J. Math. Kyoto Univ. (JMKYAZ)
39-4 (1999) 619–647

Odd generators of the mod 3 cohomology of finite H-spaces

By

Yutaka HEMMI and James P. LIN

Abstract

In this paper we derive a formula about the action of $\mathscr{P}^1$ on the odd generators of the mod 3 cohomology of a finite simply connected mod 3 H-space with associative mod 3 homology. This formula will be used in a subsequent paper to classify all possible cohomology rings that can occur as the mod 3 cohomology of such H-spaces.

0. Introduction

In this paper we study mod 3 finite H-spaces whose mod 3 homology is associative. The mod 3 cohomology of these spaces has been studied by many authors [1, 2, 5, 3, 9]. In particular, there are formulas relating the action of the Steenrod algebra on the even degree algebra generators.

However, very little is known about the action of the Steenrod algebra on the odd degree generators. Only in special cases where the H-space has no even algebra generators or when the finite H-space is homotopy associative, do we have any kinds of results. [13]

In this paper we derive results about certain odd degree generators in the general case of an associative homology ring. Notice that an odd sphere localized at the prime three is an H-space with associative mod 3 homology, so it is possible to have a cohomology generator in any odd degree that does not lie in the image of a Steenrod operation and is annihilated by any Steenrod operation.

One of the results we derive is:

Main Theorem (Theorem 5.1). *Let X be a finite simply connected* mod 3 *H-space with $H_*(X;\mathbf{F}_3)$ associative. If $\bar{x} \in QH^{6n+3}(X)$ and $\mathscr{P}^{3n+1}\bar{x} \in \operatorname{im}\mathscr{P}^2$ and $18n+16$ is not the degree of a transpotence element of $H^*(\Omega X)$ then $\mathscr{P}^1\sigma^*(x) \in \operatorname{im}\mathscr{P}^2$.*

The proof of results of this kind involves using a third order operation that is motivated by work of Zabrodsky [13]. Third order operations involve lifting into

Communicated by Prof. K. Ueno, October 29, 1998
AMS subject classification: 55S20, 55S45, 55R35, 55D45, 55G20

a three-stage Postnikov system. Here lies much of the difficulty in the proofs. Instead of lifting our H-space X into the third stage, we lift $P_3\Omega X$, the third projective space of the loop space.

The three-stage Postnikov system E_1 contains an element with nonassociative coproduct. In particular there is an element

$$v \in H^*(E_1; \mathbf{F}_3)$$

with

$$\bar{\Delta} v = u(\mathscr{P}^1 u) \otimes \mathscr{P}^1 u + \mathscr{P}^1 u \otimes u(\mathscr{P}^1 u) + (\mathscr{P}^2 u)u \otimes u + u \otimes (\mathscr{P}^2 u)u$$

where $u \in H^{6n+3}(E_1; \mathbf{F}_3)$.

We construct new invariants to study obstructions to preserving this coproduct.

In the process of proving the theorems we also calculate the cohomology of the projective spaces $P_k(\Omega X)$ for X a finite mod 3 H-space.

In a subsequent paper, results of this type will be used to give a complete classification of the the mod 3 cohomology rings of finite mod 3 H-spaces with associative mod 3 homology.

The reader may ask what is special about the prime three? Shouldn't one be able to obtain analogous results for all odd primes? We note that for primes p greater than three and all n, there is an H-space $B_n(p)$ with $H^*(B_n(p); \mathbf{F}_p) = \Lambda(x_{2n+1}, x_{2n+2p-1})$ with $P^1 x_{2n+1} = x_{2n+2p-1}$. So $\mathscr{P}^1 \sigma^*(x_{2n+1})$ is not in the image of $\mathscr{P}^2$. So the Main Theorem is not true for primes greater than three. However, we might guess that some extra homotopy associativity assumptions are needed.

The main problem seems to be the problem of lifting the mod p H-space or an appropriate projective space through p-stages of a Postnikov system. In Zabrodsky and Harper's original papers [3] they employ power space technology and the assumption that the H-space only has a few odd generators. Current developments in lifting theory allow us to lift to a third stage, but to lift to higher stages poses several problems in need of further study.

In section 1 we provide an outline of the proof so the reader has an overview of the ideas and strategy. Throughout the entire paper the symbol X will be used to denote a simply connected mod 3 finite H-space with associative mod 3 homology.

All spaces will be connected and basepointed. All homotopies will respect the basepoint. All homologies and cohomologies will be of finite type. Unless otherwise specified, the coefficients for homology and cohomology will be the field $\mathbf{F}_3$.

1. Outline of the proof

The proof of the theorem can be divided into the following steps.

Step 1. There exists a three-stage Postnikov system

$$\begin{tikzcd}
\Omega K_2 \arrow[r, "j_1"] & E_1 \arrow[r, "v"] \arrow[d, "q_1"] & K(18n+17) \\
\Omega K_1 \arrow[r, "j_0"] & E_0 \arrow[r, "w_1"'] \arrow[d, "q_0"] & K_2 \\
 & K \arrow[r, "w_0"'] & K_1
\end{tikzcd}$$

of H-spaces. All maps are H-maps except v. $v \in H^{18n+17}(E_1)$ has reduced coproduct

$$\bar{\Delta} v = u(\mathscr{P}^1 u) \otimes \mathscr{P}^1 u + \mathscr{P}^1 u \otimes u(\mathscr{P}^1 u) + (\mathscr{P}^2 u)u \otimes u + u \otimes (\mathscr{P}^2 u)u$$

where $u \in H^{6n+3}(E_1)$. See chapter 2.

Step 2. Since $H_*(X)$ is associative, there exists an even $\mathscr{A}(3)$ sub-Hopf algebra B with induced map

$$QB \to QH^*(X)$$

is isomorphism in even degrees.

If $R = \{x \in H^*(X) \mid \bar{\Delta} x \in B \otimes H^*(X)\}$, then R is an $\mathscr{A}(3)$ coalgebra and algebra generators of $H^*(X)$ have representatives in R. Further R^{odd} has no decomposables.

Step 3. Suppose $\bar{x} \in QH^{6n+3}(X)$, $\bar{y} \in QH^{18n-2}(X)$ satisfy

$$2\mathscr{P}^{3n+1}\bar{x} = \mathscr{P}^2 \bar{y}.$$

If $x, y \in R$ represent $\bar{x}, \bar{y}$ then

$$2\mathscr{P}^{3n+1}x - \mathscr{P}^2 y = 0.$$

This produces a commutative diagram (2.5)

$$\begin{tikzcd}
 & E_0 \arrow[d, "q_0"] \\
X \arrow[ur, "f_0"] \arrow[r, "f"'] & K
\end{tikzcd}$$

Step 4. X is filtered by projective spaces

$$\Sigma\Omega X \subseteq P_2\Omega X \subseteq \cdots \subseteq P_k\Omega X \subseteq \cdots \subseteq X$$

and the multiplication $\mu : X \times X \to X$ is "filtered" because there are maps $\mu_{j,k}$: $P_j\Omega X \times P_k\Omega X \to P_{j+k}\Omega X$ such that we have a commutative diagram

$$\begin{tikzcd}
\Sigma\Omega X \times \Sigma\Omega X \arrow[r, "\mu_{1,1}"] \arrow[d] & P_2\Omega X \arrow[d] \\
P_j\Omega X \times P_k\Omega X \arrow[r, "\mu_{j,k}"] \arrow[d] & P_{j+k}\Omega X \arrow[d] \\
X \times X \arrow[r, "\mu"] & X
\end{tikzcd}$$

where the vertical maps are the inclusions.

Step 5. We have

$$\begin{tikzcd}
 & & E_0 \arrow[d, "q_0"] \\
P_3\Omega X \arrow[r, "i(3)"] & X \arrow[r, "f"'] \arrow[ur, "f_0"] & K
\end{tikzcd}$$

By altering $f_0 i(3)$ by elements in the fibre, we can produce a commutative diagram

$$\begin{tikzcd}
 & E_1 \arrow[d, "q_1"] \\
 & E_0 \arrow[d, "q_0"] \\
P_3\Omega X \arrow[r, "fi(3)"'] \arrow[uur, "\bar{f}_1"] \arrow[ur, "\bar{f}_0"] & K
\end{tikzcd}$$

$H^*(P_3\Omega X)$ is computed in terms of $H^*(X)$. (Theorem 3.4)

Step 6. Using step 4, we can define maps

$$(\Sigma\Omega X)^{\wedge 3} \xrightarrow{\bar{\xi}_X} \Omega\Sigma P_3\Omega X$$

$$X^{\wedge 3} \xrightarrow{\bar{\eta}_X} \Omega\Sigma X \xrightarrow{r_X} X$$

such that the induced cohomology map satisfies $(r_X\bar{\eta}_X)^* = (\bar{\Delta}\otimes 1)\bar{\Delta}$.

Given maps

$$P_3\Omega X \xrightarrow{g} Y$$

$$X \xrightarrow{h} Y$$

we define maps

$$(\Sigma\Omega X)^{\wedge 3} \xrightarrow{c_3(g)} Y$$

$$X^{\wedge 3} \xrightarrow{D_3(h)} Y$$

If $Y = K(\mathbf{F}_p, n)$ then $D_3(h)^*(i_n) = (\bar{\Delta} \otimes 1)\bar{\Delta}h^*(i_n)$, and if $g = hi(3)$ then $c_3(g)^*(i_n) = (\sigma^* \otimes \sigma^* \otimes \sigma^*)(\bar{\Delta} \otimes 1)\bar{\Delta}h^*(i_n)$. (Lemma 4.2)

Step 7. There is a commutative diagram

$$\begin{tikzcd}
 & E_1 \arrow[d, "q_1"] \\
 & E_0 \arrow[d, "q_0"] \\
(\Sigma\Omega X)^{\wedge 3} \arrow[r, "c_3(f)"'] \arrow[ur, "c_3(\bar{f}_0)"'] \arrow[uur, "c_3(\bar{f}_1)"] & K
\end{tikzcd}$$

Step 8. Given $v \in H^{18n+17}(E_1)$, we have

$$c_3(v\bar{f}_1) = vc_3(\bar{f}_1) + D_3(v)\varphi$$

for some map φ. (Theorem 4.4) We remark that this derivation formula is by no means obvious. In fact several references claim that $D_{gf} = gD_f + D_{g(f \wedge f)}$ for the H-deviation. This formula does not hold in general. In the proof of the formula for $c_3(v\bar{f}_1)$ it should become apparent why the formula for D_{gf} does not hold in general.

Step 9. The formula for $c_3(v\bar{f}_1)$ is used to prove the Main Theorem. (Theorem 5.1)

2. Construction of the Postnikov system

In this section we construct a three-stage Postnikov system. The second stage will be an infinite loop space but the third stage will not even be homotopy associative. The third stage will be the fibre of a map with nontrivial A_3 invariant. There will be a cohomology class in the third stage with a nonassociative coproduct.

With only minor modification we are building the Postnikov tower described in [13]. The modification is that our first k-invariant is different to allow for an argument using downward induction on the degree. We recommend that the reader look at [13] for details.

To streamline our notation, all coefficients will be assumed to be $\mathbf{F}_3$ unless otherwise specified. The symbol $K(n_1, n_2, \ldots, n_k)$ will be used to denote

$$\prod_{i=1}^{k} K(\mathbf{F}_3, n_i)$$

a product of Eilenberg MacLane spaces in degrees $n_1, n_2, \ldots, n_k$.

Recall that a map between Eilenberg MacLane spaces is determined by its value on the fundamental cohomology classes. We now define E_0, the second stage of our Postnikov tower. Let n be a positive integer.

Define $w_0 : K(6n+3, 18n-1) \to K(18n+7)$ by

$$w_0^*(i_{18n+7}) = 2\mathscr{P}^{3n+1} i_{6n+3} - \mathscr{P}^2 i_{18n-1}.$$

Then let E_0 be the fibre of w_0. Note that w_0 is an infinite loop map, so E_0 is an infinite loop space.

Further

$$\mathscr{P}^1 \mathscr{P}^{3n+1} = 2P^{3n+2} \tag{2.1}$$

and $\mathscr{P}^1 \mathscr{P}^2 = 0$, so

$$\mathscr{P}^1 w_0^*(i_{18n+7}) = 0$$

We have a diagram

$$\begin{tikzcd}
K(18n+6) \arrow[r, "j_0"] & E_0 \arrow[r, "w_1"] \arrow[d, "q_0"] & K(18n+10) \\
& K(6n+3, 18n-1) \arrow[r, "w_0"'] & K(18n+7)
\end{tikzcd}$$

By (2.1) $Bw_0 : K(6n+4, 18n) \to K(18n+8)$ has

$$\begin{aligned}
\mathscr{P}^1 (Bw_0)^*(i_{18n+8}) &= \mathscr{P}^1 (2\mathscr{P}^{3n+1} i_{6n+4} - \mathscr{P}^2 i_{18n}) \\
&= \mathscr{P}^{3n+2} i_{6n+4} = i_{6n+4}^3.
\end{aligned}$$

If follows that there is an element [5, 13]

$$w_1 : E_0 \to K(18n+10)$$

with

$$(w_1 j_0)^*(i_{18n+10}) = \mathscr{P}^1 i_{18n+6}$$

$w_1^*(i_{18n+10})$ is a transpotence element and has nonzero A_3 obstruction.

Further, by altering w_1 by elements in $\operatorname{im} q_0^*$, we may choose w_1 so that

$$\mathscr{P}^2 w_1^*(i_{18n+10}) = 0.$$

In fact, We have the following homotopy commutative diagram by a similar method as in [13].

$$
\begin{tikzcd}
K(18n+6) \arrow[r, "\mathscr{P}^1"] \arrow[d, "j_0"'] & K(18n+10) \arrow[d] \arrow[dr, "\mathscr{P}^2"] & \\
E_0 \arrow[r, "h"] \arrow[ur, "w_1"] \arrow[d, "q_0"'] & \hat{E}_0 = \text{fiber of } 2\mathscr{P}^{3n+2} \arrow[r, "\hat{w}"] \arrow[d, "\hat{q}_0"] & K(18n+18) \arrow[d, "\tilde{q}_0"] \\
K(6n+3, 18n-1) \arrow[r, "proj"] \arrow[d, "w_0"'] & K(6n+3) \arrow[r, "g"] \arrow[d, "2\mathscr{P}^{3n+2}"] & \tilde{E}_0 = \text{fiber of } \mathscr{P}^2 \arrow[dl] \\
K(18n+7) \arrow[r, "\mathscr{P}^1"'] & K(18n+11) \arrow[r, "\mathscr{P}^2"'] & K(18n+19)
\end{tikzcd}
$$

By construction, $\mathscr{P}^2 w_1^*(i_{18n+10}) \in \ker j_0^* \cap PH^{18n+18}(E_0)$. So

$$\mathscr{P}^2 w_1 \simeq \hat{w}h.$$

It suffices to construct $\hat{w}$, h so that $\hat{w}h$ is null homotopic.

We have

$$\hat{E}_0 \simeq K(6n+3) \times K(18n+10) \qquad \text{as } H\text{-spaces by [13]}$$

and

$$\hat{w}^*(i_{18n+18}) = z \otimes 1 + 1 \otimes \mathscr{P}^2 i_{18n+10},$$

where $z \in PH^*(K(6n+3))$. Altering g by $\tilde{q}_0 z$ and $\hat{w}$ by z, the diagram remains commutative and

$$\hat{w}^*(i_{18n+18}) = 1 \otimes \mathscr{P}^2 i_{18n+10}.$$

Now consider $\Omega^2 E_0 \simeq F_0 \times K(6n+1)$, where F_0 is the fibre of $\mathscr{P}^2 : K(18n-3) \to K(18n+5)$

$$
\begin{tikzcd}
K(18n+4) \arrow[d, "i_0"] & \\
F_0 \arrow[d, "r_0"] & \\
K(18n-3) \arrow[r, "\mathscr{P}^2"] & K(18n+5)
\end{tikzcd}
$$

Then

$$(\sigma^*)^2 h^*(i_{18n+10}) = \gamma_0 \otimes 1 + 1 \otimes \alpha i_{6n+1} \in PH^*(\Omega^2 E_0),$$

where $i_0^*(\gamma_0) = \mathscr{P}^1 i_{18n+4}$, $\gamma_0 \in PH^*(F_0)$, $\alpha \in \mathscr{A}(3)$.

Hence $i_0^*(\mathscr{P}^2 \gamma_0) = 0$, so

$$\mathscr{P}^2 \gamma_0 = r_0^*(\delta i_{18n-3}),$$

where $\delta \in \mathscr{A}(3)$ with $\deg\delta = 19$. We may assume δ is a sum of admissible operations. Since there are no admissibles in degree 19, $\delta i_{18n-3} = 0$, so

$$\mathscr{P}^2\gamma_0 = 0.$$

Changing h by $q_0^*(\alpha i_{6n+3})$, we may assume $(\sigma^*)^2 h^*(i_{18n+10}) = \gamma_0 \otimes 1$ and

$$\mathscr{P}^2 h^*(i_{18n+10}) \in PH^{18n+18}(E_0) \cap \ker(\sigma^*)^2.$$

Further, $\mathscr{P}^2 h^*(i_{18n+10})$ is not decomposable since $H^{6n+6}(E_0) = q_0^* H^{6n+6}(K(6n+3)) = 0$.

Now $(\sigma^*)^2 : QH^{18n+18}(E_0) \to PH^{18n+16}(\Omega^2 E_0)$ is monic, so $\mathscr{P}^2 h^*(i_{18n+10}) = 0$ and $\hat{w}h$ is null homotopic. Hence we may choose w_1 so that

$$\mathscr{P}^2 w_1^*(i_{18n+10}) = 0.$$

Let E_1 be the fibre of w_1.

$$\begin{tikzcd} K(18n+9) \arrow[r, "j_1"] & E_1 \arrow[d, "q_1"] & \\ & E_0 \arrow[r, "w_1"'] & K(18n+10) \end{tikzcd}$$

Then by [13] there exists an element $v \in H^{18n+17}(E_1)$ with

$$(\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})\bar{\Delta} v = \mathscr{P}^2(u \otimes u \otimes u) \qquad \text{where } u = q_1^* q_0^*(i_{6n+3}).$$

By [13] we have

$$\bar{\Delta} v = u(\mathscr{P}^1 u) \otimes \mathscr{P}^1 u + \mathscr{P}^1 u \otimes u(\mathscr{P}^1 u) + (\mathscr{P}^2 u)u \otimes u + u \otimes (\mathscr{P}^2 u)u. \tag{2.2}$$

and $j_1^*(v) = \mathscr{P}^2 i_{18n+9}$.

It follows that

$$\begin{aligned}(\bar{\Delta} \otimes 1)\bar{\Delta} v = {} & u \otimes \mathscr{P}^1 u \otimes \mathscr{P}^1 u - \mathscr{P}^1 u \otimes u \otimes \mathscr{P}^1 u \\ & + \mathscr{P}^2 u \otimes u \otimes u - u \otimes \mathscr{P}^2 u \otimes u. \end{aligned} \tag{2.3}$$

We have the following three-stage Postnikov system

$$\begin{tikzcd} K(18n+9) \arrow[r, "j_1"] & E_1 \arrow[r, "v"] \arrow[d, "q_1"] & K(18n+17) \\ K(18n+6) \arrow[r, "j_0"] & E_0 \arrow[r, "w_1"'] \arrow[d, "q_0"] & K(18n+10) \\ & K(6n+3, 18n-1) \arrow[r, "w_0"'] & K(18n+7) \end{tikzcd}$$

Recall if X is a finite H-space with $H_*(X;\mathbf{F}_3)$ associative, we have the following results.

There is an $\mathscr{A}(3)$ subHopf algebra $B \subseteq H^*(X)$ with the induced map

$$QB \to QH^*(X)$$

an isomorphism in even degrees. Further $\sigma^*(B) = 0$ where σ^* is cohomology suspension.

If $R = \{x \in H^*(X) \mid \bar{\Delta}x \in B \otimes H^*(X)\}$, then R is an $\mathscr{A}(3)$ coalgebra and the induced map

$$R \to H^*(X) \to QH^*(X)$$

is an isomorphism in odd degrees. Therefore every odd generator has a representative in R, and there are no odd decomposables in R.

Now suppose $x \in R^{6n+3}$ and $y \in R^{18n-1}$ satisfy $2\mathscr{P}^{3n+1}x = \mathscr{P}^2 y + d$ where d is decomposable. Then since $d = 2\mathscr{P}^{3n+1}x - \mathscr{P}^2 y \in R^{18n+7}$, we must have $d = 0$.

So $2\mathscr{P}^{3n+1}x = \mathscr{P}^2 y$.

Hence $2\mathscr{P}^{3n+1}\bar{\Delta}x = \mathscr{P}^2\bar{\Delta}y$. Let

$$\bar{\Delta}x = \sum b_i \otimes r_i \qquad \text{for } b_i \in B, \quad r_i \in R$$

$$\bar{\Delta}y = \sum b_i' \otimes r_i' \qquad \text{for } b_i' \in B, \quad r_i' \in R.$$

Then $2\mathscr{P}^{3n+1}\sum b_i r_i = \mathscr{P}^2 \sum b_i' r_i'$.

Let $\hat{x} = x + \sum b_i r_i$, $\hat{y} = y + \sum b_i' r_i'$. Then

$$2\mathscr{P}^{3n+1}\hat{x} = \mathscr{P}^2\hat{y}. \tag{2.4}$$

We also note that

$$\bar{\Delta}\hat{x} \text{ and } \bar{\Delta}\hat{y} \text{ lie in } I(B)H^*(X) \otimes H^*(X) + H^*(X) \otimes I(B)H^*(X).$$

Hence if $X \xrightarrow{f} K(6n+3, 18n-1)$ satisfies $f^*(i_{6n+3}) = \hat{x}$, $f^*(i_{18n-1}) = \hat{y}$, we have a lifting

$$\begin{tikzcd} & E_0 \arrow[r, "w_0"] \arrow[d, "q_0"] & K(18n+10) \\ X \arrow[ur, "f_0"] \arrow[r, "f"'] & K(6n+3, 18n-1) & \end{tikzcd} \tag{2.5}$$

3. Lifting to E_1

We currently do not know how to lift X up to E_1. However, the H-space X is filtered by projective spaces

$$\Sigma\Omega X \subseteq P_2\Omega X \subseteq \cdots \subseteq P_j\Omega X \subseteq \cdots \subseteq X. \tag{3.1}$$

We denote $i(j) : P_j\Omega X \to X$ to be the composition of the inclusions. The goal of this chapter will be to show there is a commutative diagram

$$\begin{tikzcd}
 & & E_1 \arrow[d] \\
 & & E_0 \arrow[d] \\
P_3\Omega X \arrow[r, "i(3)"'] \arrow[uurr, "\bar{f}_1"] \arrow[urr, "\bar{f}_0"] & X \arrow[r, "f"'] & K
\end{tikzcd} \tag{3.2}$$

Note that $\bar{f}_0$ is not necessarily $f_0 i(3)$.

There are several advantages to this approach. First, $H^*(P_3\Omega X)$ has at most three fold nonzero cup products, so many decomposables in $H^*(X)$ vanish when we map them into $H^*(P_3\Omega X)$. Second, the multiplication $\mu: X \times X \to X$ is "filtered" in the following sense. There exists a commutative diagram

$$\begin{tikzcd}
\Sigma\Omega X \times \Sigma\Omega X \arrow[r, "\mu_{1,1}"] \arrow[d, "i_1 \times 1"'] & P_2\Omega X \arrow[d, "i_2"] \\
P_2\Omega X \times \Sigma\Omega X \arrow[r, "\mu_{2,1}"] \arrow[d] & P_3\Omega X \arrow[d] \\
P_j\Omega X \times P_k\Omega X \arrow[r, "\mu_{j,k}"] \arrow[d, "i(j)\times i(k)"'] & P_{j+k}\Omega X \arrow[d, "i(j+k)"] \\
X \times X \arrow[r, "\mu"'] & X
\end{tikzcd} \tag{3.3}$$

for $1 \le j$, $1 \le k$. The vertical maps are the inclusions of (3.1). The inclusion $\Sigma\Omega X \xrightarrow{\varepsilon} X$ of (3.1) induces cohomology suspension, so understanding the $\mu_{j,k}$ will allow us to control

$$[\mu(\varepsilon \times \varepsilon)]^* = (\sigma^* \otimes \sigma^*)\Delta : H^*(X) \to H^*(\Omega X) \otimes H^*(\Omega X).$$

In this chapter we construct diagram (3.2). In chapter 4 we show how (3.2) can provide us with information about the action of the Steenrod algebra on $H^*(\Omega X)$.

Third, if $\psi: X \to X$ is $\psi(x) = x^2$, we have a commutative ladder

$$\begin{tikzcd}
\Sigma\Omega X \arrow[d, "\Sigma\Omega\psi"'] & \subseteq & P_2\Omega X \arrow[d, "P_2\Omega\psi"] & \subseteq & \cdots & \subseteq & P_k\Omega X \arrow[d, "P_k\Omega\psi"] & \subseteq & \cdots & \subseteq & X \arrow[d, "\psi"] \\
\Sigma\Omega X & \subseteq & P_2\Omega X & \subseteq & & \subseteq & P_k\Omega X & \subseteq & \cdots & \subseteq & X
\end{tikzcd}$$

We will show that $P_3\Omega X$ is a space with

$$(P_3\Omega\psi)^* : H^*(P_3\Omega X) \to H^*(P_3\Omega X)$$

has properties similar to that of a power space. That is $\varphi = [(P_3\Omega\psi)^{3^t}]^*$ induces multiplication by two for some $t > 0$ on some of the algebra generators and $H^*(P_3\Omega X)$ splits as the direct sum of eigenspaces with respect to φ.

From chapter 2, (2.5) we have a commutative diagram

$$\begin{tikzcd}
 & & E_0 \arrow[r, "w_0"] \arrow[d, "q_0"] & K(18n+10) \\
P_3\Omega X \arrow[r, "i(3)"] & X \arrow[ur, "f_0"] \arrow[r, "f"'] & K &
\end{tikzcd}$$

Note that $w_0 f_0 \in H^{18n+10}(X)$.

By [9], all even generators of $H^*(X)$ lie in degrees congruent to $2 \bmod 6$. So

$$w_0 f_0 \text{ is decomposable in } H^*(X).$$

It will be useful to compute $H^*(P_3\Omega X)$. Recall that for each integer $j > 0$, there are cofibration sequences

$$P_j\Omega X \longrightarrow P_{j+1}\Omega X \xrightarrow{\alpha_j} (\Sigma\Omega X)^{\wedge j} \xrightarrow{\beta_j} \Sigma P_j\Omega X.$$

These sequences induce exact triangles

$$\begin{tikzcd}[column sep=small]
\bar{H}^*(\Omega X) \arrow[dr] & & H^*(P_2\Omega X) \arrow[ll, "i_1^*"'] \arrow[dr, "\beta_2^*"] & & H^*(P_3\Omega X) \arrow[ll, "i_2^*"'] & & H^*(X) \arrow[ll, "i(3)^*"'] \\
 & \bar{H}^*(\Omega X)^{\otimes 2} \arrow[ur, "\alpha_2^*"] & & \bar{H}^*(\Omega X)^{\otimes 3} \arrow[ur, "\alpha_3^*"] & & &
\end{tikzcd}$$

We have a short exact sequence

$$0 \longleftarrow \operatorname{im} i_2^* \longleftarrow H^*(P_3\Omega X) \xleftarrow{\alpha_3^*} \frac{\bar{H}^*(\Omega X)^{\otimes 3}}{\operatorname{im} \beta_2^*} \longleftarrow 0. \tag{3.4}$$

Further $\beta_2^*\alpha_2^* = d_1 = \bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta}$ [11]. $\beta_2^*(i_1^*)^{-1}$ represents the differential d_2 in the Eilenberg Moore spectral sequence.

For X a finite H-space, we have the Borel decomposition

$$H^*(X) \cong \Lambda(x_1, \ldots, x_\ell) \otimes \frac{\mathbf{F}_3[y_1, \ldots, y_k]}{y_1^{3^{f_1}}, \ldots, y_k^{3^{f_k}}}.$$

By [9], we may assume $y_j = \beta_1 \mathcal{P}^{s_j} x_j$ where degree $x = 2s_j + 1$, for $j = 1, \ldots k$. Then by [7], we have the following coalgebra decomposition

$$H^*(\Omega X) \cong \Gamma_3[\sigma^*(x_1) \ldots, \sigma^*(x_k)] \otimes \Gamma[\sigma^*(x_{k+1}), \ldots, \sigma^*(x_\ell)]$$
$$\otimes \Gamma[\varphi_{3^{f_1}}(y_1), \ldots, \varphi_{3^{f_k}}(y_k)].$$

Γ_3 is a divided coalgebra truncated at height three. $\varphi_{3^{f_s}}(y_s)$ is the transpotence element related to y_s truncated at height 3^{f_s}.

In the Eilenberg Moore spectral sequence with

$$E_2 = \mathrm{Cotor}_{H^*(\Omega X)}(\mathbf{F}_3, \mathbf{F}_3) \qquad \text{and} \qquad E_\infty = Gr\, H^*(X)$$

we have by [8]

$$d_{3^{f_s}-1}[\varphi_{3^{f_s}}(y_s)] = \frac{1}{3}\sum \binom{3}{i}[\sigma^*(x_s)^i|\sigma^*(x_s)^{3-i}]^{3^{f_s}} \in E_{2(3^{f_s})-1} \tag{3.5}$$

These are the only differentials in the spectral sequence. We conclude

$$\beta_2^*(i_1^*)^{-1}$$

is trivial and

$$\mathrm{im}\,\beta_2^* = \mathrm{im}\,\beta_2^*\alpha_2^* = \mathrm{im}\,d_1 = (\bar{\Delta}\otimes 1 - 1\otimes\bar{\Delta})\bar{H}^*(\Omega X)^{\otimes 2}.$$

Since $\mathrm{Cotor}^3_{H^*(\Omega X)}(\mathbf{F}_3, \mathbf{F}_3) = \ker d_1/\mathrm{im}\, d_1$ we can find a vector space summand S such that

$$\frac{H^*(\Omega X)^{\otimes 3}}{\mathrm{im}\,\beta_2^*} \cong \mathrm{Cotor}^3_{H^*(\Omega X)}(\mathbf{F}_3, \mathbf{F}_3) \oplus S$$

According to Adams, [8] if $FC^*(\Omega X)$ is the cobar construction on the cochains $C^*(\Omega X)$, then [8, p. 143]

$$H^*(FC^*(\Omega X)) \text{ is isomorphic as algebras to } H^*(X).$$

Then let

$$x_i' = i(3)^*(x_i), \qquad y_i' = i(3)^*(y_i), \qquad i_1^* i_2^*(ty_i) = \varphi_{3^{f_i}}(y_i).$$

We say an element in the cobar construction has weight j if it is represented by a linear combination of terms of the form $[u_i|\cdots|u_j]$.

Theorem 3.1.

$$H^*(P_3\Omega X) \cong \frac{\Lambda(x_i',\ldots,x_\ell') \otimes \Lambda(ty_1,\ldots,ty_k)\otimes \mathbf{F}_3[y_1',\ldots,y_k']}{\textit{elements of weight} \geq 4} \oplus \alpha_3^*(S)$$

x_i', ty_i have weight 1 and lie in odd degrees. y_i' have weight two and lie in even degrees. The x_i', ty_i, y_i' may be chosen to be eigenvectors of $\varphi = [(P_3\Omega\psi)^3]^*$. Further $\alpha_3^*(S)$ splits into a direct sum of eigenspaces of φ.

Proof. By [8] y_i' are represented in the cobar construction of elements

$$\sum \frac{1}{3}\binom{3}{i}[\sigma^*(x_i)^i|\sigma^*(x_i)^{3-i}]$$

so they have weight two. x_i' and ty_i are represented by $[\sigma^*(x_i)]$ and $[\varphi_{3^{f_i}}(y_i)]$ so they have weight 1. The product structure in $FC^*(\Omega X)$ is given by juxtaposition, so the product of a weight j and weight k element has weight $j+k$.

$\operatorname{Cotor}^3_{H^*(\Omega X)}(\mathbf{F}_3, \mathbf{F}_3)$ consists of elements of weight three.

Since they map monomorphically to $H^*(P_3\Omega X)$ by (3.4), we have nonzero weight three products. Since

$$\sigma^*(x_i) \neq 0, \qquad \sigma^*(x_i) = i_1^* i_2^* i(3)^*(x_i), \qquad \text{so } x_i' \neq 0.$$

By (3.5) $ty_i \neq 0$. Any nonzero weight two elements map nontrivially to $H^*(P_2\Omega X)$ by i_2^*. Any weight three elements lie in $\alpha_3^* \operatorname{Cotor}^3_{H^*(\Omega X)}(\mathbf{F}_3, \mathbf{F}_3)$.

Thus, we have the algebra decomposition described in the proposition.

Note if $x \in R$ and $\psi : X \to X$ is the squaring map, then ψ is the composition

$$\psi : X \xrightarrow{\Delta} X \times X \xrightarrow{\mu} X.$$

Hence $\psi^*(x) = 2x + \sum b_i r_i$ where $\bar{\Delta} x = \sum b_i \otimes r_i$ for $b_i \in B$, $r_i \in R$. It follows that

$$\begin{aligned} \psi^*\left[x + \sum b_i r_i\right] &= 2x + \sum b_i r_i + \sum \psi^*(b_i)\psi^*(r_i) \\ &= 2x + \sum b_i r_i + 4 \sum b_i r_i + d \\ &= 2x + 2 \sum b_i r_i + d \end{aligned}$$

where d is three fold decomposable in $I(B)H^*(X)$. Hence $i(3)^*(d) = 0$ since $i(3)^*(d)$ has weight greater than three.

Now we have a commutative diagram

$$\begin{tikzcd} H^*(P_3\Omega X) \arrow[d, "(P_3\Omega\psi)^*"] & H^*(X) \arrow[l, "i(3)^*"'] \arrow[d, "\psi^*"] \\ H^*(P_3\Omega X) & H^*(X) \arrow[l, "i(3)^*"] \end{tikzcd}$$

So $i(3)^*[x + \sum b_i r_i]$ is an eigenvector with respect to $(P_3\Omega\psi)^*$.

So applying this process to $x_i, y_i \in H^*(X)$ and then applying $i(3)^*$, we may assume x_i', y_i' are eigenvectors of $(P_3\Omega\psi)^*$. This process alters the original x_i', y_i' by weight three elements, so since any product with a weight three element is trivial in $H^*(P_3\Omega X)$, this does not change the algebra decomposition of $H^*(P_3\Omega X)$.

Now consider the ty_i. We have the following commutative ladder

$$\begin{tikzcd} H^*(\Omega X) \arrow[d, "(\Omega\psi)^*"] & H^*(P_2\Omega X) \arrow[l, "i_1^*"'] \arrow[d, "(P_2\Omega\psi)^*"] & H^*(P_3\Omega X) \arrow[l, "i_2^*"'] \arrow[d, "(P_3\Omega\psi)^*"] \\ H^*(\Omega X) & H^*(P_2\Omega X) \arrow[l, "i_1^*"] & H^*(P_3\Omega X) \arrow[l, "i_2^*"] \end{tikzcd}$$

with $i_1^* i_2^*(ty_i) = \varphi_{3f_i}(y_i) \in PH^*(\Omega X)$. Hence $(\Omega\psi)^*[\varphi_{3f_i}(y_i)] = 2\varphi_{3f_i}(y_i)$ so $i_1^*(P_2\Omega\psi)^*(i_2^*(ty_i)) = 2\varphi_{3f_i}(y_i) = i_1^*(2i_2^*(ty_i))$. We have an exact triangle

$$\begin{tikzcd} \bar{H}^*(\Omega X) \arrow[dr, "\bar{\Delta}"] & & H^*(P_2\Omega X) \arrow[ll, "i_1^*"'] \\ & \bar{H}^*(\Omega X)^{\otimes 2} \arrow[ur, "\alpha_2^*"'] & \end{tikzcd}$$

and $\operatorname{im}\alpha_2^*$ is in even degrees since $H^*(\Omega X)$ is even dimensional [9].

Therefore i_1^* is monic in odd degrees and

$$\begin{aligned}(P_2\Omega\psi)^*(i_2^*(ty_i)) &= 2i_2^*(ty_i) \\ &= i_2^*(P_3\Omega\psi)^*(ty_i).\end{aligned}$$

So $z = (P_3\Omega\psi)^*(ty_i) - 2ty_i \in \ker i_2^*$.

We have an exact triangle

$$\begin{tikzcd} H^*(P_2\Omega X) \arrow[dr, "\beta_2^*"] & & H^*(P_3\Omega X) \arrow[ll, "i_2^*"'] \arrow[dr, "\beta_3^*"] & \\ & \bar{H}^*(\Omega X)^{\otimes 3} \arrow[ur, "\alpha_3^*"'] & & \bar{H}^*(\Omega X)^{\otimes 4} \end{tikzcd}$$

So $z = \alpha_3^*(w)$. By (3.5) $\beta_3^*(ty_i) = 0$, so

$$\beta_3^*(z) = 0 \qquad \text{or} \qquad \beta_3^*(\alpha_3^*(w)) = d_1(w) = 0.$$

So we may consider

$$\{w\} \in \operatorname{Cotor}^3_{H^*(\Omega X)}(\mathbf{F}_3, \mathbf{F}_3)$$

which is generated by three fold products of primitives and two fold products of primitives and elements of the form

$$\frac{1}{3}\sum \binom{3}{i}[\sigma^*(x)^i|\sigma^*(x)^{3-i}].$$

So $\alpha_3^*(w)$ is a sum of three fold products of odd degree generators and two fold products in the ideal generated by $y_1', \dots, y_k'$. We have

$$\begin{tikzcd} \bar{H}^*(\Omega X)^{\otimes 3} \arrow[r, "\alpha_3^*"] \arrow[d, "(\Omega\psi)^{*\otimes 3}"] & H^*(P_3\Omega X) \arrow[d, "(P_3\Omega\psi)^*"] \\ \bar{H}^*(\Omega X)^{\otimes 3} \arrow[r, "\alpha_3^*"'] & H^*(P_3\Omega X) \end{tikzcd}$$

commutes and $H^*(\Omega X)$ is a direct sum of eigenspaces of $(\Omega\psi)^*$, so the same holds true for $H^*(\Omega X)^{*\otimes 3}$ with respect to $(\Omega\psi)^{*\otimes 3}$.

Hence $\alpha_3^*(S)$ splits into a direct sum of eigenspaces. So $z = z_1 + z_2$ where $(P_3\Omega\psi)^*(z_i) = 2^i z_i$. Following [6, p. 408], we have

$$(P_3\Omega\psi)^*(ty_i) = 2ty_i + z_1 + z_2.$$

Let

$$\overline{ty}_i = ty_i - \tfrac{1}{2}z_2.$$

Then

$$(P_3\Omega\psi)^*(\overline{ty}_i) = 2\overline{ty}_i + z_1$$

and

$$(P_3\Omega\psi)^{3^*}(\overline{ty}_i) = 2\overline{ty}_i.$$

So by changing ty_i by elements in $\operatorname{im}\alpha_3^*$ we can make ty_i an eigenvector of $(P_3\Omega\psi)^{3^*}$. Since products with elements of $\operatorname{im}\alpha_3^*$ is trivial in $H^*(P_3\Omega X)$, we retain the same algebra decomposition.

Corollary 3.2. *$H^{18n+10}(P_3\Omega X)$ is spanned by two fold products of the x_i's and ty_js.*

Proof. By [9], $QH^{even}(X)$ is concentrated in degrees congruent to $2 \bmod 6$. So $QH^{18n+10}(P_3\Omega X) = 0$. So $H^{18n+10}(P_3\Omega X)$ consists of decomposables of weight less than or equal to four. Since y_i's have weight two, and are even dimensional, all two fold products of the y_i' are zero, and the product of a y_i' with two odd generators is also zero, because they have weight four.

By Theorem 3.1 and (2.4), $i(3)^*(\hat{x})$, $i(3)^*(\hat{y})$ are eigenvectors of $\varphi = [(P_3\Omega\psi)^3]^*$. So if $\varphi_K : K \to K$ is a squaring map, we have a commutative diagram

$$\begin{array}{ccc} P_3\Omega X & \xrightarrow{fi(3)} & K \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_K^3} \\ P_3\Omega X & \xrightarrow[fi(3)]{} & K \end{array}$$

Proposition 3.3. *There exists a lifting $\tilde{f}_0 : P_3\Omega X \to E_0$ such that for some integer $t > 0$*

$$\begin{array}{ccc} P_3\Omega X & \xrightarrow{\tilde{f}_0} & E_0 \\ \downarrow{\scriptstyle (P_3\Omega\psi)^{3^t}} & & \downarrow{\scriptstyle (\varphi_0)^{3^t}} \\ P_3\Omega X & \xrightarrow[\tilde{f}_0]{} & E_0 \end{array}$$

commutes, where φ_0 is the squaring map in E_0.

Proof. The proof follows the proof in [6, Prop. A, see 48.2]. The only difference is that $H^*(P_3\Omega X)$ has generators in $\alpha_3^*(S)$ that are not eigenvectors with eigenvalue two.

In the original definition of power space, the self map induced multiplication by a fixed $\lambda \in \mathbf{F}_3$ on the module of indecomposables. The main use for this fact is that $H^*(P_3\Omega X)$ split into a direct sum of eigenspaces. This fact still holds true but some elements of $\alpha_3^*(S)$ may be eigenvectors with eigenvalues two or one.

Theorem 3.4. *There exists a lifting* $\bar{f}_0 : P_3\Omega X \to E_0$ *with* $w_0\bar{f}_0$ *null homotopic. So there is a commutative diagram*

$$\begin{tikzcd}
 & E_1 \arrow[d] \\
 & E_0 \arrow[d] \\
P_3\Omega X \arrow[r, "fi(3)"'] \arrow[uur, "\bar{f}_1"] \arrow[ur, "\bar{f}_0"'] & K
\end{tikzcd}$$

Proof. By Proposition 3.3, we have a lifting $\tilde{f}_0$ that is a power lifting. Since $\tilde{f}_0$ and $f_0i(3)$ both lift $fi(3)$, we have

$$\tilde{f}_0 = f_0 i(3) + j_0 D$$

where

$$D : P_3\Omega X \to K(18n+6).$$

Since $QH^{18n+6}(X) = 0$, we have by Theorem 3.1

$$D^*(i) = \beta + \alpha_3^*(\gamma)$$

where $\beta \in \Lambda(x_1', \dots, x_\ell') \otimes \Lambda(ty_1, \dots, ty_k)$ and $\gamma \in S$. So

$$\tilde{f}_0^* w_0^*(i) = i(3)^* f_0^* w_0^*(i) + \mathscr{P}^1\beta + \mathscr{P}^1\alpha_3^*(\gamma).$$

Since $i_2^*\alpha_3^* = 0$, we have

$$i_2^*\tilde{f}_0^* w_0^*(i) = i(2)^* f_0^* w_0^*(i) + \mathscr{P}^1 i_2^*(\beta). \tag{3.6}$$

We also have a commutative diagram

$$\begin{tikzcd}
P_2\Omega X \arrow[r, "i_2"] \arrow[d, "(P_2\Omega\psi)"] & P_3\Omega X \arrow[r, "\alpha_3"] \arrow[d, "P_3\Omega\psi"] & (\Sigma\Omega X)^{\wedge 3} \\
P_2\Omega X \arrow[r, "i_2"'] & P_3\Omega X &
\end{tikzcd}$$

By Proposition 3.3 and the fact that $w_0^*(i)$ is primitive

$$\begin{aligned}
[(P_2\Omega\psi)^{3'}]^* i_2^*\tilde{f}_0^* w_0^*(i) &= i_2^*\tilde{f}_0^*[(\varphi_0)^{3'}]^* w_0^*(i) \\
&= 2 i_2^*\tilde{f}_0^* w_0^*(i) \\
&= [(P_2\Omega\psi)^{3'}]^* i(2)^* f_0^* w_0^*(i) + \mathscr{P}^1[(P_2\Omega\psi)^{3'}]^* i_2^*(\beta) \\
&= i_2^*\tilde{f}_0^* w_0^*(i).
\end{aligned} \tag{3.7}$$

The last equality follows because β and $f_0^* w_0^*(i)$ are two fold products of eigenvectors of $[(P_3\Omega\psi)^{3'}]^*$ and $2^2 \equiv 1 \bmod 3$.

By (3.7), (3.6) we have

$$i_2^* \tilde{f}_0^* w_0^*(i) = 0 = i(2)^* f_0^* w_0^*(i) + \mathscr{P}^1 i_2^*(\beta).$$

So

$$i(2)^* f_0^* w_0^*(i) = \mathscr{P}^1[-i_2^*(\beta)]. \tag{3.8}$$

If we alter $f_0 i(3)$ by β, we obtain a new lifting $\bar{f}_0 : P_3\Omega X \to E_0$ defined by the composition

$$\bar{f}_0 : P_3\Omega X \longrightarrow P_3\Omega X \times P_3\Omega X \xrightarrow{f_0 i(3), -j_0(\beta)} E_0 \times E_0 \longrightarrow E_0.$$

Then

$$\bar{f}_0^* w_0^*(i) = i(3)^* f_0^* w_0^*(i) - \mathscr{P}^1 \beta.$$

By Corollary 3.2 $\bar{f}_0 w_0^*(i)$ is a two fold product in

$$\Lambda = \Lambda(x_1', \ldots, x_\ell') \otimes \Lambda(ty_1, \ldots ty_k).$$

Hence since $i_2^* : H^*(P_3\Omega X) \to H^*(P_2\Omega X)$ is monic on two fold products in Λ, and

$$i_2^* \bar{f}_0^* w_0^*(i) = 0$$

by (3.8), we have $\bar{f}_0^* w_0^*(i) = 0$. So $w_0 \bar{f}_0$ is null homotopic.

4. Iterated reduced coproducts

In this chapter we define maps that induce the iterated reduced coproduct $(\bar{\Delta} \otimes 1)\bar{\Delta}$. We will develop an obstruction theory which measures when a map between H-spaces preserves the iterated reduced coproduct. We generalize this obstruction theory to maps of the three-fold projective space into an H-space.

Let (Y, μ) be an H-space. For $1 \le i < j \le 3$ let $p_{ij} : Y \times Y \times Y \to Y \times Y$ be the projection to the ith and jth factor for $i < j$. Define $\gamma_Y : \Sigma(Y \times Y \times Y) \to \Sigma Y$ by

$$\begin{aligned}\gamma_Y = {} & \Sigma(\mu(\mu \times 1)) - \Sigma\mu(p_{12}) - \Sigma\mu(p_{13}) - \Sigma\mu(p_{23}) \\ & + \Sigma p_1 + \Sigma p_2 + \Sigma p_3\end{aligned}$$

where $p_i : Y \times Y \times Y \to Y$ is projection on the ith factor. Note addition and subtraction of maps is defined since $[\Sigma(Y \times Y \times Y), \Sigma Y]$ is an algebraic loop. We have

Proposition 4.1. (a) $\gamma_Y^* : \bar{H}^*(Y) \to \bar{H}^*(Y)^{\otimes 3}$ *satisfies*

$$\gamma_Y^* = (\bar{\Delta} \otimes 1)\bar{\Delta}.$$

(b) γ_Y *factors*

$$\begin{tikzcd} \Sigma(Y^3) \arrow[rr, "\gamma_Y"] \arrow[dr] & & \Sigma Y \\ & \Sigma(Y^{\wedge 3}) \arrow[ur, "\bar{\gamma}_Y"'] & \end{tikzcd}$$

Proof. This follows from the definitions.

If $\eta_Y : Y^3 \to \Omega\Sigma Y$ is adjoint to γ_Y and $\bar{\eta}_Y : Y^{\wedge 3} \to \Omega\Sigma Y$ is adjoint to $\bar{\gamma}_Y$, then we have a commutative diagram

$$\begin{tikzcd} Y^3 \arrow[rr, "\eta_Y"] \arrow[dr] & & \Omega\Sigma Y \\ & Y^{\wedge 3} \arrow[ur, "\bar{\eta}_Y"'] & \end{tikzcd}$$

For X an H-space we can define an analogous map

$$\delta_X : \Sigma(\Sigma\Omega X)^3 \to \Sigma P_3\Omega X.$$

Let $\Sigma\Omega X \xrightarrow{i_1} P_2\Omega X \xrightarrow{i_2} P_3\Omega X$ be the inclusions and let $p_i : (\Sigma\Omega X)^3 \to \Sigma\Omega X$ be the projections on the ith factor. Then we define

$$\begin{aligned} \delta_X = {} & \Sigma\mu_{2,1}(\mu_{1,1} \times 1) - \Sigma i_2\mu_{1,1}(p_{12}) - \Sigma i_2\mu_{1,1}(p_{13}) \\ & - \Sigma i_2\mu_{1,1}(p_{23}) + \Sigma i_2 i_1 p_1 + \Sigma i_2 i_1 p_2 + \Sigma i_2 i_1 p_3. \end{aligned} \tag{4.1}$$

By (4.1), δ_X induces $\bar{\delta}_X : \Sigma(\Sigma\Omega X)^{\wedge 3} \to \Sigma P_3\Omega X$. We have a commutative diagram

$$\begin{tikzcd} \Sigma(\Sigma\Omega X)^3 \arrow[rr, "\delta_X"] \arrow[dr] & & \Sigma P_3\Omega X \\ & \Sigma(\Sigma\Omega X)^{\wedge 3} \arrow[ur, "\bar{\delta}_X"'] & \end{tikzcd}$$

If ξ_X is adjoint to δ_X, $\bar{\xi}_X$ adjoint to $\bar{\delta}_X$, we have

$$\begin{tikzcd} (\Sigma\Omega X)^3 \arrow[rr, "\xi_X"] \arrow[dr] & & \Omega\Sigma P_3\Omega X \\ & (\Sigma\Omega X)^{\wedge 3} \arrow[ur, "\bar{\xi}_X"'] & \end{tikzcd}$$

Now suppose we are given a map $f : X \to Y$ between H-spaces, (X, μ_X) and (Y, μ_Y). If Z is an H-space, define $r_Z : \Omega\Sigma Z \to Z$ by $r_Z[z_1, \dots, z_t] = (\cdots(z_1 z_2)z_3 \dots)z_t$ where $[z_1, \dots, z_t]$ is the point of the James reduced product space $J(Z)$. See [12].

Consider the diagram (not necessarily commutative)

$$\begin{array}{ccccc} X^{\wedge 3} & \xrightarrow{\bar\eta_X} & \Omega\Sigma X & \xrightarrow{r_X} & X \\ \big\downarrow{\scriptstyle f^{\wedge 3}} & & \big\downarrow{\scriptstyle \Omega\Sigma f} & & \big\downarrow{\scriptstyle f} \\ Y^{\wedge 3} & \xrightarrow[\bar\eta_Y]{} & \Omega\Sigma Y & \xrightarrow[r_Y]{} & Y \end{array}$$

where r_X and r_Y are the retractions. We define $D_3'(f) : X^{\wedge 3} \to \Omega\Sigma Y$ to be $(\Omega\Sigma f)\bar\eta_X - \bar\eta_Y(f^{\wedge 3})$ and $D_3(f) : X^{\wedge 3} \to Y$ to be

$$D_3(f) = r_Y D_3'(f).$$

Now suppose we are given a map $f : P_3\Omega X \to Y$ where both X and Y are H-spaces.

Consider the diagram

$$\begin{array}{ccccc} (\Sigma\Omega X)^{\wedge 3} & \xrightarrow{\bar\xi_X} & \Omega\Sigma P_3\Omega X & & \\ \big\downarrow{\scriptstyle (fi_2i_1)^{\wedge 3}} & & \big\downarrow{\scriptstyle \Omega\Sigma f} & & \\ Y^{\wedge 3} & \xrightarrow[\bar\eta_Y]{} & \Omega\Sigma Y & \xrightarrow[r_Y]{} & Y \end{array}$$

Define $c_3(f) : (\Sigma\Omega X)^{\wedge 3} \to Y$ to be

$$c_3(f) = r_Y[(\Omega\Sigma f)\bar\xi_X - \bar\eta_Y(fi_2i_1)^{\wedge 3}].$$

Lemma 4.2. (a) *If $f : X \to K(\mathbf{F}_p, n)$ for p a prime then*

$$D_3(f)^*(i_n) = (\bar\Delta \otimes 1)\bar\Delta f^*(i_n)$$

and

$$c_3(fi(3))^*(i_n) = (\sigma^* \otimes \sigma^* \otimes \sigma^*)(\bar\Delta \otimes 1)\bar\Delta f^*(i_n).$$

(b) *If $g : P_3\Omega X \to K(\mathbf{F}_p, n)$ then*

$$c_3(g)^*(i_n) = \bar\delta_X^* g^*(i_n)$$

where $\bar\delta_X$ is defined in (4.1).

(c) *If $\bar\mu_{1,1} : \Sigma(\Sigma\Omega X)^{\wedge 2} \to \Sigma P_2\Omega X$ and*

$$\bar\mu_{2,1} : \Sigma(P_2\Omega X \wedge \Sigma\Omega X) \to \Sigma P_3\Omega X$$

are the Hopf constructions applied to $\mu_{1,1}$ and $\mu_{2,1}$ then $c_3(g)^*(i_n) = (\bar\mu_{1,1}^* \otimes 1)(\bar\mu_{2,1}^*)g^*(i_n)$.

Proof. If $K = K(\mathbf{F}_p, n)$

$$\begin{array}{ccccc} (\Sigma\Omega X)^{\wedge 3} & \xrightarrow{\bar\xi_X} & \Omega\Sigma P_3\Omega X & & \\ {\scriptstyle (gi_2i_1)^{\wedge 3}}\big\downarrow & & \big\downarrow{\scriptstyle \Omega\Sigma g} & & \\ K^{\wedge 3} & \xrightarrow[\bar\eta_K]{} & \Omega\Sigma K & \xrightarrow[r_K]{} & K \end{array}$$

Since $(r_K\bar{\eta}_K)^*(i_n) = 0$, we have

$$c_3(g) \simeq r_K(\Omega\Sigma g)\bar{\xi}_X$$

$$(r_K(\Omega\Sigma g))^*(i) = g^*(i) \in \bar{H}^*(P_3\Omega X) \subseteq H^*(\Omega\Sigma P_3\Omega X)$$

and $\bar{\xi}_X^* g^*(i) = \bar{\delta}_X^* g^*(i)$. Here we are suppressing isomorphisms due to suspension. (c) is proved since

$$\bar{\delta}_X^* g^*(i_n) = (\bar{\mu}_{1,1}^* \otimes 1)\bar{\mu}_{2,1}^* g^*(i_n).$$

We want to investigate how c_3 and D_3 behave with respect to composition of maps.

Suppose $g : Y \to Z$ is an H-map. Then $D_3(g)$ is trivial.

Proposition 4.3. *If $f : X \to Y$ and $g : Y \to Z$ is an H-map then $D_3(gf) = gD_3(f)$. If $h : P_3\Omega X \to Y$, and $g : Y \to Z$ is an H-map, then $c_3(gh) = gc_3(h)$.*

Proof. We have a diagram

$$\begin{tikzcd}
X^{\wedge 3} \arrow[r, "\bar{\eta}_X"] \arrow[d, "(f)^{\wedge 3}"'] & \Omega\Sigma X \arrow[d, "\Omega\Sigma f"] & \\
Y^{\wedge 3} \arrow[r, "\bar{\eta}_Y"] \arrow[d, "(g)^{\wedge 3}"'] & \Omega\Sigma Y \arrow[r, "r_Y"] \arrow[d, "\Omega\Sigma g"] & Y \arrow[d, "g"] \\
Z^{\wedge 3} \arrow[r, "\bar{\eta}_Z"] & \Omega\Sigma Z \arrow[r, "r_Z"'] & Z
\end{tikzcd}$$

$$\begin{aligned}
D_3(gf) &= r_Z[\Omega\Sigma(gf)\bar{\eta}_X - \bar{\eta}_Z(gf)^{\wedge 3}] \\
&\simeq r_Z[\Omega\Sigma g(\Omega\Sigma f)\bar{\eta}_X - (\Omega\Sigma g)\bar{\eta}_Y(f)^{\wedge 3} + (\Omega\Sigma g)\bar{\eta}_Y(f)^{\wedge 3} - \bar{\eta}_Z(g)^{\wedge 3}(f)^{\wedge 3}] \\
&\simeq r_Z(\Omega\Sigma g)[(\Omega\Sigma f)\bar{\eta}_X - \bar{\eta}_Y(f)^{\wedge 3}] + r_Z[(\Omega\Sigma g)\bar{\eta}_Y(f)^{\wedge 3} - \bar{\eta}_Z(g)^{\wedge 3}(f)^{\wedge 3}] \\
&\simeq gD_3(f) + D_3(g)(f)^{\wedge 3} \quad \text{since } r_Z(\Omega\Sigma g) \simeq gr_Y \text{ if } g \text{ is an } H\text{-map} \\
&\simeq gD_3(f) \quad \text{since } D_3(g) \simeq *.
\end{aligned}$$

The proof for $c_3(gh)$ is analogous.

Returning to our diagram (3.27),

$$\begin{tikzcd}
 & E_1 \arrow[d, "q_1"] \\
 & E_0 \arrow[d, "q_0"] \\
P_3\Omega X \arrow[uur, "\bar{f}_1"] \arrow[ur, "\bar{f}_0"'] \arrow[r, "fi(3)"'] & K
\end{tikzcd}$$

since q_0, q_1 are H-maps, we can apply Proposition 4.3 to get a commutative diagram

$$\begin{tikzcd} & E_1 \arrow[d, "q_1"] \\ & E_0 \arrow[d, "q_0"] \\ (\Sigma\Omega X)^{\wedge 3} \arrow[uur, "c_3(\bar{f}_1)"] \arrow[ur, "c_3(\bar{f}_0)"'] \arrow[r, "c_3(fi(3))"'] & K \end{tikzcd}$$

We have $c_3(fi(3))^*(i_{6n+3}) = (\sigma^* \otimes \sigma^* \otimes \sigma^*)(\bar{\Delta} \otimes 1)\bar{\Delta}2x = 0$ since $(\bar{\Delta} \otimes 1)\bar{\Delta}x \in B \otimes B \otimes R$ and $\sigma^*(B) = 0$. Similarly $c_3(fi(3))^*(i_{18n-1}) = 0$, so $c_3(fi(3))$ is null homotopic and $c_3(\bar{f}_0)$ factors through the fibre $K(18n+6)$:

$$\begin{tikzcd} & K(18n+6) \arrow[d, "j_0"] \\ (\Sigma\Omega X)^{\wedge 3} \arrow[ur, "\hat{c}_3(\bar{f}_0)"] \arrow[r, "c_3(\bar{f}_0)"'] & E_0 \end{tikzcd}$$

But $\hat{c}_3(\bar{f}_0) \in H^{18n+6}((\Sigma\Omega X)^{\wedge 3}) = [\bar{H}^*(\Omega X)^{\otimes 3}]^{18n+3} = 0$ since $\bar{H}^*(\Omega X)$ is even dimensional by [9].

Therefore $c_3(\bar{f}_0) \simeq *$, so $c_3(\bar{f}_1)$ factors through the fibre

$$\begin{tikzcd} & K(18n+9) \arrow[d, "j_1"] & \\ (\Sigma\Omega X)^{\wedge 3} \arrow[ur, "\hat{c}_3(\bar{f}_1)"] \arrow[r, "c_3(\bar{f}_1)"] & E_1 \arrow[r, "v"] & K(18n+17) \end{tikzcd}$$

We now want to study $c_3(v\bar{f}_1)$. Recall v is not an H-map.

Now suppose $g : Y \to Z$ is not an H-map. Then $r_Z(\Omega\Sigma g)$ is not homotopic to gr_Y.

If $J(Y)$ is the James reduced product space, then let $D'(g) : J(Y) \to Z$ be the map with

$$D'(g) + gr_Y = r_Z(\Omega\Sigma g).$$

Then $D'(g)|_Y = *$ and we have a map $D(g) : J(Y)/Y \to Z$ with

$$\begin{tikzcd} J(Y) \arrow[r, "D'(g)"] \arrow[dr] & Z \\ & J(Y)/Y \arrow[u, "D(g)"'] \end{tikzcd}$$

By definition $J_2(Y)/Y = Y \wedge Y$ and $D(g)|J_2(Y)/Y$ is the H-deviation of g.

If $Z = K(\mathbf{F}_p, n)$ then

$$\tilde{H}^*(J(Y)/Y) = \bigoplus_{i \geq 2} \tilde{H}^*(Y)^{\otimes i}$$

and $D(g)^*(i)$ is the sum of all the iterated reduced coproducts

$$\sum (\bar{\Delta} \otimes 1 \otimes \cdots \otimes 1) \cdots \bar{\Delta} g^*(i).$$

Now we have the following diagram (not necessarily commutative)

$$\begin{tikzcd}
(\Sigma\Omega X)^{\wedge 3} \arrow[r, "\xi_X"] \arrow[d, "\varphi"'] & \Omega\Sigma P_3 \Omega X \arrow[d, "\Omega\Sigma \bar{f}_1"] & \\
E_1^{\wedge 3} \arrow[r, "\bar{\eta}_{E_1}"] \arrow[d, "(v)^{\wedge 3}"'] & \Omega\Sigma E_1 \arrow[r, "r_{E_1}"] \arrow[d, "\Omega\Sigma v"] & E_1 \arrow[d, "v"] \\
K^{\wedge 3} \arrow[r, "\bar{\eta}_K"'] & \Omega\Sigma K \arrow[r, "r_K"'] & K
\end{tikzcd}$$

Here we let $K = K(18n + 17)$ and $\varphi = (\bar{f}_1 i_2 i_1)^{\wedge 3}$. Then

$$\begin{aligned}
c_3(v\bar{f}_1) &= r_K(\Omega\Sigma v)(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - r_K\bar{\eta}_K(v)^{\wedge 3}\varphi \\
c_3(\bar{f}_1) &= r_{E_1}(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - r_{E_1}\bar{\eta}_{E_1}\varphi \\
D_3(v) &= r_K(\Omega\Sigma v)\bar{\eta}_{E_1} - r_K\bar{\eta}_K(v)^{\wedge 3} \\
D'(v) &= r_K(\Omega\Sigma v) - v r_{E_1} : \Omega\Sigma E_1 \to K.
\end{aligned} \tag{4.2}$$

First we note that

$$v c_3(\bar{f}_1) \simeq v r_{E_1}(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - v r_{E_1}\eta_{E_1}\varphi. \tag{4.3}$$

To see this note that for maps $\psi_i : A \to E_1$

$$v(\psi_1 - \psi_2) \simeq v\psi_1 - v\psi_2 + D_v(\psi_1, \psi_2)$$

where $D_v : E \wedge E \to K$ is the H-deviation. If $\psi_1 = r_{E_1}(\Omega\Sigma\bar{f}_1)\bar{\xi}_X$, $\psi_2 = r_{E_1}\eta_{E_1}\varphi$, we have

$$D_v^*(i) = \bar{\Delta} v = u\mathscr{P}^1 u \otimes \mathscr{P}^1 u + \mathscr{P}^1 u \otimes u\mathscr{P}^1 u + (\mathscr{P}^2 u)u \otimes u + u \otimes (\mathscr{P}^2 u)u.$$

Now

$$\begin{aligned}
\psi_1^*(u) &= (\sigma^* \otimes \sigma^* \otimes \sigma^*)(\bar{\Delta} \otimes 1)\bar{\Delta}(x) = 0 \\
\psi_2^*(u) &= \varphi^*(\bar{\Delta} \otimes 1)\bar{\Delta} u = 0.
\end{aligned}$$

So we have $D_v(\psi_1, \psi_2)$ is null homotopic. This proves (4.3).

$$\begin{aligned} c_3(v\bar{f}_1) &= r_K(\Omega\Sigma v)(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - r_K\bar{\eta}_K(v)^{\wedge 3}\varphi \\ &\simeq [r_K(\Omega\Sigma v)(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - r_K(\Omega\Sigma v)\bar{\eta}_{E_1}\varphi] \\ &\quad + [r_K(\Omega\Sigma v)\bar{\eta}_{E_1}\varphi - r_K\bar{\eta}_K(v)^{\wedge 3}\varphi] \\ &\simeq (D'(v) + vr_{E_1})(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - (D'(v) + vr_{E_1})\bar{\eta}_{E_1}\varphi + D_3(v)\varphi \\ &\simeq D'(v)(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - D'(v)\bar{\eta}_{E_1}\varphi + vc_3(\bar{f}_1) + D_3(v)\varphi. \end{aligned} \tag{4.4}$$

The last equivalence follows from (4.3).

Theorem 4.4. *$c_3(v\bar{f}_1) \simeq vc_3(\bar{f}_1) + D_3(v)\varphi$ where $\varphi = (\bar{f}_1 i_2 i_1)^{\wedge 3}$.*

Proof. It suffices by (4.4) to prove

$$\theta = D'(v)(\Omega\Sigma\bar{f}_1)\bar{\xi}_X - D'(v)\bar{\eta}_{E_1}\varphi$$

is null homotopic.

Let's review some facts about $H^*(\Omega\Sigma Y)$. We have with field coefficients

$$H^*(\Omega\Sigma Y) = \bigoplus_{i\geq 0} \tilde{H}^*(Y)^{\otimes i}.$$

If $f : Y \to Z$ is a map, then

$$(\Omega\Sigma f)^* : H^*(\Omega\Sigma Z) \to H^*(\Omega\Sigma Y)$$

satisfies

$$(\Omega\Sigma f)^*|_{\tilde{H}^*(Z)^{\otimes i}} = (f^*)^{\otimes i}.$$

Since $H_*(\Omega\Sigma Y)$ as an algebra is the tensor algebra on $\tilde{H}_*(Y)$, we can dualize this to describe the coalgebra structure on $H^*(\Omega\Sigma Y) \equiv \bigoplus_{i\geq 0} \tilde{H}^*(Y)^{\otimes i}$.

We denote $a_1 \otimes \cdots \otimes a_k \in \tilde{H}^*(Y)^{\otimes k} \subseteq H^*(\Omega\Sigma Y)$ by $[a_1|\ldots|a_k]$.

Then the coproduct $\Delta : H^*(\Omega\Sigma Y) \to H^*(\Omega\Sigma Y) \otimes H^*(\Omega\Sigma Y)$ is defined by

$$\begin{aligned} \Delta[a_1|\ldots|a_k] = [a_1|\ldots|a_k] \otimes 1 + \sum_{i=1}^{k-1} [a_1|\ldots|a_i] \otimes [a_{i+1}|\ldots|a_k] \\ + 1 \otimes [a_1|\ldots|a_k]. \end{aligned} \tag{4.5}$$

By (4.2), (4.5) and (2.2), (2.3),

$$\begin{aligned} D'(v)^*(i) = [u(\mathcal{P}^1 u)|\mathcal{P}^1 u] + [\mathcal{P}^1 u|u(\mathcal{P}^1 u)] + [(\mathcal{P}^2 u)u|u] + [u|(\mathcal{P}^2 u)u] \\ + [u|\mathcal{P}^1 u|\mathcal{P}^1 u] - [\mathcal{P}^1 u|u|\mathcal{P}^1 u] + [\mathcal{P}^2 u|u|u] - [u|\mathcal{P}^2 u|u]. \end{aligned}$$

θ may be described as the following composition of maps

$$(\Sigma\Omega X)^{\wedge 3} \xrightarrow{\Delta} [(\Sigma\Omega X)^{\wedge 3}]^2 \xrightarrow{D'(v)(\Omega\Sigma\bar{f}_1)\xi_X, -D'(v)\bar{\eta}_{E_1}\varphi} K \times K \xrightarrow{\mu_K} K.$$

So

$$\theta^*(i) = \bar{\xi}_X^*(\Omega\Sigma\bar{f}_1)^* D'(v)^*(i) - \varphi^*\bar{\eta}_{E_1}^* D'(v)^*(i).$$

Note that $D'(v)(\Omega\Sigma\bar{f}_1)$ factors

$$\begin{tikzcd}
\Omega\Sigma(P_3\Omega X) \arrow[r, "\Omega\Sigma\bar{f}_1"] \arrow[d, "\Omega\Sigma i(3)"'] & \Omega\Sigma E_1 \arrow[r, "D'(v)"] & K \\
\Omega\Sigma X \arrow[urr, "h"'] & &
\end{tikzcd}$$

since $\bar{f}_1^*(u) = i(3)^*(\hat{x})$

$$h^*(i) = [\hat{x}(\mathcal{P}^1\hat{x})|\mathcal{P}^1\hat{x}] + [\mathcal{P}^1\hat{x}|\hat{x}(\mathcal{P}^1\hat{x})] + [(\mathcal{P}^2\hat{x})\hat{x}|\hat{x}] + [\hat{x}|(\mathcal{P}^2\hat{x})\hat{x}]$$
$$+ [\hat{x}|\mathcal{P}^1\hat{x}|\mathcal{P}^1\hat{x}] - [\mathcal{P}^1\hat{x}|\hat{x}|\mathcal{P}^1\hat{x}] + [\mathcal{P}^2\hat{x}|\hat{x}|\hat{x}] - [\hat{x}|\mathcal{P}^2\hat{x}|\hat{x}].$$

If $\mu : X \times X \to X$ is multiplication on X, $\mu_K : K \times K \to K$ is the multiplication on K. Let

$$g_1 = \Omega\Sigma\mu(\mu \times 1)$$
$$g_2 = -\Omega\Sigma\mu(p_{12})$$
$$g_3 = -\Omega\Sigma\mu(p_{13})$$
$$g_4 = -\Omega\Sigma\mu(p_{23})$$
$$g_5 = \Omega\Sigma p_1$$
$$g_6 = \Omega\Sigma p_2$$
$$g_7 = \Omega\Sigma p_3.$$

Let $k_1, \dots, k_7$ be the analogous maps for K. Let

$$g_1' = \Omega\Sigma\mu_{2,1}(\mu_{1,1} \times 1)$$
$$g_2' = -\Omega\Sigma i_2\mu_{1,1}(p_{12})$$

$$g_3' = -\Omega\Sigma i_2\mu_{1,1}(p_{13})$$

$$g_4' = -\Omega\Sigma i_2\mu_{1,1}(p_{23})$$

$$g_5' = \Omega\Sigma i_2 i_1 p_1$$

$$g_6' = \Omega\Sigma i_2 i_1 p_2$$

$$g_7' = \Omega\Sigma i_2 i_1 p_3.$$

We have the following commutative diagram

$$\begin{tikzcd}
H^*(K) \arrow[d, "D'(v)^*"] \arrow[ddr, "h^*"] \arrow[ddddddd, bend right=40, "\xi_X^*(\Omega\Sigma\bar{f}_1)^*D'(v)"'] & \\
H^*(\Omega\Sigma E_1) \arrow[d, "(\Omega\Sigma\bar{f}_1)^*"] & \\
H^*(\Omega\Sigma P_2\Omega X) \arrow[d] & H^*(\Omega\Sigma X) \arrow[l] \arrow[d] \arrow[dddd, bend left=40, "\eta_X^*"] \\
H^*(\Omega\Sigma P_3\Omega X)^{\otimes 7} \arrow[d, "(g_1')^*\otimes\cdots\otimes(g_7')"] & H^*(\Omega\Sigma X)^{\otimes 7} \arrow[l] \arrow[d, "g_1^*\otimes\cdots\otimes g_7^*"] \\
H^*(\Omega\Sigma(\Sigma\Omega X)^3)^{\otimes 7} \arrow[d] & H^*(\Omega\Sigma(X^3))^{\otimes 7} \arrow[l] \arrow[d] \\
H^*(\Omega\Sigma(\Sigma\Omega X)^3) \arrow[d] & H^*(\Omega\Sigma(X^3)) \arrow[l] \arrow[d] \\
H^*((\Sigma\Omega X)^3) & H^*(X^3) \arrow[l, "(\varepsilon\times\varepsilon\times\varepsilon)^*"']
\end{tikzcd} \tag{4.6}$$

Now $\bar{\Delta}\hat{x} \in I(B)H^*(X) \otimes H^*(X) + H^*(X) \otimes I(B)H^*(X)$ and $\sigma^*(B) = 0$. So all terms in $\eta_X^* h^*(i)$ that involve elements in B will go to zero in

$$(\sigma^* \otimes \sigma^* \otimes \sigma^*)\eta_X^* h^*(i) = \xi_X^*(\Omega\Sigma\bar{f}_1)^* D'(v)^*(i).$$

So we can treat $\hat{x}$ like it is primitive in computing

$$\xi_X^*(\Omega\Sigma\bar{f}_1)^* D'(v)^*(i).$$

$\varphi^*\eta_{E_1}^* D'(v)^*(i)$ is computed by the following diagram

$$
\begin{tikzcd}
H^*(K) \arrow[d, "D'(v)^*"] \\
H^*(\Omega\Sigma E_1) \arrow[d] \arrow[ddddd, bend right=60, "\eta^*_{E_1}"'] \\
H^*(\Omega\Sigma E_1)^{\otimes 7} \arrow[d, "k_1^*\otimes\cdots\otimes k_7^*"] \\
H^*(\Omega\Sigma(E_1^3))^{\otimes 7} \arrow[d] \\
H^*(\Omega\Sigma(E_1^3)) \arrow[d] \\
H^*(E_1^3) \arrow[d, "\varphi^*"] \\
H^*((\Sigma\Omega X)^3)
\end{tikzcd}
\tag{4.7}
$$

Comparing (4.6) and (4.7) we get

$$\xi_X^*(\Omega\Sigma\bar{f}_1)^*D'(v)^*(i) = \varphi^*\eta^*_{E_1}D'(v)^*(i).$$

Passing to the smash products we get

$$\bar{\xi}_X^*(\Omega\Sigma\bar{f}_1)^*D'(v)^*(i) = \varphi^*\bar{\eta}^*_{E_1}D'(v)^*(i).$$

Hence θ is null homotopic.

Corollary 4.5.

$$
\begin{aligned}
c_3(v\bar{f}_1)^*(i_{18n+17}) = {} & \mathcal{P}^2\hat{c}_3(\bar{f}_1)^*(i_{18n+9}) - \mathcal{P}^1 z\otimes z\otimes\mathcal{P}^1 z + z\otimes\mathcal{P}^1 z\otimes\mathcal{P}^1 z \\
& + \mathcal{P}^2 z\otimes z\otimes z - z\otimes\mathcal{P}^2 z\otimes z
\end{aligned}
$$

where $z = \sigma^*(\hat{x})$.

Proof. This follows from (2.3), Lemma 4.2(a) and $(\bar{f}_1 i_2 i_1)^*(u) = \sigma^*(\hat{x}) = z$.

5. Steenrod actions on finite H-spaces

We now prove the Main Theorem.

Theorem 5.1. *Let* $\bar{x}\in QH^{6n+3}(X)$ *with* $\mathcal{P}^{3n+1}\bar{x}\in \operatorname{im}\mathcal{P}^2$. *Suppose there are no transpotence elements in* $H^{18n+16}(\Omega X)$. *Then* $\mathcal{P}^1\sigma^*(x)\in\operatorname{im}\mathcal{P}^2$.

Proof. We may choose $x \in R^{6n+3}$ and $y \in R^{18n-1}$ with x representing $\bar{x}$ and $2\mathscr{P}^{3n+1}x = \mathscr{P}^2 y$. Hence by (2.4) $2\mathscr{P}^{3n+1}\hat{x} = \mathscr{P}^2\hat{y}$. By Theorem 3.4, we have a commutative diagram

$$\begin{tikzcd} & E_1 \arrow[r, "v"] \arrow[d] & K(18n+17) \\ & E_0 \arrow[d] & \\ P_3\Omega X \arrow[r] \arrow[ruu, "\bar{f}_1"] \arrow[ru] & K & \end{tikzcd}$$

By Corollary 4.5

$$\begin{aligned} c_3(v\bar{f}_1)^*(i_{18n+17}) &= \mathscr{P}^2\hat{c}_3(\bar{f}_1)^*(i_{18n+9}) \\ &\quad - \mathscr{P}^1 z \otimes z \otimes \mathscr{P}^1 z + z \otimes \mathscr{P}^1 z \otimes \mathscr{P}^1 z \\ &\quad + \mathscr{P}^2 z \otimes z \otimes z - z \otimes \mathscr{P}^2 z \otimes z \\ &\in PH^*(\Omega X)^{\otimes 3} + \mathscr{P}^2(H^*(\Omega X)^{\otimes 3}) \end{aligned}$$

for $z = \sigma^*(x)$.

If $\mathscr{P}^1 z \notin \operatorname{im} \mathscr{P}^2$ we can choose $s \in H_*(\Omega X)$ with

$$\langle s, \mathscr{P}^1 z \rangle = 1 \qquad \text{and} \qquad s\mathscr{P}^2 = 0.$$

Note that since z and $\mathscr{P}^1 z$ are primitive s and $s\mathscr{P}^1$ are indecomposable, then

$$\begin{aligned} &\langle s \otimes s \otimes s\mathscr{P}^1 - s\mathscr{P}^1 \otimes s \otimes s, c_3(v\bar{f}_1)^*(i_{18n+17}) \rangle \\ &\quad = \langle s \otimes s \otimes s\mathscr{P}^1 - s\mathscr{P}^1 \otimes s \otimes s, \\ &\qquad \mathscr{P}^2\hat{c}_3(\bar{f}_1)^*(i) - \mathscr{P}^1 z \otimes z \otimes \mathscr{P}^1 z + z \otimes \mathscr{P}^1 z \otimes \mathscr{P}^1 z \rangle \\ &\quad = -1. \end{aligned} \tag{5.1}$$

So

$$c_3(v\bar{f}_1)^*(i_{18n+17}) \neq 0.$$

But by Lemma 4.2(c)

$$c_3(v\bar{f}_1)^*(i) = (\bar{\mu}^*_{1,1} \otimes 1)\bar{\mu}^*_{2,1}(v\bar{f}_1)^*(i).$$

So

$$\begin{aligned} &\langle s \otimes s \otimes s\mathscr{P}^1 - s\mathscr{P}^1 \otimes s \otimes s, (\bar{\mu}^*_{1,1} \otimes 1)\bar{\mu}^*_{2,1}(v\bar{f}_1)^*(i) \rangle \\ &\quad = \langle \bar{\mu}_{2,1*}(\bar{\mu}_{1,1*} \otimes 1)(s \otimes s \otimes s\mathscr{P}^1 - s\mathscr{P}^1 \otimes s \otimes s), v\bar{f}_1^*(i) \rangle. \end{aligned}$$

We have $\langle s, \mathscr{P}^1 \sigma^*(x) \rangle = 1$ implies $\sigma_*(s) \neq 0$ and $\sigma_*(s\mathscr{P}^1) \neq 0$.

By (3.3), we have commutativity of the diagrams

$$\begin{tikzcd} \Sigma(\Sigma\Omega X)^{\wedge 2} \arrow[r, "\bar{\mu}_{1,1}"] \arrow[d, "\Sigma(\varepsilon\wedge\varepsilon)"'] & \Sigma P_2\Omega X \arrow[d, "\Sigma i(2)"] \\ \Sigma(X\wedge X) \arrow[r, "\bar{\mu}"] & \Sigma X \end{tikzcd}$$

$$\begin{tikzcd} \Sigma(P_2\Omega X\wedge\Sigma\Omega X) \arrow[r, "\bar{\mu}_{2,1}"] \arrow[d, "\Sigma(i(2)\wedge\varepsilon)"'] & \Sigma P_3\Omega X \arrow[d, "\Sigma i(3)"] \\ \Sigma(X\wedge X) \arrow[r, "\bar{\mu}"] & \Sigma X \end{tikzcd}$$

If $(v\bar{f_1})^*(i) = i(3)^*(\gamma)$, for $\gamma\in H^*(X)$. Then by Lemma 4.2(a)

$$c_3(v\bar{f_1})^*(i) = (\sigma^*\otimes\sigma^*\otimes\sigma^*)(\bar{\Delta}\otimes 1)\bar{\Delta}(\gamma)$$

$$\begin{aligned} &\langle s\otimes s\otimes\mathscr{P}^1 s - s\mathscr{P}^1\otimes s\otimes s, c_3(v\bar{f_1})^*(i)\rangle \\ &= \langle m_*(m_*\otimes 1)(\dot{\sigma}_*\otimes\sigma_*\otimes\sigma_*)(s\otimes s\otimes s\mathscr{P}^1 - s\mathscr{P}^1\otimes s\otimes s), \gamma\rangle \\ &= \langle \sigma_*(s)^2(\sigma_*(s)\mathscr{P}^1) - (\sigma_*(s)\mathscr{P}^1)\sigma_*(s)^2, \gamma\rangle \\ &= 0 \quad \text{since } \sigma_*(s)^2 = 0 \text{ by [4].} \end{aligned}$$

Therefore $(v\bar{f_1})^*(i)\notin i(3)^*H^*(X)$.

By Proposition 3.1, and for degree reasons $(v\bar{f_1})^*(i)$ must have nonzero summands of the form $(ty_i)y_j' + \alpha_3^*(\zeta)$. Further c_3 is additive by Lemma 4.2c.

But since $y_j' = i(3)^*(y_j)$ and $y_j\in B$, we have $\mu_{1,1}^*(\mu_{2,1}^*\otimes 1)(y_j') = 0$. Similarly $\bar{\mu}_{1,1}^*(\bar{\mu}_{2,1}^*\otimes 1)(ty_i) = 0$. So $c_3((ty_i)y_j') = 0$.

Finally $c_3(\alpha_3^*(\zeta))$ is a sum of permutations applied to ζ by [4]. ζ lies in a vector space complementary to $PH^*(\Omega X)^{\otimes 3}/\mathrm{im}\, d_1$ by Proposition 3.1.

So $\langle s\otimes s\otimes s\mathscr{P}^1 - s\mathscr{P}^1\otimes s\otimes s, c_3^*(\alpha_3^*\zeta)\rangle = 0$.

We conclude

$$\langle s\otimes s\otimes s\mathscr{P}^1 - s\mathscr{P}^1\otimes s\otimes s, c_3(v\bar{f_1})^*(i)\rangle = 0.$$

This contradicts (5.1) and proves the theorem.

DEPARTMENT OF MATHEMATICS
KOCHI UUIVERSITY
e-mail: hemmi@math.kochi-u.ac.jp

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA 92093-0112 USA
e-mail: jimlin@euclid.ucsd.edu

References

[1] J. Harper, *H*-spaces with Torsion, Mem. Amer. Math. Soc., **223** (1979).
[2] J. Harper, Rank 2 mod 3 *H*-space, Can. Math. Soc. Conf., Lec. Notes. Ser. 2, 1982.
[3] J. Harper and A. Zabrodsky, Evaluating a *p*-th order operation, Publicaciones Matemàtiques, Barcelona **32** (1988), 61–78.
[4] Y. Hemmi, Higher homotopy commutativity of *H*-spaces and the mod *p* torus theorem, Pac. J. Math., **149** (1991), 95–111.
[5] Y. Hemmi, Homotopy associative finite *H*-spaces and the mod 3 reduced power operations, RIMS Kyoto, **23** (1987), 1071–1084.
[6] R. Kane, The Homology of Hopf spaces, North Holland, 1988.
[7] R. Kane, On loop spaces without *p* torsion, Pac. J. Math., **71** (1977), 71–88.
[8] D. Kraines and C. Schochet, Differentials in the Eilenberg Moore spectral sequence, J. Pure App. Algebra, **2** (1972), 131–148.
[9] J. Lin, Torsion in *H*-spaces II, Ann. Math., **107** (1978), 41–88.
[10] J. Milnor and J. C. Moore, On the structure of Hopf algebras, Ann. of Math., **81** (1965), 211–264.
[11] J. Stasheff, Homotopy associativity of *H*-spaces, I, II, Trans. Amer. Math. Soc., **108** (1963), 275–292, 293–312.
[12] G. Whitehead, Elements of Homotopy Theory, Springer, vol. 61, 1978.
[13] A. Zabrodsky, Some relations in the mod 3 cohomology of *H*-spaces, Israel J. Math., **33** (1979), 59–72.