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Convergence of non-symmetric forms

By

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1. Introduction

Study on convergence of bilinear forms on a Hilbert space goes back to 1950's. Even of late years, some fundamental results have been obtained. For example, in the case of symmetric forms, it was shown that the strong convergence of associated resolvent operators is equivalent to so-called the Mosco convergence of forms (see e.g. [9]). This seems like a useful criterion in application; Kuwae and Uemura [4,5] recently developed theory of weak convergence of diffusion processes associated with Dirichlet forms with the aid of the Mosco convergence, which generalizes former results by using, for instance, the monotone convergence theorem. In the case of non-symmetric coercive closed forms, Röckner and Zhang [11] obtained strong convergence of resolvents on $L^2(\boldsymbol{R}^n)$ under weak convergences of coefficients by a purely analytical method, extending Stroock's results [13] based on detailed estimates of the transition densities of the corresponding semigroups.

In this paper, we apply Röckner and Zhang's argument to more general forms in an abstract setting, and give necessary and sufficient conditions for strong convergence of associated resolvents. We might say that these conditions are variants of the Mosco convergence. The forms we treat are the sum of a coerive closed form (in a wide sense) and a perturbation part induced by a linear operator generating a semigroup of good properties. The class of these forms includes both elliptic cases (coercive closed forms) and parabolic cases (time dependent forms). This framework is borrowed from Stannat's paper [12], in which he discussed existence of Markov processes associated with a little more conditioned forms which were called *generalized Dirichlet forms*, including time dependent processes as examples. We hope that our results will be connected with study of these types of Markov processes.

The organization of this paper is as follows : in the section 2, we set up a framework and prove preliminary lemmas. In the section 3, criteria for convergence are given. In the last section, we give a few examples.

2. Framework

In this section, we follow Section 2 of [12] for the framework, with a little

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modification to fit our context. Let $\mathscr{H}$ be a real Hilbert space with its inner product $(\cdot,\cdot)$ and norm $|\cdot|=(\cdot,\cdot)^{1/2}$. Let $\mathscr{A}$ be a bilinear form on $\mathscr{H}$ with a domain $\mathscr{V}$. $\mathscr{V}$ is not necessarily dense in $\mathscr{H}$. The symmetric part $\tilde{\mathscr{A}}$ of $\mathscr{A}$ is defined by

$$\tilde{\mathscr{A}}(u,v):=\frac{1}{2}\{\mathscr{A}(u,v)+\mathscr{A}(v,u)\}, \qquad u,v\in\mathscr{V}.$$

For $\alpha\in\boldsymbol{R}$, set $\mathscr{A}_\alpha(u,v)=\mathscr{A}(u,v)+\alpha(u,v)$. $\tilde{\mathscr{A}}_\alpha$ is similarly defined. We suppose that $(\mathscr{A},\mathscr{V})$ is a coercive closed form in a wide sense, that is, for some bound constant $\lambda\in\boldsymbol{R}$,

- $(\tilde{\mathscr{A}}_\lambda,\mathscr{V})$ is a nonnegative definite closed form,

- $(\mathscr{A}_\lambda,\mathscr{V})$ satisfies the weak sector condition: there exists a sector constant $K\geq 1$ such that

$$|\mathscr{A}_{\lambda+1}(u,v)|\leq K\mathscr{A}_{\lambda+1}(u,u)^{1/2}\mathscr{A}_{\lambda+1}(v,v)^{1/2} \qquad \text{for all } u,v\in\mathscr{V}.$$

Equipping with the norm $\|\cdot\|_{\mathscr{V}}=\tilde{\mathscr{A}}_{\lambda+1}(\cdot,\cdot)^{1/2}$, $\mathscr{V}$ becomes a Hilbert space. We denote by $\mathscr{H}^0$ the closure of $\mathscr{V}$ in $\mathscr{H}$, and by P the orthogonal projection from $\mathscr{H}$ to $\mathscr{H}^0$. Let $\mathscr{V}^*$ be the topological dual of $\mathscr{V}$. The identification of $\mathscr{H}^0$ with its dual induces the dense and continuous embedding $\mathscr{V}\subset\mathscr{H}^0\subset\mathscr{V}^*$. The pairing between $\mathscr{V}$ and $\mathscr{V}^*$ is expressed by $(\cdot,\cdot)$, the same notation as the inner product of $\mathscr{H}$.

Let Λ be a linear operator on $\mathscr{V}^*$ with a domain $D(\Lambda,\mathscr{V}^*)$. We assume the following:

- Λ generates a strongly continuous semigroup $\{U_t\}$ on $\mathscr{V}^*$.

- The restriction of $\{U_t\}$ to $\mathscr{V}$ (resp. $\mathscr{H}^0$) is a strongly continuous semigroup on $\mathscr{V}$ (resp. a strongly continuous *contraction* semigroup on $\mathscr{H}^0$).

The domain of the generator of $\{U_t|_{\mathscr{V}}\}$ is denoted by $D(\Lambda,\mathscr{V})$. Note that the adjoint operator $(\hat{\Lambda},D(\hat{\Lambda},\mathscr{V}^*))$ of $(\Lambda,D(\Lambda,\mathscr{V}))$ also satisfies the conditions above.

Set Hilbert spaces $\mathscr{F}=\mathscr{V}\cap D(\Lambda,\mathscr{V}^*)$ with norm $\|\cdot\|_{\mathscr{F}}=(\|\cdot\|_{\mathscr{V}}^2+\|\Lambda\cdot\|_{\mathscr{V}^*}^2)^{1/2}$, and $\hat{\mathscr{F}}=\mathscr{V}\cap D(\hat{\Lambda},\mathscr{V}^*)$ with $\|\cdot\|_{\hat{\mathscr{F}}}=(\|\cdot\|_{\mathscr{V}}^2+\|\hat{\Lambda}\cdot\|_{\mathscr{V}^*}^2)^{1/2}$. It holds that $\mathscr{F}$ and $\hat{\mathscr{F}}$ are dense in $\mathscr{V}$, $(\Lambda u,u)\leq 0$ for $u\in\mathscr{F}$ and $(u,\hat{\Lambda}u)\leq 0$ for $u\in\hat{\mathscr{F}}$ (cf. [12, Remark 2.1]).

For given $\mathscr{A}$ and Λ, we define a corresponding form $\mathscr{E}$ on $\mathscr{H}$ by

$$\mathscr{E}(u,v)=\begin{cases}\mathscr{A}(u,v)-(\Lambda u,v) & u\in\mathscr{F},\ v\in\mathscr{V}\\ \mathscr{A}(u,v)-(u,\hat{\Lambda}v) & u\in\mathscr{V},\ v\in\hat{\mathscr{F}}\\ \infty & \text{otherwise.}\end{cases}$$

It should be mentioned that we *define* $\mathscr{E}(u,v)=\infty$ for $u\notin\mathscr{V}$, even if $v=0$.

As usual, we define $\mathscr{E}_\alpha(u,v)=\mathscr{E}(u,v)+\alpha(u,v)$ for $\alpha\in\boldsymbol{R}$. $\mathscr{E}$ has an associated form $\mathscr{E}'$ on $\mathscr{H}^0$ which is obtained by the natural restriction of $\mathscr{E}$ to $\mathscr{H}^0$.

Proposition 2.1. *For all $\alpha>\lambda$, there exist unique linear continuous bijections $W_\alpha:\mathscr{V}^*\to\mathscr{F}$ and $\hat{W}_\alpha:\mathscr{V}^*\to\hat{\mathscr{F}}$ such that*

$$\mathscr{E}_\alpha(W_\alpha f,u)=\mathscr{E}_\alpha(u,\hat{W}_\alpha f)=(f,u),\qquad f\in\mathscr{V}^*,\ u\in\mathscr{V}.$$

Further, there exist unique, not necessarily strongly continuous resolvent $(G_\alpha)_{\alpha>\lambda}$ and coresolvent $(\hat{G}_\alpha)_{\alpha>\lambda}$ on $\mathscr{H}$ such that

$$G_\alpha(\mathscr{H})\subset\mathscr{F},\quad \hat{G}_\alpha(\mathscr{H})\subset\hat{\mathscr{F}},$$

$$\mathscr{E}_\alpha(G_\alpha f,u)=\mathscr{E}_\alpha(u,\hat{G}_\alpha f)=(f,u)\qquad \text{for all } f\in\mathscr{H},\ u\in\mathscr{V},\ \alpha>\lambda.$$

Besides, $\hat{G}_\alpha$ is adjoint of G_α, and $(\alpha-\lambda)G_\alpha$, $(\alpha-\lambda)\hat{G}_\alpha$ are contraction operators. Also, it holds that

$$\operatorname*{s\text{-}lim}_{\alpha\to\infty}(\alpha-\lambda)G_\alpha u=Pu\qquad \text{for } u\in\mathscr{H}.$$

Proof. See [6, Chapter III, Theorem 1.1] for the first assertion. Uniqueness of G_α and $\hat{G}_\alpha$ is proved by a standard argument. When $\mathscr{H}=\mathscr{H}^0$, we refer to [12, Proposition 2.6] for the proof of the remaining assertions. To treat the general case, let G'_α and $\hat{G}'_\alpha$ be the resolvent and the coresolvent of $\mathscr{E}'$, respectively. Then it is easy to check that $G'_\alpha P$ and $\hat{G}'_\alpha P$ are what are wanted as G_α and $\hat{G}_\alpha$, respectively.

Let $\Phi(u)=\sup_{\|w\|_{\mathscr{V}}=1}\mathscr{E}_{\lambda+1}(w,u)$ for $u\in\hat{\mathscr{F}}$.

Lemma 2.2. *For $u\in\hat{\mathscr{F}}$, the following hold.*

(i) $\Phi(u)\le\sqrt{2}K\|u\|_{\hat{\mathscr{F}}}$,

(ii) $\|u\|_{\mathscr{V}}\le\Phi(u)$,

(iii) $\|\hat{\Lambda}u\|_{\mathscr{V}^*}\le(K+1)\Phi(u)$.

Proof. (i): By definition,

$$\begin{aligned}\Phi(u)&\le\sup_{\|w\|_{\mathscr{V}}=1}|\mathscr{A}_{\lambda+1}(w,u)|+\sup_{\|w\|_{\mathscr{V}}=1}|(w,\hat{\Lambda}u)|\\&\le K\|u\|_{\mathscr{V}}+\|\hat{\Lambda}u\|_{\mathscr{V}^*}\le\sqrt{2}K\|u\|_{\hat{\mathscr{F}}}.\end{aligned}$$

(ii): $\|u\|^2_{\mathscr{V}}\le\mathscr{E}_{\lambda+1}(u,u)\le\|u\|_{\mathscr{V}}\Phi(u)$.

(iii): For $w\in\mathscr{V}$,

$$\begin{aligned}(w, \hat{\Lambda}u) &= \mathscr{A}_{\lambda+1}(w, u) - \mathscr{E}_{\lambda+1}(w, u) \\ &\leq K\|w\|_{\mathscr{V}}\|u\|_{\mathscr{V}} + \|w\|_{\mathscr{V}}\Phi(u) \\ &\leq (K+1)\|w\|_{\mathscr{V}}\Phi(u) \qquad \text{by (ii).}\end{aligned}$$

Hence $\|\hat{\Lambda}u\|_{\mathscr{V}^*} \leq (K+1)\Phi(u)$.

From this lemma, $\Phi(\cdot)$ defines a norm on $\hat{\mathscr{F}}$ which is equivalent to $\|\cdot\|_{\hat{\mathscr{F}}}$. We set $\Phi(u)=\infty$ for $u \notin \hat{\mathscr{F}}$ for convenience' sake. We also adopt a convention that for each norm appeared in this section, the norm of elements which are not in the domain is ∞.

Next we define approximate forms $\mathscr{E}^{(\beta)}$, $\beta > \lambda$ of $\mathscr{E}$ by

$$\mathscr{E}^{(\beta)}(u, v) = (\beta-\lambda)(u-(\beta-\lambda)G_\beta u, v) - \lambda(u, v), \qquad u, v \in \mathscr{H},$$

and set $\mathscr{E}_\alpha^{(\beta)}(u, v) = \mathscr{E}^{(\beta)}(u, v) + \alpha(u, v)$.

Proposition 2.3. (i) $\mathscr{E}_\lambda^{(\beta)}(u, v) = \mathscr{E}_\lambda((\beta-\lambda)G_\beta u, v)$ *for* $u \in \mathscr{H}$, $v \in \mathscr{V}$.

(ii) $\mathscr{E}_\lambda^{(\beta)}(u, u) = \mathscr{E}_\lambda((\beta-\lambda)G_\beta u,\ (\beta-\lambda)G_\beta u) + (\beta-\lambda)|u-(\beta-\lambda)G_\beta u|^2$ *for* $u \in \mathscr{H}$.

(iii) $\lim_{\beta\to\infty} \mathscr{E}_\lambda^{(\beta)}(u, v) = \mathscr{E}_\lambda(u, v)$ *for* $u \in \mathscr{F}$, $v \in \mathscr{V}$.

(iv) *If* $\sup_{\beta>\lambda} \mathscr{E}_{\lambda+1}^{(\beta)}(u, u) < \infty$, *then* $u \in \mathscr{V}$.

Proof. For (i)~(iii), we refer to [8, Lemma I.2.11] and [12, Proposition 2.7 (iii)].

(iv): Since $\mathscr{E}(v, v) \geq \mathscr{A}(v, v)$ for $v \in \mathscr{F}$ and $(\beta-\lambda)G_\beta$ is contractive, we have

$$\begin{aligned}\mathscr{E}_{\lambda+1}^{(\beta)}(u, u) &= \mathscr{E}_\lambda((\beta-\lambda)G_\beta u,\ (\beta-\lambda)G_\beta u) + (\beta-\lambda)|u-(\beta-\lambda)G_\beta u|^2 + |u|^2 \\ &\geq \mathscr{A}_{\lambda+1}((\beta-\lambda)G_\beta u,\ (\beta-\lambda)G_\beta u) + (\beta-\lambda)|u-(\beta-\lambda)G_\beta u|^2.\end{aligned}$$

Hence the assumption $\sup_{\beta>\lambda} \mathscr{E}_{\lambda+1}^{(\beta)}(u, u) < \infty$ implies that

$$\sup_{\beta>\lambda} \mathscr{A}_{\lambda+1}((\beta-\lambda)G_\beta u,\ (\beta-\lambda)G_\beta u) < \infty, \tag{2.1}$$

$$\sup_{\beta>\lambda} (\beta-\lambda)|u-(\beta-\lambda)G_\beta u|^2 < \infty. \tag{2.2}$$

From (2.2), $(\beta-\lambda)G_\beta u \to u$ in $\mathscr{H}$. Therefore $u \in \mathscr{H}^0$. Combining this and (2.1), we have that $u \in \mathscr{V}$ by [8, Lemma I.2.12].

Lastly, we define the associated semigroup $\{T_t\}_{t\geq 0}$ on $\mathscr{H}$. When $\mathscr{H} = \mathscr{H}^0$, the definition via the generator is well-known. In general cases, we define $T_t f = T_t' P f$, where $\{T_t'\}$ is the semigroup associated with $\mathscr{E}'$ on $\mathscr{H}^0$.

3. Criteria for convergence

Suppose that we are given forms $\{\mathscr{E}^n\}_{n=1}^{\infty}$ and $\mathscr{E}$ on $\mathscr{H}$ following the framework in the section 2 with a uniform bound constant $\lambda \in \boldsymbol{R}$. The operators and the spaces relating to $\mathscr{E}^n$ are represented by supplementing a suffix n, such as G_α^n, Φ^n, and $\mathscr{V}_n$. We emphasize that λ is taken independently of n but that the sector constants of $\mathscr{A}^n$'s need not be uniformly bounded.

We introduce the following conditions which are referred to henceforth:

(F1) If a sequence $\{u_n\}$ weakly convergent to u in $\mathscr{H}$ satisfies $\underline{\lim}_{n\to\infty}\Phi^n(u_n)<\infty$, then $u\in\mathscr{V}$.

(F2) For any sequence $\{u_n\}$ weakly covergent to u in $\mathscr{H}$ with $u_n\in\mathscr{V}_n$, $u\in\mathscr{V}$, and any $w\in\mathscr{F}$, there exists $\{w_n\}$ converging to w strongly in $\mathscr{H}$ such that $\lim_{n\to\infty}\mathscr{E}^n(w_n, u_n)=\mathscr{E}(w, u)$.

(F2′) For any sequence $\{n_k\}\uparrow\infty$ and any sequence $\{u_k\}$ weakly convergent in $\mathscr{H}$ to $u\in\mathscr{V}$ which satisfies $\sup_k\Phi^{n_k}(u_k)<\infty$, there exists a dense subset $\mathscr{C}$ of $\mathscr{F}$ for the topology of $\mathscr{F}$ such that every $w\in\mathscr{C}$ has a sequence $\{w_k\}$ converging to w strongly in $\mathscr{H}$ with $\underline{\lim}_{k\to\infty}\mathscr{E}^{n_k}(w_k, u_k)\leq\mathscr{E}(w, u)$.

(R) G_α^n converges to G_α strongly for $\alpha>\lambda$.

We also define (F1$_a$) (resp. (F1$_b$)) by replacing $\Phi^n(u_n)$ by $\|u_n\|_{\hat{\mathscr{F}}_n}$ (resp. $\|u_n\|_{\mathscr{V}_n}$) in (F1), and (F2$'_a$) (resp. (F2$'_b$)) by replacing $\Phi^{n_k}(u_{n_k})$ by $\|u_{n_k}\|_{\hat{\mathscr{F}}_{n_k}}$ (resp. $\|u_{n_k}\|_{\mathscr{V}_{n_k}}$) in (F2′).

Theorem 3.1. (F2)$\Rightarrow$(F2′), (F1)(F2′)$\Leftrightarrow$(R)$\Leftrightarrow$(F1)(F2).

We state a lemma used in the proof of Theorem 3.1.

Lemma 3.2 (cf. [2, Corollary 1.18]). *Suppose that double sequences $\{u_{i,j}\}_{i,j\in\boldsymbol{N}}\subset\mathscr{H}$, $\{a_{i,j}\}_{i,j\in\boldsymbol{N}}\subset\boldsymbol{R}$ and $u\in\mathscr{H}$, $a\in\boldsymbol{R}$ satisfy that*

$$\operatorname*{s\text{-}lim}_{i\to\infty}\operatorname*{s\text{-}lim}_{j\to\infty}u_{i,j}=u,$$

$$\lim_{i\to\infty}\lim_{j\to\infty}a_{i,j}=a\ (\textit{resp.}\ \varliminf_{i\to\infty}\varliminf_{j\to\infty}a_{i,j}\leq a).$$

Then there exists a non-decreasing, divergent sequence $\{i(j)\}$ such that

$$\operatorname*{s\text{-}lim}_{j\to\infty}u_{i(j),j}=u,$$

$$\lim_{j\to\infty}a_{i(j),j}=a\ (\textit{resp.}\ \varliminf_{j\to\infty}a_{i(j),j}\leq a).$$

Proof of Theorem 3.1. (F2)$\Rightarrow$(F2′): By letting $u_n=0$ for every n in (F2), we know that for each $w\in\mathscr{F}$ there exists $\{w_n\}$ converging to w strongly in $\mathscr{H}$ such that $w_n\in\mathscr{V}_n$ for every n. Since $\mathscr{F}$ is dense in $\mathscr{V}$, every $w\in\mathscr{V}$ has the same property by Lemma 3.2. Take an arbitrary sequence $\{n_k\}\uparrow\infty$, and $\{u_k\}$ weakly convergent to u with $u_k\in\mathscr{V}_{n_k}$, $u\in\mathscr{V}$. From the observation above, we can take $\{u'_n\}$ weakly convergent to u satisfying $u'_n\in\mathscr{V}_n$ and $u'_{n_k}=u_k$. This is enough to show that (F2′) holds.

(F1)(F2′)$\Rightarrow$(R): We follow the argument of Röckner and Zhang [11]. We may assume that $\mathscr{C}=\mathscr{F}$ by Lemma 3.2. Take $f\in\mathscr{H}$ and $\alpha>\lambda$. First we prove that $\hat{G}_\alpha^n f$ converges weakly to $\hat{G}_\alpha f$ in $\mathscr{H}$. It suffices to prove that for any sequence $\{n_k\}\uparrow\infty$ we can extract a subsequence $\{n_{k_l}\}$ such that $\hat{G}_\alpha^{n_{k_l}}f$ converges to $\hat{G}_\alpha f$ weakly. For notational convenience, we shall denote a subsequence of $\{n_k\}$ by the same symbol. Set $u_n=\hat{G}_\alpha^n f$. Since $\|\hat{G}_\alpha^n\|_{\mathrm{op}}\le(\alpha-M)^{-1}$, $\{u_n\}$ is weakly relatively compact in $\mathscr{H}$. Take a subsequence of $\{n_k\}$ such that u_{n_k} converges weakly to some $u\in\mathscr{H}$. It is easy to see that $\sup_k\Phi^{n_k}(u_{n_k})<\infty$. This implies that $u\in\mathscr{V}$ by (F1). For any $w\in\mathscr{F}$, by extracting a subsequence if necessary, we can choose $\{w_k\}$ strongly convergent to w such that $w_k\in\mathscr{V}_{n_k}$ and $\lim_{k\to\infty}\mathscr{E}^{n_k}(w_k,u_{n_k})\le\mathscr{E}(w,u)$ from (F2′). Since $\mathscr{E}_\alpha^{n_k}(w_k,u_{n_k})=(w_k,f)$, it follows that

$$0=\lim_{k\to\infty}\{\mathscr{E}_\alpha^{n_k}(w_k,u_{n_k})-(w_k,f)\}\le\mathscr{E}_\alpha(w,u)-(w,f).$$

Hence $\mathscr{E}_\alpha(w,u)\ge(w,f)$. By substituting $-w$ for w, this becomes equality. Therefore $(W_\alpha^{-1}w,u)=(W_\alpha^{-1}w,\hat{G}_\alpha f)$. Since $W_\alpha^{-1}(\mathscr{F})=\mathscr{V}^*$, it follows that $u=\hat{G}_\alpha f$. We have proved that $\hat{G}_\alpha^n$ converges to $\hat{G}_\alpha$ weakly, and as a consequence, G_α^n converges to G_α weakly. In order to prove the strong convergence, it is enough to show that $\underline{\lim}_{k\to\infty}|G_\alpha^{n_k}f|\le|G_\alpha f|$ for any sequence $\{n_k\}\uparrow\infty$. Let $v_n=G_\alpha^n f$. We can take a subsequence of $\{n_k\}$ (which is denoted by the same symbol as already mentioned) such that $-\hat{G}_\alpha^{n_k}v_{n_k}$ converges weakly to some x in $\mathscr{H}$. Since $\sup_k\Phi^{n_k}(-\hat{G}_\alpha^{n_k}v_{n_k})<\infty$, x belongs to $\mathscr{V}$ by (F1). Take $\{w_k\}$ corresponding to the case when $-\hat{G}_\alpha^{n_k}v_{n_k}$, x and $G_\alpha f$ are taken as u_k, u, and w in (F2′), respectively. We may assume that $w_k\in\mathscr{V}_{n_k}$ by further extracting a subsequence of $\{n_k\}$. Then, taking $\underline{\lim}_{k\to\infty}$ of both sides of the equation

$$\begin{aligned}|v_{n_k}|^2&=(v_{n_k},v_{n_k})+(w_k,-v_{n_k})+(w_k,v_{n_k})\\&=(f,\hat{G}_\alpha^{n_k}v_{n_k})+\mathscr{E}_\alpha^{n_k}(w_k,-\hat{G}_\alpha^{n_k}v_{n_k})+(w_k,v_{n_k}),\end{aligned}$$

we have

$$\underline{\lim}_{k\to\infty}|v_{n_k}|^2\le-(f,x)+\mathscr{E}_\alpha(G_\alpha f,x)+|G_\alpha f|^2=|G_\alpha f|^2.$$

Hence (R) follows.

(R)$\Rightarrow$(F1): Let $u_n \to u$ weakly in $\mathscr{H}$ and $M := \underline{\lim}_{n\to\infty} \Phi^n(u_n) < \infty$. Then $u \in \mathscr{H}^0$. From Proposition 2.3 (i)(ii), for $\beta > \lambda$,

$$\begin{aligned}
&(\beta-\lambda)(u-(\beta-\lambda)G_\beta^n u, u_n) + ((\beta-\lambda)G_\beta^n u, u_n) \\
&\leq \mathscr{E}_{\lambda+1}^n((\beta-\lambda)G_\beta^n u, u_n) \\
&\leq \Phi^n(u_n)\|(\beta-\lambda)G_\beta^n u\|_{\mathscr{V}_n} \\
&\leq \Phi^n(u_n)\mathscr{E}_{\lambda+1}^n((\beta-\lambda)G_\beta^n u, (\beta-\lambda)G_\beta^n u)^{1/2} \\
&\leq \Phi^n(u_n)\mathscr{E}_{\lambda+1}^{n,(\beta)}(u,u)^{1/2} \\
&= \Phi^n(u_n)\{(\beta-\lambda)(u-(\beta-\lambda)G_\beta^n u, u) + |u|^2\}^{1/2}.
\end{aligned}$$

Taking $\underline{\lim}_{n\to\infty}$ of both sides, we have

$$\mathscr{E}_{\lambda+1}^{(\beta)}(u,u) - |u|^2 + ((\beta-\lambda)G_\beta u, u) \leq M\mathscr{E}_{\lambda+1}^{(\beta)}(u,u)^{1/2}.$$

Hence $\mathscr{E}_{\lambda+1}^{(\beta)}(u,u)^{1/2} \leq \{M + \sqrt{M^2 + 4(|u|^2 - ((\beta-\lambda)G_\beta u, u))}\}/2$, which implies that $\overline{\lim}_{\beta\to\infty}\mathscr{E}_{\lambda+1}^{(\beta)}(u,u) \leq M^2$. From Proposition 2.3(iv), we obtain that $u \in \mathscr{V}$.

(R)$\Rightarrow$(F2): Let $u_n \to u$ weakly in $\mathscr{H}$, $u_n \in \mathscr{V}_n$, $u \in \mathscr{V}$, and $w \in \mathscr{F}$. It holds that

$$\operatorname*{s-lim}_{\beta\to\infty} \operatorname*{s-lim}_{n\to\infty} (\beta-\lambda)G_\beta^n w = w,$$

$$\begin{aligned}
\lim_{\beta\to\infty}\lim_{n\to\infty} \mathscr{E}_\lambda^{n,(\beta)}(w, u_n) &= \lim_{\beta\to\infty}\lim_{n\to\infty} (\beta-\lambda)(w-(\beta-\lambda)G_\beta^n w, u_n) \\
&= \lim_{\beta\to\infty} \mathscr{E}_\lambda^{(\beta)}(w,u) = \mathscr{E}_\lambda(w,u).
\end{aligned}$$

Due to Lemma 3.2, we can take a nondecreasing sequence $\{\beta_n\} \to \infty$ such that

$$\operatorname*{s-lim}_{n\to\infty} (\beta_n-\lambda)G_{\beta_n}^n w = w, \qquad \lim_{n\to\infty} \mathscr{E}_\lambda^{n,(\beta_n)}(w, u_n) = \mathscr{E}_\lambda(w,u).$$

Setting $w_n = (\beta_n - \lambda)G_{\beta_n}^n w$, we have

$$(w_n, u_n) \to (w,u), \qquad \mathscr{E}_\lambda^n(w_n, u_n) = \mathscr{E}_\lambda^{n,(\beta_n)}(w, u_n) \to \mathscr{E}_\lambda(w,u) \quad \text{as } n\to\infty.$$

Therefore, $\lim_{n\to\infty}\mathscr{E}^n(w_n, u_n) = \mathscr{E}(w,u)$.

In conjunction with Lemma 2.2, we have the following corollary.

Corollary 3.3. (i) $(\mathrm{F1}_b)(\mathrm{F2}'_b) \Rightarrow (\mathrm{R})$.
(ii) *If sector constants of $\mathscr{A}^n$'s are taken uniformly bounded, then* $(\mathrm{F1}_a)(\mathrm{F2}'_a) \Leftrightarrow (\mathrm{R})$.

Remark 3.4. Even in the case of symmetric forms, the pair of the conditions (F1) (F2) (or $(F1_a)(F2'_a)$) seems like a different type of characterization from the Mosco convergence.

We state relations to semigroup convergence.

Theorem 3.5. *When* (R) *holds, $T_t^n f$ converges strongly to $T_t f$ in $\mathscr{H}$ for every $f \in \mathscr{H}^0$. The convergence is uniform in any finite interval of $t \geq 0$. Conversely, if T_t^n converges strongly to T_t for all $t \in [0, T]$ for some $T > 0$, then* (R) *holds.*

Proof. See [3, Chapter IX, Theorem 2.16] when $\mathscr{V}_n$ and $\mathscr{V}$ are all dense in $\mathscr{H}$. We need easy modification in general cases, which is left to the reader.

4. Examples

Example 4.1 (Coercive closed forms with domains which are not dense). We refer [8, Chapter II, Section 3] for the terminology. Let E be a separable Banach space, and μ a finite positive measure on the Borel σ-field $\mathscr{B}(E)$ such that $\operatorname{supp}\mu = E$. Let E^* be a topological dual of E. Define

$$\mathscr{F}\mathscr{C}_b^\infty = \{f(l_1, \cdots, l_m) \mid m \in \boldsymbol{N}, f \in C_b^\infty(\boldsymbol{R}^m), l_1, \cdots, l_m \in E^*\},$$

where $C_b^\infty(\boldsymbol{R}^m)$ stands for the collection of infinitely differentiable functions on $\boldsymbol{R}^m$ the derivatives of which are all bounded. Suppose that there is an infinite dimensional separable Hilbert space H densely and continuously imbedded in E. For $u \in \mathscr{F}\mathscr{C}_b^\infty$, define H^*-valued function ∇u on E by

$${}_{H^*}(\nabla u(z), h)_H = \lim_{s \to 0} \{u(z + sh) - u(z)\}/s, \qquad h \in H.$$

Define a bilinear form Q on $L^2(E)$ by

$$Q(u, v) = \int_E (\nabla u, \nabla v)_{H^*} d\mu, \qquad u, v \in \mathscr{F}\mathscr{C}_b^\infty.$$

We assume that $(Q, \mathscr{F}\mathscr{C}_b^\infty)$ is closable. For example, this holds when (E, H, μ) is an abstract Wiener space. We denote the closure of $(Q, \mathscr{F}\mathscr{C}_b^\infty)$ by $(Q, D(Q))$. Take a countable subset $\{e_i\}_{i=1}^\infty$ of E^* which is a c.o.n.s. of H^* by the natural inclusion $E^* \subset H^*$. We denote by $\mathscr{F}_n$ the sub σ-field of $\mathscr{B}(E)$ generated by $\{e_i; 1 \leq i \leq n\}$, and define $\mathscr{F}\mathscr{C}_b^\infty(\mathscr{F}_n) = \mathscr{F}\mathscr{C}_b^\infty \cap \{\mathscr{F}_n\text{-measurable functions on } E\}$. Let $\mathscr{L}^\infty(H^*)$ denote the space of all bounded linear operators on H^* with the operator norm $\|\cdot\|_{\mathrm{op}}$. For $T \in \mathscr{L}^\infty(H^*)$, $\hat{T}$ stands for the adjoint operator of T. We also define $\tilde{T} = (T + \hat{T})/2$ and $\check{T} = (T - \hat{T})/2$.

Suppose that we are given strongly measurable maps $\sigma: E \to \mathscr{L}^\infty(H^*)$ and $b: E \to H^*$ such that

(i) For some $c>0$, $\tilde{\sigma}(z)\geq cI$ in the form sense for μ-a.e.z and

$$\|\tilde{\sigma}\|_{\mathrm{op}}\in L^1(E),\ \|\check{\sigma}\|_{\mathrm{op}}\in L^\infty(E).$$

(ii) $|b|_{H^*}\in L^\infty(E)$.

Define bilinear forms on $L^2(E)$ by

$$\mathscr{E}(u,v)=\int_E\{(\sigma\nabla u,\nabla v)_{H^*}-(b,\nabla u)_{H^*}v\}d\mu,\qquad u,v\in\mathscr{F}\mathscr{C}_b^\infty,$$

$$\mathscr{E}^n(u,v)=\mathscr{E}(u,v),\qquad u,v\in\mathscr{F}\mathscr{C}_b^\infty(\mathscr{F}_n).$$

Then it is easy to see that there exist constants $\lambda\geq 0$, $c>0$, and $K\geq 1$ such that

$$cQ_1(u,u)\leq\mathscr{E}_\lambda(u,u),$$
$$|\mathscr{E}_{\lambda+1}(u,v)|\leq K\mathscr{E}_{\lambda+1}(u,u)^{1/2}\mathscr{E}_{\lambda+1}(v,v)^{1/2},\qquad \text{for } u,v\in\mathscr{F}\mathscr{C}_b^\infty.$$

Hence, $\mathscr{E}$ and $\mathscr{E}^n$ are all closable and closures are coercive forms in a wide sense with a bound constant λ by a similar way of [8, Section II.3e)]. The domains of the closures are denoted by $\mathscr{V}$ and $\mathscr{V}_n$, respectively. It holds that $\mathscr{V}_1\subset\mathscr{V}_2\subset\cdots\subset\mathscr{V}\subset D(Q)$. $\mathscr{E}$ and $\mathscr{E}_n$ are in accordance with the framework in the section 2, by considering $\Lambda=\Lambda_n\equiv 0$.

Now, we will show the strong convergence of the corresponding resolvents and semigroups by checking that ($\mathrm{F1}_b$) and ($\mathrm{F2}'_b$) in the section 3 hold.

($\mathrm{F1}_b$): Let $u_n\to u$ weakly in $L^2(E)$ with $\underline{\lim}_{n\to\infty}\mathscr{E}^n_{\lambda+1}(u_n,u_n)<\infty$. Then $\underline{\lim}_{n\to\infty}\mathscr{E}_{\lambda+1}(u_n,u_n)<\infty$. This implies that a subsequence of $\{u_n\}$ converges weakly to u in $\mathscr{V}$ by [8, Lemma I.2.12]. In particular, $u\in\mathscr{V}$.

($\mathrm{F2}'_b$): Let $\{n_k\}\uparrow\infty$, $u_k\to u$ weakly with $\sup_k\mathscr{E}^{n_k}_{\lambda+1}(u_k,u_k)<\infty$, $u\in\mathscr{V}$ and $w\in\mathscr{V}$. Then $u_k\to u$ weakly in $\mathscr{V}$. Since the linear span of $\{e_i;\ i\in N\}$ is dense in E^*, there exists $\{w_k\}$ such that $w_k\in\mathscr{V}_{n_k}$ for every k and $w_k\to w$ in $\mathscr{V}$. (cf. [1, Proposition 2.10.]) Then in the equation

$$\mathscr{E}^{n_k}(w_k,u_k)=\mathscr{E}(w,u_k)+\mathscr{E}(w_k-w,u_k),$$

the first term of the right-hand side tends to $\mathscr{E}(w,u)$ as $k\to\infty$ since there exists $w'\in\mathscr{V}$ such that $\mathscr{E}(w,v)=\tilde{\mathscr{E}}_{\lambda+1}(w',v)$ for every $v\in\mathscr{V}$ from the Riesz theorem. The second term converges to 0 by the sector condition.

Remark 4.2. Since $\mathscr{E}^n$ has following another expression

$$\mathscr{E}(u,v)=\int_E\{(\sigma_n\nabla u,\nabla v)_{H^*}-(b_n,\nabla u)_{H^*}v\}d\mu,\qquad u,v\in\mathscr{F}\mathscr{C}_b^\infty(\mathscr{F}_n),$$

where $\sigma_n:E\to\mathscr{L}^\infty(H^*)$ and $b_n:E\to H^*$ are defined by

$$\sigma_n(z)(h)=\sum_{i,j=1}^{n} E_\mu[(\sigma(\cdot)e_i, e_j)_{H^*}|\mathscr{F}_n](z)(h, e_i)_{H^*}e_j, \qquad h\in H^*,$$

$$b_n=\sum_{i=1}^{n} E_\mu[(b, e_i)_{H^*}|\mathscr{F}_n]e_i,$$

$\{\mathscr{E}^n\}$ are essentially considered as finite dimensional approximations of $\mathscr{E}$. In the forthcoming paper, this example is applied to the proof of existence of invariant measures of diffusions with infinite dimensional state spaces.

Example 4.3 (Forms with time-dependent coefficients). Let $d\geq 2$. Let Ω be an open set of $\boldsymbol{R}^d$, possibly unbounded. We equip $\boldsymbol{R}$ and Ω with the Lebesgue measures, denoted by dt and dx, respectively. Let $C_0^\infty(\Omega)$ be the collection of functions on Ω which are infinitely differentiable with compact support. We define the Sobolev space V by the completion of $C_0^\infty(\Omega)$ with the norm $(\int_\Omega|\nabla\cdot|^2dx+\int_\Omega|\cdot|^2dx)^{1/2}$, where ∇ stands for the usual gradient operator and $|\cdot|$ the Euclidean norm. We consider that $\mathscr{V}:=L^2(\boldsymbol{R}\to V)$ is a subspace of $L^2(\boldsymbol{R}\times\Omega)$. Define a closed bilinear form $(Q,\mathscr{V})$ on $L^2(\boldsymbol{R}\times\Omega)$ by

$$Q(u, v)=\int_{\boldsymbol{R}} dt\int_\Omega (\nabla u, \nabla v)dx, \qquad u, v\in\mathscr{V}.$$

Suppose that we are given $\sigma_n\in L^\infty(\boldsymbol{R}\times\Omega\to(\boldsymbol{R}^d)^*\otimes\boldsymbol{R}^d)$, b_n, $d_n\in L^1_{\mathrm{loc}}(\boldsymbol{R}\times\Omega\to\boldsymbol{R}^d)$, $c_n\in L^1_{\mathrm{loc}}(\boldsymbol{R}\times\Omega)$, $n\in\boldsymbol{N}\cup\{\infty\}$ satisfying the following:

(i) There exists $\delta>0$ such that for all $n\in\boldsymbol{N}\cup\{\infty\}$, $\tilde{\sigma}_n(t,x)\geq\delta I$ in the form sense $dt\otimes dx$-a.e. Here $\tilde{\sigma}_n$ stands for the symmetrization of σ_n. Also, $\{\sigma_n\}$ are bounded in $L^\infty(\boldsymbol{R}\times\Omega\to(\boldsymbol{R}^d)^*\otimes\boldsymbol{R}^d)$.

(ii) There exists $p>d$ such that
$\{b_n\}$ and $\{d_n\}$ are bounded in $L^\infty(\boldsymbol{R}\to(L^p+L^\infty)(\Omega\to\boldsymbol{R}^d))$,
$\{c_n\}$ are bounded in $L^\infty(\boldsymbol{R}\to(L^{p/2}+L^\infty)(\Omega))$.

We define bilinear forms $\mathscr{A}^n$, $n\in\boldsymbol{N}\cup\{\infty\}$ on $L^2(\boldsymbol{R}\times\Omega)$ by

$$\mathscr{A}^n(u, v)=\int_{\boldsymbol{R}} dt\int_\Omega \{(\sigma_n\nabla u, \nabla v)+(b_n, \nabla u)v+(d_n, \nabla v)u+c_n uv\}dx, \qquad u, v\in\mathscr{V}.$$

By Sobolev's lemma, for every $\varepsilon>0$, $q\in[2,\frac{2d}{d-2})$ and $g\in V$,

$$\left(\int_\Omega |g|^q dx\right)^{2/q}\leq\varepsilon\int_\Omega |\nabla g|^2dx+C\int_\Omega g^2dx,$$

where C is a constant depending only on d, ε and q. This implies easily that there exist constants $\lambda\geq 0$, $c>0$ and $K\geq 1$, independent of n such that for every n,

$$cQ_1(u,u)\leq \mathscr{A}^n_\lambda(u,u)\leq c^{-1}Q_1(u,u),$$
$$|\mathscr{A}^n_{\lambda+1}(u,v)|\leq K\mathscr{A}^n_{\lambda+1}(u,u)^{1/2}\mathscr{A}^n_{\lambda+1}(v,v)^{1/2},\qquad u,v\in\mathscr{V}.$$

Hence $(\mathscr{A}^n,\mathscr{V})$ are all coercive closed forms with bound constant λ.

We take $\frac{d}{dt}$ as Λ in the section 2, which satisfies the required conditions for any n. Indeed, the corresponding semigroup $\{U_t\}$ on $\mathscr{V}^*\cong L^2(\boldsymbol{R}\to V^*)$ is described by $U_tf(s)=f(s+t)$, and $D(\Lambda,\mathscr{V}^*)=\{f\in\mathscr{V}^*|\frac{df}{dt}\in\mathscr{V}^*$ in the distribution sense$\}$. Then as in the section 2, corresponding forms $\mathscr{E}^n(n\in\boldsymbol{N})$ and $\mathscr{E}:=\mathscr{E}^\infty$ are defined.

Now we further assume the following convergence of coefficients:

$$\sigma_n\to\sigma_\infty=:\sigma \text{ in } L^1_{\mathrm{loc}}(\boldsymbol{R}\times\Omega\to(\boldsymbol{R}^d)^*\otimes\boldsymbol{R}^d),\qquad b_n\to b_\infty=:b \text{ in } L^1_{\mathrm{loc}}(\boldsymbol{R}\times\Omega\to\boldsymbol{R}^d),$$
$$c_n\to c_\infty=:c \text{ in } L^1_{\mathrm{loc}}(\boldsymbol{R}\times\Omega),\qquad d_n\to d_\infty=:d \text{ in } L^1_{\mathrm{loc}}(\boldsymbol{R}\times\Omega\to\boldsymbol{R}^d).$$

Then strong convergence of associated resolvents and semigroups hold by verifying (F1$_b$) and (F2$'_b$).

(F1$_b$): Let $u_n\to u$ weakly in $L^2(\boldsymbol{R}\times\Omega)$ with $\underline{\lim}_{n\to\infty}\mathscr{A}^n_{\lambda+1}(u_n,u_n)<\infty$. Then $\underline{\lim}_{n\to\infty}Q_1(u_n,u_n)<\infty$. From [8, Lemma I.2.12], a subsequence of $\{u_n\}$ converges weakly to u in $\mathscr{V}$ and in particular, $u\in\mathscr{V}$.

(F2$'_b$): Let $\{n_k\}\uparrow\infty$, $u_k\to u$ weakly with $\sup_k\mathscr{A}^{n_k}_{\lambda+1}(u_k,u_k)<\infty$, $u\in\mathscr{V}$. Then $u_k\to u$ weakly in $\mathscr{V}$. We take $\mathscr{C}=C_0^\infty(\boldsymbol{R}\to V)$ (cf. [6, Chapter 1, Theorem 2.1]). We may assume that $\sigma_{n_k}\to\sigma$, $b_{n_k}\to b$, $d_{n_k}\to d$, and $c_{n_k}\to c$ $dt\otimes dx$-a.e. by taking a subsequence if necessary. Take $w\in\mathscr{C}$, and let $I\subset\boldsymbol{R}$ be a compact set such that the support of w is contained in I. Then

$$\begin{aligned}
&|\mathscr{E}^{n_k}(w,u_k)-\mathscr{E}(w,u)|\\
&=|\mathscr{E}^{n_k}(w,u_k)-\mathscr{E}(w,u_k)+\mathscr{E}(w,u_k-u)|\\
&\leq\left|\int_I dt\int_\Omega((\sigma_{n_k}-\sigma)\nabla w,\nabla u_k)dx\right|+\left|\int_I dt\int_\Omega(b_{n_k}-b,\nabla w)u_kdx\right|\\
&\quad+\left|\int_I dt\int_\Omega(d_{n_k}-d,\nabla u_k)wdx\right|+\left|\int_I dt\int_\Omega(c_{n_k}-c)wu_kdx\right|+|\mathscr{E}(w,u_k-u)|\\
&=:J_1+J_2+J_3+J_4+J_5.
\end{aligned}$$

Since $(\sigma_{n_k}-\sigma)\nabla w\to0$ strongly in $L^2(I\times\Omega\to\boldsymbol{R}^d)$, we have that $J_1\to0$ as $k\to\infty$. Fix $p'\in(d,p)$. For any $\varepsilon>0$, we choose a compact set $K_\varepsilon\subset\Omega$ such that $\int_Idt\int_{\Omega\setminus K_\varepsilon}|\nabla w|^2dx<\varepsilon^2$ and $\int_I\left\{(\int_{\Omega\setminus K_\varepsilon}|w|^{2p'/(p'-2)}dx)^{(p'-2)/2p'}+(\int_{\Omega\setminus K_\varepsilon}|w|^2dx)^{1/2}\right\}^2dt<\varepsilon^2$. Below, C_i stands for a constant which is independent of k and ε. We have

$$J_2\leq\int_I\left\{C_1\left(\int_{K_\varepsilon}|b_{n_k}-b|^{p'}dx\right)^{1/p'}\left(\int_\Omega|\nabla w|^2dx\right)^{1/2}\|u_k\|_V\right.$$

$$
\begin{aligned}
&+C_2\|b_{n_k}-b\|_{(L^p+L^\infty)(\Omega)}\left(\int_{\Omega\setminus K_\varepsilon}|\nabla w|^2dx\right)^{1/2}\|u_k\|_V\Bigg\}dt\\
&\le C_3Q_1(u_k,u_k)^{1/2}\left[\left\{\int_I\left(\int_{K_\varepsilon}|b_{n_k}-b|^{p'}dx\right)^{2/p'}dt\right\}^{1/2}\|\|w\|_V\|_{L^\infty(I)}\right.\\
&\left.\quad+\|\|b_{n_k}-b\|_{(L^p+L^\infty)(\Omega)}\|_{L^\infty(I)}\varepsilon\right].
\end{aligned}
$$

Since $\sup_k\|\|b_{n_k}-b\|_{L^p(K_\varepsilon)}\|_{L^\infty(I)}<\infty$, we have that $J_2\to 0$ by letting $k\to\infty$ and $\varepsilon\to 0$. Next we have

$$
\begin{aligned}
J_3\le&\int_I\left[C_4\left(\int_{K_\varepsilon}|d_{n_k}-d|^{p'}dx\right)^{1/p'}\left(\int_\Omega|\nabla u_k|^2dx\right)^{1/2}\|w\|_V\right.\\
&+\|d_{n_k}-d\|_{(L^p+L^\infty)(\Omega)}\left(\int_\Omega|\nabla u_k|^2dx\right)^{1/2}\\
&\left.\times\left\{\left(\int_{\Omega\setminus K_\varepsilon}|w|^{2p'/(p'-2)}dx\right)^{(p'-2)/2p'}+\left(\int_{\Omega\setminus K_\varepsilon}|w|^2dx\right)^{1/2}\right\}\right]dt\\
\le& Q_1(u_k,u_k)^{1/2}\left[C_4\left\{\int_I\left(\int_{K_\varepsilon}|d_{n_k}-d|^{p'}dx\right)^{2/p'}dt\right\}^{1/2}\|\|w\|_V\|_{L^\infty(I)}\right.\\
&\left.+\|\|d_{n_k}-d\|_{(L^p+L^\infty)(\Omega)}\|_{L^\infty(I)}\varepsilon\right].
\end{aligned}
$$

Letting $k\to\infty$ and $\varepsilon\to 0$, we get that $J_3\to 0$. Similarly, we have that $J_4\to 0$. That $J_5\to 0$ follows from the weak convergence of u_k in $\mathscr{V}$.

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