J. Math. Kyoto Univ. (JMKYAZ)
39-4 (1999) 763–782

Numerical classification of singular fibers in genus 3 pencils

By

Kazuhiro UEMATSU

1. Introduction

Let $\pi : X \to D$ be a proper surjective holomorphic map of a complex manifold X of dimension 2 to a small open disc $D = \{t \in \mathbf{C} \mid |t| < \varepsilon\}$. We assume that π is smooth over a punctured disc $D' = D - \{0\}$. Moreover we assume that for every $t \in D'$ the fiber $X_t = \pi^{-1}(t)$ is a non-singular curve of genus g and that X contains no exceptional curves of the first kind. By L_t, we denote the effective divisor in X defined by the equation $\pi = t$ $(t \in D)$. We call the divisor L_0 the *singular fiber of* π. For every $t \in D'$ we call the divisor L_t a *generic fiber*. We write the singular fiber L_0 as

$$L_0 = \sum_{i=1}^{r} n_i \Gamma_i, \tag{1.1}$$

where Γ_i is an irreducible reduced component of L_0 and n_i is its multiplicity. By $p(\Gamma_i)$, we denote the arithmetic genus of the component Γ_i. The combination of integers $\{r, n_i, p(\Gamma_i), \Gamma_i \cdot \Gamma_j \ (1 \leq i < j \leq r)\}$ is called a *numerical type of the singular fiber* L_0.

In the study of elliptic surfaces [2], Kodaira showed that there exist only ten types of singular fibers of pencils of curves of genus one. Iitaka [1] and Ogg [5] gave a numerical classification of singular fibers of curves of genus 2. Namikawa and Ueno [3], [4] classified their numerical types completely, constructed all their singular fibers and calculated the monodromies around them.

In our previous paper [6], we studied the numerical properties of singular fibers in pencils of curves of genus g (≥ 2) and gave a method to classify all the numerical types of the singular fibers. In this article, by using this method, we give the complete numerical classification of singular fibers in pencils of curves of genus three.

If the number of irreducible components is more than one, we have $\Gamma^2 < 0$ and $\Gamma \cdot K_X \geq 0$, where K_X is a canonical divisor. If $\Gamma \cdot K_X > 0$, we call this component Γ *a trunk*. If $\Gamma \cdot K_X = 0$, then we have $\Gamma^2 = -2$. Thus we call this component Γ a (-2)-*curve*. Further we call a connected component consisting of (-2)-curves in a singular fiber a *branch*. Our method of numerical classification

Communicated by Prof. K. Ueno, May 14, 1999

of singular fibers is as follows. First, we determine all the combinations of trunks, the number of which is finite. Secondly, we classify the possible branches in the singular fiber in pencils of curves of genus 3. Finally by combining trunks with branches, we classify all the singular fibers. In our calculation there exist 4343 types of singular fibers in pencils of curves of genus 3, while on the other hand, in the case of genus 2 there exist 140 types.

I would like to express my hearty thanks to Professor Kenji Ueno, who introduced me to this problem and gave me incessant encouragement and advices. I would also like to thank Professor Masahiko Saito for their useful and stimulating discussions.

2. General numerical properties of a singular fiber

We consider a family $\pi : X \to D$ as above. The following theorem is well-known (See for example; Ogg [5], Kodaira, [2]). Here, if the support of the divisor D is connected, we simply say that D is *connected.* If the greatest common divisor of multiplicities n_i's in (1.1) is more than one, we call the singular fiber L_0 a *multiple fiber.*

Theorem 2.1. *For a singular fiber L_0 in* (1.1), *we have the following*:

(1) *L_0 is connected,*

(2) $L_0 \cdot \Gamma_i = 0$ $(i = 1, 2, \dots, r)$, $L_0 \cdot K_X = 2(g-1)$,

(3) $\Gamma_i^2 + \Gamma_i \cdot K_X \in 2\mathbf{Z}$,

(4) *and if L_0 is simply connected, L_0 is not a multiple fiber.*

3. Classification

In the case of genus 3 by Theorem 2.1, we have the following equalities:

$$(3.1) \qquad \sum_{i=1}^{r} n_i \Gamma_i \cdot \Gamma_j = 0 \qquad (j = 1, 2, \dots, r), \qquad \sum_{i=1}^{r} n_i \Gamma_i \cdot K_X = 4.$$

We have the two cases as follows.

3.1. Case I—X_0 has only one irreducible component. Set $L_0 = n\Gamma$. By (3.1), we have $n\Gamma \cdot \Gamma = 0$, $n\Gamma \cdot K_X = 4$. Noticing the statement (3) of Theorem 2.1, we have $(n, \Gamma \cdot K_X) = (2, 2)$ or $(1, 4)$. In case $(n, \Gamma \cdot K_X) = (1, 4)$, we have $L_0 = \Gamma$ and $p(\Gamma) = \frac{1}{2}(\Gamma^2 + \Gamma \cdot K_X) + 1 = 3$. In case $(n, \Gamma \cdot K_X) = (2, 2)$, we have $L_0 = 2\Gamma$ and $p(\Gamma) = \frac{1}{2}(\Gamma^2 + \Gamma \cdot K_X) + 1 = 2$. In this case, by the statement (4) of Theorem 2.1, the component Γ is not simply connected.

3.2. Case II—X_0 has more than one irreducible component.

3.2.1. The Type of components. Set $L_0 = \sum_{i=1}^{r} n_i \Gamma_i$ $(r \geq 2)$. Then the following lemma holds (see Ogg [5]).

Lemma 3.1. *If we set $L_0 = D + n_i\Gamma_i$, where D is a divisor which does not contain the component Γ_i, we have $\Gamma_i^2 < 0$, and $\Gamma_i \cdot K_X \geq 0$.*

From this lemma and the equality (3.1), L_0 has the following types of components.

Table 1. The Types of components of singular fibers ($g = 3$)

Type	A	B	C	D	E	F	G	H	I	J	K
$\Gamma_i \cdot K_X$	4	4	4	3	3	3	2	2	1	1	0
Γ_i^2	−2	−4	−6	−1	−3	−5	−2	−4	−1	−3	−2
$p(\Gamma_i)$	2	1	0	2	1	0	1	0	1	0	0

In the above table the component of Type K is a (-2)-curve and others are trunks. From the equality (3.1), only the following combinations of trunks appear.

[A], [B], [C], [D,I], [D,J], [E,I], [E,J], [F,I], [F,J], [G,G], [2G], [G,H], [G,I,I], [G,2I], [G,I,J], [G,J,J], [G,2J], [H,H], [2H], [H,I,I], [H,2I], [H,I,J], [H,J,J], [H,2J], [I,I,I,I], [I,I,2I], [2I,2I], [I,3I], [4I], [I,I,I,J], [I,2I,J], [3I,J], [I,I,J,J], [2I,J,J], [I,I,2J], [2I,2J], [I,J,J,J], [I,J,2J], [I,3J], [J,J,J,J], [J,J,2J], [2J,2J], [J,3J], [4J].

3.2.2. The numerical properties of (-2)-curves. The classification of branches is more complicated. In this subsection, we review the numerical properties of (-2)-curves in singular fibers. In the following lemmas, we use the dual graphs for the configuration of curves in singular fibers. In the dual graphs a double circle represents a trunk and a single circle with a number represents a (-2)-curve and its multiplicity. Also the number below a line represents its intersection number. Further the number in the box which is connected to a circle represents the intersection number between this (-2)-curve and the divisor sum of all the trunks in the singular fiber.

The following lemmas holds (see Uematsu [6]).

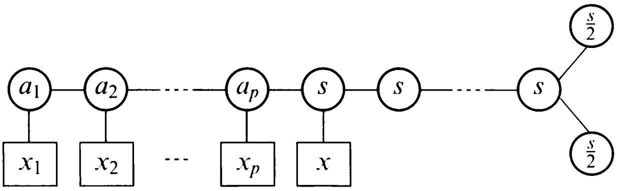
Lemma 3.2. *Let L be a singular fiber in pencils of curves of genus g (≥ 2).*

(1) *Suppose Γ_1 and Γ_2 are irreducible reduced (-2)-curves in the singular fiber L. Then the intersecton number $\Gamma_1 \cdot \Gamma_2$ is not greater than one.*

(2) *Suppose $L = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + D$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are (-2)-curves and D is an effective divisor which does not contain $\Gamma_1, \Gamma_2, \Gamma_3$. In addition we assume $\Gamma_1 \cdot \Gamma_2 = \Gamma_2 \cdot \Gamma_3 = 1$. Then $a_2 \geq \frac{1}{2}(a_1 + a_3)$. Moreover, the equalty holds if and only if $\Gamma_2 \cdot D = 0$.*

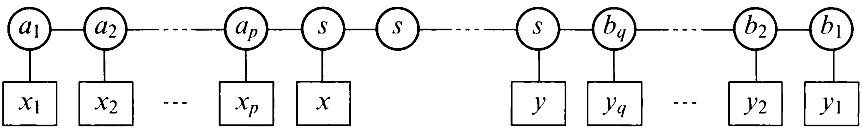
(3) *The configuration of (-2)-curves of branches in the singular fiber L is classified into the following three types:*

(a) Type (I)



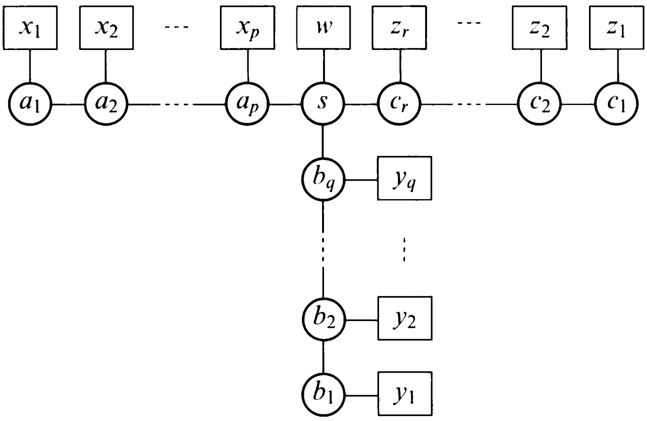
where s is an even integer and $p \geq 0$, $x > 0$.

(b) Type (II)



where $p \geq 0$, $q \geq 0$ and $x > 0$, $y > 0$.

(c) Type (III)



where $p \geq 1$, $q \geq 1$, $r \geq 1$. Moreover, at least one of these numbers x_i, y_j, z_k, w is positive and $a_p < s$, $b_q < s$, $c_r < s$.

(4) *Let s be the largest number of multiplicities of (-2)-curves in the above branches. Then we have*
   (a) *in case of* Type (I), (II), $s \leq 6\ (g-1)$,
   (b) *in case of* Type (III), $s \leq 12\ (g-1)$.

The following lemmas are useful for the classification of branches.

Lemma 3.3. *Let L be a singular fiber and we set*

$$L = kX + a_1\Gamma_1 + a_2\Gamma_2 + \cdots + a_l\Gamma_l + \cdots,$$

where X is a trunk and Γ_i $(1 \leq i \leq l)$ are (-2)-curves. We assume that $X \cdot \Gamma_1 = \alpha$, $\Gamma_1 \cdot \Gamma_2 = \Gamma_2 \cdot \Gamma_3 = \cdots = \Gamma_{l-1} \cdot \Gamma_l = 1$ and that Γ_i $(1 \leq i \leq l-1)$ intersects only two components Γ_{i-1} and Γ_{i+1}, where we set $X = \Gamma_0$. We call the divsor

$a_1\Gamma_1 + a_2\Gamma_2 + \cdots + a_l\Gamma_l$ *a chain of* (-2)*-curves. Then the sequence*

$$k\alpha, a_1, a_2, \ldots, a_l$$

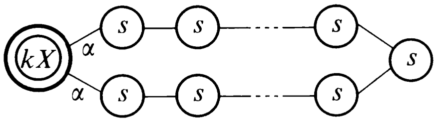
is an arithmetical progression. Moreover, if Γ_l *intersects only the component* Γ_{l-1}, *then the sequence*

$$k\alpha, a_1, a_2, \ldots, a_l, 0$$

is an arithmetical progression.

Proof. This lemma follows from the equalities $L \cdot \Gamma_i = 0$ $(1 \le i \le l)$.

Lemma 3.4. *Let L be a singular fiber and let kX be a trunk with multiplicity k in L. We assume that a branch* Σ *in L contains* (-2)*-curves* $a\Gamma, b\Gamma'$, *both of which intersect the trunk kX. Suppose that* $X \cdot \Gamma = \alpha$, $X \cdot \Gamma' = \beta$ *and* $1 \le \alpha \le \beta$. *Then we have the inequality* $a \ge k\alpha$. *Moreover, if this equality holds, then the equalities* $\alpha = \beta$ *and* $a = b = k\alpha$ *hold and the branch has the following configuration*:



where $s = k\alpha$.

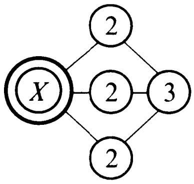
Proof. Since the branch Σ is connected, there exists a sequence of (-2)-curves $a_1\Gamma_1, a_2\Gamma_2, \ldots, a_m\Gamma_m$ in Σ which satisfy the equalities $\Gamma_{i-1} \cdot \Gamma_i = 1$ $(2 \le i \le m)$. Here we set $a = a_1, b = a_m$, $\Gamma = \Gamma_1$ and $\Gamma' = \Gamma_m$. Then by Lemma 3.2 (2) and the equality $L \cdot \Gamma_1 = L \cdot \Gamma_m = 0$, we have the following inequalities:

$$a_i \ge \frac{a_{i-1} + a_{i+1}}{2} \qquad (1 \le i \le m),$$

where we put $a_0 = k\alpha$ and $a_{m+1} = k\beta$. The lemma follows from these inequalities.

In case the genus g is three, the following corollaries hold.

Corollary 3.5. *Let kX be a trunk with the multiplicity k in a singular fiber L in a pencil of curves of genus three. Let* Σ *be a branch in L. Then the number of the* (-2)*-curves in* Σ *which intersect X is less than or equal to three. Moreover, if the number of these components is three, then the singular fiber L is the following type*:



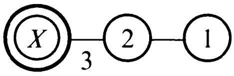
where the trunk X is of Type C.

Proof. We put $\Sigma = a_1\Gamma_1 + a_2\Gamma_2 + \cdots + a_l\Gamma_l$, where Γ_i $(1 \leq i \leq l)$ is a (-2)-curve. We assume $X \cdot \Gamma_1 \geq 1$, $X \cdot \Gamma_2 \geq 1$, $X \cdot \Gamma_3 \geq 1$. Then we have

$$0 = L \cdot X = kX^2 + a_1 X \cdot \Gamma_1 + a_2 X \cdot \Gamma_2 + a_3 X \cdot \Gamma_3 + \cdots \geq kX^2 + a_1 + a_2 + a_3.$$

Hence, $a_1 + a_2 + a_3 \leq -kX^2$. From Lemma 3.4, if one of these three numbers a_1, a_2, a_3 equals k, then the only two curves in Σ intersect the trunk kX. This contradicts our assumption. Hence we have $a_i \geq k+1$ $(i = 1,2,3)$. Considering Table 1 and the combinations of trunks, if $k = 1$, then $-X^2 \leq 6$, hence $a_1 + a_2 + a_3 \leq 6$. Since $a_1 \geq 2$, $a_2 \geq 2$, $a_3 \geq 2$, we obtain $a_1 = a_2 = a_3 = 2$ and the other components don't intersect the trunk kX. If $k = 2$, then $-X^2 \leq 4$. If $k = 3,4$, then $-X^2 \leq 3$. Therefore in case $k = 2,3,4$, the inequalities $3(k+1) \leq -kX^2$ don't hold.

Corollary 3.6. *Let kX be a trunk in a singular fiber L with multiplicity k. Let $a\Gamma$ be a (-2)-curve with multiplicity a which is contained in L. Then $X \cdot \Gamma \leq 3$. Moreover, the equality holds if and only if $k = 1$, $a = 2$, $X^2 = -6$ and in this case the singular fiber is as follows*:



where the trunk X is of Type C.

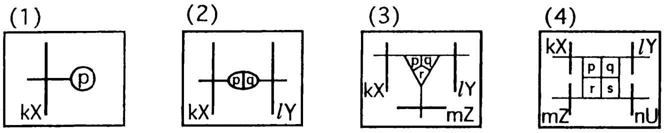
Proof. We set $L = kX + a\Gamma + \cdots$. Since $L \cdot X = L \cdot \Gamma = 0$, we have $kX^2 + aX \cdot \Gamma \leq 0$ and $kX \cdot \Gamma - 2a \leq 0$. If we put $X \cdot \Gamma = \alpha$, then $k\alpha^2 \leq 2a\alpha \leq -2kX^2$. Hence $\alpha^2 \leq -2X^2$. By Table 1, if $k = 1$, we have $\alpha^2 \leq 12$. Thus $\alpha = 1,2,3$. In case $\alpha = 3$, we have $9 \leq 6a \leq 12$. This implies $a = 2$ and $X^2 = -6$. If $k = 2,3,4$, then we have $\alpha^2 \leq 8$. Hence $\alpha = 1,2$.

3.2.3. The calculation of branches. Let L be a singular fiber in a genus 3 pencil. We set

$$L = m_1 \Lambda_1 + m_2 \Lambda_2 + \cdots + m_h \Lambda_h + \Sigma + \Delta,$$

where Λ_i $(1 \leq i \leq h)$ are trunks in L, Σ is a branch and Δ is the other effective divisor which consists of (-2)-curves. We put $\Sigma \cdot \Lambda_i = p_i$ $(1 \leq i \leq h)$. We can easily determine all the possible sets of the positive integers $\{h, m_i, p_i\ (1 \leq i \leq h)\}$ which the branch Σ in singular fibers in genus three pencils may have. Then we calculate the configuration of the branch as follows. First we fix the integers $\{h, m_i, p_i\ (1 \leq i \leq h)\}$. Secondly we fix the numbers and multiplicities of the (-2)-curves in Σ which intersect each trunk and we also fix the intersection numbers between those (-2)-curves and the trunks. Thirdly we extend each chain of (-2)-curves whose one end is a (-2)-curve intersecting each trunk so that the multiplicities of (-2)-curves in the chain may form an arithmetical progression. Finally we join the other edges so that the equations (3.1) may hold. Using the lemmas in the subsection 3.2.2., we classified all the possible branches in singular fibers in pencils of curves of genus g (≤ 3). In order to show this classification, we introduce the following symbols:

Definition 3.1. Let kX, lY, mZ, nU be trunks with multiplicities k, l, m, n respectively. We represent a branch Σ by the following symbols:



(1) By the symbol (1), we denote a branch Σ which intersects a trunk kX and which satisfies $\Sigma \cdot X = p$.
(2) By the symbol (2), we denote a branch Σ which intersects two trunks kX, lY and which satisfies $\Sigma \cdot X = p$ and $\Sigma \cdot Y = q$.
(3) By the symbol (3), we denote a branch Σ which intersects three trunks kX, lY, mZ and which satisfies $\Sigma \cdot X = p$, $\Sigma \cdot Y = q$ and $\Sigma \cdot Z = r$.
(4) By the symbol (4), we denote a branch Σ which intersects four trunks kX, lY, mZ, nU and which satisfies $\Sigma \cdot X = p$, $\Sigma \cdot Y = q$, $\Sigma \cdot Z = r$ and $\Sigma \cdot U = s$.

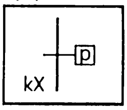
Moreover we regard the intersection among those trunks as a kind of branch. For example, in case that three trunks kX, lY, mZ intersect each other in one point with the intersection number $X \cdot Y = \alpha$, $Y \cdot Z = \beta$, $Z \cdot X = \gamma$, we regard this intersection as a branch Σ of type (3) as above, where Σ satisfies the following equations: $p = \Sigma \cdot X = l\alpha + m\gamma$, $q = \Sigma \cdot Y = k\alpha + m\beta$ and $r = \Sigma \cdot Z = k\gamma + l\beta$. (We call these *branches in a wide sense*.)

In Figure 2, we show all the branches in a wide sense which can appear in singular fibers of genus three pencils. We remark that this classification of a branch Σ is independent of the numerical types of a trunk Λ intersecting Σ, i.e., the selfintersection number Λ^2 and the arithmetic genus $p(\Lambda)$. Thus, this classification of branches can be used in the numerical classifications of singular fibers in pencils of curves of arbitrary genus.

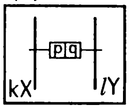
3.2.4. The numerical classification of singular fibers. We construct the singular fibers by combining the trunks and the above branches in a wide sense. We also define the following symbols to simplify the representation of singular fibers.

Definition 3.2. Let kX, lY be trunks with multiplicities k, l respectively.

(1)



(2)



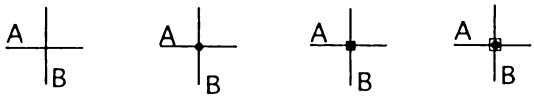
(1) By the symbol (1), we denote the combination of the branches $\Sigma_1, \Sigma_2, \dots, \Sigma_t$ which intersect a trunk kX and which satisfy $\Sigma_1 \cdot X + \Sigma_2 \cdot X + \cdots + \Sigma_t \cdot X = p$.

(2) By the symbol (2), we denote the combination of the branches in a wide sense $\Sigma_1, \Sigma_2, \dots, \Sigma_t$ which intersect two trunks kX, lY and which satisfy $\Sigma_1 \cdot X + \Sigma_2 \cdot X + \cdots + \Sigma_t \cdot X = p$ and $\Sigma_1 \cdot Y + \Sigma_2 \cdot Y + \cdots + \Sigma_t \cdot Y = q$.

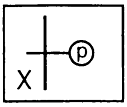
Using this symbols, we classify all the numerical types of singular fibers of genus three pencils in Figure 3. In this figure, the number at the right-hand side of each box is the number of numerical types of singular fibers which belongs to the symbol in the box. Since we followed the tradition, our numerical classification is a little finer than the exact numerical classification. Note that all these singular fibers really exist (see Winters [7]). By our calculation, the number of numerical types of singular fibers in genus three pencils is 4343, while in case of genus two pencils the number of numerical types of singular fibers is 140.

Figure 2. The classification of branches

In this figure the letter X,Y,Z,W (thick lines) represent trunks. Other lines represent (-2)-curves. The number on or by each curve means its multiplicity. We also denote the intersection number by the following symbols.

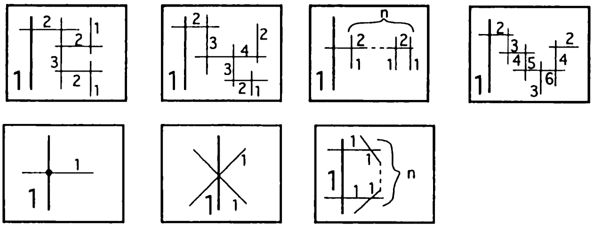


A·B = 1   A·B = 2   A·B = 3   A·B = 4

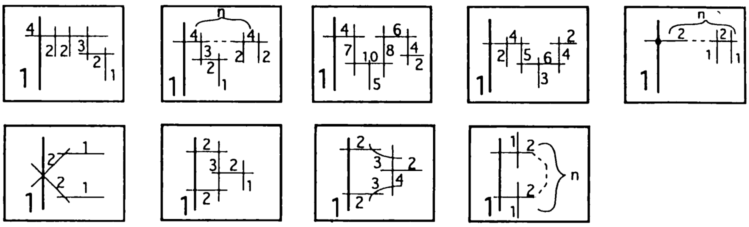
(a) 

· p=1,3,5   none

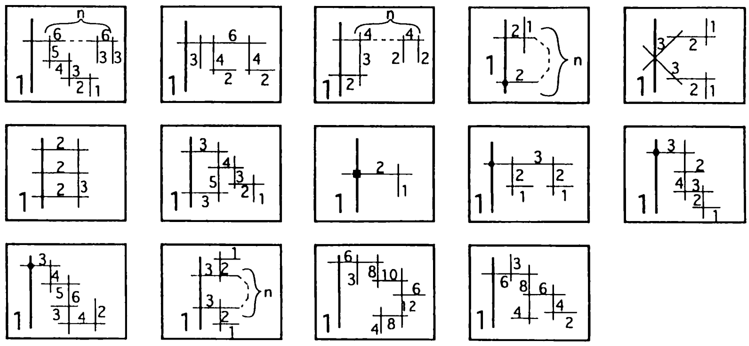
· p=2



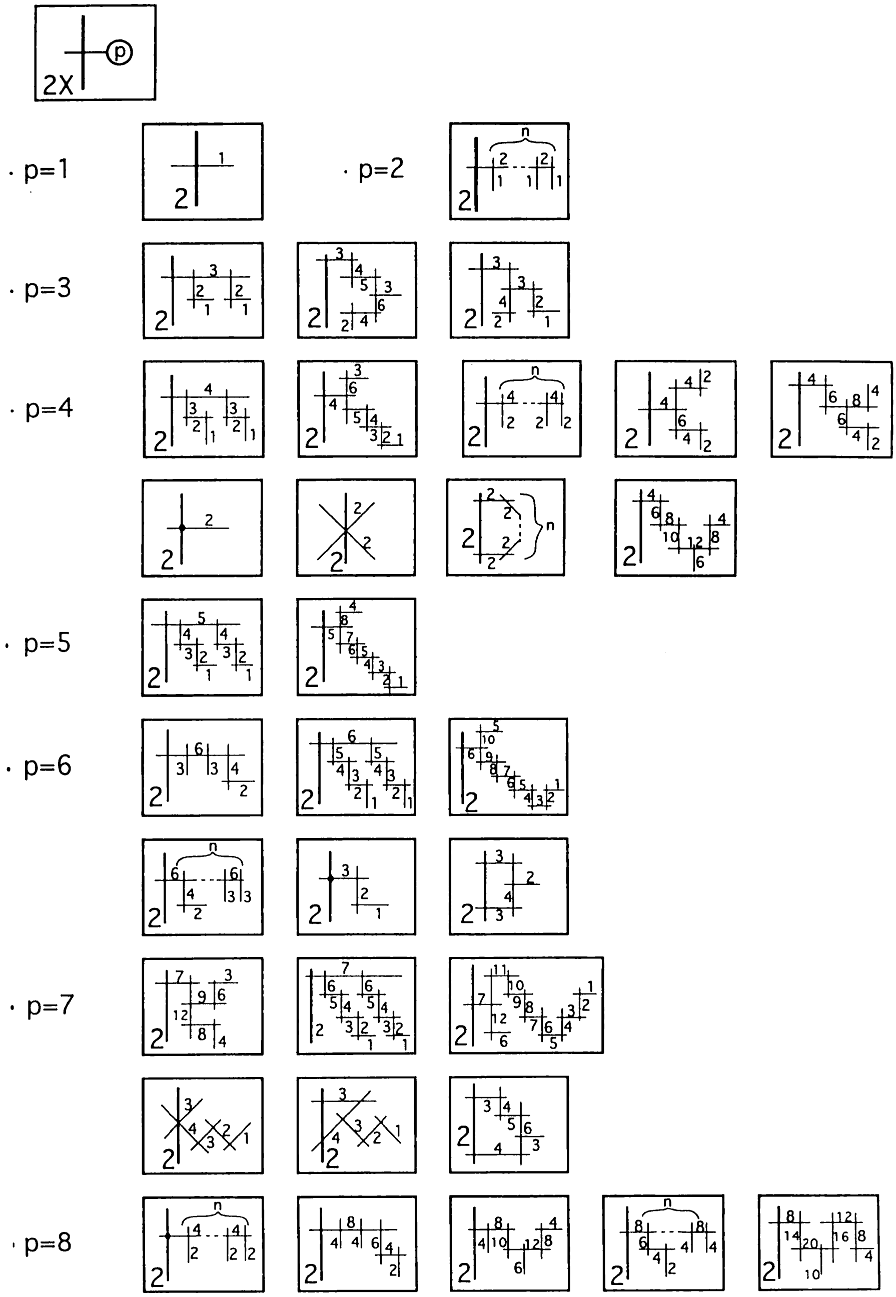
· p=4

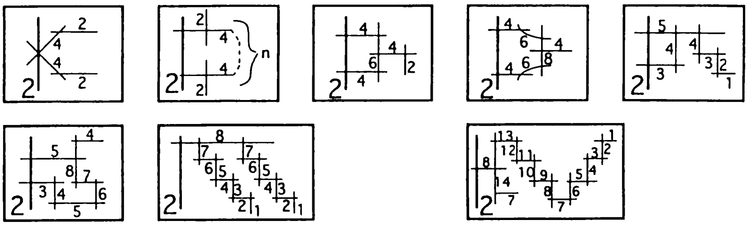


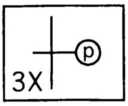
· p=6



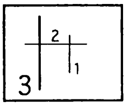
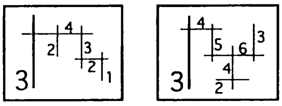
(b)

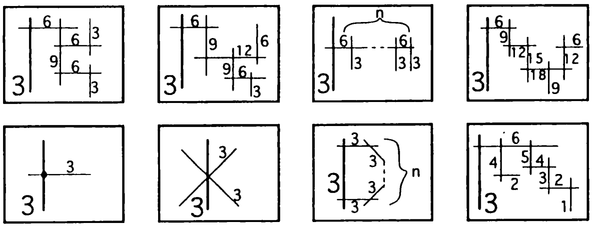


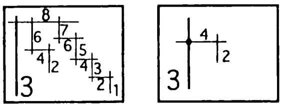


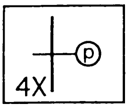
(c) 

· p=1,3,5,7,9 <u>none</u>

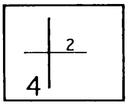
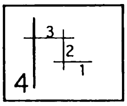
· p=2  · p=4 

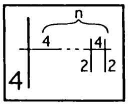
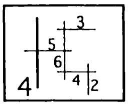
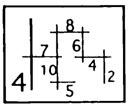
· p=6 

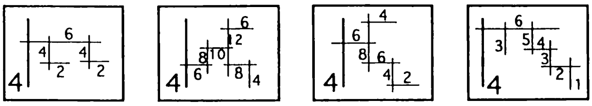
· p=8 

(d) 

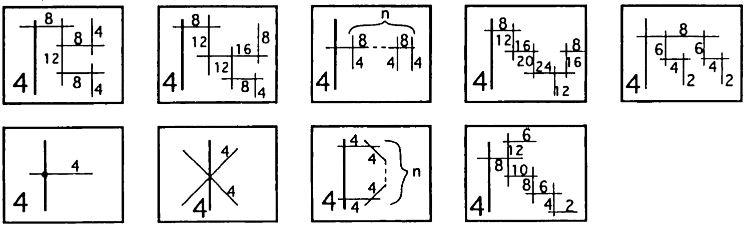
· p=1 <u>none</u>

· p=2  · p=3 

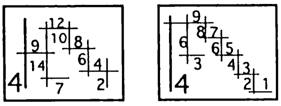
· p=4  · p=5  · p=7 

· p=6 

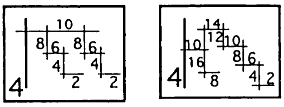
· p=8



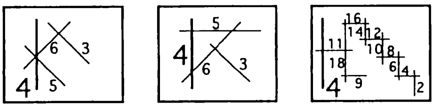
· p=9



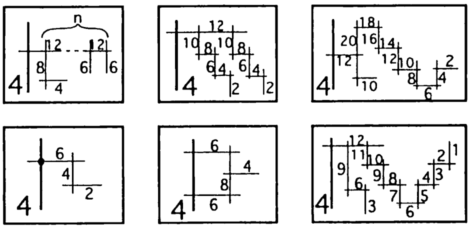
· p=10



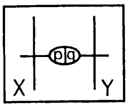
· p=11



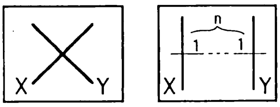
· p=12



(e)

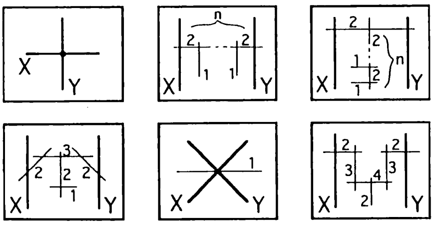


· —11—

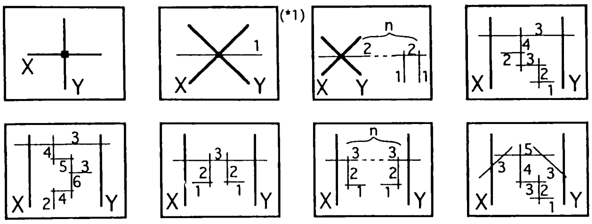


· —21— none

· —22—



· —33—

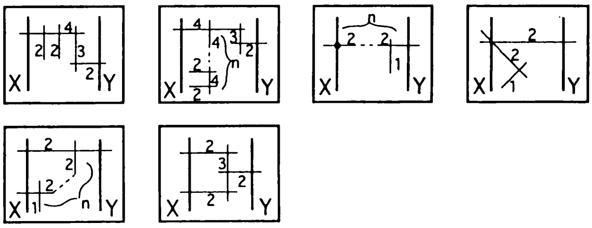


(*1) The intersection number X·Y is two, and the others are all one.

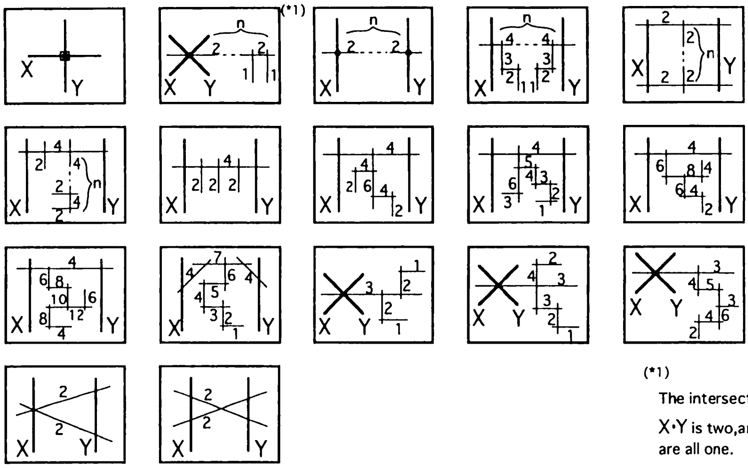
· —39— none (q=1,2)

· —49— none (q=1,3)

· —(4|2)—



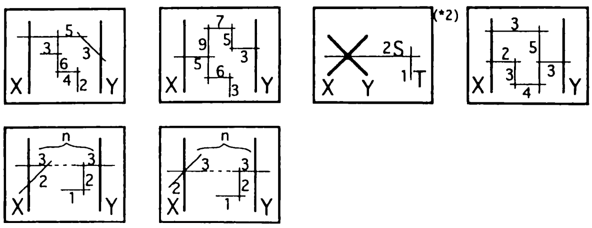
· —(4|4)—



(*1)
The intersection number
X·Y is two,and the others
are all one.

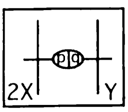
· —(5|q)— none (q=1,2)

· —(5|3)—



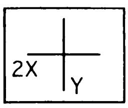
(*2)
The intersection number
X·S is two, and the others
are all one.

(f)

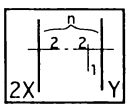


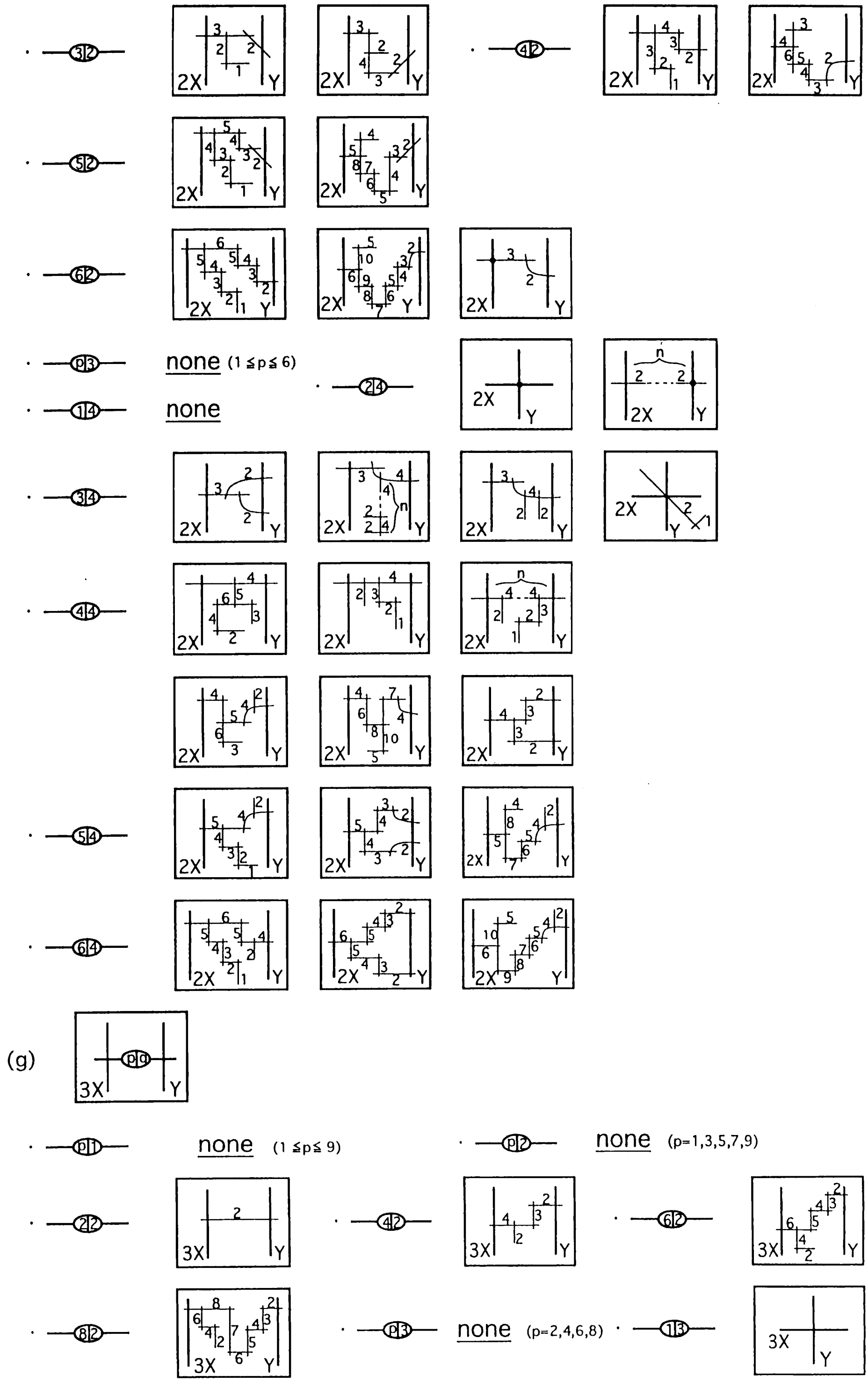
· —(p|1)— none (1 ≦ p ≦ 6)

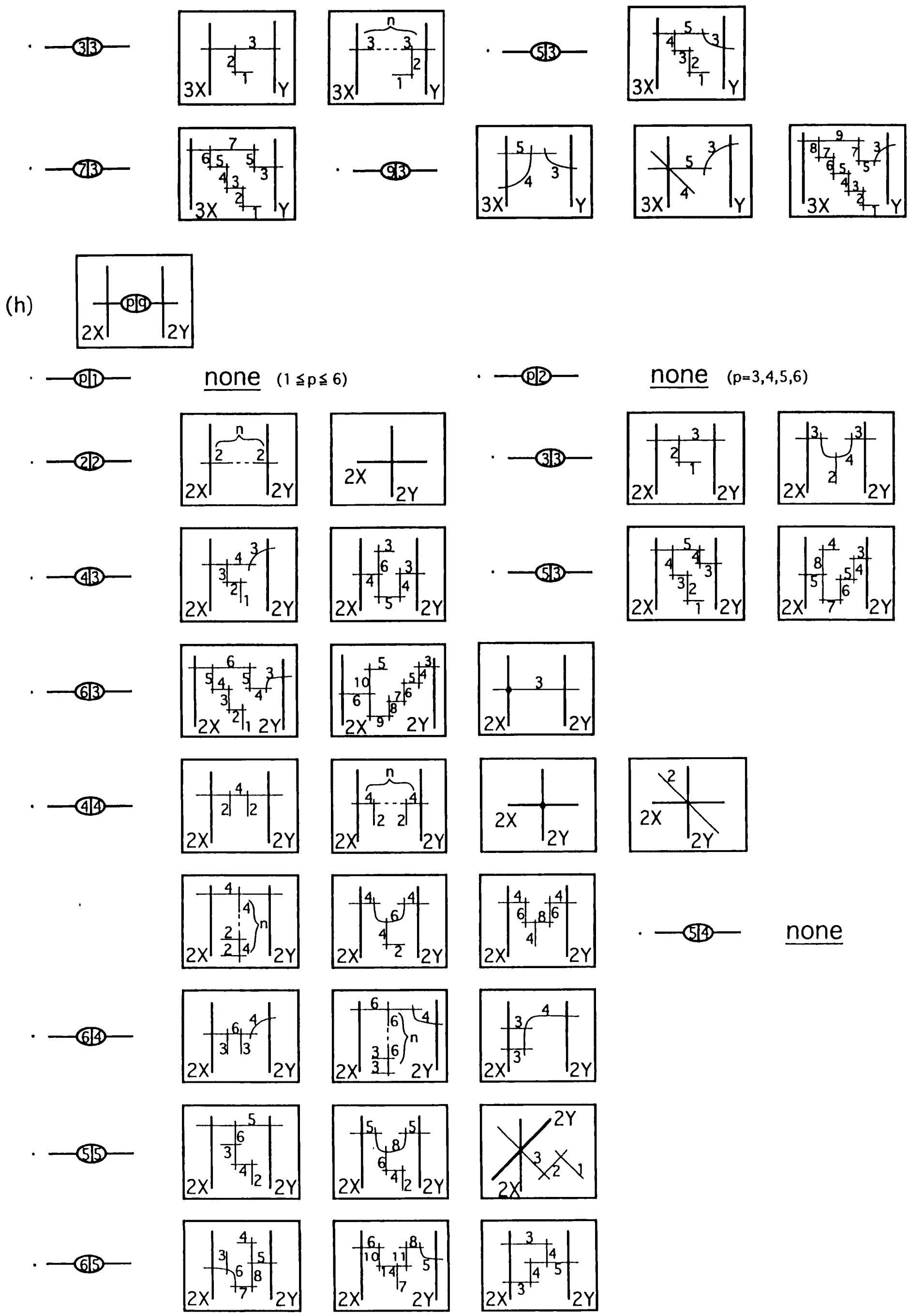
· —(1|2)—

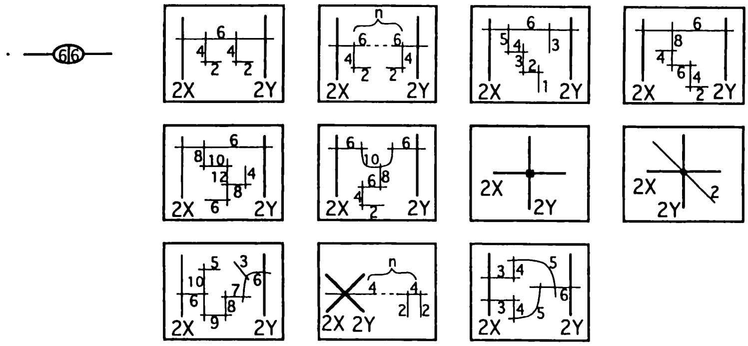


· —(2|2)—




none (1 ≤ p ≤ 6)
none
(g)
none (1 ≤ p ≤ 9)
none (p=1,3,5,7,9)
none (p=2,4,6,8)

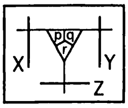

(h)
none (1 ≤ p ≤ 6)
none (p=3,4,5,6)
none

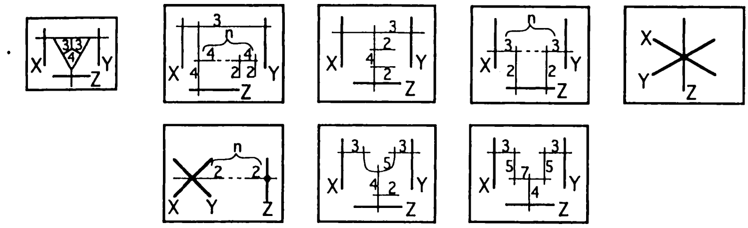


(*1)

(*1) The intersection number X·Y is two, and the others are all one.

(i)

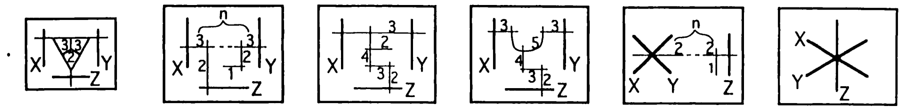




(*2)

(*2) X·Y = 1

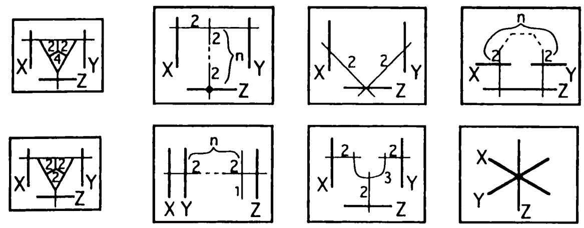
X·Z = Y·Z = 2



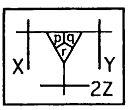
(*3)

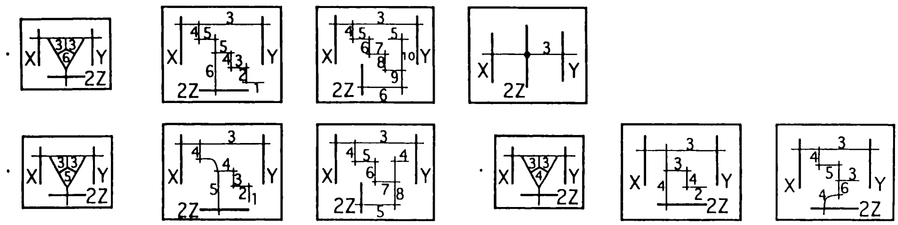
(*3) X·Y = 2

X·Z = Y·Z = 1



(j)





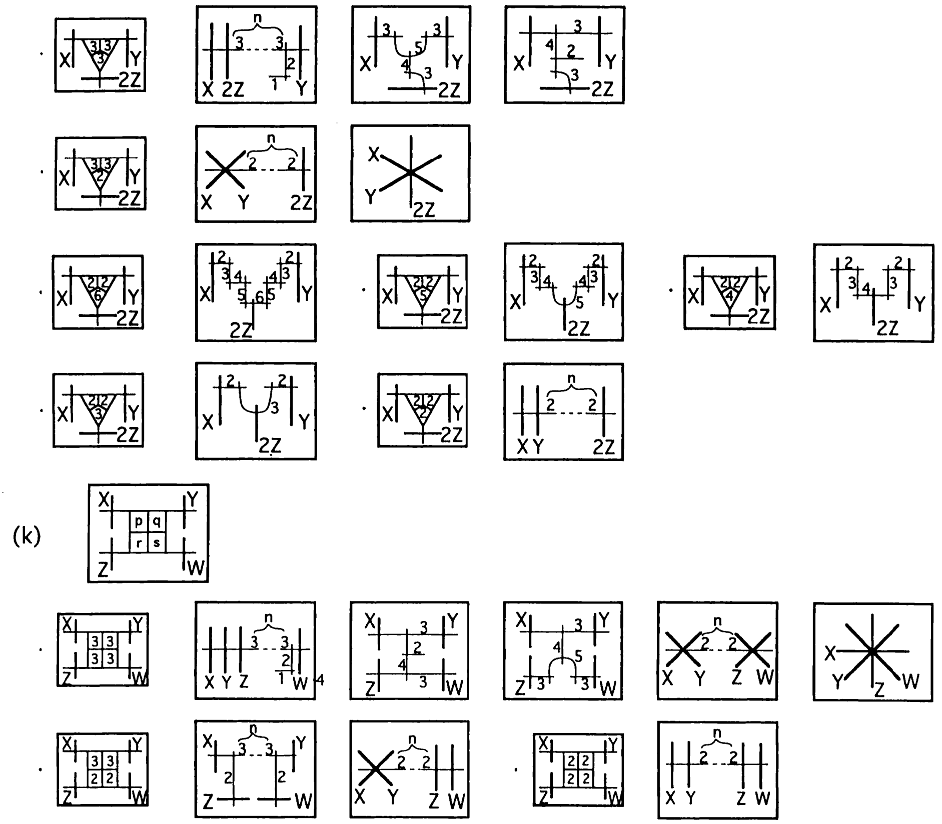
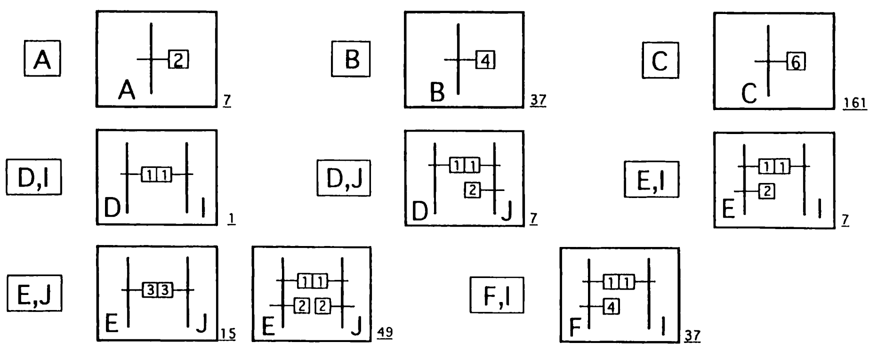
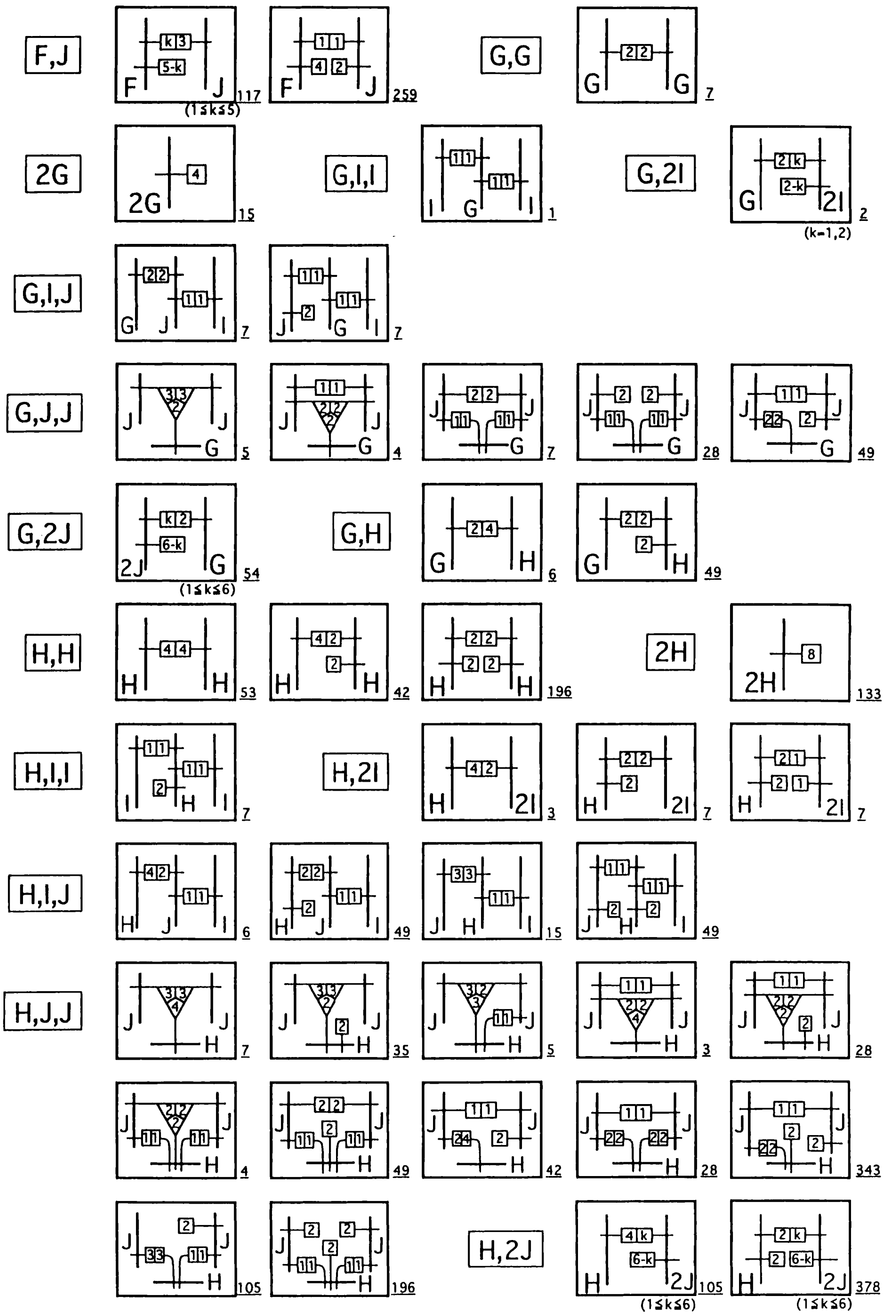
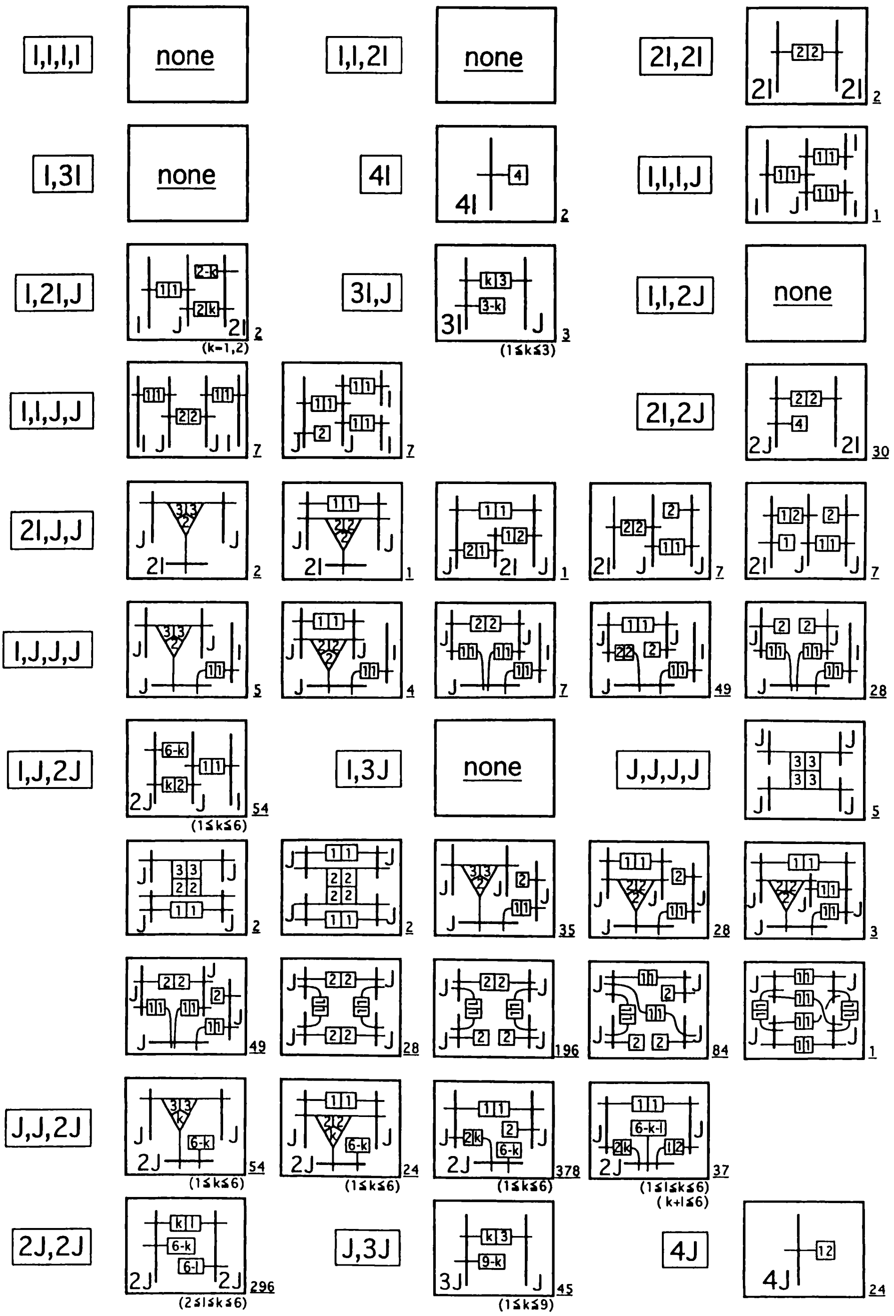


Figure 3. The classification of singular fibers





F,J
G,G
2G
G,I,I
G,2I
G,I,J
G,J,J
G,2J
G,H
H,H
2H
H,I,I
H,2I
H,I,J
H,J,J
H,2J


I,I,I,I
none
I,I,2I
none
2I,2I
2
I,3I
none
4I
2
I,I,I,J
1
I,2I,J
2
(k=1,2)
3I,J
3
(1≤k≤3)
I,I,2J
none
I,I,J,J
7
7
2I,2J
30
2I,J,J
2
1
1
7
7
I,J,J,J
5
4
7
49
28
I,J,2J
54
(1≤k≤6)
I,3J
none
J,J,J,J
5
2
2
35
28
3
49
28
196
84
1
J,J,2J
54
(1≤k≤6)
24
(1≤k≤6)
378
(1≤k≤6)
37
(1≤l≤k≤6)
(k+l≤6)
2J,2J
296
(2≤l≤k≤6)
J,3J
45
(1≤k≤9)
4J
24

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