lemma 5.2. From (5.1), we get

$$G_{Q_E(R)}(u, w)$$
$$=G_{Q_E(W(D,4d_k))|Q_E(R)\backslash Q_E(W(D,4d_k))}(u, w)$$
$$-\int_{\bar{\gamma}} \partial_{n_{z_1}} G_{Q_E(W(D,4d_k))|Q_E(R)\backslash Q_E(W(D,4d_k))}(u, z_1) G_{Q_E(R)}(z_1, w)dz_1$$
$$=G_{Q_E(W(D,4d_k))}(u, w)-\int_{\bar{\gamma}} \partial_{n_{z_1}} G_{Q_E(W(D,4d_k))}(u, z_1) G_{Q_E(R)}(z_1, w)dz_1.$$

Because $z_1 \in \bar{\gamma} \subset Q_E(Q)$, it follows

$$\begin{aligned}&G_{Q_E(R)}(z_1, w)\\&= G_{Q_E(Q)|Q_E(B)}(z_1, w) - \int_\gamma \partial_{n_{z_2}} G_{Q_E(Q)|Q_E(B)}(z_1, z_2) G_{Q_E(R)}(z_2, w)\, dz_2\\&= -\int_\gamma \partial_{n_{z_2}} G_{Q_E(Q)}(z_1, z_2) G_{Q_E(R)}(z_2, w)\, dz_2.\end{aligned}$$

Hence we get

$$\begin{aligned}&G_{Q_E(R)}(u, w)\\&= G_{Q_E(W(D,4d_k))}(u, w)\\&+ \int_{\bar{\gamma}} \partial_{n_{z_1}} G_{Q_E(W(D,4d_k))}(u, z_1) \int_\gamma \partial_{n_{z_2}} G_{Q_E(Q)}(z_1, z_2) G_{Q_E(R)}(z_2, w)\, dz_2 dz_1.\end{aligned}$$

Because $z_2 \in \gamma \subset Q_E(W(D, 4d_k))$, it follows

$$\begin{aligned}&G_{Q_E(R)}(z_2, w)\\&= G_{Q_E(W(D,4d_k))}(z_2, w) - \int_{\bar{\gamma}} \partial_{n_{z_3}} G_{Q_E(W(D,4d_k))}(z_2, z_3) G_{Q_E(R)}(z_3, w)\, dz_3.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&G_{Q_E(R)}(u, w)\\&= G_{Q_E(W(D,4d_k))}(u, w)\\&+ \int_{\bar{\gamma}} \partial_{n_{z_1}} G_{Q_E(W(D,4d_k))}(u, z_1) \int_\gamma \partial_{n_{z_2}} G_{Q_E(Q)}(z_1, z_2) G_{Q_E(W(D,4d_k))}(z_2, w)\, dz_2 dz_1\\&- \int_{\bar{\gamma}} \partial_{n_{z_1}} G_{Q_E(W(D,4d_k))}(u, z_1) \int_\gamma \partial_{n_{z_2}} G_{Q_E(Q)}(z_1, z_2)\\&\quad \times \int_{\bar{\gamma}} \partial_{n_{z_3}} G_{Q_E(W(D,4d_k))}(z_2, z_3) G_{Q_E(R)}(z_3, w)\, dz_3 dz_2 dz_1.\end{aligned}$$

Inductively we obtain

$$\begin{aligned}(5.8)\quad &G_{Q_E(R)}(u, w)\\&= G_{Q_E(W(D,4d_k))}(u, w)\\&+ \sum_{n=1}^{N} \overbrace{\int_{\bar{\gamma}}\int_\gamma \cdots \int_{\bar{\gamma}}\int_\gamma}^{2n} \left(\prod_{j=0}^{n-1} \partial_{n_{z_{2j+1}}} G_{Q_E(W(D,4d_k))}(z_{2j}, z_{2j+1}) \partial_{n_{z_{2j+2}}} G_{Q_E(Q)}(z_{2j+1}, z_{2j+2}) \right)\\&\quad \times G_{Q_E(W(D,4d_k))}(z_{2n}, w) \prod_{j=1}^{2n} dz_j\\&+ \overbrace{\int_{\bar{\gamma}}\int_\gamma \cdots \int_{\bar{\gamma}}\int_\gamma}^{2(N+1)} \left(\prod_{j=0}^{N} \partial_{n_{z_{2j+1}}} G_{Q_E(W(D,4d_k))}(z_{2j}, z_{2j+1}) \partial_{n_{z_{2j+2}}} G_{Q_E(Q)}(z_{2j+1}, z_{2j+2}) \right)\\&\quad \times G_{Q_E(R)}(z_{2N+2}, w) \prod_{j=1}^{2N+2} dz_j\end{aligned}$$

where $z_0 = u$. From Lemma A.6, it follows

(5.9)
$$|G_{Q_E(W(D,4d_k))}(E+i\varepsilon;\, t,\, s)| \le \frac{1}{|t-s|} + \frac{c_4}{\operatorname{dist}(\sigma(H^D_{Q_E(W(D,4d_k))}),\, E+i\varepsilon)} \le \frac{1}{|t-s|} + c_4 \exp(d_k^{\frac{1}{2}}).$$

The last inequality follows from Condition A(k) (d). If $t \in \gamma \cup Q_E(B)$ and $s \in \bar{\gamma}$, it follows

$$|t-s| > |t-s|-1 \ge d_k L(E) - 1 > \frac{1}{5} L(E) d_k > 1$$

for sufficiently small $E>0$. Hence by Lemma A.5 and (5.9) we have

(5.10)
$$|\partial_{n_s} G_{Q_E(W(D,4d_k))}(t,\, s)| \le c_3\Big(\frac{1}{|t-s|-1} + c_4 \exp(d_k^{\frac{1}{2}}\Big) \le c_3 (5L(E)^{-1} d_k^{-1} + c_4 \exp(d_k^{\frac{1}{2}})) \le c_5 \exp(d_k^{\frac{1}{2}})$$

for $t \in \gamma \cup Q_E(B)$ and $s \in \bar{\gamma}$. From (5.9) we have

(5.11)
$$\Big|\int_\gamma G_{Q_E(W(D,4d_k))}(z_{2n}, w)\, dz_{2n}\Big| \le \int_\gamma \Big(\frac{1}{|z_{2n}-w|} + c_4 \exp(d_k^{\frac{1}{2}})\Big) dz_{2n} = \int_\gamma \frac{1}{|z_{2n}-w|} dz_{2n} + c_4 \exp(d_k^{\frac{1}{2}})|\gamma|.$$

By the shape of $Q_E(B) = \cup_j Q_E(j)$, we can estimate as follows:

$$\int_\gamma \frac{1}{|z_{2n}-w|} dz_{2n} \le c_6 |\gamma|$$

where c_6 is independent of $\gamma = \partial Q_E(B)$ and w. Then the last member of (5.11) is bounded by

(5.12)
$$c_7 \exp(d_k^{\frac{1}{2}})|\gamma|.$$

Since $z_{2j+1} \in \bar{\gamma}$ and $z_{2j+2} \in \gamma$, it follows

(5.13)
$$|\partial_{n_{z_{2j+2}}} G_{Q_E(Q)}(z_{2j+1}, z_{2j+2})| \le c_3 \sup_{|s-z_{2j+2}|\le 1} |G_{Q_E(Q)}(z_{2j+1},\, s)| \le c_3 \exp\Big(-m_k \frac{L(E)}{5} d_k\Big)$$

by Lemma A.5 and (5.5). From Lemma A.6 it follows

$$|G_{Q_E(R)}(E+i\varepsilon;\, t,\, s)| = |G_{Q_E(R)}(t,\, s)| \le \frac{1}{|t-s|} + \frac{1}{\varepsilon}$$

and then

$$\int_{\gamma}|G_{Q_E(R)}(z_{2(N+1)}, w)|dz_{2(N+1)} \leq \left(c_6+\frac{1}{\varepsilon}\right)|\gamma|. \tag{5.14}$$

From (5.8)-(5.14), we obtain

$$\begin{aligned} |G_{Q_E(R)}(u, w)| \leq & |G_{Q_E(W(D,4d_k))}(u, w)| \\ & + \sum_{n=1}^{N}\left(c_3c_5|\bar{\gamma}||\gamma|\exp(d_k^{\frac{1}{2}})\exp\left(-m_k\frac{L(E)}{5}d_k\right)\right)^n c_7 \exp(d_k^{\frac{1}{2}}) \\ & + \left(c_3c_5|\bar{\gamma}||\gamma|\exp(d_k^{\frac{1}{2}})\exp\left(-m_k\frac{L(E)}{5}d_k\right)\right)^{N+1}\left(c_6+\frac{1}{\varepsilon}\right). \end{aligned} \tag{5.15}$$

Since $|\bar{\gamma}||\gamma| \leq c_8 d_k^6$ and it follows

$$cd_k^6 \exp(d_k^{\frac{1}{2}})\exp\left(-m_k\frac{L(E)}{5}d_k\right) < \frac{1}{2}$$

if $E>0$ is sufficiently small, then the last term of the right hand side of (5.15) converges to 0 as $N\to\infty$. Therefore we obtain

$$\begin{aligned} |G_{Q_E(R)}(u, w)| & \leq |G_{Q_E(W(D,4d_k))}(u, w)| + c \exp(d_k^{\frac{1}{2}}) \\ & \leq \frac{1}{|u-w|} + c_4' \exp(d_k^{\frac{1}{2}}) \end{aligned} \tag{5.16}$$

by (5.9). We have thus proved Lemma 5.2.

We return to the proof of Lemma 5.1. Noting the continuity of the Green function, it follows from Lemma 5.2

$$|G_{Q_E(R)}(z, z')| \leq \frac{1}{|z-z'|} + c_4 \exp(d_k^{\frac{1}{2}}) \tag{5.17}$$

for $z, z' \in \gamma$. Using (5.4)-(5.7) and (5.17), we get

$$\begin{aligned} & |G_{Q_E(R)}(x, y)| \\ & \leq \exp(-m_k(E)|x-y|) \\ & \quad + (c_3')^2 \int_{\gamma}\int_{\gamma} \frac{\exp(-m_k(E)|x-z|)\exp(-m_k(E)|z'-y|)}{|z-z'|} dz dz' \\ & \quad + c_4 (c_3')^2 \exp(d_k^{\frac{1}{2}}) \int_{\gamma}\int_{\gamma} \exp(-m_k(E)|x-z|)\exp(-m_k(E)|z'-y|) dz dz' \\ & \leq \exp(-m_k(E)|x-y|) \\ & \quad \times \left\{1 + c_9 \int_{\gamma}\int_{\gamma} \frac{\exp m_k(E)(|x-y|-|x-z|-|z'-y|)}{|z-z'|} dz dz' \right. \\ & \quad \left. + c_{10} \exp(d_k^{\frac{1}{2}}) \int_{\gamma}\int_{\gamma} \exp m_k(E)(|x-y|-|x-z|-|z'-y|) dz dz' \right\}. \end{aligned} \tag{5.18}$$

Since $z, z' \in \gamma = \partial Q_E(B)$, it follows

$$|x-y|-|x-z|-|z'-y| \leq |z-z'| \leq \sqrt{3}(7d_k+1)L(E) \leq 14 d_k L(E).$$

And there exists positive constants c_{11}, c_{12} which are independent of γ, such that

$$\int_{\gamma}\int_{\gamma}\frac{1}{|z-z'|}dzdz' \leq c_{11}|\gamma|^2 \leq c_{12}d_k^6.$$

Then the right hand side of (5.18) is bounded by

$$\begin{aligned}(5.19)\qquad & \exp(-m_k(E)|x-y|) \\ & \times \{1+c_{13}|\gamma|^2 \exp(d_k^{\frac{1}{2}})\exp(m_k(E)14d_kL(E))\} \\ & \leq \exp(-m_k(E)|x-y|)c_{14}\exp(2d_k^{\frac{1}{2}})\exp(m_k(E)14d_kL(E)).\end{aligned}$$

Since there exists $\delta>0$ such that

$$m_kL(E)\geq\delta>0$$

uniformly in k and sufficiently small $E>0$, it follows

$$\begin{aligned}(5.20)\qquad & c_{14}\exp(2d_k^{\frac{1}{2}})\exp(m_k(E)14d_kL(E)) \\ & \leq \exp(15m_k(E)d_kL(E)) = \exp(\mu_k(E)\frac{1}{5}d_{k+1}L(E)).\end{aligned}$$

Here we used the definition of $\mu_k(E)=75m_k(E)d_k^{1-\alpha}$. By (5.18), (5.19) and (5.20), we obtain

$$|G_{Q_E(R)}(x, y)|\leq\exp(-(m_k(E)-\mu_k(E))|x-y|)$$

if $\text{dist}_E(j(x), j(y))\geq\frac{1}{5}d_{k+1}$ for sufficientely small $E>0$ uniformly in k.

Case (i.2). in the case of $x\in Q_E(Q)$ and $y\in Q_E(B)$, the proof is in a similar fashion in case 1.

We have completed the proof of Lemma 5.1.

Next we shall study the case (ii).

Lemma 5.3. *Let $A\subset\boldsymbol{Z}^3(E)$ be a k-admissible set such that $A\cap S_{k+1}=\emptyset$, $A\cap S_k\neq\emptyset$ and $\text{diam}_E A>\frac{3}{2}d_{k+1}$. If $x,y\in Q_E(A)$ and $\text{dist}_E(j(x), j(y))\geq\frac{1}{5}d_{k+1}$, then we have*

$$|G_{Q_E(A)}(E+i\varepsilon; x, y)|\leq\exp(-m_{k+1}(E)|x-y|).$$

Proof. Let $p_1, p_2\in Q_E(A)$ such that $\text{dist}_E(j(p_1), j(p_2))\geq\frac{1}{5}d_{k+1}$. If $\text{dist}_E(j(p_1), j(p_2))\leq\frac{1}{2}d_{k+1}$, then we put

$$D_1=\left\{z\in A\,\middle|\,\text{dist}_E(z, j(p_1))\leq\frac{29}{40}d_{k+1}\right\}$$

and

$$D_2=\left\{z\in A\,\middle|\,\mathrm{dist}_E(z,j(p_1))\leq\frac{3}{4}d_{k+1}\right\}.$$

If $\mathrm{dist}_E(j(p_1),j(p_2))>\frac{1}{2}d_{k+1}$, then we put

$$D_1=\left\{z\in A\,\middle|\,\mathrm{dist}_E(z,j(p_1))\leq\frac{1}{4}d_{k+1}\right\}$$

and

$$D_2=\left\{z\in A\,\middle|\,\mathrm{dist}_E(z,j(p_1))\leq\frac{11}{40}d_{k+1}\right\}.$$

For sufficientely small $E>0$, from Lemma B.1 it follows that there exists *k-admissible* set $R_{p_1}\subset A$ such that

$$p_1\in R_{p_1},\ \mathrm{dist}_E(\{j(p_1),j(p_2)\},\ \partial R_{p_1}\backslash A)\geq\frac{17}{80}d_{k+1} \tag{5.21}$$

and

$$\mathrm{diam}_E R_{p_1}\leq\frac{3}{2}d_{k+1}. \tag{5.22}$$

Let $B_{p_1}=A\backslash R_{p_1}$. Then it follows $Q_E(A)=Q_E(R_{p_1})+Q_E(B_{p_1})$. Since $\mathrm{dist}_E(j(x),j(y))\geq\frac{1}{5}d_{k+1}$, by putting $p_1=x$, $p_2=y$ there exists a *k–admissible* set $R_x\subset A$ satisfying (5.21), (5.22). From (5.1) we have

$$\begin{aligned}&G_{Q_E(A)}(x,y)\\&=G_{Q_E(R_x)|Q_E(B_x)}(x,y)-\int_{\partial Q_E(R_x)}\partial_{n_{z_1}}G_{Q_E(R_x)|Q_E(B_x)}(x,z_1)G_{Q_E(A)}(z_1,y)dz_1\\&=G_{Q_E(R_x)|Q_E(B_x)}(x,y)-\int_{\partial Q_E(R_x)\backslash\partial Q_E(A)}\partial_{n_{z_1}}G_{Q_E(R_x)}(x,z_1)G_{Q_E(A)}(z_1,y)dz_1\end{aligned}$$

where we used the fact $G_{Q_E(A)}(z_1,y)=0$ if $z_1\in\partial Q_E(A)$. Because $z_1\in\partial Q_E(R_x)\backslash\partial Q_E(A)$, we have $\mathrm{dist}_E(j(z_1),j(y))\geq\frac{1}{5}d_{k+1}$. Hence by putting $p_1=z_1$, $p_2=y$ there exists a *k-admissible* set $R_{z_1}\subset A$ satistying (5.21), (5.22). We have

$$\begin{aligned}&G_{Q_E(A)}(x,y)\\&=G_{Q_E(R_x)|Q_E(B_x)}(x,y)-\int_{\partial Q_E(R_x)\backslash\partial Q_E(A)}\partial_{n_{z_1}}G_{Q_E(R_x)}(x,z_1)G_{Q_E(R_{z_1})|Q_E(B_{z_1})}(z_1,y)dz_1\\&\quad+\int_{\partial Q_E(R_x)\backslash\partial Q_E(A)}\partial_{n_{z_1}}G_{Q_E(R_x)}(x,z_1)dz_1\\&\quad\times\int_{\partial Q_E(R_{z_1})\backslash\partial Q_E(A)}\partial_{n_{z_2}}G_{Q_E(R_{z_1})}(z_1,z_2)G(_{Q_E(A)}(z_2,y)dz_2.\end{aligned}$$

Inductively we have

$$\begin{aligned}&G_{Q_E(A)}(x,y)\\&=G_{Q_E(R_x)|Q_E(B_x)}(x,y)\end{aligned} \tag{5.23}$$

$$+\sum_{n=1}^{N}(-1)^n\prod_{j=0}^{n-1}\left(\int_{\partial Q_E(R_{z_j})\backslash\partial Q_E(A)}\partial_{n_{z_{j+1}}}G_{Q_E(R_{z_j})}(z_j, z_{j+1})dz_{j+1}\right)$$
$$\times G_{Q_E(R_{z_n})|Q_E(B_{z_n})}(z_n, y)$$
$$+(-1)^{N+1}\prod_{j=0}^{N}\left(\int_{\partial Q_E(R_{z_j})\backslash\partial Q_E(A)}\partial_{n_{z_{j+1}}}G_{Q_E(R_{z_j})}(z_j, z_{j+1})dz_{j+1}\right)$$
$$\times G_{Q_E(A)}(z_{N+1}, y)$$
$$= \mathrm{I}+\mathrm{II}+\mathrm{III}$$

where $z_0=x$. First we estimate I. If $y\notin Q_E(R_x)$, then we have $\mathrm{I}=0$. Hence we have only to study the case where $y\in Q_E(R_x)$. We have $G_{Q_E(R_x)|Q_E(B_x)}(x, y)=G_{Q_E(R_x)}(x, y)$. By the definition of R_x, R_x is *k-admissible* set and $R_x\cap S_{k+1}=\phi$ because of $A\cap S_k=\emptyset$. Therefore if $R_x\cap S_k=\emptyset$, then by the assumption of induction we have

$$(5.24)\qquad |G_{Q_E(R_x)}(x, y)|\leq\exp(-m_k(E)|x-y|)\leq\exp(-m'_k(E)|x-y|)$$

and if $R_x\cap S_k\neq\emptyset$ then by Lemma 5.1 we have

$$(5.25)\qquad |G_{Q_E(R_x)}(x, y)|\leq\exp(-m'_k(E)|x-y|)$$

where $m'_k(E)=m_k(E)-\mu_k(E)$. Next we estimate II. By Lemma A.5 we have

$$(5.26)\qquad |\partial_{n_{z_{j+1}}}G_{Q_E(R_{z_j})}(z_j, z_{j+1})|\leq c_3\sup_{|z_{j+1}-u|\leq 1}|G_{Q_E(R_{z_j})}(z_j, u)|.$$

From (5.21) we have

$$(5.27)\qquad \begin{aligned}&\mathrm{dist}_E(j(z_j), j(u))\\&\geq\mathrm{dist}_E(j(z_j), j(z_{j+1}))+\mathrm{dist}_E(j(z_{j+1}), j(u))\\&\geq\frac{17}{80}d_{k+1}-1>\frac{1}{5}d_{k+1}\end{aligned}$$

for sufficientely small $E>0$. From (5.26), (5.27), Lemma 5.1 and the assumption of induction as we obtained (5.24) and (5.25) we have

$$(5.28)\qquad |\partial_{n_{z_{j+1}}}G_{Q_E(R_{z_j})}(z_j, z_{j+1})\leq c_3\sup_{|z_{j+1}-u|\leq 1}\exp(-m'_k(E)|z_j-u|).$$

Since $|z_j-z_{j+1}|\geq|z_j-u|-1$ and there exists $\delta>0$ such that $m'_k(E)\leq\delta$ uniformly in k for sufficiently small $E>0$, we have

$$(5.29)\qquad |\partial_{n_{z_{j+1}}}G_{Q_E(R_{z_j})}(z_j, z_{j+1})|\leq c'_3\exp(-m'_k(E)|z_j-z_{j+1}|).$$

Since $\mathrm{dist}_E(j(z_n), j(y))\geq\frac{1}{5}d_{k+1}$, it follows similary to (5.24) and (5.25) that

$$(5.30)\qquad \left|G_{Q_E(R_{z_n})|Q_E(B_{z_n})}(z_n, y)\right|\leq\exp(-m'_k(E)|z_n-y|).$$

From (5.29), we have

$$(5.31)\qquad |\mathrm{II}|$$

$$\leq \sum_{n=1}^{N} \prod_{j=0}^{n-1} \left(\int_{\partial Q_E(R_{z_j}) \setminus \partial Q_E(A)} c_3' \exp(-m_k'(E)|z_j - z_{j+1}|) dz_{j+1} \right) \exp(-m_k'(E)|z_n - y|).$$

Let $\nu_k(E) = m_k(E) d_k^{1-\alpha}$ and $m_k''(E) = m_k'(E) - \nu_k(E)$. We have

(5.32) $\exp(-m_k'(E)|z_j - z_{j+1}|) = \exp(-m_k''(E)|z_j - z_{j+1}|) \exp(-\nu_k(E)|z_j - z_{j+1}|)$

and

(5.33)

$$\left(\prod_{j=0}^{n-1} \exp(-m_k''(E)|z_j - z_{j+1}|) \right) \exp(-m''(E)|z_n - y|) \leq \exp(-m_k''(E)|x - y|).$$

Since

$$\begin{aligned} &|z_j - z_{j+1}| \\ &\quad \geq (\mathrm{dist}_E(j(z_j), j(z_{j+1})) - 1) L(E) \\ &\quad \geq \left(\frac{17}{80} d_{k+1} - 1 \right) L(E) > \frac{1}{5} d_{k+1} L(E) \end{aligned}$$

for stfficiently small E>0, we have

(5.34) $$\exp(-\nu_k(E)|z_j - z_{j+1}|) \leq \exp\left(-\nu_k(E) \frac{1}{5} d_{k+1} L(E)\right).$$

From (5.31)-(5.34) we have

(5.35) $|\mathrm{II}|$

$$\leq \sum_{n=1}^{N} \prod_{j=0}^{n-1} \left(c_3' |\partial Q_E(R_{z_j}) \setminus \partial Q_E(A)| \exp\left(-\nu_k(E) \frac{1}{5} d_{k+1} L(E)\right) \right) \exp(-m_k''(E)|x - y|).$$

By (5.22) we have $|\partial Q_E(R_{z_j}) \setminus \partial Q_E(A)| \leq 6L(E)^2 \left(\frac{3}{2} d_{k+1}\right)^3 = c_{15} L(E)^2 d_{k+1}^3$. Since there exists $\delta > 0$ such that $\nu_k(E) L(E) d_{k+1} > \delta d_k$ uniformly in k for sufficiently small $E > 0$, there exists $0 < \delta' < 1$ such that

(5.36) $$\begin{aligned} &c_3' |\partial Q_E(R_{z_j}) \setminus \partial Q_E(A)| \exp\left(-\nu_k(E) \frac{1}{5} d_{k+1} L(E)\right) \\ &\leq c_3' c_{15} L(E)^2 d_{k+1}^3 \exp\left(-\nu_k(E) \frac{1}{5} d_{k+1} L(E)\right) \\ &< \delta' \end{aligned}$$

uniformly in k for sufficiently small $E > 0$. From (5.35) and (5.36) we have

(5.37) $$|\mathrm{II}| \leq c_{16} \exp(-m_k''(E)|x - y|)$$

for sufficiently small $E > 0$. Finally we estimate III. From Lemma A.6 we have

(5.38) $$\begin{aligned} |G_{Q_E(A)}(z_{n+1}, y)| &= |G_{Q_E(A)}(E + i\varepsilon; z_{n+1}, y)| \\ &\leq \frac{1}{|z_{n+1} - y|} + c_4 \varepsilon^{-1}. \end{aligned}$$

In a fashion similar to that used to estimate II, we have

$$
\begin{aligned}
(5.39)\quad & |\text{III}| \\
& \le \prod_{j=0}^{N} \Big(\int_{Q_E(R_{t_j}) \setminus \partial Q_E(A)} \Big| \partial_{n_{t_{j+1}}} G_{Q_E(R_{t_j})}(z_j, z_{j+1}) \Big| \Big) dz_{j+1} \Big) |G_{Q_E(A)}(z_{N+1}, y)| \\
& \le \Big(c_3' c_{15} L(E)^2 d_{k+1} \exp\big(-m_k'(E) \tfrac{1}{5} d_{k+1} L(E)\big) \Big)^{N+1} \Big(\frac{1}{\frac{1}{5} d_{k+1} L(E)} + c_4 \varepsilon^{-1} \Big) \\
& \le (\delta')^{N+1} \Big(\frac{1}{\frac{1}{5} d_{k+1} L(E)} + c_4 \varepsilon^{-1} \Big)
\end{aligned}
$$

where we used (5.38). Since $0 < \delta' < 1$, the last member of (5.39) converges to 0 as $N \to 0$. Hence from (5.23), (5.24), (5.25) and (5.37) we have

$$
\begin{aligned}
(5.40)\qquad & |G_{Q_E(A)}(x, y)| \\
& \le \exp(-m_k'(E)|x-y|) + c_{16} \exp(-m_k''(E)|x-y|).
\end{aligned}
$$

Since $m_{k+1}(E) = m_k'(E) - 2\nu_k(E) = m_k''(E) - \nu_k(E)$, from (5.40) we have

$$
\begin{aligned}
(5.41)\quad & |G_{Q_E(A)}(x, y)| \\
& \le (\exp(-\nu_k(E)|x-y|) + c_{16}) \exp(-\nu_k(E)|x-y|) \exp(-m_{k+1}(E)|x-y|).
\end{aligned}
$$

Since

$$
(5.42)\qquad \nu_k(E)|x-y| \ge m_k(E) d_k^{1-\alpha} \frac{1}{5} d_{k+1} L(E) = \frac{1}{5} m_k(E) d_k L(E),
$$

we have

$$
(\exp(-\nu_k(E)|x-y|) + c_{16}) \exp(-\nu_k(E)|x-y|) \le 1
$$

uniformly in k for sufficiently small $E > 0$. We have thus proved Lemma 5.3.

From Lemma 5.1 and Lemma 5.2, we complete the proof of step 2.

As a result we complete the induction and then we have proved Theorem 5.1.

6. Proof of Theorem 1.3

For $l > 0$, we denote by B_l the following condition on $A \subset \mathbf{Z}^3(E)$:

$$
\frac{l}{2} \le \min_{b \in \partial A} |b|_E \le \max_{b \in \partial A} |b|_E \le l.
$$

Let c_1 be as in Theorem 5.1, $m(E) = c_1 E^{\frac{1}{2}}$ and $0 < \varepsilon \le E$. Let

$F_l = \cup_{k=0}^{\infty} \{\omega \in \Omega \,|$ There exists a *k-admissible* set $0 \in A \subset \mathbf{Z}^3(E)$ satisfying B_l and

$$|G_{Q_E(A)}(\omega, E+i\varepsilon; x, y)|\leq \exp(-m(E)|x-y|) \text{ for } |x-y|\geq L(E)l^{\gamma}\}.$$

and $\alpha>\gamma>0$. We can prove Theorem 1.3 by the following theorem.

Theorem 6.1. *For any $p>0$, there exists $E^*>0$ such that if $0<E\leq E^*$, then we have*

$$P(F_l)\geq 1-l^{-p}$$

for $l\geq\left(\frac{1}{5}d_0\right)^{\frac{1}{\gamma}}$.

Proof. Let $p'>\frac{\alpha}{\gamma}(3+p)$ be fixed. Let $E'>0$, $E_1>0$ be the constant which is given in Theorem 4.1 with $p=p'$ and Theorem 5.1 respectively. For $0<E\leq\min(E', E_1)$, let $k=k(l)$ be the largest natural number such that $l^{\gamma}\geq\frac{1}{5}d_k$. Let

$$F_l'=\{\omega\in\Omega|\text{ There exists a } (k-1)\text{-}admissible \text{ set } 0\in A\subset \boldsymbol{Z}^3(E) \text{ satisfying } B_l \text{ and } A\cap S_k=\emptyset\}.$$

Because of $\frac{1}{5}L(E)d_k\leq L(E)l^{\gamma}$ and Theorem 5.1, we have

$$P(F_l')\leq P(F_l). \tag{6.1}$$

Since $\frac{l}{2}\geq\frac{1}{2}\left(\frac{1}{5}\right)^{\frac{1}{\gamma}}d_{k-1}^{\frac{\alpha}{\gamma}}>12d_{k-1}$ for sufficiently small $E>0$, from Lemma B.1 we have

$$\begin{aligned}\boldsymbol{P}(F_l')&\geq\boldsymbol{P}(\{\omega|B\cap S_k=\emptyset \text{ for any } B\subset\boldsymbol{Z}^3(E) \text{ satisfying } B_l\})\\ &\geq\boldsymbol{P}\Big(\bigcap_{\substack{x\in Z^3(E)\\ |x|_E\leq l}}\{\omega|x\notin S_k\}\Big)\\ &=1-\boldsymbol{P}\Big(\bigcup_{\substack{x\in Z^3(E)\\ |x|_E\leq l}}\{\omega|x\in S_k\}\Big)\\ &\geq 1-(2l+1)^3\boldsymbol{P}(\{\omega|0\in S_k\})\end{aligned} \tag{6.2}$$

where we use

$$\boldsymbol{P}(\{\omega|x\in S_k\})=\boldsymbol{P}(\{\omega|0\in S_k\}) \tag{6.3}$$

which follows from the translation invariance of $\boldsymbol{P}$. We have

$$\boldsymbol{P}(\{\omega|0\in S_k\})\leq\sum_{j=k}^{\infty}\boldsymbol{P}(\{\omega|0\in S_k^{\sharp}\})+\boldsymbol{P}(\{\omega|0\in\cap_{k=0}^{\infty}S_k\}). \tag{6.4}$$

We need the following Lemma which is proved in a similar fashion as in [1] and [5].

Lemma 6.1. *We have $\boldsymbol{P}(\{\omega|0\in\cap_{k=0}^{\infty}S_k\})=0$.*

By the definition of k, we have $d_{k+1}=d_k^{\alpha}>5l^{\gamma}$ and then

(6.5) $$d_k \geq c l^{\frac{\gamma}{\alpha}}.$$

From Lemma 6.1, (6.4), (6.5) and Theorem 4.1, we have

(6.6) $$\boldsymbol{P}(\{\omega | 0 \in S_k\}) \leq \sum_{j=k}^{\infty} d_j^{-p'} \leq c d_k^{-p'} \leq c' l^{-\frac{\gamma}{\alpha}p'}$$

where c' is independent of $E \in (0, E^*]$ and l. Therefore from $p' > \frac{\alpha}{\gamma}(3+p)$, there exists $L > 0$ independent of $E \in (0, E^*]$ such that if $l > L$, then

(6.7) $$c' l^3 l^{-\frac{\gamma}{\alpha}p'} \leq l^{-p}.$$

From (6.2), (6.6) and (6.7) we have

$$\boldsymbol{P}(F'_l) \geq 1 - c' l^3 l^{-\frac{\gamma}{\alpha}p'} \geq 1 - l^{-p}$$

for $l > L$. We have proved the Theorem by (6.1).

Proof of Theorem 1.3. Let $p > 0$ be given. For this p, let E^* be the constants which are given Theorem 6.1. For $N \in \boldsymbol{N}$ we fix a constant $R > 1$ satisfying

(6.8) $$RL(E^*) - \sqrt{3} \geq (4^2 R)^{\gamma} L(E^*),$$

(6.9) $$R > \frac{1}{5} L(E)$$

and

(6.10) $$2^3 k (NR4^k)^3 \exp(-D_0 (NR4^k)^{\gamma}) \leq 1 \text{ for any } k \in \boldsymbol{N}$$

where $D_0 = \inf_{0 < E \leq E^*} m(E) L(E) > 0$. We put

(6.11) $$l_j = NR4^j \text{ for } j = 0, 1, 2 \cdots$$

We note that from (6.8) it follows

(6.12) $$l_j L(E) - \sqrt{3} \geq l_{j+2}^{\gamma} L(E) \text{ for } j = 0, 1, 2 \cdots$$

For $0 < E \leq E^*$ and $\varepsilon \neq 0$, we put

$F_{l_j} = \{\omega |$There exists *k-admissible* set $0 \in A \subset \boldsymbol{R}^3$ satisfying B_{l_j} and $|G_{Q_E(A)}(\omega, E + i\varepsilon; x, y)| \leq \exp(-m(E)|x-y|)$ for $|x-y| \geq L(E) l_j^{\frac{3}{4}}\}$.

Since $l_j > \frac{1}{5} L(E)$ from (6.9) and $0 < E \leq E^*$, by Theorem 6.1 we have

(6.13) $$P(F_{l_j}) \geq 1 - l_j^{-p} \text{ for } j = 0, 1, 2, \cdots$$

Hence we have

(6.14) $$\boldsymbol{P}\Big(\bigcap_{j=0}^{\infty} F_{l_j}\Big) \geq 1 - \sum_{j=0}^{\infty} l_j^{-p}$$

$$=1-N^{-p}R^{-p}\sum_{j=0}^{\infty}4^{-pj}=1-\frac{K_{p,1}}{N^{p}}$$

where $K_{p,1}\ (E)\ =\ R^{-p}\sum_{j=0}^{\infty}4^{-pj}$. We fix $\omega\in\cap_{j=0}^{\infty}F_{l_j}$. Then there exists a *k-admissible* set $0\in A_j\subset\boldsymbol{Z}^3(E)$ satisfying B_{l_j} and

$$|G_{Q_E(A_j)}(\omega,E+i\varepsilon;x,y)|\leq\exp(-m(E)|x-y|) \tag{6.15}$$

for $|x-y|\geq L(E)l^{\gamma}{}_j$. We put

$$\Lambda=\{x\in\boldsymbol{R}^3\,\|x|>l_0L(E)\}.$$

For $x\in\Lambda$, let j_0 be the smallest natural number satisfying

$$|x|\leq\frac{1}{2}l_{j_0}L(E). \tag{6.16}$$

By (5.1) we have inductively

$$\begin{aligned}G(x,y)&\\ =&G_{Q_E(A_{j_0+1})}(x,y)\\ &+\sum_{n=j_0+1}^{M}(-1)^{n-j_0}\prod_{k=j_0+1}^{n}\Big(\int_{\gamma_k}\partial_{n_{z_k}}G_{Q_E(A_k)}(z_{k-1},z_k)\,dz_k\Big)G_{Q_E(A_{n+1})}(z_n,y)\\ &+(-1)^{M-j_0+1}\prod_{k=j_0+1}^{M+1}\Big(\int_{\gamma_k}\partial_{n_{z_k}}G_{Q_E(A_k)}(z_{k-1},z_k)\,dz_k\Big)G(z_{M+1},y)\end{aligned} \tag{6.17}$$

for $y\in[0,1)^3$, where $\gamma_j=\partial Q_E(A_j)$, $G(u,v)=G(\omega,E+i\varepsilon;u,v)$, $G_\Lambda(u,v)=G_\Lambda(\omega,E+i\varepsilon;u,v)$ and $z_{j_0}=x$. Since

$$|y-x|\geq l_{j_0-1}L(E)-\sqrt{3}\geq l_{j_0+1}^{\gamma}L(E),$$

we have

$$|G_{Q_E(A_{j_0+1})}(x,y)|\leq\exp(-m(E)|x-y|). \tag{6.18}$$

Since from (6.12) it follows for $|u-z_k|\leq 1$

$$\begin{aligned}|z_{k-1}-u|&\geq|z_{k-1}-z_k|-1\\ &\geq\frac{1}{2}l_kL(E)-l_{k-1}L(E)-1\\ &\geq l_{k-1}L(E)-1\geq l^{\gamma}{}_kL(E)\end{aligned}$$

for $k\geq j_0+1$, by Lemma A.5 we have

$$\begin{aligned}&|\partial_{n_{z_k}}G_{Q_E(A_k)}(z_{k-1},z_k)|\\ &\leq\sup_{|u-z_k|\leq1}|G_{Q_E(A_k)}(z_{k-1},u)|\leq\sup_{|u-z_k|\leq1}\exp(-m(E)|z_{k-1}-u|)\\ &\leq\exp(-m(E)(|z_{k-1}-z_k|-1))\leq\exp(-m(E)l_k^{\gamma}L(E))\end{aligned} \tag{6.19}$$

for $k \geq j_0+1$. Similary we have

(6.20) $$|G_{Q_E(A_n)}(z_n, y)| \leq \exp(-m(E)|z_n-y|) \leq \exp(-m(E)|x-y|).$$

By Lemma A.4, we have

(6.21) $$|G(z_{M+1}, y)| \leq \frac{1}{|z_{M+1}-y|} + \frac{c}{\varepsilon} \leq l_{M+1}^{-\gamma} L(E)^{-1} + \frac{c}{\varepsilon}.$$

From (6.19), (6.20) and $|\gamma_k| \leq 2^3 l_k^3 L(E)^2$, we have

(6.22) $$\prod_{k=j_0+1}^{n} \int_{\gamma_k} |\partial n_{z_k} G_{Q_E(A_k)}(z_{k-1}, z_k)| dz_k$$
$$\leq \prod_{k=j_0+1}^{n} 2^3 l_k^3 L(E)^2 \exp(-m(E) l_k^\gamma L(E)) \leq \frac{(L(E)^2)^{n-j_0}}{(n-j_0)!}$$

where we used (6.10). From (6.20) and (6.22) we have

(6.23) $$\left| \sum_{n=j_0+1}^{M} \prod_{k=j_0+1}^{n} \left(\int_{\gamma_k} \partial_{n_{z_k}} G_{Q_E(A_k)}(z_{k-1}, z_k) dz \right) G_{Q_E(A_n)}(z_n, y) \right|$$
$$\leq \exp(-m(E)|x-y|) \sum_{n=j_0+1}^{M} \frac{(L(E)^2)^{n-j_0}}{(n-j_0)!}$$
$$= \exp(-m(E)|x-y|) \sum_{n=1}^{M-j_0} \frac{(L(E)^2)^n}{n!}.$$

From (6.19), (6.21) and (6.10), we have

(6.24) $$\left| \prod_{k=j_0+1}^{M+1} \left(\int_{\gamma_k} \partial_{n_{z_k}} G_{Q_E(A_k)}(z_{k-1}, z_k) dz_k \right) G(z_{M+1}, y) \right|$$
$$\leq \left(\prod_{k=j_0+1}^{M+1} 2^3 l_k^3 L(E)^2 \exp(-m(E) l_k^\gamma L(E)) \right) \left(l_{M+1}^{-\gamma} L(E)^{-1} + \frac{c}{\varepsilon} \right)$$
$$\leq \frac{(L(E)^2)^{M+1-j_0}}{(M+1-j_0)!} \left(l_{M+1}^{-\gamma} L(E)^{-1} + \frac{c}{\varepsilon} \right) \to 0 \text{ as } M \to \infty.$$

From (6.17), (6.18), (6.23) and (6.24), we have

(6.25) $$|G(x, y)| \leq \exp(L(E)^2) \exp(-m(E)|x-y|)$$

for any $x \in \Lambda$ and $y \in [0, 1)^3$.
For $x \in \boldsymbol{R}^3 \backslash \Lambda$ and $y \in [0, 1)^3$, we have from (5.1)

(6.26) $$G(x, y) = G_{Q_E(A_1)}(x, y) + \sum_{n=1}^{M} (-1)^n \prod_{k=1}^{n} \left(\int_{\gamma_k} \partial_{n_{z_k}} G_{Q_E(A_k)}(z_{k-1}, z_k) dz_k \right) G_{Q_E(A_{n-1})}(z_n, y)$$

$$+(-1)^{M+1}\prod_{k=1}^{M+1}\left(\int_{\gamma_k}\partial_{n_{z_k}}G_{Q_E(A_k)}(z_{k-1}, z_k)\,dz_k\right)G(z_{M+1}, y).$$

We put

$$F=\{\omega|\operatorname{dist}(\sigma(H_{Q_E(A_1)}(\omega)), E)\geq \exp(-m(E)NL(E)^2)\}.$$

From Proposition 4.1, we have

$$\boldsymbol{P}(F)\geq 1-c(2NRL(E))^6\exp(-m(E)NL(E)^2) \tag{6.27}$$

where c is independent of $N\in\boldsymbol{N}$ and $0<E\leq E^*$. Since there exist positive numbers δ_1, δ_2 and δ_3 such that

$$D_0\leq m(E)L(E)\leq\delta_1 \tag{6.28}$$

and

$$\delta_2E^{-\frac{1}{2}}\leq L(E)\leq\delta_3E^{-\frac{1}{2}} \tag{6.29}$$

for $E>0$, we have

$$\exp(-m(E)NL(E)^2)\leq\exp(-D_0\delta_2NE^{-\frac{1}{2}}). \tag{6.30}$$

From (6.29) and (6.30) there $N_1\in\boldsymbol{N}$ and $K_{p,2}>0$ such that $N\geq N_1$, then we have

$$\begin{aligned} &c(2NRL(E))^6\exp(-m(E)NL(E)^2)\\ &\leq c(2NR\delta_3E^{-\frac{1}{2}})^6\exp(-D_0\delta_2NE^{-\frac{1}{2}})\\ &\leq\frac{K_{p,2}}{N^p}\end{aligned} \tag{6.31}$$

for any $0<E\leq E^*$. Let $\omega\in F\cap(\cap_{j=0}^{\infty}F_{l_j})$ be fixed. From Lemma A.6, we have

$$|G_{Q_E(A_1)}(x, y)|\leq\frac{1}{|x-y|}+c\ \exp(m(E))NL(E)^2). \tag{6.32}$$

In a similar fashion as in (6.23) and (6.24), we have

$$\begin{aligned}&\left|\sum_{n=1}^{M}\prod_{k=1}^{n}\left(\int_{\gamma_k}\partial_{n_{z_k}}G_{Q_E(A_k)}(z_{k-1}, z_k)\,dz_k\right)G_{Q_E(A_{n+1})}(z_n, y)\right|\\ &\leq\exp(L(E)^2)\exp(-m(E)|x-y|)\end{aligned} \tag{6.33}$$

and

$$\lim_{M\to\infty}\prod_{k=1}^{M+1}\left(\int_{\gamma_k}\partial_{n_{z_k}}G_{Q_E(A_k)}(z_{k-1}, z_k)\,dz_k\right)G(z_{M+1}, y)=0. \tag{6.34}$$

From (6.32), (6.33) and (6.34), we have

$$(6.35)\quad |G(x,y)| \leq \frac{1}{|x-y|} + c\exp(m(E)NL(E)^2) + \exp(L(E)^2 - m(E)|x-y|)$$

for $0<E\leq E^*$ and $N\geq N_1$. There exists $E^*\geq E_2>0$ such that if $0<E\leq E_2$, , then we have

$$(6.36)\quad \frac{L(E)^2}{2} - L(E) > 2R.$$

In the following let $0<E\leq E_2$. There exists $N_2\geq N_1$ such that if $N\geq N_2$, then it follows

$$(6.37)\quad |x-y| \leq NRL(E) + \sqrt{3} \leq 2NRL(E).$$

Hence by (6.36) there exists $N_3\geq N_2$ such that if $N\geq N_3$, then we have

$$(6.38)\quad \begin{aligned} &\exp(m(E)(NL(E)^3 - |x-y|)) \\ &\geq \exp(m(E)(NL(E)^3 - 2NRL(E))) \\ &\geq \exp(D_0 N(L(E)^2 - 2R) \geq 3. \end{aligned}$$

There exists $N_4\geq N_3$ such that if $N\geq N_4$, then

$$(6.39)\quad 3c \leq \frac{\exp\left(D_0 \frac{N}{2} L(E)^2\right)}{2NRL(E)} \leq \frac{\exp\left(\frac{N}{2} m(E) L(E)^3\right)}{2NRL(E)}$$

where c is as in (6.35). Then we have

$$(6.40)\quad \begin{aligned} &3c\exp(m(E)NL(E)^2) \\ &\leq \frac{\exp(m(E)(NL(E)^3 - 2NRL(E)))}{2NRL(E)} \quad \text{(by (154) and (151))} \\ &\leq \frac{\exp(m(E)(NL(E)^3 - |x-y|))}{|x-y|}. \quad \text{(by (152))} \end{aligned}$$

In a similar fashion we have

$$(6.41)\quad 3\exp(L(E)^2 - m(E)|x-y|) \leq \frac{\exp(m(E)(NL(E)^3 - |x-y|))}{|x-y|}$$

for sufficiently large $N>0$. From (6.35), (6.38), (6.40) and (6.41), we have

$$(6.42)\quad |G(x, y)| \leq \frac{\exp(m(E)(NL(E)^3 - |x-y|))}{|x-y|}$$

for any $x\in \boldsymbol{R}^3\backslash \Lambda$ and any $y\in [0, 1)^3$ satisfying $|x-y|\geq 1$ for sufficiently large $N>0$ and $0<E\leq E_2$. From (6.25), (6.42), there exists $N_5>0$ such that if $\omega\in F\cap\left(\cap_{j=0}^{\infty} F_{l_j}\right)$, $0<E\leq E_2$ and $N\geq N_5$, then we have

$$(6.43)\quad |G(x, y)| \leq \exp(m(E)(NL(E)^3 - |x-y|)) \max\left\{1, \frac{1}{|x-y|}\right\}$$

for any $x \in \boldsymbol{R}^3$ and any $y \in [0, 1)^3$. From (6.14), (6.27), (6.31) and (6.43), Theorem 1.3 is proved.

A. Appendix 1

Proof of the last equality of (2.1). Let $\phi \in L^2(\boldsymbol{R}^3)$ and let $X_l(x)$ be the characteristic function of $\{x \in \boldsymbol{R}^3 \mid |x| \leq l\}$ for $l > 0$. We have:

$$\begin{aligned}&(\phi, X_l|x| \int_0^\infty e^{-\varepsilon t} e^{i\lambda t} e^{-itH_\omega} \Psi_\omega dt)\\&= \int_0^\infty e^{-i\lambda t} (\phi, X_l|x| e^{-itH_\omega} \Psi_\omega) dt\end{aligned}$$

where $\Psi_\omega = g_E(H_\omega)\phi(x)$. Therefore by using the Plancherel theorem, we get:

$$\begin{aligned}&\int_0^\infty |(\phi, X_l|x| e^{-\varepsilon t} e^{-itH_\omega} \Psi_\omega)|^2 dt\\&= \frac{1}{2\pi} \int_{-\infty}^\infty |(\phi, X_l|x| \int_0^\infty e^{i\lambda t} e^{-\varepsilon t} e^{-itH_\omega} \Psi_\omega dt)|^2 d\lambda\\&= \frac{1}{2\pi} \int_{-\infty}^\infty |(\phi, X_l|x| R_\omega(\lambda + i\varepsilon) \Psi_\omega)|^2 d\lambda.\end{aligned}$$

Let $\{\phi_n\}_{n=1}^\infty$ be a complete orthonormal system of $L^2(\boldsymbol{R}^3)$. By putting $\phi_n = \phi$ in the above equation and summing up with respect to n, we have

$$\begin{aligned}&\frac{1}{2\pi} \int_{-\infty}^\infty \|X_l|x| R_\omega(\lambda + i\varepsilon) \Psi_\omega\|^2 d\lambda\\&= \int_0^\infty e^{-2\varepsilon t} \|X_l|x| e^{-itH_\omega} \Psi_\omega\|^2 dt.\end{aligned} \tag{A.1}$$

If we let $l \to \infty$ and integrate the both sides of (A.1) with respect to P, then it follows from the monotone convergence theorem and the definition of $r_E^2(t)$ that:

$$\begin{aligned}&\int_0^\infty e^{-2\varepsilon t} r_E^2(t) dt\\&= \frac{1}{2\pi} \int_{-\infty}^\infty \boldsymbol{E}[\||x| R_\omega(\lambda + i\varepsilon) \Psi_\omega\|^2] d\lambda.\end{aligned}$$

Lemma A.1. *Let $V \geq 0$ be a bounded function on $\boldsymbol{R}^3$ and $H = -\Delta + V$ on $L^2(\boldsymbol{R}^3)$. Then there exists a constant $c > 0$ such that for $f \in C_0^\infty(\boldsymbol{R})$ we have*

$$\begin{aligned}&\|f(H)\|_{L_2^2 \to L_2^2}\\&\leq c |\mathrm{supp} f|^{\frac{1}{2}} \left(\|f\|_\infty + \left\| \frac{d^3}{dx^3} f \right\|_\infty + \|h\|_\infty + \left\| \frac{d^3}{dx^3} h \right\|_\infty + \|k\|_\infty + \left\| \frac{d^3}{dx^3} k \right\|_\infty \right),\end{aligned}$$

where $h(x) = xf$ and $k = x^2 f(x)$.

Proof of Lemma. Since

$$f(H)=\langle x\rangle^{-2}\langle x\rangle^{2}f(H)\langle x\rangle^{-2}\langle x\rangle^{2},$$

$\langle x\rangle^{2}$ is unitary operator from L^2_2 to L^2 and $\langle x\rangle^{-2}$ is a unitary operator from L^2 to L^2_2, we have

$$\|f(H)\|_{L^2_2\to L^2_2}=\|\langle x\rangle^{2}f(H)\langle x\rangle^{-2}\|_{L^2\to L^2}. \tag{A.2}$$

Let $g(\lambda)=(1+\lambda)^2 f(\lambda)\in C_0^\infty(\boldsymbol{R})$. We have

$$\begin{aligned}
&\langle x\rangle^{2}f(H)\langle x\rangle^{-2}\\
&=\langle x\rangle^{2}f(H)(H+1)^{2}(H+1)^{-2}\langle x\rangle^{-2}\\
&=\langle x\rangle^{2}g(H)(H+1)^{-2}\langle x\rangle^{-2}\\
&=\frac{1}{\sqrt{2\pi}}\int_{\boldsymbol{R}}\hat{g}(\lambda)\langle x\rangle^{2}e^{i\lambda H}(H+1)^{-2}\langle x\rangle^{-2}d\lambda,
\end{aligned} \tag{A.3}$$

where $\hat{g}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{\boldsymbol{R}}e^{-i\lambda x}g(x)dx$. We have

$$\begin{aligned}
&\langle x\rangle^{2}e^{i\lambda H}(H+1)^{-2}\langle x\rangle^{-2}\\
&=[\langle x\rangle^{2},e^{i\lambda H}](H+1)^{-2}\langle x\rangle^{-2}\\
&+e^{i\lambda H}\langle x\rangle^{2}(H+1)^{-2}\langle x\rangle^{-2},
\end{aligned} \tag{A.4}$$

where [,] is commutator. First we shall estimate the 2nd term of right hand side of (A.4). We shall show that $\langle x\rangle^{2}(H+1)^{-2}\langle x\rangle^{-2}$ is a bounded operator on $L^2(\boldsymbol{R}^3)$. Since we have

$$\begin{aligned}
&[\langle x\rangle^{2},(H+1)^{-1}]\\
&=(H+1)^{-1}[H,x^2](H+1)^{-1}\\
&=6(H+1)^{-2}-4(H+1)^{-1}\nabla\cdot x(H+1)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
&[x,(H+1)^{-1}]=(H+1)^{-1}[H,x](H+1)^{-1}\\
&=-2(H+1)^{-1}\nabla(H+1)^{-1},
\end{aligned}$$

it follows immediately that $\langle x\rangle^{2}(H+1)^{-2}\langle x\rangle^{-2}$ is a bounded operator on $L^2(\boldsymbol{R}^3)$. Next we shall show that

$$\|[\langle x\rangle^{2},e^{i\lambda H}](H+1)^{-2}\langle x\rangle^{-2}\|\le c(1+\lambda^2). \tag{A.5}$$

We have

$$\begin{aligned}
&[\langle x\rangle^{2},e^{i\lambda H}]\\
&=e^{i\lambda H}(e^{-i\lambda H}\langle x\rangle^{2}e^{i\lambda H}-\langle x\rangle^{2})\\
&=ie^{i\lambda H}\int_0^\lambda e^{-i\mu H}[x^2,H]e^{i\mu H}d\mu
\end{aligned} \tag{A.6}$$

$$=ie^{i\lambda H}\int_0^{\lambda}\Big(-6+4e^{-i\mu H}\nabla\cdot xe^{i\mu H}\Big)d\mu$$

and

(A.7)
$$\begin{aligned}&e^{-i\mu H}\nabla\cdot xe^{i\mu H}(H+1)^{-2}\langle x\rangle^{-2}\\ &\quad=e^{-i\mu H}\nabla\cdot x(H+1)^{-1}e^{i\mu H}(H+1)^{-1}\langle x\rangle^{-2}\\ &\quad=e^{-i\mu H}\nabla\cdot([x,(H+1)^{-1}]+(H+1)^{-1}x)\\ &\qquad\times e^{i\mu H}(H+1)^{-1}\langle x\rangle^{-2}.\end{aligned}$$

Since $[x,(H+1)^{-1}]=(H+1)^{-1}(-2\nabla)(H+1)^{-1}$, it suffices to show that

(A.8)
$$\|xe^{i\mu H}(H+1)^{-1}\langle x\rangle^{-2}\|\le c(1+\mu).$$

Since

$$\begin{aligned}&[x,e^{i\mu H}]\\ &\quad=ie^{i\mu H}\int_0^{\mu}e^{-i\tau H}[x,H]e^{i\tau H}d\tau\end{aligned}$$

and $[x,H]=2\nabla$, we have

(A.9)
$$\|[x,e^{i\mu H}](H+1)^{-1}\|\le c\mu.$$

Since

$$\begin{aligned}&e^{i\mu H}x(H+1)^{-1}\langle x\rangle^{-2}\\ &\quad=e^{i\mu H}\Big([x,(H+1)^{-1}]+(H+1)^{-1}x\Big)\langle x\rangle^{-2},\end{aligned}$$

we have $\|e^{i\mu H}x(H+1)^{-1}\langle x\rangle^{-2}\|\le c$. Then (A.8) is shown. From (A.6), (A.7) and (A.8), we have (A.5). Therefore we have

(A.10)
$$\begin{aligned}&\|\langle x\rangle^2 f(H)\langle x\rangle^{-2}\|_{L^2\to L^2}\\ &\quad\le c\int_R|\hat{g}(\lambda)|(1+\lambda^2)d\lambda\\ &\quad=c\Big(\int_R\frac{1}{1+\lambda^2}d\lambda\Big)^{\frac12}\Big(\int_R|\hat{g}(\lambda)|^2(1+\lambda^2)^3d\lambda\Big)^{\frac12}\\ &\quad\le c\Big(\int_R\frac{1}{1+\lambda^2}d\lambda\Big)^{\frac12}\Big(\int_R|\hat{g}(\lambda)|^2(1+\lambda^6)d\lambda\Big)^{\frac12}.\end{aligned}$$

We have

(A.11)
$$\begin{aligned}&\int_R|\hat{g}(\lambda)|^2d\lambda\\ &\quad=\int_R|g(x)|^2dx=\int_R|f(x)(1+x)^2|^2dx\\ &\quad\le c|\mathrm{supp}f|(|f|_\infty+|h|_\infty+|k|_\infty)^2\end{aligned}$$

and

$$\int_R|\hat{g}(\lambda)\lambda^3|^2d\lambda$$

$$= \int_{\boldsymbol{R}} \left| \left(\frac{d}{dx} \right)^3 f(x) \, (1+x)^2 \right|^2 dx$$

$$\leq c |\mathrm{supp} f| \left(\left\| \left(\frac{d}{dx} \right)^3 f \right\|_\infty + \left\| \left(\frac{d}{dx} \right)^3 h \right\|_\infty + \left\| \left(\frac{d}{dx} \right)^3 k \right\|_\infty \right)^2.$$

Then we have proved Lemma A.1.

Lemma A.2. *Let $V \geq 0$ be bounded function on $\boldsymbol{R}^3$ and $H = -\Delta + V$ on $L^2(\boldsymbol{R}^3)$. If $f \in C_0^\infty(\boldsymbol{R})$, then $f(H)$ is a bounded operator from L_2^2 to L_2^∞.*

Proof. Noting that $H^2(\boldsymbol{R}^3) \subset L^\infty(\boldsymbol{R}^3)$, for $u \in L_2^2(\boldsymbol{R}^3)$ we have

$$\begin{aligned} \text{(A.12)} \quad & \| \langle x \rangle^2 f(H) u \|_{L^\infty} \\ & \leq c \| (-\Delta + 1) \langle x \rangle^2 f(H) u \|_{L^2} \\ & \leq c \| (H+1) \langle x \rangle^2 f(H) \langle x \rangle^{-2} \langle x \rangle^2 u \|_{L^2} + c \| V \langle x \rangle^2 f(H) \langle x \rangle^{-2} \langle x \rangle^2 u \|_{L^2}. \end{aligned}$$

Since it follows from Lemma A.1, $V \langle x \rangle^2 f(H) \langle x \rangle^{-2}$ is a bounded operator on L^2, we have only to show that $(H+1) \langle x \rangle^2 f(H) \langle x \rangle^{-2}$ is a bounded operator on $L^2(\boldsymbol{R}^3)$. From Lemma A.1, we have that $\langle x \rangle^2 (H+1) f(H) \langle x \rangle^{-2}$ is a bounded operator on $L^2(\boldsymbol{R}^3)$. It is sufficient to show that $[H, \langle x \rangle^2] f(H) \langle x \rangle^{-2}$ is a bounded operator on $L^2(\boldsymbol{R}^3)$. Noting that $[H, \langle x \rangle^2] = 6 - 4\Delta \cdot x$, we have only to study $\nabla \cdot x f(H) \langle x \rangle^{-2}$. We have

$$\nabla \cdot x f(H) \langle x \rangle^{-2} = \nabla f(H) x \langle x \rangle^{-2} + \nabla [x, f(H)] \langle x \rangle^{-2}.$$

Let $f(H) = (H+1)^{-1} g(H)$, we have

$$[x, f(H)] = (H+1)^{-1} [H, x] (H+1)^{-1} g(H) + (H+1)^{-1} (x g(H) - g(H) x)$$

and $x g(H) \langle x \rangle^{-2}$ is a bounded operator on $L^2(\boldsymbol{R}^3)$. Hence $\nabla \cdot x f(H) \langle x \rangle^{-2}$ is bounded operator on $L^2(\boldsymbol{R}^3)$.

Lemma A.3. *Let $\Omega \subset \boldsymbol{R}^3$ be a domain and the let $0 \leq V$ be a bounded function on $\boldsymbol{R}^3$. Let $H^D = -\Delta + V$ with Dirichlet boundary conditions on $L^2(\Omega)$. If* $\inf \sigma(H^D) \geq 2E > 0$, *then we have*

$$|(H^D - E - i\varepsilon)^{-1}(x, y)| \leq 5 \exp\left(-\frac{\sqrt{E}}{4} |x - y| \right)$$

for $x, y \in \Omega$, $|x - y| \geq 1$ and $E \geq |\varepsilon|$. Here $(H^D - E - i\varepsilon)^{-1}(x, y)$ is the Green function of $H^D - E - i\varepsilon$.

Proof. Using the resolvent equation twice, we get

$$\begin{aligned} \text{(A.13)} \quad & (H^D - E - i\varepsilon)^{-1} \\ & = (H^D + E + i\varepsilon)^{-1} + 2(E + i\varepsilon)(H^D + E + i\varepsilon)^{-1}(H^D - E - i\varepsilon)^{-1} \\ & = (H^D + E + i\varepsilon)^{-1} + 2(E + i\varepsilon)(H^D + E + i\varepsilon)^{-2} \\ & \quad + 4(E + i\varepsilon)^2 (H^D + E + i\varepsilon)^{-1} (H^D - E - i\varepsilon)^{-1} (H^D + E + i\varepsilon)^{-1}. \end{aligned}$$

First we shall estimate the first and second terms of (A.13). We have

$$|(H^{D}+E+i\varepsilon)^{-1}(x,y)|\leq\frac{\exp(-\sqrt{E}|x-y|)}{4\pi|x-y|} \tag{A.14}$$

and

$$|(H^{D}+E+i\varepsilon)^{-2}(x,y)|\leq\frac{\exp\left(-\sqrt{\frac{E}{2}}|x-y|\right)}{2\pi E|x-y|}. \tag{A.15}$$

In fact by Feynman-Kac formura, we have

$$0\leq\exp\left(-tH^{D}\right)(x,y)$$

$$\leq\exp(-tH_{0}^{D})(x,y)\leq\exp(-tH_{0})(x,y)=\frac{\exp\left(-\frac{|x-y|^{2}}{4t}\right)}{(4\pi t)^{\frac{3}{2}}}.$$

Here $H_{0}=-\Delta$ on $L^{2}(\boldsymbol{R}^{3})$. Therefore it follows that

$$|(H^{D}+E+i\varepsilon)^{-1}(x,y)|\leq\int_{0}^{\infty}\exp(-Et)\exp(-tH^{D})(x,y)dt$$

$$\leq\int_{0}^{\infty}\exp(-Et)\frac{\exp\left(-\frac{|x-y|^{2}}{4t}\right)}{(4\pi t)^{\frac{3}{2}}}dt$$

$$=(H_{0}+E)^{-1}(x,y)=\frac{\exp(-\sqrt{E}|x-y|)}{4\pi|x-y|}.$$

and

$$|(H^{D}+E+i\varepsilon)^{-2}(x,y)|\leq\int_{0}^{\infty}t\exp(-Et)\exp(-tH^{D})(x,y)dt$$

$$\leq\int_{0}^{\infty}t\exp(-Et)\frac{\exp\left(-\frac{|x-y|^{2}}{4t}\right)}{(4\pi t)^{\frac{3}{2}}}dt$$

$$\leq\frac{2}{E}\left(H_{0}+\frac{E}{2}\right)^{-1}(x,y)=\frac{\exp\left(-\sqrt{\frac{E}{2}}|x-y|\right)}{2\pi E|x-y|}.$$

since $t\exp\left(-\frac{E}{2}t\right)\leq\frac{2}{E}$. Thus we have (A.14) and (A.15).

Next we shall estimate the third term of (A.13). Let Ψ be a bounded and C^{∞}-function such that $|\nabla\Psi|\leq 1$ and $(\partial/\partial x)^{\beta}\Psi$ are bounded for all multi-index $|\beta|\leq 2$ and let $\alpha\in\boldsymbol{C}$. Noting that $\exp(\alpha\Psi)$ is bounded, we estimate the norm of the following operator:

$$e^{-\alpha\Psi}(H^{D}+E+i\varepsilon)^{-1}e^{\alpha\Psi}e^{-\alpha\Psi}(H^{D}-E-i\varepsilon)^{-1}e^{\alpha\Psi}e^{-\alpha\Psi}(H^{D}+E+i\varepsilon)^{-1}e^{\alpha\Psi}: L^{1}(\Omega)\rightarrow L^{\infty}(\Omega)$$

Since $|\nabla \Psi| \leq 1$, it follows that $|\Psi(x) - \Psi(y)| \leq |x-y|$. Then by (A.14), we have

$$(A.16)\qquad \begin{aligned} &|e^{-\alpha\Psi}(H^D+E+i\varepsilon)^{-1}e^{\alpha\Psi}(x, y)| \\ &\leq \frac{\exp(-\Re(\alpha(\Psi(x)-\Psi(y))))\exp(-\sqrt{E}|x-y|)}{4\pi|x-y|} \\ &\leq \frac{\exp(-(\sqrt{E}-|\alpha|)|x-y|)}{4\pi|x-y|} \\ &\leq \frac{\exp\left(-\frac{\sqrt{E}}{2}|x-y|\right)}{4\pi|x-y|} \equiv G(x-y) \end{aligned}$$

if $|\alpha| \leq \frac{\sqrt{E}}{2}$. We have

$$(A.17)\qquad \|e^{-\alpha\Psi}(H^D+E+i\varepsilon)^{-1}e^{\alpha\Psi}\|_{L^1(\Omega)\to L^2(\Omega)} \leq \|G\|_{L^2} = (4\pi)^{-\frac{1}{2}}E^{-\frac{1}{4}}$$

and

$$(A.18)\qquad \|e^{-\alpha\Psi}(H^D+E+i\varepsilon)^{-1}e^{\alpha\Psi}\|_{L^2(\Omega)\to L^\infty(\Omega)} \leq \|G\|_{L^2} = (4\pi)^{-\frac{1}{2}}E^{-\frac{1}{4}}.$$

Next we estimate the norm of the following operator:

$$e^{-\alpha\Psi}(H^D-E-i\varepsilon)^{-1}e^{\alpha\Psi} : L^2(\Omega) \to L^2(\Omega).$$

Noting that the operator $e^{-\alpha\Psi}$ is bijective and bounded on $Dom(H^D) = H_0^1(\Omega) \cap H^2(\Omega)$, for $u \in Dom(H^D)$ we have

$$(A.19)\qquad \begin{aligned} &\|(e^{-\alpha\Psi}H^De^{\alpha\Psi}-E-i\varepsilon)u\|\|u\| \\ &\geq |((e^{-\alpha\Psi}H^De^{\alpha\Psi}-E-i\varepsilon)u,u)| \geq \Re\left((e^{-\alpha\Psi}H^De^{\alpha\Psi}-E-i\varepsilon)u, u\right) \\ &= \Re((\nabla e^{\alpha\Psi}u, \nabla e^{-\alpha\Psi}u) + (Vu, u) - (E-\varepsilon i)\|u\|^2) \end{aligned}$$

Here it follows

$$(A.20)\qquad \begin{aligned} &\left(\nabla e^{\alpha\Psi}u, \nabla e^{-\alpha\Psi}u\right) \\ &= (\nabla u, \nabla u) + (\alpha(\nabla \Psi)u, -\alpha(\nabla \Psi)u) \\ &+ \{(\alpha(\nabla \Psi)u, \nabla u) + (\nabla u, -\alpha(\nabla \Psi)u)\} \end{aligned}$$

and

$$(\alpha(\nabla \Psi)u, -\alpha(\nabla \Psi)u) = -|\alpha|^2\||\nabla \Psi|u\|^2 \geq -|\alpha|^2\|u\|^2.$$

Since the third term of the right hand side of (A.20) is pure imaginary, the last member of (A.19) is bounded from below by

$$(A.21)\qquad \begin{aligned} &(\nabla u, \nabla u) + (Vu, u) - |\alpha|^2\|u\|^2 - E\|u\|^2 \\ &\geq (\inf\sigma(H^D) - |\alpha|^2 - E)\|u\|^2 \\ &\geq (E-|\alpha|^2)\|u\|^2. \end{aligned}$$

Therefore if $|\alpha|<\sqrt{\frac{E}{2}}$, we have

$$\|e^{-\alpha\Psi}(H^D-E-i\varepsilon)e^{\alpha\Psi}u\|\geq\frac{E}{2}\|u\|$$

and the operator $e^{-\alpha\Psi}(H^D-E-i\varepsilon)e^{\alpha\Psi}$ is surjective on $L^2(\Omega)$. Then

$$\|e^{-\alpha\Psi}(H^D-E-i\varepsilon)^{-1}e^{\alpha\Psi}\|_{L^2(\Omega)\to L^2(\Omega)}\leq\frac{2}{E}. \tag{A.22}$$

Hence from (A.17), (A.18) and (A.22), it follows

$$\|e^{-\alpha\Psi}(H^D+E+i\varepsilon)^{-1}(H^D-E-i\varepsilon)^{-1}(H^D+E+i\varepsilon)^{-1}e^{\alpha\Psi}\|_{L^1(\Omega)\to L^\infty(\Omega)}\leq\frac{1}{2\pi E^{\frac{3}{2}}}.$$

From this we have

$$|e^{-\alpha(\Psi(x)-\Psi(y))}(H^D+E+i\varepsilon)^{-1}(H^D-E-i\varepsilon)^{-1}(H^D+E+i\varepsilon)^{-1}(x,y)|\leq\frac{1}{2\pi E^{\frac{3}{2}}}$$

and then

$$\begin{aligned}&|(H^D+E+i\varepsilon)^{-1}(H^D-E-i\varepsilon)^{-1}(H^D+E+i\varepsilon)^{-1}(x,y)|\\&\leq\frac{1}{2\pi E^{\frac{3}{2}}}\exp\Re(\alpha(\Psi(x)-\Psi(y)))\end{aligned}$$

for any $\alpha\in\boldsymbol{C}$ such that $|\alpha|<\frac{\sqrt{E}}{2}$ and any bounded function $\Psi\in C^\infty(\boldsymbol{R})$, $|\nabla\Psi|\leq1$ and $(\partial/\partial x)^\beta\Psi$ are bounded for all multi-index $|\beta|\leq2$. Therefore since for fixed x and y we have

$$\inf_{\alpha,\Psi}\exp\Re(\alpha(\Psi(x)-\Psi(y)))=\exp\left(-\frac{\sqrt{E}}{2}|x-y|\right),$$

we have

(A.23)

$$|(H^D+E+i\varepsilon)^{-1}(H^D-E-i\varepsilon)^{-1}(H^D+E+i\varepsilon)^{-1}(x,y)\Big|\leq\frac{1}{2\pi E^{\frac{3}{2}}}\exp\left(-\frac{\sqrt{E}}{2}|x-y|\right).$$

From (A.13), (A.14), (A15) and (A.23), we obtain

$$|(H^D-E-i\varepsilon)^{-1}(x,y)|\leq5\exp\left(-\frac{\sqrt{E}}{4}|x-y|\right) \tag{A.24}$$

for any x and y such that $|x-y|\geq1$ and $0<\varepsilon\leq E$. We have thus proved the lemma.

Lemma A.4. *Let E^* and $\bar{E}$ be two positive numbers such that $\bar{E}\leq E^*$. For any $z\in\boldsymbol{C}$ such that* $\mathrm{Re}(z)\in[\bar{E},E^*]$, $\mathrm{Im}(z)\neq0$ *and* $|\mathrm{Im}(z)|<1$ *it follows that*

$$|(H_\omega - z)^{-1}(x, y)| \leq \frac{1}{|x-y|} + c\frac{1}{|\mathrm{Im}(z)|}$$

where c is independent of z and ω.

Proof. As is shown in the proof of Lemma A.3, we have

$$(A.25) \qquad |(H_\omega - w)^{-1}(x, y)| \leq \frac{\exp(-\sqrt{|w|}|x-y|)}{4\pi|x-y|}$$

and

$$(A.26) \qquad |(H_\omega - w)^{-2}(x, y)| \leq \frac{\exp\left(-\sqrt{\frac{|w|}{2}}|x-y|\right)}{2\pi|w||x-y|}$$

for $w<0$. From (A.24) and (A.25) in a similar fashion to that used in the proof of Lemma A.3 we have

$$(A.27) \qquad |(H_\omega - w)^{-1}(H_\omega - z)^{-1}(H_\omega - w)^{-1}(x, y)| \leq \frac{1}{\sqrt{|w|}|\mathrm{Im}(z)|}.$$

Using the resolvent equation twice, we get

$$\begin{aligned}(A.28) \qquad & (H_\omega - z)^{-1} \\ &= (H_\omega - w)^{-1} + (z-w)(H_\omega - w)^{-1}(H_\omega - z)^{-1} \\ &= (H_\omega - w)^{-1} + (z-w)(H_\omega - w)^{-2} \\ &\quad + (z-w)^2(H_\omega - w)^{-2}(H_\omega - z)^{-1}.\end{aligned}$$

From (A.24)-(A.27), we have

$$|(H_\omega - z)^{-1}(x, y)| \leq \frac{1}{|x-y|} + c\frac{1}{|\mathrm{Im}(z)|}.$$

Lemma A.5. *Let $v \in \partial\Lambda$ be not one of the corners. Then*

$$|\partial_{nv} G_\Lambda(E+i\varepsilon; u, v)| \leq c_3 \sup_{|v'-v|\leq 1} G_\Lambda(E+i\varepsilon; u, v')|$$

for any u such that $|u-v| \geq 1$. Here c_3 is independent of Λ, E and ε.

Proof. This lemma has be shown in [5] (Lemma 3.1).

Lemma A.6. *Let $\Lambda \subset \boldsymbol{R}^3$, $0 \leq V$ be a bounded function on $\boldsymbol{R}^3$ and $H = -\Delta + V$. Let $H_\Lambda = H|_{L^2(\Lambda)}$ with Dirichlet boundary conditions on $L^2(\Lambda)$. If $u, w \in \Lambda$, then it follows*

$$|G_\Lambda(E+i\varepsilon; u, v)| \leq \frac{1}{|u-v|} + \frac{c_4}{\mathrm{dist}(\sigma(H_\Lambda), E+i\varepsilon)}$$

where c_4 is independent of Λ, u, v, E and ε.

Proof. This lemma is shown a fashion similar to that used in the proof of

Lemma A.4.

B. Appendix 2

Lemma B.1. *There exists $E''>0$ such that for $0<E\leq E''$ it follows that if D_1, $D_2\subset \boldsymbol{Z}^3(E)$, $D_1\subset D_2$ and* $\mathrm{dist}_E(D_1, D^c{}_2)\geq 12d_k$, *then there exists a k–admissible set A such that $D_1\subset A\subset D_2$.*

Proof. We denote by P_k in the following assertion:
If $D_1\subset D_2\subset \boldsymbol{Z}^3(E)$ and $\mathrm{dist}_E(D_1, D_2^c)\geq 12d_k$, then there exists a *k-admissible* set A such that $D_1\subset A\subset D_2$.

We shall prove P_k for $k\geq 0$ by induction.

Step 1. *Proof of P_0.*
We have only to show the case that there exists a component D_0^κ such that

$$D_1\cap W(D_0^\kappa, 4d_0)\neq\emptyset.$$

Let

$$K=\{\kappa|W(D_0^\kappa, 4d_0)\cap D_1)\neq\emptyset\}$$

and

$$A=D_1\cup\cup_{\kappa\in K}W(D_0^\kappa, 4d_0+1).$$

Then A is *0-admissible* and $D_1\subset A\subset D_2$ by Condition $A(0)$.

Step 2. *Proof of P_{k+1} under the assumption of P_k.*
By the assumption of P_k, there exists *k-admissible* set A such that $D_1\subset A\subset W(D_1, 12d_k)$. Let

$$K=\{\kappa|A\cap W(D_{k+1}^\kappa, 4d_{k+1})\neq\emptyset\}$$

For $\kappa\in K$, by the assumption of P_k, there exists *k-admissible* set A^κ such that

$$W(D_{k+1}^\kappa, 4d_{k+1}+1)\subset A^\kappa\subset W(D_{k+1}^\kappa, 4d_{k+1}+12d_k).$$

Let $A'=A\cup\cup_{\kappa\in K}A^\kappa$. Then by Condition $A(k+1)$, A' satisfies the assertion of P_{k+1}.

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