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Complements of resultants and homotopy types

By

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§ 1. Introduction

For $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, we denote by $Q^d_{(n)}(\mathbf{K})$ the space consisting of all n-tuples $(p_1(z), \dots, p_n(z)) \in \mathbf{K}[z]^n$ of $\mathbf{K}$-coefficients monic polynomials of degree d such that $p_1(z) = p_2(z) = \cdots = p_n(z) = 0$ have no common *real* roots (but may have common *complex* roots). This space has an interesting topology and it has been studied from many different points of view. A. Kozlowski and the author already determined its homotopy type explicitely ([8]) except for the case $(\mathbf{K}, n) = (\mathbf{R}, 3)$. In this paper we shall investigate its homotopy type for this case. For this purpose, let us consider the map $j^d_{(n)} : Q^d_{(n)}(\mathbf{R}) \to \Omega_{[n]_2}\mathbf{R}\mathrm{P}^{n-1} \simeq \Omega S^{n-1}$ which is defined by

$$j^d_{(n)}(p_1(z), \dots, p_n(z))(t) = \begin{cases} [p_1(t) : \cdots : p_n(t)] & \text{if } t \in \mathbf{R} \\ [1 : 1 : \cdots : 1] & \text{if } t = \infty \end{cases}$$

for $(p_1(z), \dots, p_n(z)) \in Q^d_{(n)}(\mathbf{R})$ and $t \in S^1 = \mathbf{R} \cup \infty$, where $[d]_2$ denotes the number modulo 2. Here, we shall call a map $f : X \to Y$ as a *homotopy equivalence* (or *homology equivalence*) up to dimension d, if the induced homomorphism $f_* : \pi_j(X) \to \pi_j(Y)$ (or $f_* : H_j(X, \mathbf{Z}) \to H_j(Y, \mathbf{Z})$) is bijective when $j < d$ and surjective when $j = d$. In [8] we already obtained the following homological informations.

Theorem 1.1 ([8]). (1) *The map $j^d_{(3)} : Q^d_{(3)}(\mathbf{R}) \to \Omega S^2$ is a homology equivalence up to dimension d.*

(2)
$$H^j(Q^d_{(3)}(\mathbf{R}), \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } j = 0, 1, 2, \dots, d \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathrm{RRat}_d(2)$ denote the space consisting of all triples $(p_1(z), p_2(z), p_3(z)) \in \mathbf{R}[z]^3$ of monic real coefficients polynomials of degree d such that $p_1(z) = p_2(z) = p_3(z) = 0$ have no common roots. Note that $\mathrm{RRat}_d(2) \subset Q^d_{(3)}(\mathbf{R})$ and each $p = (p_1(z), p_2(z), p_3(z)) \in \mathrm{RRat}_d(2)$ uniquely determines the map $p : S^1 \to \mathbf{R}\mathrm{P}^2$ in a natural way. However, since this is dense in $\Omega_{[d]_2}\mathbf{R}\mathrm{P}^2$ but is not closed in $\Omega_{[d]_2}\mathbf{R}\mathrm{P}^2$, we shall study the closure $\mathrm{rat}_d(2) = \overline{\mathrm{RRat}_d(2)} \subset \Omega_{[d]_2}\mathbf{R}\mathrm{P}^2$. Then recently J. Mostovoy obtained the following interesting result in [10].

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Theorem 1.2 (Mostovoy, [10]). *The natural inclusion map*

$$\mathrm{rat}_d(2) \xrightarrow{\subset} \Omega_{[d]_2}\mathbf{RP}^2 \simeq \Omega S^2$$

is a homotopy equivalence up to dimension d.

It is known that there is a homotopy equivalence $\mathrm{rat}_d(2) \simeq Q^d_{(3)}(\mathbf{R})$, and we also obtain

Theorem 1.3. *The map $j^d_{(3)} : Q^d_{(3)}(\mathbf{R}) \to \Omega S^2$ is a homotopy equivalence up to dimension d.*

It follows from theorem 1.1 that the above homotopy stability dimension is best possible. Moreover, we may regard the space $Q^d_{(3)}(\mathbf{R})$ as a finite dimensional model for the infinite dimensional space ΩS^2. However, we can only know its low dimensional information and we would like to investigate the homotopy type of it explicitely. This is the main purpose of this paper and we shall prove the following 2 results:

Theorem A. *There is a homotopy equivalence $Q^{2m+1}_{(3)}(\mathbf{R}) \simeq S^1 \times J_m(\Omega S^3)$, where $J_m(\Omega S^3) = s^2 \cup e^4 \cup \cdots \cup e^{2(m-1)} \cup e^{2m} \subset \Omega S^3$ denotes the m-th stage James filtration of ΩS^3* ([7]).

Theorem B. *There is a homotopy equivalence $\sum Q^{2m}_{(3)}(\mathbf{R}) \simeq \Sigma(\Omega S^2)^{[2m]}$, where Σ denotes the reduced suspension and $X^{[d]}$ denotes the d dimensional skelton of a CW complex X.*

Remark. If d is an even integer, the fundamental group action of $Q^d_{(3)}(\mathbf{R})$ is non-trivial for higher dimensional homotopy groups and it may be not easy to classify its homotopy type directly.

It is easy to see the following 2 results by using theorem A.

Corollary C. *The univeral covering of the space $Q^{2m+1}_{(3)}(\mathbf{R})$ is homotopy equivalent to $J_m(\Omega S^3)$.*

Corollary D. *There is a ring isomorphism*

$$H^*(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) = E[x_1] \otimes \Gamma[y_2, y_4, \ldots, y_{2m}]/(y_i \cdot y_j : i + j \geq 2(m+1))$$

where $E[x_1]$ and $\Gamma[y_2, \ldots, y_{2m}]$ denote the exterior algebra over $\mathbf{Z}$ generated by x_1 and the divided polynomial algebra over $\mathbf{Z}$ generated by $y_2, \ldots, y_{2m}$, respectively. Here $|x_1| = 1$ and $|y_{2j}| = 2j$.

The main method of this paper is to use the results given in [4] and [8], which are obtained by Segal's scanning method ([4], [11]) and by the computations of spectral sequences ([8], [13]). Moreover, we shall also give the independent proof of theorem 1.3, and so we do not need theorem 1.2 in this paper.

The plan of this paper is as follows. In §2, we shall recall the general properties of π_1 action on homotopy groups and in § 3, we shall give the proof of theorem 1.3. Finally in § 4, we shall prove theorems A and B.

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§ 2. Basic principle of the homotopical stability

Recall several properties of π_1 actions on homotopy groups.

Definition. Let X be a connected space. Then the space X is called *n-simple* when $\pi_1(X)$ acts trivially on $\pi_j(X)$ for any $1 \leq j \leq n$. In particular, a space X is called *simple* when X is n-simple for any $n \geq 1$.

Since $\pi_1(X)$ action on $\pi_1(X)$ is given by the conjugation, a connected space X is 1-simple if and only if $\pi_1(X)$ is abelian. It is known that any connected H-space is simple. Moreover, if X is n-simple and Y is simply connected, $X \times Y$ is n-simple. For example, if X is a H-space and Y is simply connected, $X \times Y$ is simple.

Lemma 2.1. *Let $n, m \geq 1$ be integers and let $f : X \to Y$ be a continuous map such that $f_* : \pi_j(X) \to \pi_j(Y)$ is surjective for any $j \leq m$. Then if X is n-simple, Y is* $\min(n, m)$*-simple.*

Proof. Let $k = \min(n, m)$ and let $1 \leq j \leq k$ be any integer. It suffices to prove that $\pi_1(Y)$ action on $\pi_j(Y)$ is trivial. Let $y_1 \in \pi_1(Y)$ and $\pi_j(Y)$ be any elements. Since $f_* : \pi_s(X) \to \pi_s(Y)$ is surjective for $s = 1$ or $s = j$, there are elements $x_1 \in \pi_1(X)$ and $x \in \pi_j(X)$ such that, $y_1 = f_*(x_1)$ and $y = f_*(x)$. Since the diagram

$$\begin{array}{ccc} \pi_1(X) \times \pi_j(X) & \xrightarrow{f_* \times f_*} & \pi_1(Y) \times \pi_j(Y) \\ \Big\downarrow \cdot & & \Big\downarrow \cdot \\ \pi_j(X) & \xrightarrow{f_*} & \pi_j(Y) \end{array}$$

is commutative, $y_1 \cdot y = f_*(x_1) \cdot f_*(x) = f_*(x_1 \cdot x) = f_*(x) = y$. Hence $\pi_1(Y)$ action on $\pi_j(Y)$ is trivial.

The following lemma is well-known and we omit the proof.

Lemma 2.2. *Let $f : X \to Y$ be a homology equivalence up to dimension d. Then if X and Y are both $(d-1)$-simple, then the map f is a homotopy equivalence up to dimension d.*

Corollary 2.3. *Let $n, d \geq 1$ be integers and let $f : X \to Y$ be a homology equivalence up to dimension d. If X is n-simple, the map f is a homotopy equivalence* *up to dimension* $D(n;d)$, *where*

$$D(n;d) = \begin{cases} n+1 & (\text{if } n < d) \\ d & (\text{if } n \geq d) \end{cases}$$

Proof. Let $k = \min(n,d)$. From lemma 1.1, Y is k-simple. If $k \leq d-1$, then $k = n < d$ and it follows from lemma 2.2 that f is a homotopy equivalence up to dimension $k+1 = n+1$. If $k \geq d$, then $n \geq d$ and it follows from lemma 2.2 that f is a homotopy equivalence up to dimension d.

§3. Spaces $Q^d_{(3)}(\mathbf{R})$ and their homotopy stability

In this section, first recall several results which were already obtained in [8]. For this purpose, recall the definition of stabilization map $s_d : Q^d_{(3)}(\mathbf{R}) \to Q^{d+1}_{(3)}(\mathbf{R})$.

Definition. Let $Q^d_{(3)}(|z| < d)$ denote the subspace of $Q^d_{(3)}(\mathbf{R})$ consisting of all triples $(p_1(z), p_2(z), p_3(z)) \in Q^d_{(3)}(\mathbf{R})$ such that, for each $1 \leq j \leq 3$, any root $\alpha \in \mathbf{C}$ of $p_j(z) = 0$ satisfies the condition $|\alpha| < d$. Then we can identify $Q^d_{(3)}(\mathbf{R}) \cong Q^d_{(3)}(|z| < d)$. Let us choose 3 mutually distinct real numbers $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ such that $|\alpha_j| > d$ for each $1 \leq j \leq 3$. Then define the stabilization map $s_d : Q^d_{(3)}(\mathbf{R}) \to Q^{d+1}_{(3)}(\mathbf{R})$ by

$$\begin{array}{ccc} Q^d_{(3)} \xrightarrow{\cong} Q^d_{(3)}(|z| < d) & \longrightarrow & Q^{d+1}_{(3)}(\mathbf{R}) \\ (p_1(z), p_2(z), p_3(z)) & \to & ((z-\alpha_1)p_1(z), (z-\alpha_2)p_2(z), (z-\alpha_3)p_3(z)) \end{array}$$

Then recall the following result:

Proposition 3.1. (1) $s_d : Q^d_{(3)}(\mathbf{R}) \to Q^{d+1}_{(3)}(\mathbf{R})$ *is a homology equivalence up to dimension* d.

(2) $(s_d)_* : \pi_1(Q^d_{(3)}(\mathbf{R})) \xrightarrow{\cong} \pi_1(Q^{d+1}_{(3)}(\mathbf{R})) \xrightarrow{\cong} \mathbf{Z}$ *is bijective.*

(3) *The maps* $j^d_{(3)}$ *induces a homotopy equivalence*

$$j = \lim_{d\to\infty} j^d_{(3)} : \lim_{d\to\infty} Q^d_{(3)}(\mathbf{R}) \xrightarrow{\simeq} \Omega S^2$$

Proof. The assertions (1) and (3) follows from [8] and it suffices to prove (2). Analogous method given in appendix of [3] easily proves that $\pi_1(Q^d_{(3)}(\mathbf{R}))$ is abelian. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(Q^d_{(3)}(\mathbf{R})) & \xrightarrow{(s_d)_*} & \pi_1(Q^{d+1}_{(3)}(\mathbf{R})) \\ h \downarrow \cong & & h' \downarrow \cong \\ \mathbf{Z} = H_1(Q^d_{(3)}(\mathbf{R}), \mathbf{Z}) & \underset{\cong}{\xrightarrow{(s_d)_\#}} & H_1(Q^{d+1}_{(3)}(\mathbf{R}), \mathbf{Z}) = \mathbf{Z} \end{array}$$

Note that h and h' are isomorphisms by Hurewicz theorem. If $d \geq 2$, since s_d is a homology equivalence up to dimension d by (1), $(s_d)_{\#}$ is bijective. Similarly if $d = 1$, $(s_1)_{\#}$ is surjective. However, since $H_1(Q^1_{(3)}(\mathbf{R}), \mathbf{Z}) \cong H_1(Q^2_{(3)}(\mathbf{R}), \mathbf{Z}) \cong \mathbf{Z}$, $(s_1)_{\#}$ is also bijective. Therefore $(s_d)_* : \pi_1(Q^d_{(3)}(\mathbf{R})) \xrightarrow{\cong} \pi_1(Q^{d+1}_{(3)}(\mathbf{R})) \cong \mathbf{Z}$ is bijective for any d.

Remark. Define the map $\gamma_1 : S^1 \to Q^1_{(3)}(\mathbf{R})$ by

$$\gamma_1(e^{i\theta}) = (z + \cos\theta, z + \sin\theta, z) \qquad \text{for } e^{i\theta} \in S^1 = \{\alpha \in \mathbf{C} : |\alpha| = 1\}.$$

For $d \geq 2$, let $\gamma_d : S^1 \to Q^d_{(3)}(\mathbf{R})$ be the composite of maps

$$S^1 \xrightarrow{\gamma_1} Q^1_{(3)}(\mathbf{R}) \xrightarrow{s_1} Q^2_{(3)}(\mathbf{R}) \xrightarrow{s_2} \cdots \xrightarrow{s_{d-2}} Q^{d-1}_{(3)}(\mathbf{R}) \xrightarrow{s_{d-1}} Q^d_{(3)}(\mathbf{R})$$

Then it is easy to see that γ_d is the generator of $\pi_1(Q^d_{(3)}(\mathbf{R})) = \mathbf{Z}$.

Lemma 3.2. *There is a homotopy equivalence $Q^{2m+1}_{(3)}(\mathbf{R}) \simeq S^1 \times X_m$ for some simply connected space X_m.*

Remark. Mostovoy proves in [10] that there is a homotopy equivalence $\mathrm{rat}_{2m+1}(2) \simeq S^1 \times X_m$ for some simply connected space X_m. Since it is known that $\mathrm{rat}_d(2) \simeq Q^d_{(3)}(\mathbf{R})$, the result easily follows. However, here we shall give another proof, which is inspired by the discussion with A. Kozlowski.

Proof. Denote by $Q^d_{(n)}$ the space consisting of all n-tuples $(p_1(z), \ldots, p_n(z)) \in \mathbf{R}[z]^n$ of real coefficients polynomials satisfying the following 3 conditions:
(1) $\max\{\deg p_k(z) : 1 \leq k < n\} < \deg p_n(z)$.
(2) $p_n(z)$ is a monic polynomial of degree d.
(3) $p_1(z) = \cdots = p_n(z) = 0$ have no common *real* roots.
Then there is a homeomorphism $Q^d_{(n)}(\mathbf{R}) \cong Q^d_{(n)}$ and it suffices to show that there is a homotopy equivalence $Q^{2m+1}_{(3)} \simeq S^1 \times X_m$ for some simply connected space X_m. For this purpose, define the free S^1 action on $Q^{2m+1}_{(3)}$ by the rotation

$$\begin{array}{ccc} Q^{2m+1}_{(3)} \times S^1 & \longrightarrow & Q^{2m+1}_{(3)} \\ ((p_1, p_2, p_3), e^{i\theta}) & \longrightarrow & (p_1 \cos\theta - p_2 \sin\theta, p_1 \sin\theta + p_2 \cos\theta, p_3) \end{array}$$

Note that there is a fibration $S^1 \xrightarrow{\gamma} Q^{2m+1}_{(3)} \xrightarrow{q} Q^{2m+1}_{(3)}/S^1 = X_m$. It follows from the definition of the above action that the map $\gamma : S^1 \to Q^{2m+1}_{(3)}$ can be given by $\gamma(e^{i\theta}) = (z + \cos\theta, z + \sin\theta, z(z^2+1)^m)$. Since the map γ is homotopic to a generator of $\pi_1(Q^{2m+1}_{(3)})$, X_{2m+1} is simply connected. Next we would like to define the map $R : Q^d_{(3)} \to S^1$ such that $R \circ \gamma = \mathrm{id} : S^1 \to S^1$. If such a map R exists, the assertion easily follows.

Let $(p_1(z), p_2(z), p_3(z)) \in Q^{2m+1}_{(3)}$ be any element. Because $p_3(z) \in \mathbf{R}[z]$ is a monic polynomial of degree $2m+1$, it is expressed as

$$p_3(z) = \prod_{j=1}^{l}(z - \alpha_j) \cdot g(z)$$

where $\begin{cases} \text{each } \alpha_j \in \mathbf{R}, \quad \alpha_1 \le \alpha_2 \cdots \le \alpha_{l-1} \le \alpha_l, \quad \text{and} \\ g(z) \in \mathbf{R}[z] \text{ is a monic polynomial such that it has no real roots.} \end{cases}$

Let us consider the product

$$R_1(p_1, p_2, p_3) = \prod_{j=1}^{l}(p_1(\alpha_j) + ip_2(\alpha_j))^{\varepsilon_j} \in \mathbf{C}^* \text{ (where } \varepsilon_j = (-1)^{j-1}) \tag{3.3}$$

Since the product $(p_1(\alpha_{j-1}) + ip_2(\alpha_{j-1}))^{\varepsilon_{j-1}}(p_1(\alpha_j) + ip_2(\alpha_j))^{\varepsilon_j} = 1$ if $\alpha_{j-1} = \alpha_j$, the map $R_1 : Q_{(3)}^{2m+1} \to \mathbf{C}^*$ is well-defined and continuous. Define the map $R : Q_{(3)}^{2m+1} \to S^1$ by $R(p_1, p_2, p_3) = R_1(p_1, p_2, p_3)/|R_1(p_1, p_2, p_3)|$. A direct computation shows that $R \circ \gamma = \mathrm{id} : S^1 \to S^1$. This completes the proof.

Remark. The above proof is failed if d is an even integer. The main reason is that if d is an even integer some monic polynomial $f(z) \in \mathbf{R}[z]$ does not have any real roots. So R_1 is not well-defined when d is even.

Corollary 3.4. *If $d = 2m + 1$, then $Q_{(3)}^{2m+1}(\mathbf{R})$ is simple.*

Corollary 3.5. *If $d = 2m$, then the space $Q_{(3)}^{2m}(\mathbf{R})$ is $(2m - 1)$-simple.*

Proof. From the above corollary 3.4, $Q_{(3)}^{2m-1}(\mathbf{R})$ is simple. Since the stabilization map $Q_{(3)}^{2m-1}(\mathbf{R}) \to Q_{(3)}^{2m}(\mathbf{R})$ is a homology equivalence up to dimension $(2m - 1)$, the assertion follows from lemma 3.1.

Although theorem 1.3 can be obtained by Mostovoy's theorem 1.2, we give independent proof of it.

Proof of theorem 1.3. Consider the stabilization map $s_d : Q_{(3)}^d(\mathbf{R}) \to Q_{(3)}^{d+1}(\mathbf{R})$. It follows from (3.4), (3.5) that both spaces $Q_{(3)}^d(\mathbf{R})$ and $Q_{(3)}^{d+1}(\mathbf{R})$ are $(d-1)$-simple. Recall that (from (3) of proposition 3.1) the maps $j_{(3)}^d$ induces the homotopy equivalence

$$j : \lim_{d\to\infty} Q_{(3)}^d(\mathbf{R}) \xrightarrow{\simeq} \Omega S^2$$

Hence the map $j_{(3)}^d$ is a homotopy equivalence up to dimension d.

§4. Homotopy types

In this section, we give the proofs of theorems A and B. For this purpose, note the following results:

Lemma 4.1. *Let $X^{[d]}$ denote the d-skelton of a CW complex X. Then there is a homotopy equivalence $(\Omega S^2)^{[2m+1]} \simeq S^1 \times J_m(\Omega S^3)$.*

Proof. Since there is a homotopy equivalence $\Omega S^2 \simeq S^1 \times \Omega S^3$, this is clear.

Corollary 4.2. *The space* $(\Omega S^2)^{[2m+1]} \simeq S^1 \times J_m(\Omega S^3)$ *is simple.*

Now we can prove theorem A.

Proof of theorem A. It follows from the cellular approximation theorem that there is a cellular map $f_d : Q^d_{(3)}(\mathbf{R}) \to \Omega S^2$ such that f_d is homotopic to $j^d_{(3)}$. Consider the case $d = 2m+1$. Since $f_{2m+1}(Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]}) \subset (\Omega S^2)^{[2m+1]}$, define the map $g_{2m+1} : Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]} \to (\Omega S^2)^{[2m+1]}$ by the restriction

$$g_{2m+1} = f_{2m+1}|Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]} : Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]} \to (\Omega S^2)^{[2m+1]}.$$

However, since $H^j(Q^{2m+1}_{(3)}(\mathbf{R})) = 0$ for any $j \geq 2m+2$ and $(\Omega S^2)^{[2m+1]}$ is simple, using the obstruction theory, the map g_{2m+1} extends to the map

$$\bar{g}_{2m+1} : Q^{2m+1}_{(3)}(\mathbf{R}) \to (\Omega S^2)^{[2m+1]}$$

Consider the homotopy commutative diagram

$$(4.3)\qquad \begin{array}{ccc}
Q^{2m+1}_{(3)}(\mathbf{R}) & \xrightarrow{\bar{g}_{2m+1}} & (\Omega S^2)^{[2m+1]} \\
\cup\big\uparrow i & & =\big\downarrow \\
Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]} & \xrightarrow{g_{2m+1}} & (\Omega S^2)^{[2m+1]} \\
\cap\big\downarrow i & & \cap\big\downarrow i' \\
Q^{2m+1}_{(3)}(\mathbf{R}) & \xrightarrow{j^{2m+1}_{(3)}} & \Omega S^2
\end{array}$$

If $j \leq 2m$, the above diagram induces the commutative diagram

$$(4.4)\qquad \begin{array}{ccc}
H_j(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) & \xrightarrow{(\bar{g}_{2m+1})_*} & H_j((\Omega S^2)^{[2m+1]}, \mathbf{Z}) \\
\cong\big\uparrow i_* & & =\big\downarrow \\
H_j(Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]}, \mathbf{Z}) & \xrightarrow{(g_{2m+1})_*} & H_j((\Omega S^2)^{[2m+1]}, \mathbf{Z}) \\
\cong\big\downarrow i_* & & \cong\big\downarrow i'_* \\
H_j(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) & \xrightarrow[\cong]{(j^{2m+1}_{(3)})_*} & H_j(\Omega S^2, \mathbf{Z})
\end{array}$$

where vertical homomorphisms are all isomorphisms. Since $(j^{2m+1}_{(3)})_*$ is bijective by theorem 1.1, the induced homomorphism $(\bar{g}_{2m+1})_* : H_j(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) \xrightarrow{\cong} H_j((\Omega S^2)^{[2m+1]}, \mathbf{Z})$ is bijective when $j \leq 2m$.

Similarly, when $j = 2m+1$, the diagram (4.3) induces the commutative diagram

$$
\begin{array}{ccc}
\mathbf{Z} = H_{2m+1}(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) & \xrightarrow{(\bar{g}_{2m+1})_*} & H_{2m+1}((\Omega S^2)^{[2m+1]}, \mathbf{Z}) = \mathbf{Z} \\
\text{surjective} \uparrow i_* & & \downarrow = \\
\mathbf{Z} = H_{2m+1}(Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]}, \mathbf{Z}) & \xrightarrow{(g_{2m+1})_*} & H_{2m+1}((\Omega S^2)^{[2m+1]}, \mathbf{Z}) = \mathbf{Z} \\
\text{surjective} \downarrow i_* & & \text{surjective} \downarrow i'_* \\
\mathbf{Z} = H_{2m+1}(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) & \underset{\text{surjective}}{\xrightarrow{(j^{2m+1}_{(3)})_*}} & H_{2m+1}(\Omega S^2, \mathbf{Z}) = \mathbf{Z}
\end{array}
\tag{4.5}
$$

Since $(j^{2m+1}_{(3)})_* : \mathbf{Z} = H_{2m+1}(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) \to H_{2m+1}(\Omega S^2, \mathbf{Z}) = \mathbf{Z}$ is surjective by proposition 3.1, it is bijective. By the same reason, induced homomorphisms

$$
\begin{cases}
i_* : \mathbf{Z} = H_{2m+1}(Q^{2m+1}_{(3)}(\mathbf{R})^{[2m+1]}, \mathbf{Z}) \xrightarrow{\cong} H_{2m+1}(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) = \mathbf{Z} \\
j_* : \mathbf{Z} = H_{2m+1}((\Omega S^2)^{[2m+1]}, \mathbf{Z}) \xrightarrow{\cong} H_{2m+1}(\Omega S^2, \mathbf{Z}) = \mathbf{Z}
\end{cases}
$$

are also isomorphisms. Hence from the diagram (4.5), the induced homomorphism $(\bar{g}_{2m+1})_* : H_j(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) \xrightarrow{\cong} H_j((\Omega S^2)^{[2m+1]}, \mathbf{Z})$ is an isomorphism for $j = 2m+1$. Therefore, the induced homomorphism $(\bar{g}_{2m+1})_* : H_j(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) \xrightarrow{\cong} H_j((\Omega S^2)^{[2m+1]}, \mathbf{Z})$ is bijective for any $j \leq 2m+1$.

However, since $H_j(Q^d_{(3)}(\mathbf{R}), \mathbf{Z}) = H_j((\Omega S^2)^{[2m+1]}, \mathbf{Z}) = 0$ for any $j \geq 2m+2$ by theorem 1.1, $(\bar{g}_{2m+1})_* : H_j(Q^{2m+1}_{(3)}(\mathbf{R}), \mathbf{Z}) \xrightarrow{\cong} H_j((\Omega S^2)^{[2m+1]}, \mathbf{Z})$ is an isomorphism for any integer j.

Note that $(\Omega S^2)^{[2m+1]} \simeq S^1 \times J_m(\Omega S^3)$ and $Q^{2m+1}_{(3)}(\mathbf{R})$ are simple. Hence, by the Whitehead theorem, $\bar{g}_{2m+1}$ is a homotopy equivalence. Thus there is a homotopy equivalence

$$
Q^{2m+1}_{(3)}(\mathbf{R}) \underset{\simeq}{\xrightarrow{\bar{g}_{2m+1}}} (\Omega S^2)^{[2m+1]} \simeq S^1 \times J_m(\Omega S^3)
$$

Next consider the homotopy type of $Q^{2m}_{(3)}(\mathbf{R})$. However, since it is not simple, it seems difficult to classify its homotopy type explicitely. So we shall consider its single suspension.

Lemma 4.6. *There are cofibre sequences*

$$
\begin{cases}
(a) \quad Q^{2m-1}_{(3)}(\mathbf{R}) \xrightarrow{s_{2m-1}} Q^{2m}_{(3)}(\mathbf{R}) \xrightarrow{p} S^{2m} \\
(b) \quad (\Omega S^2)^{[2m-1]} \xrightarrow{\subset} (\Omega S^2)^{[2m]} \xrightarrow{q} S^{2m}
\end{cases}
$$

Proof. Since the proofs are similar, we only give the proof of (a). Let C_m denote the mapping cone of the map s_{2m-1}. Then from theorem 1.1, note that

$$
\tilde{H}_j(C_m, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } j = 2m \\ 0 & \text{if } j \neq 2m \end{cases}
$$

Since $\pi_1(s_{2m-1})$ is bijective from (3.1), C_m is simply connected. Hence there is a homotopy equivalence $C_m \simeq S^{2m}$.

Now we can prove theorem B.

Proof of theorem B. Consider the commutative diagram

$$\begin{tikzcd} S^{2m} \arrow[r] \arrow[d, "="'] & \Sigma Q^{2m-1}_{(3)}(\mathbf{R}) \arrow[r, "\Sigma s_{2m-1}"] \arrow[d, "\Sigma \bar{g}_{2m-1}"', "\simeq"] & \Sigma Q^{2m}_{(3)}(\mathbf{R}) \\ S^{2m} \arrow[r] & \Sigma(\Omega S^2)^{[2m-1]} \arrow[r, "\subset"] & \Sigma(\Omega S^2)^{[2m]} \end{tikzcd}$$

where horizontal sequences are cofibre sequences. Hence there is a map $h_{2m} : \Sigma Q^{2m}_{(3)}(\mathbf{R}) \to \Sigma(\Omega S^2)^{[2m]}$ such that the diagram

$$\begin{tikzcd} S^{2m} \arrow[r] \arrow[d, "="'] & \Sigma Q^{2m-1}_{(3)}(\mathbf{R}) \arrow[r, "\Sigma s_{2m-1}"] \arrow[d, "\Sigma \bar{g}_{2m-1}"', "\simeq"] & \Sigma Q^{2m}_{(3)}(\mathbf{R}) \arrow[d, "h_{2m}"'] \\ S^{2m} \arrow[r] & \Sigma(\Omega S^2)^{[2m-1]} \arrow[r, "\subset"] & \Sigma(\Omega S^2)^{[2m]} \end{tikzcd}$$

is homotopy commutative. Since $\Sigma \bar{g}_{2m-1}$ is a homotopy equivalence and $\Sigma Q^{2m}_{(3)}(\mathbf{R})$ and $\Sigma(\Omega S^2)^{[2m]}$ are simply connected, h_{2m} is a homotopy equivalence.

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