J. Math. Kyoto Univ. (JMKYAZ)
43-2 (2003), 261–303

Space-time regularity for linear stochastic evolution equations driven by spatially homogeneous noise

By

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Abstract

In this paper we study space-time regularity of solutions of the following linear stochastic evolution equation in $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered distributions on $\mathbb{R}^d$:

$$(*)\qquad \begin{aligned} du(t) &= Au(t)\,dt + dW(t), \qquad t \geqslant 0, \\ u(0) &= 0. \end{aligned}$$

Here A is a pseudodifferential operator on $\mathscr{S}'(\mathbb{R}^d)$ whose symbol $q : \mathbb{R}^d \to \mathbb{C}$ is symmetric and bounded above, and $\{W(t)\}_{t\geqslant 0}$ is a spatially homogeneous Wiener process with spectral measure μ. We prove that for any $p \in [1, \infty)$ and any nonnegative weight function $\varrho \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the following assertions are equivalent:

(1) The problem $(*)$ admits a unique $L^p(\varrho)$-valued solution;

(2) The weight ϱ is integrable and

$$\int_{\mathbb{R}^d} \frac{1}{C - \operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty$$

for sufficiently large C.

Under stronger integrability assumptions we prove that the $L^p(\varrho)$-valued solution has a continuous, resp. Hölder continuous version.

1. Introduction

In this paper we study space-time regularity of weak solutions of linear stochastic partial differential equations. Apart from their interest in their own right, linear models (such as the Laplace equation or the Stokes equations) serve as a first step towards understanding more complicated nonlinear models (such as nonlinear elliptic equations or the Navier-Stokes equations).

2000 *Mathematics Subject Classification(s).* Primary 60H15; Secondary 35R15, 47D06, 60B11, 60G15, 60G60, 60H05
Received November 7, 2001
Revised May 1, 2003

In the theory of stochastic PDE's there are two basic linear model equations: the Langevin equation and the Zakai equation. In the present paper we will be concerned with the former one, which can be written as

$$\begin{aligned} du(t) &= Au(t)\,dt + dW(t), \qquad t \geqslant 0, \\ u(0) &= 0. \end{aligned} \tag{1.1}$$

Here A is some linear operator acting in a vector space E and $\mathbb{W} = \{W(t)\}_{t\geqslant 0}$ is some type of Wiener process. There is an extensive literature on equation (1.1), see e.g. the monographs by Itô [13] and Da Prato and Zabczyk [5], [6].

In a recent paper [2] the authors have obtained necessary and sufficient conditions for existence and uniqueness of weak solutions to equation (1.1) in the situation where E is an arbitrary separable real Banach space, A is the generator of a C_0-semigroup of bounded linear operators on E, and $\mathbb{W}$ is a cylindrical Wiener process with a given Cameron-Martin space H which is assumed to be continuously embedded in E.

A different approach to equation (1.1) was introduced by Dawson and Salehi [9] for modelling the growth of populations in a random environment; see also [19]. In this approach $\mathbb{W}$ is interpreted as a homogeneous Wiener process on $\mathbb{R}^d$, and the equation admits a natural formulation in the space $\mathscr{S}'$ of tempered distributions on $\mathbb{R}^d$. In the context of $\mathscr{S}'$-valued solutions it is natural to ask for conditions under which an $\mathscr{S}'$-valued solution actually takes values in some space of functions. For the stochastic wave equation in dimension $d = 2$, this problem was investigated by Dalang and Frangos [8], who obtained conditions for the existence of a function-valued solution in terms of the spectral measure associated with $\mathbb{W}$. These results have been extended to higher dimensions and to a wider class of equations by many authors [17], [6], [22], [23], [3], [7], [15], [16], [21].

Consider, as a concrete example, the stochastic heat equation

$$\begin{aligned} du(t) &= \Delta u(t)\,dt + dW(t), \qquad t \geqslant 0, \\ u(0) &= 0. \end{aligned} \tag{1.2}$$

As is well-known, this equations has a unique weak solution in $\mathscr{S}'$, which is given by the stochastic convolution integral

$$u(t) = \int_0^t e^{(t-s)\Delta}\,dW(s). \tag{1.3}$$

Let $0 \leqslant \varrho \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be given and let $L^2(\varrho)$ denote the associated weighted L^2-space. Let μ denote the spectral measure of the homogeneous Wiener process $\mathbb{W}$ and denote by H_μ the Hilbert space of all tempered distributions of the form $\mathscr{F}^{-1}(\phi\mu)$ for some symmetric $\phi \in L^2_{\mathbb{C}}(\mu)$ (see Section 3 for more details). It is shown in [15] that the following assertions are equivalent:

(i) For all $t \geqslant 0$ we have $\int_0^t \|S(s)\|^2_{\mathscr{L}_2(H_\mu, L^2(\varrho))}\,ds < \infty$;
(ii) The weight ϱ is integrable and $\int_{\mathbb{R}^d} 1/(1+|\xi|^2)d\mu(\xi) < \infty$.

In (i), $\|\cdot\|_{\mathscr{L}_2(H_\mu,L^2(\varrho))}$ denotes the Hilbert-Schmidt norm.

An extension of this result to a class of pseudodifferential operators A including, e.g., the fractional Laplacians $-(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$, was obtained subsequently in [16]. Prior to [15], the integrability condition (ii) was discovered in [23] to imply the existence of $L^2(\varrho)$-valued solutions for certain nonlinear stochastic problems under more restrictive assumptions on the weight ϱ.

The finitiness of the integral in (i) implies that for each $t \geqslant 0$ the stochastic integral on the right hand side of (1.3) converges in $L^2(\varrho)$. For this reason it makes sense to view the resulting $L^2(\varrho)$-valued process an $L^2(\varrho)$-valued solution of (1.2). This notion of solution is a formal one, because $L^2(\varrho)$ does not always embed into $\mathscr{S}'$:

Example 1.1. Let $\varrho(x) = \exp(-\|x\|)$. Then the function $\exp((1/4)\|x\|)$ belongs to $L^2(\varrho)$, but this function does not define a tempered distribution.

In order to get around this problem, we think of both $\mathscr{S}'$ and $L^2(\varrho)$ as being embedded in $\mathscr{D}'$, the space of distributions on $\mathbb{R}^d$. This motivates the following definition. If E is a real Banach space, continuously embedded in $\mathscr{D}'$, then a predictable E-valued process $\{U(t)\}_{t\geqslant 0}$ will be called an *E-valued solution* of the problem (7.3) if for all $t \geqslant 0$ we have $U(t) = u(t)$ in $\mathscr{D}'$ a.s. For the stochastic heat equation, our main result now reads as follows (cf. Theorem 9.1):

Theorem 1.2. *Let $0 \leqslant \varrho \in L^1_{\rm loc}$ and $1 \leqslant p < \infty$ be arbitrary and fixed. The following assertions are equivalent*:

(1) *The problem* (1.2) *admits a unique $L^p(\varrho)$-valued solution*;
(2) *The weight ϱ is integrable and $\int_{\mathbb{R}^d} 1/(1+|\xi|^2)\,d\mu(\xi) < \infty$.*

In fact, we prove a more general version of this result for a class of pseudodifferential operators A generating a C_0-semigroup in $\mathscr{S}'$. We also show that the $L^p(\varrho)$-valued solution has a continuous modification if condition (2) is slightly strengthened.

The implication (1) ⇒ (2) is an extension of the above implication (i) ⇒ (ii). The main difficulty is to show that (1) actually implies the integrability condition (i). In the setting of an arbitrary separable Banach space E, this is achieved by proving that the existence of an E-valued solution implies a certain E-valued integral operator to be γ-radonifying (Theorem 7.3), hence Hilbert-Schmidt if E is a Hilbert space.

The implication (2) ⇒ (1) extends the implication (ii) ⇒ (i) above to arbitrary values of $p \in [1,\infty)$. This extension is nontrivial and has three main ingredients: a characterization of γ-radonifying operators taking values in weighted L^p-spaces (Theorem 2.3), a factorization theorem (Theorem 4.9) and the theory of stochastic integration in separable Banach spaces as developed in [2].

A particular feature of our approach that we would like to stress is that we do *not* require the semigroup generated by A to act in $L^p(\varrho)$, even when discussing the existence of a continuous modification of the solution.

After the completion of this paper, Professor Dalang kindly pointed out to us that a result closely related to our Theorem 9.1 is proved in [7, Theorem 11]. In this theorem, linear stochastic PDE's with constant coefficients are considered under a mild assumption on the Fourier transform of the Green's function, and a necessary and sufficient condition is obtained for existence of a locally square integrable random field solution. This condition is essentially equivalent to the integrability condition on the spectral measure μ in Theorem 9.1.

Finally, we modify our framework in order to be able to study the stochastic Schrödinger equation. In this case, we have (Theorem 11.1):

Theorem 1.3. *Let $1 \leqslant p < \infty$ and $0 \leqslant \varrho \in L^1_{\rm loc}$ be arbitrary and fixed. The following assertions are equivalent*:

(1) *Problem* (11.1) *admits an $L^p(\varrho)$-valued solution*;

(2) *μ is a finite measure and ϱ is integrable*;

2. γ-Radonifying operators

In this preliminary section we recall some facts about reproducing kernel Hilbert spaces and γ-radonifying operators that will be needed later. For proofs and unexplained terminology we refer to [2], [5], [18], [24], [25].

Reproducing kernel Hilbert spaces. Let E be a real Banach space. We call a bounded linear operator $Q \in \mathscr{L}(E^*, E)$ *positive* if

$$\langle Qx^*, x^*\rangle \geqslant 0, \qquad x^* \in E^*,$$

and *symmetric* if

$$\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle, \qquad x^*, y^* \in E^*.$$

If Q is positive and symmetric, then

$$(Qx^*, Qy^*) \mapsto \langle Qx^*, y^*\rangle, \qquad x^*, y^* \in E^*,$$

defines a real inner product on the range of Q. The completion H_Q of range Q with respect to this inner product is a real Hilbert space, the *reproducing kernel Hilbert space* (RKHS) associated with Q. If E is separable, then so is H_Q. The inclusion mapping from range Q into E extends to a continuous injection i_Q from H_Q into E, and we have the operator identity

$$Q = i_Q \circ i_Q^*.$$

Conversely, if $i : H \hookrightarrow E$ is a continuous embedding of a Hilbert space H into E, then $Q := i \circ i^*$ is positive and symmetric. As subsets of E we have $H = H_Q$ and the map $i^*x^* \mapsto i_Q^* x^*$ defines an isometrical isomorphism of H onto H_Q.

On various occasions we shall encounter the situation where we have an inclusion operator $i : H \hookrightarrow E$ and an embedding $k : E \hookrightarrow F$, where F is another real Banach space. Defining $Q := i \circ i^*$ and $R := (k \circ i) \circ (k \circ i)^*$, we obtain two positive symmetric operators, in $\mathscr{L}(E^*, E)$ and in $\mathscr{L}(F^*, F)$ respectively. One may now ask in which way their RKHS's H_Q and H_R are related. The answer is given in the following proposition:

Proposition 2.1. *Under the above assumptions, the identity map*

$$H_Q \ni i^*(k^*x^*) \mapsto (k \circ i)^* x^* \in H_R \qquad (x^* \in F^*),$$

extends uniquely to an unitary operator from H_Q onto H_R. In particular, as subsets of F we have equality

$$(k \circ i)(H_Q) = k(H_R).$$

Proof. This follows from

$$\|i^*(k^*x^*)\|_{H_Q}^2 = \langle Qk^*x^*, k^*x^* \rangle = \langle Rx^*, x^* \rangle = \|(k^* \circ i^*)x^*\|_{H_R}^2$$

and the fact that range i^* and range $(k \circ i)^*$ are dense in H_Q and H_R, respectively. □

γ-Radonifying operators. The standard cylindrical Gaussian measure of a separable real Hilbert space H will be denoted by γ_H. This is the unique finitely additive measure on the field of cylindrical subsets of H whose image with respect to every orthogonal finite rank projection P is a standard Gaussian measure on the finite dimensional range of P. The following well-known result links the concepts of Gaussian measure, reproducing kernel Hilbert space, and standard cylindrical measure.

Proposition 2.2. *Let E be a separable real Banach space and let $Q \in \mathscr{L}(E^*, E)$ be positive and symmetric. The following assertions are equivalent*:

(1) *Q is the covariance of a centred Gaussian measure ν_Q on E*;

(2) *The image cylindrical measure $i_Q(\gamma_{H_Q})$ extends to a centred Gaussian measure ν on E.*

In this situation, $\nu_Q = \nu$.

Let E be a separable real Banach space. A bounded operator $T \in \mathscr{L}(H, E)$ is called *γ-radonifying* if $T(\gamma_H)$ extends to a Gaussian measure on E. With this terminology we can rephrase Proposition 2.2 as follows: a positive symmetric operator Q is a covariance operator if and only if the associated embedding $i_Q : H_Q \hookrightarrow E$ is γ-radonifying. There is an extensive literature on γ-radonifying operators; we refer to [24], [25] and [1] and the references given there.

We will need the following well-known facts:

- If $T : H \to E$ is γ-radonifying and $S : E \to F$ is bounded, then also $S \circ T : H \to F$ is γ-radonifying.
- If $T : H_1 \to E$ is γ-radonifying and $S : H_0 \to H_1$ is bounded, then $T \circ S : H_0 \to E$ is γ-radonifying [1].

• If $T_1 : H_1 \to E$ is bounded and $U : H_0 \to H_1$ is unitary, then T is γ-radonifying if and only if $T_1 \circ U : H_0 \to F$ is γ-radonifying.

• If $H = H_0 \oplus H_1$ and $T_1 : H_1 \to E$ is bounded, then T_1 is γ-radonifying if and only if $T_1 \circ P_1 : H \to E$ is γ-radonifying, where P_1 is the orthogonal projection of H onto H_1.

• If E is a Hilbert space, then $T : H \to E$ is γ-radonifying if and only if T is Hilbert-Schmidt.

In Section 9 it will be important to know when certain operators taking values in weighted L^p-spaces are γ-radonifying. In this direction we have the following general result.

Theorem 2.3. *Suppose H is a separable real Hilbert space and let $p \in [1, \infty)$ be fixed. Let $(\mathscr{O}, \mathscr{F}, \nu)$ be a σ-finite measure space. For a bounded linear operator $K : H \to L^p(\mathscr{O})$ the following assertions are equivalent*:

(1) *K is γ-radonifying*;

(2) *There exists a ν-measurable function $\kappa : \mathscr{O} \to H$ with*

$$\int_{\mathscr{O}} \|\kappa(x)\|_H^p \, d\nu(x) < \infty$$

such that for all ν-almost all $x \in \mathscr{O}$ we have

$$(K(h))(x) = [\kappa(x), h]_H, \qquad h \in H.$$

Proof. Let $(e_j)_{j\geqslant 1}$ be an orthonormal basis for H and let $(\beta_j)_{j\geqslant 1}$ be a sequence of independent identically distributed real-valued standard Gaussian random variables.

It is well known (cf. [25, Section V.5.4], [5, Theorem 2.12]) that K is γ-radonifying if and only if the series $\sum_{j=1}^\infty \beta_j K e_j$ converges in $L^p(\mathscr{O})$ almost surely.

(1) ⇒ (2): By the almost sure convergence of $\sum_{j=1}^\infty \beta_j K e_j$ and Fernique's theorem,

$$\mathbb{E} \left\| \sum_{j=1}^\infty \beta_j K e_j \right\|_{L^p(\mathscr{O})}^p < \infty.$$

The map

$$(\omega, x) \mapsto \sum_{j=1}^\infty \beta_j(\omega)(K e_j)(x)$$

is measurable from $\Omega \times \mathscr{O}$ to $\mathbb{R}$, each term $\beta_j(\omega)(K e_j)(x)$ being measurable. Hence by Fubini's theorem,

$$\begin{aligned} \mathbb{E} \left\| \sum_{j=1}^\infty \beta_j K e_j \right\|_{L^p(\mathscr{O})}^p &= \int_{\mathscr{O}} \mathbb{E} \left| \sum_{j=1}^\infty \beta_j (K e_j)(x) \right|^p d\nu(x) \\ &= c_p \int_{\mathscr{O}} \left(\sum_{j=1}^\infty |(K e_j)(x)|^2 \right)^{\frac{p}{2}} d\nu(x) \end{aligned}$$

with $c_p > 0$ a constant depending on p only; cf. [25, Lemma V.5.2]. In particular,

$$\sum_{j=1}^{\infty} |(Ke_j)(x)|^2 < \infty$$

for ν-almost all $x \in \mathscr{O}$. It follows that there exists a measurable $\tilde{\mathscr{O}} \subset \mathscr{O}$ with $\nu(\mathscr{O} \setminus \tilde{\mathscr{O}}) = 0$ such that for all $x \in \tilde{\mathscr{O}}$ the map $\kappa_x : H \to \mathbb{R}$,

$$\kappa_x h := (Kh)(x)$$

is Hilbert-Schmidt, hence bounded. By the Riesz representation theorem, we obtain a function $\kappa : \tilde{\mathscr{O}} \to H$ such that

$$\kappa_x h = [\kappa(x), h]_H, \qquad h \in H, \ x \in \tilde{\mathscr{O}}.$$

Noting that

$$[\kappa(\cdot), e_j]_H = Ke_j(\cdot)|_{\tilde{\mathscr{O}}}$$

we see that $x \mapsto [\kappa(x), e_j]_H$ is measurable for each j, and therefore $x \mapsto \kappa(x)$ is measurable by Pettis's measurability theorem and the separability of H. By the Parseval formula,

$$\sum_{j=1}^{\infty} |(Ke_j)(x)|^2 = \sum_{j=1}^{\infty} |[\kappa(x), e_j]_H|^2 = \|\kappa(x)\|_H^2, \qquad x \in \tilde{\mathscr{O}}.$$

We extend κ to a function on $\mathscr{O}$ by extending it identically zero on $\mathscr{O} \setminus \tilde{\mathscr{O}}$. Combining everything, we find

$$c_p \int_{\mathscr{O}} \|\kappa(x)\|_H^p \, d\nu(x) = \mathbb{E} \Big\| \sum_{j=1}^{\infty} \beta_j K e_j \Big\|_{L^p(\mathscr{O})}^p < \infty.$$

(2) $\Rightarrow$ (1): This is a special case of a result due to Kwapień ([25, Proposition II.2.1 and Theorem VI.5.4]); the following short direct proof is a modification of [3, Proposition 2.1].

Using the Kahane-Khinchine inequality, for some constant C_p and all $1 \leqslant M \leqslant N$ we have

$$\begin{aligned}
\Bigg(\mathbb{E} \Big\| \sum_{j=M}^{N} \beta_j K e_j \Big\|_{L^p(\mathscr{O})}^2 \Bigg)^{\frac{p}{2}} &\leqslant C_p^p \mathbb{E} \Big\| \sum_{j=M}^{N} \beta_j K e_j \Big\|_{L^p(\mathscr{O})}^p \\
&= C_p^p \mathbb{E} \int_{\mathscr{O}} \Big| \sum_{j=M}^{N} \beta_j [\kappa(x), e_j]_H \Big|^p \, d\nu(x) \\
&= c_2^p C_p^p \int_{\mathscr{O}} \Bigg(\sum_{j=M}^{N} [\kappa(x), e_j]_H^2 \Bigg)^p \, d\nu(x).
\end{aligned}$$

Here c_p is the constant from the first part of the proof. By assumption the right hand side tends to 0 as $M,\ N \to \infty$. Thus the series $\sum_{j=1}^{\infty} \beta_j K e_j$ converges in $L^2(\Omega; L^p(\mathscr{O}))$ and, by the Itô-Nisio theorem, almost surely. This means that K is γ-radonifying. ☐

The following example will be relevant in later sections:

Example 2.4. Let μ be a nonnegative symmetric tempered measure on $\mathbb{R}^d$. Let $0 \leqslant \varrho \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a nonnegative locally integrable function. For $1 \leqslant p < \infty$ we denote by $L^p(\varrho)$ the associated weighted L^p-space. Let $H := L^2((0,T); L^2_{(s)}(\mu))$ (see the beginning of Section 3 for the definition of $L^2_{(s)}(\mu)$). Let $q : \mathbb{R}^d \to \mathbb{C}$ be symmetric, i.e. $\overline{q(-\xi)} = q(\xi)$ for all $\xi \in \mathbb{R}^d$, and assume that $\sup_{\xi\in\mathbb{R}^d} \operatorname{Re} q(\xi) < \infty$. Define $\kappa : \mathbb{R}^d \to H$ by

$$(\kappa(x))(t) = e^{-i\langle x,\,\cdot\rangle} e^{tq(-\,\cdot)}.$$

Then

$$\begin{aligned} \|\kappa(x)\|_H^2 &= \int_0^T \int_{\mathbb{R}^d} \left| e^{-i\langle x,\xi\rangle} e^{tq(-\xi)} \right|^2 d\mu(\xi)\,dt \\ &= \int_0^T \int_{\mathbb{R}^d} e^{2t\ \operatorname{Re} q(\eta)}\, d\mu(\eta)\,dt \end{aligned} \tag{2.1}$$

is independent of $x \in \mathbb{R}^d$. Therefore,

$$\int_{\mathbb{R}^d} \|\kappa(x)\|_H^p \varrho(x)\,dx = \int_0^T \int_{\mathbb{R}^d} e^{2t\ \operatorname{Re} q(\eta)}\, d\mu(\eta)\,dt \int_{\mathbb{R}^d} \varrho(x)\,dx$$

is finite if and only if both $\int_0^T \int_{\mathbb{R}^d} e^{2t\ \operatorname{Re} q(\eta)}\, d\mu(\eta)\,dt$ and $\int_{\mathbb{R}^d} \varrho(x)\,dx$ are finite. In particular, the operator $K : H \to L^p(\varrho)$ with an integral kernel κ is γ-radonifying if and only if both of these conditions hold. Below (Proposition 4.3) we will give a necessary and sufficient condition for the first integral to be finite.

3. The Hilbert space H_μ associated with a symmetric measure μ

Throughout the rest of this paper, $d \geqslant 1$ is a fixed integer. We denote by $\mathscr{S} = \mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}_{\mathbb{C}} = \mathscr{S}(\mathbb{R}^d; \mathbb{C})$ the Schwartz spaces of real-valued and complex-valued, rapidly decreasing functions on $\mathbb{R}^d$, respectively. Their topological duals $\mathscr{S}'$ and $\mathscr{S}'_{\mathbb{C}}$ are the spaces of real and complex tempered distributions on $\mathbb{R}^d$. A *tempered measure* is a Radon measure μ on $\mathbb{R}^d$ that is also a tempered distribution. A nonnegative Radon measure μ is tempered whenever there exists $N \geqslant 0$ such that

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^N}\, d\mu(\xi) < \infty.$$

If μ is a nonnegative tempered measure and $f \in L^2_{\mathbb{C}}(\mu) = L^2(\mathbb{R}^d, \mu; \mathbb{C})$, then the map

$$\phi \mapsto \int_{\mathbb{R}^d} \phi f \, d\mu, \qquad \phi \in \mathscr{S}_{\mathbb{C}},$$

defines a tempered distribution $f\mu \in \mathscr{S}'_{\mathbb{C}}$. Note that we do not take complex conjugates in this identification; this convention should be kept in mind in the definition of the Fourier transform of a tempered distribution below.

The *Fourier transform* of a function $\phi \in \mathscr{S}_{\mathbb{C}}$ is defined by

$$(\mathscr{F}\phi)(\xi) := \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} \phi(x) \, dx, \qquad \xi \in \mathbb{R}^d,$$

where dx represents the *normalized* Lebesgue measure on $\mathbb{R}^d$. Thanks to this normalization the inverse Fourier transform is given by

$$(\mathscr{F}^{-1}\phi)(\xi) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \phi(x) \, dx, \qquad \xi \in \mathbb{R}^d.$$

The Fourier transform on the space of tempered distributions is defined by duality, i.e. for $\Phi \in \mathscr{S}'_{\mathbb{C}}$ we take

$$\langle \phi, \mathscr{F}\Phi \rangle := \langle \mathscr{F}\phi, \Phi \rangle, \qquad \phi \in \mathscr{S}_{\mathbb{C}}.$$

The inverse Fourier transform on $\mathscr{S}'_C$ is then given by

$$\langle \phi, \mathscr{F}^{-1}\Phi \rangle = \langle \mathscr{F}^{-1}\phi, \Phi \rangle, \qquad \phi \in \mathscr{S}_{\mathbb{C}}.$$

For a function $f : \mathbb{R}^d \to \mathbb{C}$ we define $\check{f} : \mathbb{R}^d \to \mathbb{C}$ by

$$\check{f}(x) = \overline{f(-x)}, \qquad x \in \mathbb{R}^d.$$

If $\check{f} = f$ we say that f is *symmetric*. We define

$$L^2_{(s)}(\mu) = \left\{ f \in L^2_{\mathbb{C}}(\mu) : \ \check{f} = f \right\}.$$

This is a closed linear subspace of $L^2_{\mathbb{C}}(\mu)$.

Now let μ be a nonnegative *symmetric* tempered measure on $\mathbb{R}^d$. Then for any two $f, g \in L^2_{(s)}(\mu)$ we have

$$\begin{aligned}\overline{[f,g]_{L^2_{(s)}(\mu)}} &= \int_{\mathbb{R}^d} \overline{f(\xi)} g(\xi) \, d\mu(\xi) = \int_{\mathbb{R}^d} f(-\xi)\overline{g(-\xi)} \, d\mu(\xi) \\ &= \int_{\mathbb{R}^d} f(\eta)\overline{g(\eta)} \, d\mu(\eta) = [f,g]_{L^2_{(s)}(\mu)},\end{aligned}$$

and therefore the inner product on $L^2_{(s)}(\mu)$ is real-valued. Thus, $L^2_{(s)}(\mu)$ is a separable real Hilbert space in a natural way.

It is easily checked that $\mathscr{F}\phi \in L^2_{(s)}(\mu)$ for all $\phi \in \mathscr{S}$. This observation motivates the following definition:

Definition 3.1. Let μ be a nonnegative symmetric tempered measure on $\mathbb{R}^d$. We define $\mathscr{H}_\mu$ to be the separable real Hilbert space obtained as the completion of $\mathscr{S}$ with respect to the inner product

$$[\phi, \psi]_{\mathscr{H}_\mu} := [\mathscr{F}\phi, \mathscr{F}\psi]_{L^2_{(s)}(\mu)}, \qquad \phi, \psi \in \mathscr{S}.$$

The space $\mathscr{H}_\mu$ will be used below to describe the covariance structure of a spatially homogeneous Wiener process in $\mathscr{S}'$ with spectral measure μ.

For all $f \in L^2_{(s)}(\mu)$, the tempered distribution $\mathscr{F}^{-1}(f\mu)$ is real. Indeed, a simple computation shows that $\langle \phi, \mathscr{F}^{-1}(f\mu)\rangle$ is real-valued for all $\phi \in \mathscr{S}$. This motivates the following definition:

Definition 3.2. Let μ be a nonnegative symmetric tempered measure on $\mathbb{R}^d$. We define H_μ to be the linear subspace of all tempered distributions of the form $\mathscr{F}^{-1}(f\mu)$ with $f \in L^2_{(s)}(\mu)$. With respect to the inner product

$$[\mathscr{F}^{-1}(f\mu), \mathscr{F}^{-1}(g\mu)]_{H_\mu} := [f, g]_{L^2_{(s)}(\mu)}, \tag{3.1}$$

this is a separable real Hilbert space.

The space H_μ will turn out to be invariant under the action of semigroups in $\mathscr{S}'$ generated by certain pseudodifferential operators in $\mathscr{S}'$ introduced in the next section. This is the key fact in our analysis of E-valued solutions in Section 7 below.

The relation between the spaces $\mathscr{H}_\mu$ and H_μ is described in the following proposition.

Proposition 3.3. *The mapping*

$$U_\mu \phi := \mathscr{F}^{-1}((\mathscr{F}\phi)\mu), \qquad \phi \in \mathscr{S},$$

extends to a unitary operator from $\mathscr{H}_\mu$ *onto* H_μ.

Proof. For all $\phi, \psi \in \mathscr{S}$ we have

$$\begin{aligned}[U_\mu^* U_\mu \phi, \psi]_{\mathscr{H}_\mu} &= [U_\mu \phi, U_\mu \psi]_{H_\mu} = [\mathscr{F}^{-1}((\mathscr{F}\phi)\mu), \mathscr{F}^{-1}((\mathscr{F}\psi)\mu)]_{H_\mu} \\ &= [\mathscr{F}\phi, \mathscr{F}\psi]_{L^2_{(s)}(\mu)} = [\phi, \psi]_{\mathscr{H}_\mu}.\end{aligned}$$

Hence $U_\mu^* U_\mu = I$.

Next, for all $f \in L^2_{(s)}(\mu) \cap \mathscr{S}_\mathbb{C}$ and $\phi \in \mathscr{S}$ we have

$$\begin{aligned}[U_\mu^*(\mathscr{F}^{-1}(f\mu)), \phi]_{\mathscr{H}_\mu} &= [\mathscr{F}^{-1}(f\mu), \mathscr{F}^{-1}((\mathscr{F}\phi)\mu)]_{H_\mu} \\ &= [f, \mathscr{F}\phi]_{L^2_{(s)}(\mu)} = [\mathscr{F}^{-1}f, \phi]_{\mathscr{H}_\mu}.\end{aligned}$$

Hence

$$U_\mu^*(\mathscr{F}^{-1}(f\mu)) = \mathscr{F}^{-1}f, \qquad f \in L^2_{(s)}(\mu) \cap \mathscr{S}_\mathbb{C}.$$

It follows that for all $f, g \in L^2_{(s)}(\mu) \cap \mathscr{S}_{\mathbb{C}}$ we have

$$[U_\mu U_\mu^*(\mathscr{F}^{-1}(f\mu)), (\mathscr{F}^{-1}(g\mu))]_{H_\mu} = [\mathscr{F}^{-1}f, \mathscr{F}^{-1}g]_{\mathscr{H}_\mu} \\ = [f, g]_{L^2_{(s)}(\mu)} = [\mathscr{F}^{-1}(f\mu), \mathscr{F}^{-1}(g\mu)]_{H_\mu}.$$

We claim that $L^2_{(s)}(\mu) \cap \mathscr{S}_{\mathbb{C}}$ is dense in $L^2_{(s)}(\mu)$. Once we know this, it follows that

$$\{\mathscr{F}^{-1}(f\mu) : \ f \in L^2_{(s)}(\mu) \cap \mathscr{S}_{\mathbb{C}}\}$$

is dense in H_μ and therefore $U_\mu U_\mu^* = I$.

Given a function $f \in L^2_{(s)}(\mu)$ we choose a sequence $(g_n) \in \mathscr{S}_{\mathbb{C}}$ such that $g_n \to f$ in $L^2_{\mathbb{C}}(\mu)$ (we could even take complex-valued compactly supported smooth functions). Define $f_n \in L^2_{(s)}(\mu)$ by

$$f_n := \frac{1}{2}(g_n + \breve{g}_n).$$

Then $\lim_{n\to\infty} f_n = (1/2)(f + \breve{f}) = f$ as desired. □

Let us denote by $i_{\mathscr{S},\mathscr{H}_\mu} : \mathscr{S} \hookrightarrow \mathscr{H}_\mu$ and $i_{H_\mu,\mathscr{S}'} : H_\mu \hookrightarrow \mathscr{S}'$ the natural inclusion mappings. We then have the following sequence of mappings:

$$\mathscr{S} \overset{i_{\mathscr{S},\mathscr{H}_\mu}}{\hookrightarrow} \mathscr{H}_\mu \xrightarrow{U_\mu} H_\mu \overset{i_{H_\mu,\mathscr{S}'}}{\hookrightarrow} \mathscr{S}'.$$

The following proposition relates these three mappings:

Proposition 3.4. *We have $i_{\mathscr{S},\mathscr{H}_\mu} = (i_{H_\mu,\mathscr{S}'} \circ U_\mu)^*$ and $i_{H_\mu,\mathscr{S}'} = (U_\mu \circ i_{\mathscr{S},\mathscr{H}_\mu})^*$.*

Proof. In the proof we will make no identifications and write out all inclusion mappings.

Let $\phi, \psi \in \mathscr{S}$ be arbitrary and fixed. Then $i^*_{H_\mu,\mathscr{S}'}$ maps ϕ onto an element $i^*_{H_\mu,\mathscr{S}'}\phi \in H_\mu$. By definition of H_μ there exists a function $f \in L^2_{(s)}(\mu)$ such that $i^*_{H_\mu,\mathscr{S}'}\phi = \mathscr{F}^{-1}(f\mu)$. Then,

$$\begin{aligned}\langle \phi, i_{H_\mu,\mathscr{S}'} U_\mu i_{\mathscr{S},\mathscr{H}_\mu}\psi\rangle &= [\mathscr{F}^{-1}(f\mu), U_\mu i_{\mathscr{S},\mathscr{H}_\mu}\psi]_{H_\mu} \\ &= [\mathscr{F}^{-1}(f\mu), \mathscr{F}^{-1}((\mathscr{F}\psi)\mu)]_{H_\mu} \\ &= [f, \mathscr{F}\psi]_{L^2_{(s)}(\mu)} = [\mathscr{F}\psi, f]_{L^2_{(s)}(\mu)} \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle}\psi(x)\,dx\,\overline{f(\xi)}\,d\mu(\xi) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle x,\eta\rangle} f(\eta)\,d\mu(\eta)\,\psi(x)\,dx \\ &= \langle \psi, i_{H_\mu,\mathscr{S}'}\mathscr{F}^{-1}(f\mu)\rangle \\ &= \langle \psi, i_{H_\mu,\mathscr{S}'} i^*_{H_\mu,\mathscr{S}'}\phi\rangle \\ &= \langle \phi, i_{H_\mu,\mathscr{S}'} i^*_{H_\mu,\mathscr{S}'}\psi\rangle.\end{aligned}$$

This shows that

$$i_{H_\mu,\mathscr{S}'} \circ U_\mu \circ i_{\mathscr{S},\mathscr{H}_\mu} = i_{H_\mu,\mathscr{S}'} \circ i^*_{H_\mu,\mathscr{S}'}.$$

Since $i_{H_\mu,\mathscr{S}'}$ is injective, it follows that

$$U_\mu \circ i_{\mathscr{S},\mathscr{H}_\mu} = i^*_{H_\mu,\mathscr{S}'}. \tag{3.2}$$

Multiplying both sides in (3.2) from the left with U_μ^* gives the first identity; dualizing (3.2) gives the second identity. □

4. A C_0-semigroup on H_μ associated with a symmetric symbol q

Throughout the rest of this paper, it will be a standing assumption that $q : \mathbb{R}^d \to \mathbb{C}$ is a measurable function satisfying

$$q = \check{q}, \tag{4.1}$$

$$q^* := \sup_{\xi\in\mathbb{R}^d} \operatorname{Re} q(\xi) < \infty. \tag{4.2}$$

We fix a nonnegative symmetric tempered measure μ on $\mathbb{R}^d$ and let H_μ denote the separable real Hilbert space from Definition 3.2. We define a semigroup of bounded linear operators $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ on H_μ by

$$S(t)(\mathscr{F}^{-1}(f\mu)) = \mathscr{F}^{-1}(e^{tq(\cdot)}f(\cdot)\mu).$$

Since q is symmetric and $\operatorname{Re} q$ is bounded from above, the function $e^{tq(\cdot)}f(\cdot)$ belongs to $L^2_{(s)}(\mu)$, which shows that the operators $S(t)$ are well-defined.

Example 4.1. We give some examples of functions q satisfying the conditions (4.1) and (4.2).

(1) The function $q(\xi) = i\xi$ ($\xi \in \mathbb{R}$). The semigroup $\mathbf{S}$ is the restriction to H_μ of the left translation semigroup on $\mathscr{S}'$ in dimension $d = 1$.

(2) The symbol q of an elliptic operator with constant coefficients. For $q(\xi) = -|\xi|^2$, $\mathbf{S}$ is the restriction of H_μ of the heat semigroup.

(3) The function $q(\xi) = |\xi|^2 - |\xi|^4$. It arises in connection with the beam equation.

(4) The function $q(\xi) = -|\xi|^{2\gamma}$ with $\gamma > 0$. This example was considered in [11]. For $\gamma = 1/2$, $\mathbf{S}$ is the restriction of H_μ of the Poisson semigroup.

Notice that the function $q(\xi) = (i/2)|\xi|^2$, which corresponds to the Schrödinger semigroup, satisfies (4.2), but not (4.1). In the final section of this paper we will return to this example.

Proposition 4.2. *The semigroup $\{S(t)\}_{t\geqslant 0}$ is strongly continuous on H_μ and satisfies*

$$\|S(t)\|_{H_\mu} \leqslant e^{tq^*}, \qquad t \geqslant 0. \tag{4.3}$$

Proof. The inequality $|e^{tq(\xi)}| \leqslant e^{tq^*}$ shows that $S(t)$ satisfies the estimate (4.3). It remains to prove strong continuity of $\{S(t)\}_{t\geqslant 0}$ in H_μ. By the dominated convergence theorem, for $\Phi = \mathscr{F}^{-1}(f\mu)$ and $\Psi = \mathscr{F}^{-1}(g\mu)$ we have

$$\lim_{t\downarrow 0} [S(t)\Phi, \Psi]_{H_\mu} = \lim_{t\downarrow 0} \int_{\mathbb{R}^d} e^{tq(\xi)} f(\xi)\overline{g(\xi)}\, d\mu(\xi)$$
$$= \int_{\mathbb{R}^d} f(\xi)\overline{g(\xi)}\, d\mu(\xi) = [\Phi, \Psi]_{H_\mu}.$$

This proves that $\{S(t)\}_{t\geqslant 0}$ is weakly continuous as a semigroup in H_μ, and therefore strongly continuous by a standard result from semigroup theory [20]. □

Under an appropriate integrability condition, the semigroup $\{S(t)\}_{t\geqslant 0}$ maps H_μ into BUC (here we identify both H_μ and BUC with linear subspaces of $\mathscr{S}'$). This will be derived as a consequence of the following proposition.

Proposition 4.3. *Fix $T > 0$ and $C > q^*$. Then*

$$\int_{\mathbb{R}^d} \frac{1}{C - \operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty \tag{4.4}$$

if and only if

$$\int_0^T \int_{\mathbb{R}^d} e^{2t \operatorname{Re} q(\xi)}\, d\mu(\xi)\, dt < \infty. \tag{4.5}$$

Proof. Clearly,

$$I := \int_0^T \int_{\mathbb{R}^d} e^{2t \operatorname{Re} q(\xi)}\, d\mu(\xi)\, dt < \infty$$

is finite if and only if I_0 and I_∞ are both finite, where

$$I_0 := \int_0^T \int_{|\operatorname{Re} q| \leqslant |C|} e^{2t\ \operatorname{Re} q(\xi)}\, d\mu(\xi)\, dt$$

and

$$I_\infty := \int_0^T \int_{\operatorname{Re} q < -|C|} e^{2t\ \operatorname{Re} q(\xi)}\, d\mu(\xi)\, dt.$$

Since the function $(t, \xi) \mapsto e^{2t\ \operatorname{Re} q(\xi)}$ is bounded away from 0 on $[0, T] \times \{|\operatorname{Re} q| \leqslant |C|\}$, it is clear that $I_0 < \infty$ if and only if

$$\mu\{|\operatorname{Re} q| \leqslant |C|\} < \infty.$$

In view of

$$0 < C - q^* \leqslant C - \operatorname{Re} q(\xi) \leqslant 2|C|,$$

the right most inequality being valid whenever $|\operatorname{Re} q(\xi)| \leqslant |C|$, this happens if and only if

$$\int_{|\operatorname{Re} q| \leqslant |C|} \frac{1}{C - \operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty.$$

Concerning I_∞ we note that

$$I_\infty = \int_{\operatorname{Re} q < -|C|} \frac{1}{-2\operatorname{Re} q(\xi)} (1 - e^{2T\operatorname{Re} q(\xi)}) d\mu(\xi).$$

Hence we can estimate

$$(1 - e^{-2T|C|}) \int_{\operatorname{Re} q < -|C|} \frac{1}{-2\operatorname{Re} q(\xi)}\, d\mu(\xi) \leqslant I_\infty \leqslant \int_{\operatorname{Re} q < -|C|} \frac{1}{-2\operatorname{Re} q(\xi)}\, d\mu(\xi).$$

Hence $I_\infty < \infty$ if and only if

$$\int_{\operatorname{Re} q < -|C|} \frac{1}{-\operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty. \tag{4.6}$$

If $C \geqslant 0$, then for all $\xi \in \mathbb{R}^d$ we have

$$-\operatorname{Re} q(\xi) \leqslant C - \operatorname{Re} q(\xi) \leqslant -2\operatorname{Re} q(\xi).$$

If $C < 0$ we choose $\varepsilon > 0$ such that $(1-\varepsilon)q^* \leqslant C$. Then for all $\xi \in \mathbb{R}^d$ we have $(1-\varepsilon)\operatorname{Re} q(\xi) \leqslant (1-\varepsilon)q^* \leqslant C$ and therefore

$$-\varepsilon \operatorname{Re} q(\xi) \leqslant C - \operatorname{Re} q(\xi) \leqslant -\operatorname{Re} q(\xi).$$

In both cases it follows that (4.6) holds if and only if

$$\int_{\operatorname{Re} q < -|C|} \frac{1}{C - \operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty.$$

□

Remark 4.4. If (4.4) holds, then in particular we have

$$\int_{\mathbb{R}^d} e^{2t\operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty, \qquad t > 0. \tag{4.7}$$

This can be deduced from (4.5) or by simply observing that for every $t > 0$ there exists a constant $M_t \geqslant 0$ such that $e^{2ts} \leqslant M_t/(C - s)$ for all $s \leqslant q^*$.

The following Hypothesis, expressing that the equivalent statements of Proposition 4.3 hold, will play an important rôle:

Hypothesis (H). There exists a constant $C > q^*$ such that

$$\int_{\mathbb{R}^d} \frac{1}{C - \operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty.$$

Let $\mathscr{C}_0$ and BUC denote the Banach spaces of continuous real-valued functions on $\mathbb{R}^d$ vanishing at infinity, respectively which are bounded and uniformly continuous. Both spaces are endowed with the supremum norm. In our next result we identify $\mathscr{C}_0$ and BUC with a linear subspace $\mathscr{S}'$ in the natural way.

Proposition 4.5. *Assume* **(H)**. *For all $t > 0$, the operator $S(t)$ maps H_μ into BUC and we have*

$$\|S(t)\|_{\mathscr{L}(H_\mu, BUC)} \leqslant \left(\int_{\mathbb{R}^d} e^{2t \operatorname{Re} q(\xi)}\, d\mu(\xi)\right)^{\frac{1}{2}}. \tag{4.8}$$

If μ is absolutely continuous with respect to the Lebesgue measure, then $S(t)$ maps H_μ into $\mathscr{C}_0$.

Proof. Let $f \in L^2_{(s)}(\mu)$ be fixed. By (4.7) we have $e^{tq} \in L^2_{(s)}(\mu)$, so $e^{tq} f \in L^1_{(s)}(\mu)$ by the Cauchy-Schwarz inequality. From the identity $S(t)(\mathscr{F}^{-1}(f\mu)) = \mathscr{F}^{-1}(e^{tq} f\mu)$ it follows that $S(t)(\mathscr{F}^{-1}(f\mu))$ can be represented by the bounded function

$$x \mapsto \mathscr{F}^{-1}(e^{tq} f\mu)(x) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} e^{tq(\xi)} f(\xi)\, d\mu(\xi). \tag{4.9}$$

This function is real-valued because $e^{tq} f$ is symmetric. Moreover,

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} |S(t)\mathscr{F}^{-1}(f\mu)(x)| &\leqslant \|e^{tq} f\|_{L^1_{(s)}(\mu)} \leqslant \|e^{tq}\|_{L^2_{(s)}(\mu)} \|f\|_{L^2_{(s)}(\mu)} \\ &= \left(\int_{\mathbb{R}^d} e^{2t \operatorname{Re} q(\xi)}\, d\mu(\xi)\right)^{\frac{1}{2}} \|\mathscr{F}^{-1}(f\mu)\|_{H_\mu}.\end{aligned}$$

The proof that the function representing $S(t)(\mathscr{F}^{-1}(f\mu))$ is uniformly continuous is standard, and is included just for the convenience of the reader. Given $\varepsilon > 0$, for large enough R we have $\int_{|\xi|>R} |e^{tq(\xi)} f(\xi)|\, d\mu(\xi) < \varepsilon$ and therefore

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} (e^{i\langle x,\xi\rangle} - e^{i\langle x',\xi\rangle}) e^{tq(\xi)} f(\xi)\, d\mu(\xi)\right| \\ &\quad\leqslant \max_{|\xi|\leqslant R} |e^{i\langle x,\xi\rangle} - e^{i\langle x',\xi\rangle}| \cdot \int_{|\xi|\leqslant R} |e^{tq(\xi)} f(\xi)|\, d\mu(\xi) + 2\varepsilon \\ &\quad\leqslant \max_{|\xi|\leqslant R} |1 - e^{i\langle x-x',\xi\rangle}| \cdot \int_{\mathbb{R}^d} |e^{tq(\xi)} f(\xi)|\, d\mu(\xi) + 2\varepsilon.\end{aligned}$$

From this estimate we deduce that the function in (4.9) is uniformly continuous. The previous estimate shows that $S(t)$, as an operator from H_μ into BUC, is bounded with norm given by (4.8).

The final assertion is a consequence of the Riemann-Lebesgue lemma. □

As an operator in $\mathscr{L}(H_\mu, BUC)$, we denote $S(t)$ by $S_{BUC}(t)$. We will study the operators $S_{BUC}(t)$ in more detail next. In the results that follow, the rôle of BUC may be replaced by $\mathscr{C}_0$ if μ is absolutely continuous with respect to the Lebesgue measure.

Lemma 4.6. *Assume* **(H)**. *For all $T > 0$ and $g \in L^2((0,T); H_\mu)$ the BUC-valued function $t \mapsto S_{BUC}(t)g(t)$ is Bochner integrable on $(0,T)$ and we have*

$$\left\| \int_0^T S_{BUC}(t)g(t)\,dt \right\|_{BUC} \leqslant \left(\int_0^T \int_{\mathbb{R}^d} e^{2t\,\mathrm{Re}\,q(\xi)}\,d\mu(\xi)\,dt \right)^{\frac{1}{2}} \|g\|_{L^2((0,T);H_\mu)}.$$

Proof. For each fixed $h \in H_\mu$, the BUC-valued function $t \mapsto S_{BUC}(t)h$ is right continuous on $(0,\infty)$. To see this, fix $h \in H_\mu$ and $t_0 > 0$. Then, by the strong continuity of $\{S(t)\}_{t\geqslant 0}$ in H_μ,

$$\lim_{\varepsilon\downarrow 0} S_{BUC}(t_0+\varepsilon)h = \lim_{\varepsilon\downarrow 0} S_{BUC}(t_0)(S(\varepsilon)h) = 0. \tag{4.10}$$

It follows that $t \mapsto S_{BUC}(t)g(t)$ is strongly measurable on $(0,\infty)$ for all step functions $g \in L^2((0,T); H_\mu)$. Since the step functions are dense in $L^2((0,T);$ $H_\mu)$, it follows that $t \mapsto S_{BUC}(t)g(t)$ is strongly measurable on $(0,\infty)$ for all $g \in L^2((0,T); H_\mu)$.

By (4.8),

$$\|S_{BUC}(t)g(t)\|_{BUC} \leqslant \left(\int_{\mathbb{R}^d} e^{2t\,\mathrm{Re}\,q(\xi)}\,d\mu(\xi) \right)^{\frac{1}{2}} \|g(t)\|_{H_\mu}.$$

Hence by Hölder's inequality,

$$\int_0^T \|S_{BUC}(t)g(t)\|_{BUC}\,dt \leqslant \left(\int_0^T \int_{\mathbb{R}^d} e^{2t\,\mathrm{Re}\,q(\xi)}\,d\mu(\xi)\,dt \right)^{\frac{1}{2}} \|g\|_{L^2((0,T);H_\mu)},$$

which is finite by Proposition 4.3. It follows that $t \mapsto S_{BUC}(t)g(t)$ is Bochner integrable in BUC and that the desired estimate holds. □

Proposition 4.7. *Assume* **(H)**. *For all $\varphi \in BUC^*$ the H_μ-valued function $t \mapsto S^*_{BUC}(t)\varphi$ is strongly measurable on $(0,\infty)$ and for all $T > 0$ we have*

$$\int_0^T \|S^*_{BUC}(t)\varphi\|^2_{H_\mu}\,dt \leqslant \left(\int_0^T \int_{\mathbb{R}^d} e^{2t\,\mathrm{Re}\,q(\xi)}\,d\mu(\xi)\,dt \right) \|\varphi\|^2_{BUC^*}.$$

Proof. The H_μ-valued function $t \mapsto S^*_{BUC}(t)\varphi$ is weakly measurable and separably valued, and therefore strongly measurable by the Pettis measurability theorem [10]. For $T > 0$ let us define the bounded operator J_T : $L^2((0,T); H_\mu) \to BUC$ by

$$J_T g := \int_0^T S_{BUC}(t)g(t)\,dt.$$

For all $g \in L^2((0,T);H_\mu)$ and $\varphi \in BUC^*$ we have

$$\langle g, J_T^*\varphi\rangle = \left\langle \int_0^T S_{BUC}(t)g(t)\,dt, \varphi \right\rangle = \int_0^T [g(t), S_{BUC}^*(t)\varphi]_{H_\mu}\,dt$$
$$= [g(\cdot), S_{BUC}^*(\cdot)\varphi]_{L^2((0,T);H_\mu)}.$$

It follows that $J_T^*\varphi = S_{BUC}^*(\cdot)\varphi$ and consequently,

$$\int_0^T \|S_{BUC}^*(t)\varphi\|_{H_\mu}^2\,dt = \|J_T^*\varphi\|_{L^2((0,T);H_\mu)}^2 \leqslant \|J_T\|^2\|\varphi\|_{BUC^*}^2.$$

Finally, by Lemma 4.6,

$$\|J_T\| \leqslant \left(\int_0^T \int_{\mathbb{R}^d} e^{2t\operatorname{Re} q(\xi)}\,d\mu(\xi)\,dt\right)^{\frac{1}{2}}.$$

□

Before proceeding with the main line of developement, we insert a related proposition which will be needed in Section 8 when we study time regularity of weak solutions.

Proposition 4.8. *Assume* **(H)**. *For all* $\varphi \in BUC^*$ *and all* $0 \leqslant s \leqslant t$ *we have*

$$\int_0^s \|S_{BUC}^*(t-s+r)\varphi - S_{BUC}^*(r)\varphi\|_{H_\mu}^2\,dr$$
$$\leqslant \left(\int_0^s \int_{\mathbb{R}^d} |e^{(t-s+r)q(\xi)} - e^{rq(\xi)}|^2\,d\mu(\xi)\,dr\right)\|\varphi\|_{BUC^*}^2.$$

Proof. For $g \in L^2((0,T),H_\mu)$ we define

$$J_{s,t}g = \int_0^s S_{BUC}(t-s+r)g(r) - S_{BUC}(r)g(r)\,dr.$$

We now write $g(r) = \mathscr{F}^{-1}(f(r)\mu)$ with $f(r) \in L^2_{(s)}(\mu)$. Using (4.9) and estimating as above with Hölder's inequality, we obtain

$$\|J_{s,t}\| \leqslant \left(\int_0^s \int_{\mathbb{R}^d} \left|e^{(t-s+r)q(\xi)} - e^{tq(\xi)}\right|^2\,d\mu(\xi)\,dr\right)^{\frac{1}{2}}.$$

As in Proposition 4.7, our inequality now follows by considering the adjoint of $J_{s,t}$. □

By Proposition 4.7, for every $T > 0$ we may define a bounded linear operator $Q_T \in \mathscr{L}(BUC^*, BUC)$ by

$$Q_T\varphi := \int_0^T S_{BUC}(t)S_{BUC}^*(t)\varphi\,dt, \qquad \varphi \in BUC^*, \tag{4.11}$$

where the integral converges in BUC as a Bochner integral. For this operator we have the following factorization result, which we obtain as an application of RKHS techniques.

Theorem 4.9. *Assume* **(H)**. *Define* $\kappa : \mathbb{R}^d \to L^2((0,T); L^2_{(s)}(\mu))$ *by*

$$(\kappa(x))(t)(\xi) = e^{-i\langle x,\xi\rangle} e^{tq(-\xi)}.$$

Then:

(1) *For all* $f \in L^2((0,T); L^2_{(s)}(\mu))$ *the function*

$$x \mapsto [f, \kappa(x)]_{L^2((0,T);L^2_{(s)}(\mu))}$$

is bounded and uniformly continuous.

(2) *The linear operator* $K_T : L^2((0,T); L^2_{(s)}(\mu)) \to BUC$ *defined by*

$$K_T f = [f, \kappa(\cdot)]_{L^2((0,T);L^2_{(s)}(\mu))}$$

is bounded and satisfies the operator identity

$$Q_T = K_T \circ K_T^*.$$

Proof. Let $f \in L^2((0,T); L^2_{(s)}(\mu))$ be arbitrary and fixed. For all $x \in \mathbb{R}^d$ we have, recalling that $\overline{q(-\xi)} = q(\xi)$,

$$\begin{aligned}
&|[f, \kappa(x)]_{L^2((0,T);L^2_{(s)}(\mu))}| \\
&\quad = \left| \int_0^T \int_{\mathbb{R}^d} (f(t))(\xi) \overline{e^{-i\langle x,\xi\rangle} e^{tq(-\xi)}} \, d\mu(\xi) \, dt \right| \\
&\quad \leqslant \int_0^T \int_{\mathbb{R}^d} |(f(t))(\xi) e^{tq(\xi)}| \, d\mu(\xi) \, dt \\
&\quad \leqslant \|f\|_{L^2((0,T);L^2_{(s)}(\mu))} \left(\int_0^T \int_{\mathbb{R}^d} e^{2t \operatorname{Re} q(\xi)} \, d\mu(\xi) \, dt \right)^{\frac{1}{2}}.
\end{aligned}$$

The double integral being finite, this shows that $x \mapsto [f, \kappa(x)]_{L^2((0,T);L^2_{(s)}(\mu))}$ is bounded. The uniform continuity of this map is proved as in Proposition 4.5. Hence K_T is well-defined as a linear operator from $L^2((0,T); L^2_{(s)}(\mu))$ into BUC, and the above estimate shows that K_T is bounded.

For the proof of the identity $Q_T = K_T \circ K_T$ we will set up a commutative diagram as follows:

$$\begin{array}{ccccc}
L^2((0,T); L^2_{(s)}(\mu)) & \xrightarrow[V_T]{} & L^2((0,T); H_\mu) & \xrightarrow[J_T]{} & BUC \\
\big\downarrow \mathscr{P}_T & & \big\downarrow P_T & & \big\uparrow i_T \\
\mathscr{Z}_{\mu,T} & \xrightarrow[V_{\mu,T}]{} & Z_{\mu,T} & \xrightarrow[I_{\mu,T}]{} & H_T
\end{array}$$

The meaning of the spaces and operators involved will be explained next. To start with, H_T denotes the RKHS associated with Q_T and $i_T : H_T \hookrightarrow BUC$ denotes the inclusion mapping; cf. Section 2. Recall that $Q_T = i_T \circ i_T^*$.

As before, J_T denotes the bounded operator from $L^2((0,T); H_\mu)$ into BUC defined by

$$J_T g = \int_0^T S_{BUC}(t) g(t)\, dt, \qquad g \in L^2((0,T); H_\mu).$$

Its adjoint is given by

$$J_T^* \varphi = S_{BUC}^*(\cdot)\varphi, \qquad \varphi \in BUC^*.$$

Let $Z_{\mu,T}$ denote the closure in $L^2((0,T); H_\mu)$ of the linear subspace of all functions of the form $g = S_{BUC}^*(\cdot)\varphi$ with $\varphi \in BUC^*$. Then $Z_{\mu,T} = \overline{\operatorname{range} J_T^*}$ and therefore $\ker J_T = (\operatorname{range} J_T^*)^\perp = (Z_{\mu,T})^\perp$. It follows that

$$L^2((0,T); H_\mu) = Z_{\mu,T} \oplus \ker J_T. \tag{4.12}$$

For all $\varphi \in BUC^*$ we have

$$J_T(S_{BUC}^*(\cdot)\varphi) = Q_T \varphi. \tag{4.13}$$

Identifying H_T and its image $i_T(H_T)$ in BUC, we have $Q_T\varphi \in H_T$ and

$$\begin{aligned} \|Q_T\varphi\|_{H_T}^2 = \langle Q_T\varphi, \varphi\rangle &= \int_0^T \langle S_{BUC}(t) S_{BUC}^*(t)\varphi, \varphi\rangle\, dt \\ &= \int_0^T \|S_{BUC}^*(t)\varphi\|_{H_\mu}^2\, dt = \|S_{BUC}^*(\cdot)\varphi\|_{L^2((0,T);H_\mu)}^2. \end{aligned}$$

It follows from these equalities and (4.13) that J_T maps $L^2((0,T); H_\mu)$ onto H_T and that its restriction to $Z_{\mu,T}$ is unitary. As an operator from $L^2((0,T); H_\mu)$ onto H_T we denote J_T by I_T. The restriction of I_T to $Z_{\mu,T}$ will be denoted by $I_{\mu,T}$; this is a unitary operator from $Z_{\mu,T}$ onto H_T.

Summarizing our discussion so far, we see that a function $f \in BUC$ belongs to H_T if and only if there exists a function $g \in Z_{\mu,T}$ such that

$$f = \int_0^T S_{BUC}(t) g(t)\, dt = J_T g;$$

moreover by (4.12),

$$\|f\|_{H_T} = \inf\{\|g\|_{L^2((0,T);H_\mu)} : \ g \in L^2((0,T); H_\mu),\ J_T g = f\}.$$

Define $J_{\mu,T} := i_T \circ I_{\mu,T}$. Then $Q_T = J_{\mu,T} \circ J_{\mu,T}^*$. Next we define

$$\mathscr{Z}_{\mu,T} = \{\varphi \in L^2((0,T); L^2_{(s)}(\mu)) : \ \mathscr{F}^{-1}(\varphi(\cdot)\mu) \in Z_{\mu,T}\}.$$

The operators $V_T : L^2((0,T);L^2_{(s)}(\mu)) \to L^2((0,T);H_\mu)$ and $V_{\mu,T} : \mathscr{Z}_{\mu,T} \to Z_{\mu,T}$ defined by

$$V_T\varphi := \mathscr{F}^{-1}(\varphi(\cdot)\mu)$$

and $V_{\mu,T} := V_T|_{\mathscr{Z}_{\mu,T}}$ are unitary. Therefore,

$$Q_T = (J_{\mu,T} \circ V_{\mu,T}) \circ (J_{\mu,T} \circ V_{\mu,T})^*.$$

Finally let $\mathscr{P}_T : L^2((0,T);H_\mu) \to Z_{\mu,T}$ and $P_T : L^2((0,T);L^2_{(s)}(\mu)) \to \mathscr{Z}_{\mu,T}$ denote the orthogonal projections. Then $\mathscr{P}_T \circ \mathscr{P}_T^* = I_{\mathscr{Z}_{\mu,T}}$, the identity operator on $\mathscr{Z}_{\mu,T}$. Therefore,

$$Q_T = (J_{\mu,T} \circ V_{\mu,T} \circ \mathscr{P}_T) \circ (J_{\mu,T} \circ V_{\mu,T} \circ \mathscr{P}_T)^*.$$

But for all $g \in L^2((0,T);L^2_{(s)}(\mu))$ we have

$$(J_{\mu,T} \circ V_{\mu,T} \circ \mathscr{P}_T)g = (J_T \circ V_T)g = \int_0^T S_{BUC}(t)(\mathscr{F}^{-1}(g(t)\mu)\,dt.$$

We will prove next that the right hand side equals K_Tg. Once we know this it follows that $Q_T = K_T \circ K_T^*$.

Identifying BUC with a linear subspace of $\mathscr{S}'$, for all $\phi \in \mathscr{S}$ we have

$$\begin{aligned}
\left\langle \phi, \int_0^T S_{BUC}(t)(\mathscr{F}^{-1}(g(t)\mu)\,dt \right\rangle &= \int_0^T \langle \mathscr{F}^{-1}\phi, e^{tq}g(t)\mu\rangle\,dt \\
&= \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\phi(x)\,dx\right) e^{tq(\xi)}(g(t))(\xi)\,d\mu(\xi)\,dt \\
&= \int_{\mathbb{R}^d} \phi(x) \left(\int_0^T \int_{\mathbb{R}^d} (g(t))(\xi)\,\overline{e^{-i\langle x,\xi\rangle}e^{tq(-\xi)}}\,d\mu(\xi)\,dt\right) dx \\
&= \int_{\mathbb{R}^d} \phi(x) \left(\int_0^T [g(t),(\kappa(x))(t)]_{L^2_{(s)}(\mu)}\,dt\right) dx \\
&= \int_{\mathbb{R}^d} \phi(x)\,[g,\kappa(x)]_{L^2((0,T);L^2_{(s)}(\mu))}\,dx \\
&= \langle \phi, K_Tg\rangle.
\end{aligned}$$

□

Now let E be a real Banach space in which BUC is embedded by means of a continuous embedding $i_{BUC,E} : BUC \hookrightarrow E$. Assuming **(H)**, for $t > 0$ we denote by $S_E(t) : H_\mu \to E$ the composition of $S_{BUC}(t)$ with the inclusion mapping $i_{BUC,E}$:

$$S_E(t) = i_{BUC,E} \circ S_{BUC}(t).$$

For every $T \geqslant 0$ we then define a bounded operator $Q_T \in \mathscr{L}(E^*,E)$ by

$$Q_T^E x^* := \int_0^T S_E(t)S_E^*(t)x^*\,dt, \qquad x^* \in E^*. \tag{4.14}$$

Note that $Q_T^E = i_{BUC,E} \circ Q_T \circ i_{BUC,E}^*$, where $Q_T : BUC^* \to BUC$ is the operator defined by (4.11).

Similarly we define

$$K_T^E(t) = i_{BUC,E} \circ K_T.$$

For the sake of simplicity, we will omit the embedding $i_{BUC,E}$ from our notations whenever it is convenient.

Proposition 4.10. *Assume* **(H)**. *Under the above assumptions, for every fixed $T > 0$ the following assertions are equivalent*:

(1) *The operator K_T^E is γ-radonifying from $L^2((0,T); L^2_{(s)}(\mu))$ into E*;

(2) *The operator Q_T^E is the covariance of a centred Gaussian measure on E.*

Proof. By Proposition 2.1 the RKHS's of Q_T and Q_T^E are canonically isometrically isomorphic, and identical as subsets of E. For this reason we will not distinguish these spaces from each other, and denote both by H_T.

From Section 2 we recall that Q_T^E is a covariance if and only if the associated embedding $i_T : H_T \hookrightarrow E$ is γ-radonifying. Clearly,

$$i_T = J_{\mu,T}^E \circ I_{\mu,T}^{-1},$$

where $J_{\mu,T}^E := i_{BUC,E} \circ J_{\mu,T}$; here $I_{\mu,T}$ and $J_{\mu,T}$ are the operators introduced in the proof of Theorem 4.9. From this we see that $i_T : H_T \hookrightarrow E$ is γ-radonifying if and only if $J_{\mu,T}^E : Z_{\mu,T} \to E$ is γ-radonifying, and this is the case if and only if $J_{\mu,T}^E \circ V_{\mu,T} : \mathscr{Z}_{\mu,T} \to E$ is γ-radonifying. Finally, since $\mathscr{P}_T$ is an orthogonal projection, for the standard Gaussian cylindrical measures $\gamma_{L^2((0,T);L^2_{(s)}(\mu))}$ and $\gamma_{\mathscr{Z}_{\mu,T}}$ of $L^2((0,T); L^2_{(s)}(\mu))$ and $\mathscr{Z}_{\mu,T}$ respectively we have

$$\mathscr{P}_T \left(\gamma_{L^2((0,T);L^2_{(s)}(\mu))} \right) = \gamma_{\mathscr{Z}_{\mu,T}},$$

and therefore $J_{\mu,T}^E \circ V_{\mu,T} : \mathscr{Z}_{\mu,T} \to E$ is γ-radonifying if and only if $K_T^E = J_{\mu,T}^E \circ V_{\mu,T} \circ \mathscr{P}_T : L^2((0,T); L^2_{(s)}(\mu)) \to E$ is γ-radonifying. □

5. Cylindrical Wiener processes

Let E be a separable real Banach space in which BUC is continuously embedded by means on an embedding $i_{BUC,E}$:

$$i_{BUC,E} : BUC \hookrightarrow E.$$

In this section we will use the estimates from the previous section to give a meaning to the stochastic integral

$$\int_0^t S_E(t-s)\, dW_{H_\mu}(s), \qquad t \geqslant 0,$$

where $\{W_{H_\mu}(t)\}_{t\geqslant 0}$ is a cylindrical Wiener process with Cameron-Martin space H_μ and

$$S_E(t) := i_{BUC,E} \circ S_{BUC}(t), \qquad t > 0.$$

Let us first state the definition of a cylindrical Wiener process:

Definition 5.1. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete filtered probability space, and let H be a separable real Hilbert space. A *cylindrical Wiener process* on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with *Cameron-Martin space* H is a one-parameter family $\mathbb{W}_H = \{W_H(t)\}_{t\geqslant 0}$ of bounded linear operators from H into $L^2(\mathbb{P})$ with the following properties:

(1) For each $h \in H$, $\{W_H(t)h\}_{t\geqslant 0}$ is an adapted real-valued Brownian motion;

(2) For all $g, h \in H$ and $t, s \geqslant 0$ we have

$$\mathbb{E}(W_H(t)g \cdot W_H(s)h) = (t \wedge s)[g, h]_H.$$

In [2] a theory for stochastic convolution of certain operator-valued functions with respect to a cylindrical Wiener process has been developed. We will briefly recall its mean features. Let H be a separable real Hilbert space, E a separable real Banach space, and let $F : (0,T) \to \mathscr{L}(H,E)$ be a function with the following property: for each $x^* \in E^*$, the function $t \mapsto F^*(t)x^*$ is strongly measurable and

$$\int_0^T \|F^*(t)x^*\|_H^2 \, dt < \infty, \qquad x^* \in E^*. \tag{5.1}$$

Under this assumption, for all $x^* \in E^*$ the function $t \mapsto F(t)F^*(t)x^*$ is Pettis integrable [2, Proposition 2.2]. Thus we may define a bounded operator $Q_T \in \mathscr{L}(E^*, E)$ by

$$Q_T x^* = \int_0^T F(t)F^*(t)x^* \, dt.$$

The following result is a reformulation of [2, Theorem 3.3], where it is stated in terms of convolutions. For Hilbert spaces E, the result is well-known. A detailed treatment of the stochastic Itô integral in Hilbert spaces may be found in the book [5].

Proposition 5.2. *Let E be a separable real Banach space and let $\mathbb{W}_H$ be a cylindrical Wiener process with Cameron-Martin space H. Then the following assertions are equivalent*:

(1) *Q_T is the covariance of a centred Gaussian measure ν_T on E*;

(2) *There exists an $\mathscr{F}_T$-measurable E-valued random variable X_T such that*

$$\langle X_T, x^*\rangle = \int_0^T \langle F(t)\, dW_H(t), x^*\rangle, \qquad x^* \in E^*.$$

In this situation, X_T is centred Gaussian and ν_T is its distribution; *in particular,*

$$\mathbb{E}(\langle X_T, x^*\rangle^2) = \int_0^T \|F^*(t)x^*\|_H^2 \, dt, \qquad x^* \in E^*. \tag{5.2}$$

The scalar stochastic integral in (2) is defined in the natural way: for a simple function $F:(0,T)\to\mathscr{L}(H,E)$ of the form

$$F(t)=F(t_k),\qquad t\in[t_k,t_{k+1});\quad k=0,\dots,m-1,$$

with $0<t_0<\dots<t_m=T$, we define

$$\int_0^T \langle F(t)\,dW_H(t),x^*\rangle := \sum_{k=0}^{m-1}(W_H(t_{k+1})-W_H(t_k))F^*(t_k)x^*.$$

If the assumptions of the theorem are satisfied for $t=T$, then by tightness they are satisfied for all $t\in[0,T]$. Thus we obtain an adapted E-valued process $\{X_t\}_{t\in[0,T]}$. In what follows, we will use the notation $\int_0^t F(s)\,dW_H(s)$ to denote the random variables X_t.

Let us now assume that (**H**) holds and that we have a continuous embedding $i_{BUC,E}: BUC\hookrightarrow E$. We define, for $t>0$, the bounded linear operators $S_E(t): H_\mu\to E$ by $S_E(t):=i_{BUC,E}\circ S_{BUC}(t)$. Thanks to Proposition 4.8, for all $x^*\in E^*$ we have

$$\int_0^T \|S_E^*(t)x^*\|^2\,dt<\infty.$$

By Proposition 4.10, the operator $Q_T^E: E^*\to E$ defined by

$$Q_T^E x^* := \int_0^T S_E(t)S_E^*(t)x^*\,ds$$

is the covariance of a centred Gaussian measure on E if and only if the operator K_T^E introduced in Theorem 4.9 is γ-radonifying from $L^2((0,T);L^2_{(s)}(\mu))$ into E. If this is the case, we obtain an E-valued process $\{u(t)\}_{t\in[0,T]}$ by stochastic convolution with a cylindrical Wiener process $\mathbb{W}_{H_\mu}$:

$$u(t):=\int_0^t S_E(t-s)\,dW_{H_\mu}(s). \tag{5.3}$$

6. Spatially homogeneous Wiener processes

Our next aim is to show that it makes sense to regard the process $\{u(t)\}_{t\in[0,T]}$ defined by (5.3) as an E-valued 'solution' of the problem

$$\begin{aligned} du(t)&=Au(t)\,dt+dW_\mu(t),\qquad t\geqslant 0,\\ u(0)&=0,\end{aligned}$$

where $\{W_\mu(t)\}_{t\geqslant 0}$ is a spatially homogeneous Wiener process whose spectral measure is μ, and A is defined *formally* by

$$A\Phi=\mathscr{F}^{-1}(e^{tq}\mathscr{F}\Phi),\qquad \Phi\in\mathscr{S}'.$$

This aim will be achieved in the next section. In order to be able to state the precise results, in this section we will study spatially homogeneous Wiener process and their relationship with cylindrical Wiener processes.

Definition 6.1. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete filtered probability space. A *spatially homogeneous Wiener process* on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is a continuous, adapted $\mathscr{S}'$-valued process $\mathbb{W} = \{W(t)\}_{t\geqslant 0}$ with the following properties:

(1) For each $\phi \in \mathscr{S}$, $\{\langle \phi, W(t)\rangle\}_{t\geqslant 0}$ is an adapted real-valued Brownian motion;

(2) For each $t \geqslant 0$ the distribution of $W(t)$ is invariant with respect to all translations $\tau'_h : \mathscr{S}' \to \mathscr{S}'$, where $\tau_h : \mathscr{S} \to \mathscr{S}$ is given by

$$\tau_h\phi(x) = \phi(x+h), \qquad x, h \in \mathbb{R}^d, \quad \phi \in \mathscr{S}.$$

We refer to [3], [15], [22], [23] for more infomation. By [12, Theorem 6, p. 169, Theorem 1′, p. 264], for a process $\mathbb{W}$ satisfying condition (1), condition (2) is equivalent to:

(2′) There exists a nonnegative symmetric tempered measure μ on $\mathbb{R}^d$ such that for all $\phi, \psi \in \mathscr{S}$ and $t, s \geqslant 0$ we have

$$\mathbb{E}(\langle \phi, W(t)\rangle \cdot \langle \psi, W(s)\rangle) = (t \wedge s)[\phi, \psi]_{\mathscr{H}_\mu}.$$

The measure μ occurring in condition (2′) is uniquely determined by $\mathbb{W}$ and is called the *spectral measure* of the process $\mathbb{W}$. We will sometimes use the notation $\mathbb{W}_\mu$ for a spatially homogeneous Wiener process with spectral measure μ.

It is possible to integrate certain operator-valued processes with respect to a spatially homogeneous Wiener process $\mathbb{W}_\mu$. Let $\mathscr{L}(\mathscr{S}')$ denote the space of all continuous linear operators from $\mathscr{S}'$ into itself. A mapping $F : (0,T)\times\Omega \to \mathscr{L}(\mathscr{S}')$ is called *simple* if there exist $0 < t_0 < t_1 < \cdots < t_m = T$ and $\mathscr{F}_{t_k}$-measurable random variables $F(t_k) : \Omega \to \mathscr{L}(\mathscr{S}')$ taking finitely many values only, such that

$$F(t,\omega) = F(t_k,\omega), \qquad t \in [t_k, t_{k+1}); \quad k = 0, \dots, m-1.$$

For a simple $F : (0,T)\times\Omega \to \mathscr{L}(\mathscr{S}')$ of this form we define the stochastic integral with respect to $\mathbb{W}$ by

$$\int_0^T F(t)\, dW_\mu(t) := \sum_{k=0}^{m-1} F(t_k)(W_\mu(t_{k+1}) - W_\mu(t_k)).$$

An easy computation shows that

$$\mathbb{E}\left|\left\langle \phi, \int_0^T F(t)\, dW_\mu(t)\right\rangle\right|^2 = \mathbb{E}\int_0^T \|F'(t)\phi\|^2_{\mathscr{H}_\mu}\, dt, \qquad \phi \in \mathscr{S}. \tag{6.1}$$

Here $F'(t) : \mathscr{S} \to \mathscr{S}$ is the adjoint of $F(t) : \mathscr{S}' \to \mathscr{S}'$ and $\mathscr{H}_\mu$ is the Hilbert space introduced in Definition 3.1. By a standard approximation argument, the stochastic integral defined in this way extends to the space of all predictable functions $F : (0,T)\times\Omega \to \mathscr{L}(\mathscr{S}')$ for which

$$\mathbb{E}\int_0^T \|F'(t)\phi\|^2_{\mathscr{H}_\mu}\, dt < \infty, \qquad \phi \in \mathscr{S}.$$

Here, measurability of $\mathscr{L}(\mathscr{S}')$-valued functions is understood in the sense of [13], [14], where more details are given.

We will investigate next the relationship between the stochastic integral introduced above and the one from the previous section. To this end we consider the situation where a spatially homogeneous Wiener process $\mathbb{W}_\mu$ with spectral measure μ is given.

There ia a canonical way to associate a cylindrical Wiener process with a given spatially homogeneous Wiener process, cf. [14, Proposition 2.5], [22]:

Proposition 6.2. *Let $\mathbb{W}_\mu$ be a spatially homogeneous Wiener process. For each $t \geqslant 0$, the mapping*

$$W_{H_\mu}(t): \ U_\mu \phi \mapsto \langle \phi, W_\mu(t)\rangle \qquad \phi \in \mathscr{S},$$

extends uniquely to a bounded linear operator $W_{H_\mu}(t): H_\mu \to L^2(\mathbb{P})$, and $\mathbb{W}_{H_\mu}$ is a cylindrical Wiener process with Cameron-Martin space H_μ.

Proof. Just note that

$$\begin{aligned}\mathbb{E}(W_{H_\mu}(t)U_\mu\phi \cdot W_{H_\mu}(s)U_\mu\psi) &= \mathbb{E}(\langle \phi, W_\mu(t)\rangle \cdot \langle \phi, W_\mu(s)\rangle)\\ &= (t\wedge s)\,[\phi,\psi]_{\mathscr{H}_\mu} = (t\wedge s)\,[U_\mu\phi, U_\mu\psi]_{H_\mu}.\end{aligned}$$

□

We denote by $\mathbb{W}_{H_\mu}$ the associated cylindrical Wiener process with Cameron-Martin space H_μ from Proposition 6.2.

Proposition 6.3. *Let E be a separable real Banach space, continuously embedded in $\mathscr{D}'$. Let $F:(0,T)\to\mathscr{L}(\mathscr{S}')$ be a function for which the stochastic integral*

$$\int_0^T F(t)\,dW_\mu(t)$$

is well-defined in the sense described above.

Let $F_E:(0,T)\to\mathscr{L}(H_\mu, E)$ be a function for which the stochastic integral

$$\int_0^T F_E(t)\,dW_{H_\mu}(t)$$

is well-defined in the sense described above.

If for all $h\in H_\mu$ and $t\in(0,T)$ we have

$$F(t)h = F_E(t)h,$$

the equality being understood in the space $\mathscr{D}'$, then in $\mathscr{D}'$ we have

$$\int_0^T F(t)\,dW_\mu(t) = \int_0^T F_E(t)\,dW_{H_\mu}(t) \quad \textit{almost surely.} \tag{6.2}$$

Proof. We shall denote the inclusion mappings $E \hookrightarrow \mathscr{D}'$ and $\mathscr{S}' \hookrightarrow \mathscr{D}'$ by $i_{E,\mathscr{D}'}$ and $i_{\mathscr{S}',\mathscr{D}'}$, respectively. The compatibility assumption on $F(t)$ and $F_E(t)$ then reads

$$i_{\mathscr{S}',\mathscr{D}'} F(t) i_{H_\mu,\mathscr{S}'} h = i_{E,\mathscr{D}'} F_E(t) h, \qquad h \in H_\mu.$$

In order to prove the proposition it suffices to consider two step functions of the form $1_{(a,b)} \otimes F$ and $1_{(a,b)} \otimes F_E$ where $F \in \mathscr{L}(\mathscr{S}')$ and $F_E \in \mathscr{L}(H_\mu, E)$ are related by

$$i_{\mathscr{S}',\mathscr{D}'} \circ F \circ i_{H_\mu,\mathscr{S}'} = i_{E,\mathscr{D}'} \circ F_E.$$

Noting that $i^*_{S',\mathscr{D}'} = i_{\mathscr{D},\mathscr{S}}$, this can be rewritten as

$$F_E^* \circ i^*_{E,\mathscr{D}'} = i^*_{H_\mu,\mathscr{S}'} \circ F' \circ i_{\mathscr{D},\mathscr{S}}.$$

To prove (6.2), note that for all $\psi \in \mathscr{D}$ we have

$$\begin{aligned}
\left\langle \psi, i_{E,\mathscr{D}'} \int_0^T 1_{(a,b)} \otimes F_E \, dW_{H_\mu}(t) \right\rangle &= \int_0^T \langle 1_{(a,b)} \otimes F_E \, dW_{H_\mu}(t), i^*_{E,\mathscr{D}'}\psi \rangle \\
&= (W_{H_\mu}(b) - W_{H_\mu}(a)) F_E^* i^*_{E,\mathscr{D}'} \psi \\
&= (W_{H_\mu}(b) - W_{H_\mu}(a)) U_\mu U_\mu^* F_E^* i^*_{E,\mathscr{D}'} \psi \\
&= (W_{H_\mu}(b) - W_{H_\mu}(a)) U_\mu U_\mu^* i^*_{H_\mu,\mathscr{S}'} F' i_{\mathscr{D},\mathscr{S}} \psi \\
&= (W_{H_\mu}(b) - W_{H_\mu}(a)) U_\mu F' i_{\mathscr{D},\mathscr{S}} \psi \\
&= \langle F' i_{\mathscr{D},\mathscr{S}} \psi, W_\mu(b) - W_\mu(a) \rangle \\
&= \langle i_{\mathscr{D},\mathscr{S}} \psi, F(W_\mu(b) - W_\mu(a)) \rangle \\
&= \left\langle \psi, i_{\mathscr{S}',\mathscr{D}'} \int_0^T \langle 1_{(a,b)} \otimes F \, dW_\mu(t) \right\rangle.
\end{aligned}$$

where we used Proposition 3.4 and suppressed the inclusion mapping $i_{\mathscr{S},\mathscr{H}_\mu}$ from our notations in the same way as we did in Proposition 6.2. ☐

7. E-valued weak solutions

Up to this point, it has been a standing assumption that the function q satisfies the conditions (4.1) and (4.2). In the remaining sections we will always assume the additional condition:

(7.1) q is smooth and all of its derivatives have at most polynomial growth.

Then for all $t \geqslant 0$ the function $e^{tq(\cdot)}$ is a multiplier in $\mathscr{S}'$. More precisely, by (4.2) and (7.1) for each $t \geqslant 0$ we may define a continuous linear operator $S_{\mathbb{C}}(t) \in \mathscr{L}(\mathscr{S}'_{\mathbb{C}})$ by

$$S_{\mathbb{C}}(t)\Phi := \mathscr{F}^{-1}(e^{tq(\cdot)} \mathscr{F}\Phi), \qquad \Phi \in \mathscr{S}'_{\mathbb{C}}, \quad t \geqslant 0. \tag{7.2}$$

Condition (4.1) ensures that $S_{\mathbb{C}}(t)$ maps $\mathscr{S}'$ into itself. Denoting the restriction of the operator $S_{\mathbb{C}}(t)$ to $\mathscr{S}'$ by $S(t)$, the family $\{S(t)\}_{t\geqslant 0}$ is a C_0-semigroup on $\mathscr{S}'$ in the sense of [26]. Its infinitesimal generator is the pseudodifferential operator A with symbol q:

$$A\Phi = \mathscr{F}^{-1}(q(\cdot)\mathscr{F}\Phi), \qquad \Phi \in \mathscr{D}(A),$$

where the domain $\mathscr{D}(A)$ consists of all $\Phi \in \mathscr{S}'$ such that $q(\cdot)\mathscr{F}\Phi \in \mathscr{S}'$. If μ is a positive symmetric tempered measure, then the operator $S(t)$ map H_μ into itself and the restricted semigroup is precisely the semigroup studied in Section 4. It follows that we may apply Proposition 6.3 and conclude that

$$\int_0^t S_E(t-s)\, dW_{H_\mu}(t) = \int_0^t S(t-s)\, dW_\mu(t)$$

whenever both integrals are defined.

In $\mathscr{S}'$ we now consider the following linear stochastic Cauchy problem:

$$\begin{aligned} du(t) &= Au(t)\, dt + dW_\mu(t), \qquad t \geqslant 0, \\ u(0) &= 0. \end{aligned} \tag{7.3}$$

Here $\{W_\mu(t)\}_{t\geqslant 0}$ is a given spatially homogeneous Wiener process with spectral measure μ. A *weak solution* of (7.3) is a predictable $\mathscr{S}'$-valued process $\{u(t)\}_{t\geqslant 0}$ such that for all $\phi \in \mathscr{D}(A)$ we have $s \mapsto \langle A\phi, u(s)\rangle \in L^1_{\mathrm{loc}}[0,\infty)$ a.s. and

$$\langle \phi, u(t)\rangle = \int_0^t \langle A\phi, u(s)\rangle\, ds + \langle \phi, W_\mu(t)\rangle \ \text{ a.s.}, \qquad t \geqslant 0.$$

With the use of the stochastic integral in $\mathscr{S}'$ discussed in Section 6, it is possible to show that

$$u(t) := \int_0^t S(t-s)\, dW_\mu(s) \tag{7.4}$$

is a weak solution of (7.3) and that up to modification this solution is unique.

Let us think for the moment of $u(\cdot)$ as taking values in $\mathscr{D}'$ rather than in $\mathscr{S}'$. We will be interested in finding conditions ensuring that $u(t)$ actually takes values in some smaller Banach space E that is continuously embedded in $\mathscr{D}'$. In order to make this idea precise, we introduce the following terminology.

Definition 7.1. Let E be a real Banach space, continuously embedded in $\mathscr{D}'$. A predictable E-valued process $\{U(t)\}_{t\geqslant 0}$ will be called an *E-valued weak solution* of the problem (7.3) if for all $t \geqslant 0$ we have $U(t) = u(t)$ in $\mathscr{D}'$ almost surely.

Clearly, an E-valued weak solution, if it exists, is unique up to modification.

Proposition 7.2. *Let E be a real Banach space that is continuously embedded in $\mathscr{D}'$, and let $\{U(t)\}_{t\geqslant 0}$ be an E-valued weak solution of* (7.3). *Then as an E-valued process, $\{U(t)\}_{t\geqslant 0}$ is Gaussian.*

The covariance operator R_T^E of the distribution of the random variable $U(T)$ satisfies

$$i_{E,\mathscr{D}'} \circ R_T^E \circ i_{E,\mathscr{D}'}^* = i_{H_\mu,\mathscr{D}'} \circ R_T^\mu \circ i_{H_\mu,\mathscr{D}'}^*,$$

where $R_T^\mu \in \mathscr{L}(H_\mu)$ is defined by

$$R_T^\mu h := \int_0^T S(t)S^*(t)h\,dt, \qquad h \in H_\mu.$$

Proof. Each random variable $U(t)$, being strongly measurable, takes its values in a separable closed subspace E_t of E almost surely. The joint distribution of $(U(t_1),\dots,U(t_m))$ is a Radon probability measure $\mu_{t_1,\dots,t_m}$ supported in $E_{t_1}\oplus\cdots\oplus E_{t_m}$. We claim that this measure is Gaussian. Once we know this, it follows that $\mu_{t_1,\dots,t_m}$ is Gaussian as a measure on E^m and the proposition is proved.

Let $i_{t_1,\dots,t_m} : E_{t_1}\oplus\cdots\oplus E_{t_m} \hookrightarrow \mathscr{D}'\oplus\cdots\oplus\mathscr{D}'$ denote the inclusion mapping. Then $i_{t_1,\dots,t_m}(\mu_{t_1,\dots,t_m}) = \nu_{t_1,\dots,t_m}$, the distribution of the $\mathscr{D}'$-valued random variable $(u(t_1),\dots,u(t_m))$ defined by (7.4). Hence for all $\phi_1,\dots,\phi_m \in \mathscr{D}$ we have

$$\langle \mu_{t_1,\dots,t_m}, i_{t_1,\dots,t_m}^*(\phi_1,\dots,\phi_m)\rangle = \langle \nu_{t_1,\dots,t_m}, (\phi_1,\dots,\phi_m)\rangle,$$

where we use brackets to denote image measures along linear functionals. The process $\{u(t)\}_{t\geqslant 0}$ being Gaussian in $\mathscr{D}'$, it follows that the image measures

$$\langle \mu_{t_1,\dots,t_m}, i_{t_1,\dots,t_m}^*(\phi_1,\dots,\phi_m)\rangle$$

are Gaussian on $\mathbb{R}^m$. Since $i_{t_1,\dots,t_m}^*$ has weak*-dense range in $(E_t\oplus\cdots\oplus E_{t_m})^*$, [2, Corollary 1.3] implies that the measure $\mu_{t_1,\dots,t_m}$ is Gaussian on $E_t\oplus\cdots\oplus E_{t_m}$.

Let μ_T^E denote the distribution of $U(T)$. Using Proposition 3.4, for all $\phi \in \mathscr{D}$ we have:

$$\begin{aligned}
\langle R_T^E i_{E,\mathscr{D}'}^*\phi, i_{E,\mathscr{D}'}^*\phi\rangle &= \mathbb{E}\langle (U(T), i_{E,\mathscr{D}'}^*\phi\rangle^2 = \mathbb{E}\langle i_{\mathscr{S}',\mathscr{D}'}^*\phi, u(t)\rangle^2 \\
&= \int_0^T \|i_{\mathscr{S},\mathscr{H}_\mu} S'(t) i_{\mathscr{S}',\mathscr{D}'}^*\phi\|_{\mathscr{H}_\mu}^2\,dt \\
&= \int_0^T \|U_\mu^* i_{H_\mu,\mathscr{S}'}^* S'(t) i_{\mathscr{S}',\mathscr{D}'}^*\phi\|_{\mathscr{H}_\mu}^2\,dt \\
&= \int_0^T \|i_{H_\mu,\mathscr{S}'}^* S'(t) i_{\mathscr{S}',\mathscr{D}'}^*\phi\|_{H_\mu}^2\,dt \\
&= \int_0^T \|S^*(t) i_{H_\mu,\mathscr{S}'}^* i_{\mathscr{S}',\mathscr{D}'}^*\phi\|_{H_\mu}^2\,dt \\
&= \int_0^T \|S^*(t) i_{H_\mu,\mathscr{D}'}^*\phi\|_{H_\mu}^2\,dt
\end{aligned}$$

$$= \int_0^T [S(t)S^*(t)i^*_{H_\mu,\mathscr{D}'}\phi, i^*_{H_\mu,\mathscr{D}'}\phi]_{H_\mu}\,dt$$
$$= [R^\mu_T i^*_{H_\mu,\mathscr{D}'}\phi, i^*_{H_\mu,\mathscr{D}'}\phi]_{H_\mu}.$$

□

The following result gives a necessary condition for the existence of an E-valued weak solution. It will play an important rôle in our discussion of weighted L^p-solutions below.

Theorem 7.3. *Let E be a separable real Banach space and let $E \hookrightarrow \mathscr{D}'$ be a continuous embedding. If the problem* (7.3) *admits a weak E-valued solution, then the operators $K^E_T : L^2((0,T); L^2_{(s)}(\mu)) \to E$ are well-defined and γ-radonifying.*

Proof. Let $\{U(t)\}_{t\geqslant 0}$ be an E-valued weak solution of the problem (7.3). Let $T > 0$ be fixed. The RKHS's of the operators R^E_T and R^μ_T will be denoted by (i^E_T, H^E_T) and (i^μ_T, H^μ_T), respectively. In view of the previous result, for all $\phi \in \mathscr{D}'$ we have

$$\|(i^E_T)^* i^*_{E,\mathscr{D}'}\phi\|^2_{H^E_T} = \langle R^E_T i^*_{E,\mathscr{D}'}\phi, i^*_{E,\mathscr{D}'}\phi\rangle$$
$$= \langle R^\mu_T i^*_{H_\mu,\mathscr{D}'}\phi, i^*_{H_\mu,\mathscr{D}'}\phi\rangle = \|(i^\mu_T)^* i^*_{H_\mu,\mathscr{D}'}\phi\|^2_{H^\mu_T}.$$

Since $(i^\mu_T)^* \circ i^*_{H_\mu,\mathscr{D}'}$ has dense range in H_μ, this shows that the operator

$$U : (i^\mu_T)^*(i^*_{H_\mu,\mathscr{D}'}\phi) \mapsto (i^E_T)^*(i^*_{E,\mathscr{D}'}\phi), \qquad \phi \in \mathscr{D},$$

uniquely extends to an isometry from H^μ_T into H^E_T. Since $(i^E_T)^* \circ i^*_{E,\mathscr{D}'}$ has dense range in E^*, this isometry is actually a unitary operator. Noting that by definition we have

$$U \circ (i^\mu_T)^* \circ i^*_{H_\mu,\mathscr{D}'} = (i^E_T)^* \circ i^*_{E,\mathscr{D}'},$$

by dualizing we obtain

$$i_{H_\mu,\mathscr{D}'} \circ i^\mu_T \circ U^* = i_{E,\mathscr{D}'} \circ i^E_T.$$

Multiply both sides from the right with U. Since U is unitary, this gives

$$i_{H_\mu,\mathscr{D}'} \circ i^\mu_T = i_{E,\mathscr{D}'} \circ i^E_T \circ U. \tag{7.5}$$

Define $J^\mu_T : L^2((0,T); H_\mu) \to H_\mu$ by

$$J^\mu_T f := \int_0^T S(t)f(t)\,dt.$$

By general results on RKHS's, J^μ_T takes values in H^μ_T. The resulting operator from $L^2((0,T); H_\mu)$ into H^μ_T will be denoted by j^μ_T. Thus, $J^\mu_T = i^\mu_T \circ j^\mu_T$ and from (7.5) we obtain

$$i_{H_\mu,\mathscr{D}'} \circ J^\mu_T = i_{H_\mu,\mathscr{D}'} \circ i^\mu_T \circ j^\mu_T = i_{E,\mathscr{D}'} \circ i^E_T \circ U \circ j^\mu_T.$$

Let $V_T : L^2((0,T); L^2_{(s)}(\mu)) \to L^2((0,T); H_\mu)$ be defined by $V_T g = \mathscr{F}^{-1}(g\mu)$. If g is a step function taking values in $L^2_{(s)}(\mu) \cap C_c(\mathbb{R}^d; \mathbb{C})$, then

$$
\begin{aligned}
(i_{E,\mathscr{D}'} \circ (i_T^E \circ U \circ j_T^\mu \circ V_T))g &= i_{H_\mu,\mathscr{D}'}((J_T^\mu \circ V_T)g) \\
&= i_{H_\mu,\mathscr{D}'} \int_0^T S(t)\mathscr{F}^{-1}(g(t)\mu)\, dt \\
&= i_{H_\mu,\mathscr{D}'} \int_0^T \mathscr{F}^{-1}(e^{tq} g(t)\mu)\, dt \\
&= i_{H_\mu,\mathscr{D}'} \int_0^T \int_{R_d} e^{i\langle\cdot,\xi\rangle} e^{tq(\cdot)}(g(t))(\cdot)\, d\mu(\xi)\, dt \\
&= (i_{E,\mathscr{D}'} \circ i_{BUC,E})K_T g \\
&= i_{E,\mathscr{D}'} K_T^E g.
\end{aligned}
$$

Hence for such g we obtain

$$(i_T^E \circ U \circ j_T^\mu \circ V_T)g = K_T^E g.$$

The subspace of all such g being dense, we have shown that K_T^E extends to a bounded linear operator from $L^2((0,T); L^2_{(s)}(\mu))$ into E.

Since $R_T^E = i_T^E \circ (i_T^E)^*$ is a covariance operator, i_T^E is γ-radonifying. It follows that the operator $K_T^E = i_T^E \circ (U \circ j_T^\mu \circ V_T)$ is γ-radonifying as an operator from $L^2((0,T); L^2_{(s)}(\mu))$ into E. □

We do not know whether the existence of an E-valued solution implies Hypothesis **(H)**. Below, we will give an affirmative answer to this question when E is a weighted L^p-space.

If we assume that Hypothesis **(H)** holds and that BUC embeds into E, we can represent E-valued solutions as stochastic convolutions in E:

Theorem 7.4. *Assume that* **(H)** *holds. Let E be a separable real Banach space for which we have continuous embeddings $BUC \hookrightarrow E \hookrightarrow \mathscr{D}'$. If* (7.3) *admits an E-valued weak solution $\{U(t)\}_{t\geqslant 0}$, then for all $t \geqslant 0$ we have*

$$U(t) = \int_0^t S_E(t-s)\, dW_{H_\mu}(s) \tag{7.6}$$

in $\mathscr{D}'$ almost surely, where $\{W_{H_\mu}(t)\}_{t\geqslant 0}$ is the cylindrical Wiener process associated with μ.

Proof. The assumptions imply that the operators $S_E(t)$ are well-defined. Fix $T > 0$ and define as before the operator $Q_T^E : E^* \to E$ by

$$Q_T^E x^* := \int_0^T S_E(t) S_E^*(t) x^*\, dt, \qquad x^* \in E^*.$$

For all $x^* \in E^*$ and $\psi \in \mathscr{D}$ we have, using Proposition 3.4 and the definition of an E-valued weak solution,

$$\begin{aligned}
\langle Q_T^E i_{E,\mathscr{D}'}^* \phi, i_{E,\mathscr{D}'}^* \psi\rangle &= \int_0^T \|S_E^*(t)(i_{E,\mathscr{D}'}^*\psi)\|_{H_\mu}^2\,dt \\
&= \int_0^T \|U_\mu^* S_E^*(t)(i_{E,\mathscr{D}'}^*\psi)\|_{\mathscr{H}_\mu}^2\,dt \\
&= \int_0^T \|U_\mu^* i_{H_\mu,\mathscr{S}'}^* S'(t) i_{\mathscr{S}',\mathscr{D}'}^*\psi\|_{\mathscr{H}_\mu}^2\,dt \\
&= \int_0^T \|i_{\mathscr{S},\mathscr{H}_\mu}^* S'(t) i_{\mathscr{S}',\mathscr{D}'}^*\psi\|_{\mathscr{H}_\mu}^2\,dt \\
&= \mathbb{E}\,\langle u(T), i_{\mathscr{S}',\mathscr{D}'}^*\psi\rangle^2 \\
&= \mathbb{E}\,\langle U(T), i_{E,\mathscr{D}'}^*\psi\rangle^2 \\
&= \langle R_T^E i_{E,\mathscr{D}'}^*\phi, i_{E,\mathscr{D}'}^*\psi\rangle.
\end{aligned}$$

By a density argument, it follows that $Q_T^E = R_T^E$. In particular, Q_T^E is a covariance operator, and therefore the stochastic convolution in (7.6) is well-defined. By Proposition 6.3, in $\mathscr{D}'$ we have

$$U(T) = u(T) = \int_0^T S(T-t)\,dW_\mu(t) = \int_0^T S_E(T-t)\,dW_{H_\mu}(t)$$

almost surely. □

We conclude with a result that gives sufficient conditions for the existence of an E-valued solution:

Theorem 7.5. *Let Hypothesis* **(H)** *hold. Let E be a separable real Banach space for which we have continuous embeddings $BUC \hookrightarrow E \hookrightarrow \mathscr{D}'$. If for all $T > 0$ the operator K_T^E is γ-radonifying from $L^2((0,T); L^2_{(s)}(\mu))$ into E, then the problem* (7.3) *admits a unique E-valued weak solution $\{U(t)\}_{t\geqslant 0}$, and this solution is given by*

$$U(t) = \int_0^t S_E(t-s)\,dW_{H_\mu}(s).$$

Proof. By Propositions 4.7 and 4.10, we may apply Propositon 5.2 to define, for every $t \geqslant 0$, an E-valued random variable $U(t)$ by

$$U(t) := \int_0^t S_E(t-s)\,dW_{H_\mu}(s).$$

By Proposition 6.3, for all $t \geqslant 0$ we have

$$U(t) = u(t) := \int_0^t S(t-s)\,dW_\mu(s)$$

in $\mathscr{D}'$ almost surely. This shows that $\{U(t)\}_{t\geqslant 0}$ is an E-valued weak solution of (7.3). Uniqueness has already been shown. $\square$

8. Existence of a continuous version

In this section we will show that an E-valued solution, if it exists, has a continuous E-valued modification if the following integrability condition is satisfied:

Hypothesis ($\mathbf{H}_\alpha$). There exists a constant $C > q^*$ such that

$$\int_{\mathbb{R}^d} \frac{1+|q(x)|^\alpha}{C - \operatorname{Re} q(x)}\, d\mu(x) < \infty.$$

Note that this hypothesis stronger than **(H)**. Hence in particular, Hypothesis ($\mathbf{H}_\alpha$) implies that the operators $S_E(t)$ are well-defined.

Lemma 8.1. *Assume that* ($\mathbf{H}_\alpha$) *holds for some* $\alpha > 0$ *and let* $T > 0$ *be fixed. Then*:

(1) *There exists a constant* $c \geqslant 0$ *such that for all* $x^* \in E^*$ *and* $t \in [0,T]$ *we have*

$$\int_0^t \|S_E^*(s)x^*\|_{H_\mu}^2\, ds \;\leqslant\; ct^\alpha \|x^*\|^2;$$

(2) *There exists a constant* $c \geqslant 0$ *such that for all* $x^* \in E^*$ *and* $s, t \in [0,T]$ *with* $s < t$ *we have*

$$\int_0^s \|S_E^*(t-s+u)x^* - S_E^*(u)x^*\|_{H_\mu}^2\, du \;\leqslant\; c(t-s)^\alpha \|x^*\|^2.$$

Proof. Without loss of generality we assume that $\alpha \in (0,1]$. For the constant C in Hypothesis ($\mathbf{H}_\alpha$) we assume without loss of generality that $C > \max\{0, q^*\}$.

We start with the proof of (1). Fix $0 < t \leqslant T$. By Proposition 4.7, for all $x^* \in E^*$ we have

$$\int_0^t \|S_E^*(s)x^*\|_{H_\mu}^2\, ds \leqslant \int_0^t \int_{\mathbb{R}^d} e^{2s \operatorname{Re} q(\xi)} d\mu(\xi)\, ds \cdot \|x^*\|^2. \tag{8.1}$$

We will estimate the double integral on the right hand side by splitting the inner integral into two parts corresponding to the sets where $|\operatorname{Re} q| \leqslant C$ and where $\operatorname{Re} q < -C$. We have

$$\begin{aligned}\int_0^t \int_{|\operatorname{Re} q|\leqslant C} e^{2s \operatorname{Re} q(\xi)}\, d\mu(\xi)\, ds &\leqslant te^{2TC}\mu\{|\operatorname{Re} q| \leqslant C\} \\ &\leqslant t^\alpha T^{1-\alpha} e^{2TC}\mu\{|\operatorname{Re} q| \leqslant C\}.\end{aligned} \tag{8.2}$$

Note that $\mu\{|\operatorname{Re} q| \leqslant C\} < \infty$. Indeed, for all $\xi \in \mathbb{R}^d$ with $|\operatorname{Re} q(\xi)| \leqslant C$ we have $C - \operatorname{Re} q(\xi) \leqslant 2C$, and therefore

$$\mu\{|\operatorname{Re} q| \leqslant C\} \leqslant 2C \int_{|\operatorname{Re} q|\leqslant C} \frac{1}{C - \operatorname{Re} q(\xi)}\, d\mu(\xi) < \infty.$$

Next, by Fubini's theorem,

$$\int_0^t \int_{\operatorname{Re} q < -C} e^{2s \operatorname{Re} q(\xi)} \, d\mu(\xi) \, ds = \int_{\operatorname{Re} q < -C} \frac{1 - e^{2t \operatorname{Re} q(\xi)}}{-2 \operatorname{Re} q(\xi)} \, d\mu(\xi).$$

Using the inequality $0 \leqslant 1 - e^{-2t\beta} \leqslant \min\{1, 2t\beta\}$ $(\beta \geqslant 0)$ and recalling that $0 < \alpha \leqslant 1$, we now estimate:

$$\begin{aligned} \int_{\operatorname{Re} q < -C} \frac{1 - e^{2t \operatorname{Re} q(\xi)}}{-2 \operatorname{Re} q(\xi)} \, d\mu(\xi) &\leqslant \int_{\operatorname{Re} q < -C} \frac{(1 - e^{2t \operatorname{Re} q(\xi)})^\alpha}{-2 \operatorname{Re} q(\xi)} \, d\mu(\xi) \\ &\leqslant (2t)^\alpha \int_{\operatorname{Re} q < -C} \frac{(-\operatorname{Re} q(\xi))^\alpha}{-2 \operatorname{Re} q(\xi)} \, d\mu(\xi) \\ &\leqslant (2t)^\alpha \int_{\mathbb{R}^d} \frac{|q(\xi)|^\alpha}{C - \operatorname{Re} q(\xi)} \, d\mu(\xi). \end{aligned} \tag{8.3}$$

The right hand side integral is finite by assumption. Combining the estimates (8.2) and (8.3) with (8.1) we see that (1) is proved.

For the proof of (2) we fix $0 \leqslant s \leqslant t \leqslant T$. By Proposition 4.8, for all $x^* \in E^*$ we have

$$\begin{aligned} &\int_0^s \|S_E^*(t - s + u)x^* - S_E^*(u)x^*\|_{H_\mu}^2 \, du \\ &\qquad \leqslant \int_0^s \int_{\mathbb{R}^d} |e^{(t-s+u)q(\xi)} - e^{uq(\xi)}|^2 \, d\mu(\xi) \, du \cdot \|x^*\|^2. \end{aligned} \tag{8.4}$$

We are going to estimate the double integral on the right hand side. First,

$$|e^{(t-s)q(\xi)} - 1| = \left| q(\xi) \int_0^{t-s} e^{uq(\xi)} \, du \right| \leqslant (t - s)|q(\xi)| \, e^{TC}.$$

Recalling that $0 < \alpha \leqslant 1$, we choose $M \geqslant 0$ such that $r^2 \leqslant Mr^\alpha$ for all $r \in [0, 2e^{TC}]$. Then,

$$\begin{aligned} &\int_0^s \int_{\mathbb{R}^d} |e^{(t-s+u)q(\xi)} - e^{uq(\xi)}|^2 \, d\mu(\xi) \, du \\ &\qquad \leqslant M \int_0^s \int_{\mathbb{R}^d} |e^{(t-s+u)q(\xi)} - e^{uq(\xi)}|^\alpha \, d\mu(\xi) \, du \\ &\qquad \leqslant M \int_0^s \int_{\mathbb{R}^d} e^{\alpha u \operatorname{Re} q(\xi)} |e^{(t-s)q(\xi)} - 1|^\alpha \, d\mu(\xi) \, du \\ &\qquad \leqslant M(t - s)^\alpha e^{\alpha TC} \int_0^s \int_{\mathbb{R}^d} e^{\alpha u (\operatorname{Re} q(\xi)} |q(\xi)|^\alpha d\mu(\xi) \, du \\ &\qquad \leqslant M(t - s)^\alpha e^{2\alpha TC} \int_0^s \int_{\mathbb{R}^d} e^{-\alpha u (C - \operatorname{Re} q(\xi))} |q(\xi)|^\alpha d\mu(\xi) \, du \\ &\qquad = \alpha^{-1} M(t - s)^\alpha e^{2\alpha TC} \int_{\mathbb{R}^d} \frac{1 - e^{-\alpha s (C - \operatorname{Re} q(\xi))}}{C - \operatorname{Re} q(\xi)} |q(\xi)|^\alpha d\mu(\xi) \\ &\qquad \leqslant \alpha^{-1} M(t - s)^\alpha e^{2\alpha TC} \int_{\mathbb{R}^d} \frac{1}{C - \operatorname{Re} q(\xi)} |q(\xi)|^\alpha d\mu(\xi). \end{aligned} \tag{8.5}$$

The integral in the right hand side is finite by assumption and the proof is completed. □

Theorem 8.2. *Suppose there exist $C > q^*$ and $\alpha > 0$ such that*

$$\int_{\mathbb{R}^d} \frac{1 + |q(x)|^\alpha}{C - \operatorname{Re} q(x)}\, d\mu(x) < \infty.$$

Let E be a separable real Banach space for which we have continuous embeddings $BUC \hookrightarrow E \hookrightarrow \mathscr{D}'$. If (7.3) *admits an E-valued weak solution, then this solution has a continuous E-valued modification.*

Proof. Thanks to the estimates in Lemma 8.1 we can apply [2, Proposition 4.3] to the operator-valued function $S_E(\cdot)$ on each interval $(0, T]$ and obtain a continuous modification (depending on T) of $\{u(t)\}_{t\in[0,T]}$. By applying this to a sequence $T_n \to \infty$ we obtain a continuous version of $\{u(t)\}_{t\geqslant 0}$. □

It seems reasonable to expect that if $(\mathbf{H}_\alpha)$ holds, the E-valued solution has a Hölder continuous modification. Under the additional assumption that $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on E, in the next section we prove that this is indeed the case if E is a weighted L^p-space.

9. Weighted L^p-solutions

In this section we are going to apply our results to weighted L^p-spaces and prove our main result, which was stated in the Introduction for $A = \Delta$. We will always assume (4.1), (4.2), and (7.1).

Let $0 \leqslant \varrho \in L^1_{\mathrm{loc}}$ be a nonnegative locally integrable function. For $1 \leqslant p < \infty$ we denote by $L^p(\varrho)$ the Banach space of all real functions on $\mathbb{R}^d$ for which

$$\|f\|^p_{L^p(\varrho)} := \int_{\mathbb{R}^d} |f(x)|^p\, \varrho(x)\, dx < \infty.$$

As usual we identify functions that are equal $\varrho(x)\,dx$-almost everywhere. Clearly we have a continuous inclusion $L^p(\varrho) \hookrightarrow \mathscr{D}'$, and if ϱ is integrable we also have a continuous inclusion $BUC \hookrightarrow L^p(\varrho)$.

Theorem 9.1. *Let $1 \leqslant p < \infty$ and $0 \leqslant \varrho \in L^1_{\mathrm{loc}}$ be arbitrary and fixed. The following assertions are equivalent*:

(1) *Problem* (7.3) *admits an $L^p(\varrho)$-valued solution*;

(2) *Hypothesis* **(H)** *holds and ϱ is integrable.*

Proof. If we have an $L^p(\varrho)$-valued solution, then the operator $K_T^{L^p(\varrho)}$ is well-defined from $L^2((0,T); L^2_{(s)}(\mu))$ into $L^p(\varrho)$, and γ-radonifying by Theorem 7.3. We now apply Theorem 2.3 to the function $\kappa(x) = e^{-i\langle x,\cdot\rangle} e^{tq(-\,\cdot)}$. In combination with (2.1) we find that

$$\left(\int_0^T \int_{\mathbb{R}^d} e^{2t \operatorname{Re} q(\xi)}\, d\mu(\xi)\, dt\right)^{\frac{p}{2}} \cdot \int_{\mathbb{R}^d} \varrho(x)\, dx < \infty.$$

By Proposition 4.3, the finiteness of the first double integral is equivalent to Hypothesis **(H)**.

For the converse we first note that the conditions in (2) imply that BUC embeds into $L^p(\varrho)$ and that the operators $S_{L^p(\varrho)}(t)$ are well-defined. Hence the operator $K_T^{L^p(\varrho)}$ is well-defined. By Theorem 2.3, applied in the converse direction, $K_T^{L^p(\varrho)}$ is γ-radonifying. Hence by Proposition 4.10, $Q_T^{L^p(\varrho)}$ is the covariance of a Gaussian measure on $L^p(\varrho)$. It follows that we may apply Proposition 6.3 to obtain that

$$U(t) := \int_0^t S_{L^p(\varrho)}(t-s)\, dW_{H_\mu}(s) = \int_0^t S(t-s)\, dW_\mu(s)$$

a.s. in $\mathscr{D}'$ for all $t \geqslant 0$. This shows that $\{U(t)\}_{t\geqslant 0}$ is an $L^p(\varrho)$-valued weak solution of the problem (7.3). □

From Theorem 8.2 we obtain:

Theorem 9.2. *Let $1 \leqslant p < \infty$ and $0 \leqslant \varrho \in L^1$ be arbitrary and fixed. If there exist $C > q^*$ and $\alpha > 0$ such that*

$$\int_{\mathbb{R}^d} \frac{1 + |q(x)|^\alpha}{C - \operatorname{Re} q(x)}\, d\mu(x) < \infty,$$

then problem (7.3) *admits a continuous $L^p(\varrho)$-valued weak solution.*

Remark 9.3. The implication (2) ⇒ (1) in Theorem 9.1 does not really depend upon the fact that $S_{L^p(\varrho)}(t) : H_\mu \to L^p(\varrho)$ factors through BUC. In order to derive Theorem 9.1 as quickly as possible, we could prove directly that $S(t)$ maps H_μ into $L^p(\varrho)$ and give all subsequent estimates in the $L^p(\varrho)$-norm.

10. Hölder continuity of the $L^p(\varrho)$-valued solution

It turns out that under an invariance condition, the $L^p(\varrho)$-valued solution has a Hölder continuous version. Throughout this section we assume that (4.1), (4.2), and (7.1) hold.

We begin with a simple observation.

Lemma 10.1. *Let $\alpha \in (0,1)$ and $C > q^*$ be given. For all $t > 0$ there exists a constant $M \geqslant 0$ such that*

$$\int_0^t \int_{\mathbb{R}^d} s^{-\alpha} e^{2s \operatorname{Re} q(\xi)}\, d\mu(\xi)\, ds \leqslant M \int_{\mathbb{R}^d} \frac{1}{(C - \operatorname{Re} q(\xi))^{1-\alpha}}\, d\mu(\xi).$$

Proof. By elementary calculations, for all $t > 0$ and $-\infty < \eta < \zeta < \infty$ we have

$$\int_0^t s^{-\alpha} e^{2s\eta}\, ds \leqslant (\zeta - \eta)^{\alpha-1} e^{2t\zeta} \int_0^\infty s^{-\alpha} e^{-2s}\, ds. \tag{10.1}$$

By taking $\eta = \operatorname{Re} q(\xi)$, $\zeta = C$ and integrating, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \int_0^t s^{-\alpha} e^{2s \operatorname{Re} q(\xi)} \, ds \, d\mu(\xi) \\ \leqslant e^{2tC} \Big(\int_0^\infty s^{-\alpha} e^{-2s} \, ds \Big) \Big(\int_{\mathbb{R}^d} \frac{1}{(C - \operatorname{Re} q(\xi))^{1-\alpha}} \, d\mu(\xi) \Big).\end{aligned} \tag{10.2}$$

This gives the desired estimate, with $M = e^{2tC} \int_0^\infty s^{-\alpha} e^{-2s} \, ds$. □

Motivated by this observation we introduce the following hypothesis.

Hypothesis ($\mathbf{H}^\alpha$). There exists a constant $C > q^*$ such that

$$\int_{\mathbb{R}^d} \frac{1}{(C - \operatorname{Re} q(\xi))^{1-\alpha}} \, d\mu(\xi) < \infty.$$

Note that **($\mathbf{H}^\alpha$)** trivially implies **($\mathbf{H}_\alpha$)**.

We have the following analogue of Lemma 4.6.

Lemma 10.2. *Assume* **($\mathbf{H}^\alpha$)** *holds for some* $\alpha \in (0,1)$. *For all* $t > 0$ *and* $g \in L^2((0,T); H_\mu)$, *the* BUC*-valued function* $r \mapsto (t-r)^{-\alpha/2} S_{BUC}(t-r) g(r)$ *is Bochner integrable on* $(0,t)$ *and we have*

$$\begin{aligned}\Big\| \int_0^t (t-r)^{-\frac{\alpha}{2}} S_{BUC}(t-r) g(r) \, dr \Big\|_{BUC} \\ \leqslant \Big(\int_0^t \int_{\mathbb{R}^d} s^{-\alpha} e^{2s \operatorname{Re} q(\xi)} \, d\mu(\xi) \, ds \Big)^{\frac{1}{2}} \cdot \|g\|_{L^2((0,T);H_\mu)}.\end{aligned}$$

Proof. For step functions g, the strong measurability of $r \mapsto (t-r)^{-\alpha/2} S_{BUC}(t-r) g(r)$ follows from Lemma 4.6; the general case follows by approximation.

By (4.8) and the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_0^t (t-r)^{-\frac{\alpha}{2}} \|S_{BUC}(t-r) g(r)\|_{BUC} \, dr \\ \leqslant \Big(\int_0^t \int_{\mathbb{R}^d} (t-r)^{-\alpha} e^{2(t-r) \operatorname{Re} q(\xi)} \, d\mu(\xi) \, dr \Big)^{\frac{1}{2}} \cdot \|g\|_{L^2((0,t);H_\mu)} \\ = \Big(\int_0^t \int_{\mathbb{R}^d} s^{-\alpha} e^{2s \operatorname{Re} q(\xi)} \, d\mu(\xi) \, ds \Big)^{\frac{1}{2}} \cdot \|g\|_{L^2((0,t);H_\mu)}.\end{aligned}$$

The repeated integral in the right hand side is finite by Lemma 10.1. □

Arguing as at the end of the proof of Theorem 4.9 we deduce the following representation for the above integral:

Lemma 10.3. *Assume* $(\mathbf{H}^\alpha)$ *holds for some* $\alpha \in (0,1)$. *For* $t > 0$ *define* $\kappa_t^{1-\alpha/2} : \mathbb{R}^d \to L^2((0,T); H_\mu)$ *by*

$$\kappa_t^{1-\frac{\alpha}{2}}(x)(r) = (t-r)^{-\frac{\alpha}{2}} \mathscr{F}(e^{i\langle x,\cdot\rangle} e^{(t-r)q(-\cdot)}\mu)\mathbf{1}_{(0,t)}(r).$$

Then for all $g \in L^2((0,T); H_\mu)$ *and* $t > 0$ *we have*

$$\int_0^t (t-r)^{-\frac{\alpha}{2}} S(t-r)g(r)\,dr = [\kappa_t^{1-\frac{\alpha}{2}}(\,\cdot\,), g]_{L^2((0,T);H_\mu)}. \tag{10.3}$$

By a direct computation we obtain the following identity: for all $x \in \mathbb{R}^d$,

$$\|\kappa_t^{1-\frac{\alpha}{2}}(x)\|_{L^2((0,T);H_\mu)} = \left(\int_0^t \int_{\mathbb{R}^d} s^{-\alpha} e^{2s\,\mathrm{Re}\,q(\xi)}\,d\mu(\xi)\,ds\right)^{\frac{1}{2}}. \tag{10.4}$$

In particular, the norm is independent of $x \in \mathbb{R}^d$.

From this point on, we assume that $(\mathbf{H}^\alpha)$ holds for some fixed $\alpha \in (0,1)$. We fix $1 \leqslant p < \infty$ and a weight function $0 \leqslant \varrho \in L^1$. Since BUC embeds into $L^p(\varrho)$, for $t > 0$ we may define $(\Lambda_T^{1-\alpha/2} f)(t) \in L^p(\varrho)$ by

$$(\Lambda_T^{1-\frac{\alpha}{2}} f)(t) := \frac{1}{\Gamma(1-\frac{\alpha}{2})} \int_0^t (t-r)^{-\frac{\alpha}{2}} S_{L^p(\varrho)}(t-r)f(r)\,dr.$$

Then,

$$(\Lambda_T^{1-\frac{\alpha}{2}} f)(t) = \frac{1}{\Gamma(1-\frac{\alpha}{2})} [\kappa_t^{1-\frac{\alpha}{2}}(\,\cdot\,), f]_{L^2((0,T);H_\mu)}.$$

From the above estimates we find

$$\begin{aligned}
&\|(\Lambda_T^{1-\frac{\alpha}{2}} f)(t)\|_{L^p(\varrho)} \\
&\quad\leqslant \frac{1}{\Gamma(1-\frac{\alpha}{2})} \|\varrho\|_{L^1}^{\frac{1}{p}} \left(\sup_{x\in\mathbb{R}^d} \|\kappa_t^{1-\frac{\alpha}{2}}(x)\|_{L^2((0,T);H_\mu)}\right) \|f\|_{L^2((0,T);H_\mu)} \\
&\quad\leqslant \frac{1}{\Gamma(1-\frac{\alpha}{2})} \|\varrho\|_{L^1}^{\frac{1}{p}} \left(\int_0^t \int_{\mathbb{R}^d} s^{-\alpha} e^{2s\,\mathrm{Re}\,q(x)}\,d\mu(x)\,ds\right)^{\frac{1}{2}} \|f\|_{L^2((0,T);H_\mu)}.
\end{aligned}$$

In particular, for each $f \in L^2((0,T); H_\mu)$, the function $t \mapsto (\Lambda_T^{1-\alpha/2} f)(t)$ defines an element of $L^\infty((0,T); L^p(\varrho))$, and hence of $L^p((0,T); L^p(\varrho))$. In this way we obtain bounded operators $\Lambda_T^{1-\alpha/2} : L^2((0,T); H_\mu) \to L^p((0,T); L^p(\varrho))$. Arguing as in Example 2.4 we obtain:

Proposition 10.4. *Assume* $(\mathbf{H}^\alpha)$ *holds for some* $\alpha \in (0,1)$ *and let* $1 \leqslant p < \infty$. *Then* $\Lambda_T^{1-\alpha/2}$ *is* γ*-radonifying from* $L^2((0,T); H_\mu)$ *into* $L^p((0,T); L^p(\varrho))$.

In what follows, given a separable real Banach space X and a real number $\beta \in (0,1)$, the little Hölder space $c_0^\beta([0,T]; X)$ is the (separable) Banach space of all real-valued continuous functions $f : [0,T] \to X$ such that $f(0) = 0$ and

$$
(10.5)\qquad \begin{aligned}
&\|f\| := \sup_{t\in[0,T]} |f(t)| + \sup_{0\leqslant s<t\leqslant T} \frac{|f(t)-f(s)|}{(t-s)^\beta} < \infty,\\
&\lim_{\delta\downarrow 0} \sup_{0<t-s\leqslant\delta} \frac{|f(t)-f(s)|}{(t-s)^\beta} = 0.
\end{aligned}
$$

Proposition 10.5. *Assume* $(\mathbf{H}^\alpha)$ *holds for some* $\alpha\in(0,1)$*. Let* $2/\alpha < r < \infty$ *and* $\beta \in (0, \alpha/2 - 1/r)$ *be given and assume that the semigroup* $\{S(t)\}_{t\geqslant 0}$ *restricts to a* C_0*-semigroup on* $L^r(\varrho)$*. Then the operator* $\Lambda_T : L^2((0,T);H_\mu) \to L^r((0,T);L^r(\varrho))$ *defined by*

$$
(\Lambda_T f)(t) := \int_0^t S_{L^r(\varrho)}(t-\tau)f(\tau)\,d\tau
$$

takes values in the space $c_0^\beta([0,T];L^r(\varrho))$*. As an operator from* $L^2((0,T);H_\mu)$ *into* $c_0^\beta([0,T];L^r(\varrho))$*,* Λ_T *is* γ*-radonifying.*

Proof. By a result of Da Prato, Kwapień and Zabczyk [4], the invariance of $L^r(\varrho)$ implies that

$$
\Lambda_T^{\frac{\alpha}{2}} f := \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^t (t-r)^{-1+\frac{\alpha}{2}} S(t-r)f(r)\,dr
$$

defines a bounded operator from $L^r((0,T);L^r(\varrho))$ into $c_0^\beta[0,T];L^r(\varrho))$. By standard arguments we have the factorization

$$
\Lambda_T = \Lambda_T^{\frac{\alpha}{2}}\Lambda_T^{1-\frac{\alpha}{2}}.
$$

The result now follows from Proposition 10.4 and the left ideal property of γ-radonifying operators mentioned in Section 2. □

After these preparations we can state and prove the main result of this section:

Theorem 10.6. *Assume that there exist* $0<\alpha<1$ *and a constant* $C > q^*$ *such that*

$$
(10.6)\qquad \int_{\mathbb{R}^d} \frac{1}{(C-\operatorname{Re} q(x))^{1-\alpha}}\,d\mu(x) < \infty.
$$

Let $1\leqslant p<\infty$ *and* $0\leqslant\varrho\in L^1$ *be fixed. If the semigroup* $\{S(t)\}_{t\geqslant 0}$ *restricts to a* C_0*-semigroup on* $L^r(\varrho)$ *for all sufficiently large* r*, then for all* $\beta\in(0,\alpha/2)$ *the* $L^p(\varrho)$*-valued solution of Theorem 9.1 has a* β*-Hölder continuous version.*

Proof. Choose $\delta>0$ such that $\beta+\delta\in(0,\alpha/2)$. Choose $r>\max\{2/\alpha,p\}$ sufficiently large, and subject to the additional conditions that $1/r<\delta$ and $\beta+\delta<\alpha/2-1/r$.

Consider the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$, where $\tilde{\Omega} = c_0^{\beta+\delta}([0,T]; L^r(\varrho))$, $\mathscr{F}$ is the Borel σ-algebra of $\tilde{\Omega}$, and $\tilde{\mathbb{P}} := \Lambda_T(\gamma_{H_{\mu,T}})$ is the image measure whose σ-additivity is guaranteed by Proposition 10.5. We define an $L^r(\varrho)$-valued process $\{\xi_t\}_{t\in[0,T]}$ on this probability space by

$$\xi(t, \tilde{\omega}) := \tilde{\omega}(t), \qquad t \in [0,T], \quad \tilde{\omega} \in \tilde{\Omega}.$$

It is routine to check that the joint distributions of this process are given by

$$\mathscr{L}(\xi(s), \xi(t)) = \mathscr{L}(u(s), u(t)), \qquad 0 \leqslant s, t \leqslant T,$$

where, for the moment, we think of $\{u(t)\}_{t\in[0,T]}$ as an $L^r(\varrho)$-valued process (which is justified by Theorem 9.1 applied to $L^r(\varrho)$). Hence for any fixed $0 \leqslant s \neq t \leqslant T$,

$$\mathbb{E}\left(\frac{\|u(t) - u(s)\|^r_{L^r(\varrho)}}{|t-s|^{(\beta+\delta)r}}\right) = \tilde{\mathbb{E}}\left(\frac{\|\xi(t) - \xi(s)\|^r_{L^r(\varrho)}}{|t-s|^{(\beta+\delta)r}}\right) \leqslant \tilde{\mathbb{E}}\|\xi\|^r_{c_0^{\beta+\delta}([0,T];L^r(\varrho))}.$$

By Fernique's theorem,

$$\tilde{\mathbb{E}}\|\xi\|^r_{c_0^{\beta+\delta}([0,T];L^r(\varrho))} < \infty.$$

It follows that there exists a finite constant K such that

$$\tilde{\mathbb{E}}\|u(t) - u(s)\|^r_{L^r(\varrho)} \leqslant K|t-s|^{(\beta+\delta)r}, \qquad 0 \leqslant s \neq t \leqslant T.$$

By the Kolmogorov continuity theorem, the process $\{u(t)\}_{t\in[0,T]}$ has a η-Hölder continuous version for each

$$\eta < \frac{(\beta+\delta)r - 1}{r} = \beta + \delta - \frac{1}{r}.$$

Since by assumption we have $1/r < \delta$, the existence of a β-Hölder continuous version of $\{u(t)\}_{t\in[0,T]}$, as an $L^r(\varrho)$-valued process, is proved.

Since by assumption we have $r > p$, the integrability of ϱ implies that $L^r(\varrho)$ is continuously embedded in $L^p(\varrho)$. Hence as an $L^p(\varrho)$-valued process, $\{u(t)\}_{t\in[0,T]}$ has a β-Hölder continuous version as well. ☐

Example 10.7. Suppose q satisfies a uniformly ellipticity condition of order $2m$. Then the invariance condition is automatically satisfied for the weight functions

$$\varrho(x) = e^{-b|x|} \qquad (b > 0)$$

and

$$\varrho(x) = (1 + |x|^2)^{-b} \qquad (b > 0).$$

This is the content of [3, Lemma 3.1].

We return to the functions q from Example 4.1.

(1) $q(x) = -ix$, $x \in \mathbb{R}$ $(d = 1)$. Then (10.6) reduces to the condition that μ is a finite measure.

(2) $q(x) = -|x|^2$, $x \in \mathbb{R}^d$. Then (10.6) reduces to

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^{1-\alpha}}\, d\mu(x) < \infty.$$

We see that in the case (1), if the condition (10.6) is satisfied for $\alpha = 0$ then it is also satisfied for any $\alpha \in (0,1)$. The following example will show that this in the case (2) the situation is quite different. In fact, we will provide an example of a measure μ for which the condition (10.6) is true with $\alpha = 0$ but not with any $\alpha > 0$.

Example 10.8. In dimension $d = 1$, consider the following tempered measure

$$d\mu(x) = \frac{|x|}{(1 + (\ln|x|)^2)}\, dx.$$

For this measure we have

$$\int_{\mathbb{R}} \frac{1}{1+|x|^2}\, d\mu(x) < \infty,$$

but for all $\varepsilon > 0$,

$$\int_{\mathbb{R}} \frac{1}{(1+|x|^2)^{1-\varepsilon}}\, d\mu(x) = \infty.$$

11. The stochastic Schrödinger equation

The stochastic Schrödinger equation requires some modifications to the assumptions that have been made up to this point. Let us list the changes:

(1) The function $q : \mathbb{R}^d \to \mathbb{C}$ is of class $\mathscr{C}^\infty$ and satisfies (4.2) and (7.1), but not necessarily (4.1).

(2) The measure μ is assumed to be nonnegative and tempered but not necessarily symmetric.

(3) All spaces are replaced by their complex counterparts. In particular, this applies to the spaces $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{D}$, $\mathscr{D}'$, $L^p(\varrho)$ and $L^2(\mu)$. For notational convenience, we will not explicitly express this in our notations. For example, in this section $L^2(\mu)$ will always denote the space of complex-valued square μ-integrable functions.

(4) The rôle of $L^2_{(s)}(\mu)$ is replaced by $L^2(\mu)$.

(5) All operators are complex. This applies in particular to the semigroup $\{S(t)\}_{t\geqslant 0}$ whose symbol is q.

In Definition 6.1, condition (1) is replaced by

(1c) For each $\phi \in \mathscr{S}$, the process $\{\langle \phi, W(t)\rangle\}_{t\geqslant 0}$ is an adapted complex-valued Brownian motion.

As in Section 6, by [12, Theorem 6, p. 169, Theorem 1′, p. 264] for a process $\mathbb{W}$ satisfying conditions (1c) and (2) is equivalent to:

(2c′) There exists a nonnegative tempered measure μ on $\mathbb{R}^d$ such that for all $\phi, \psi \in \mathscr{S}$ and $t, s \geqslant 0$ we have

$$\mathbb{E}(\langle \phi, W(t)\rangle \, \overline{\langle \psi, W(s)\rangle}) = (t \wedge s)[\phi, \psi]_{\mathscr{H}_\mu}.$$

Similarly, in Definition 5.1 the conditions (1) and (2) are replaced by:

(1c) For each $h \in H$, $\{W_H(t)h\}_{t\geqslant 0}$ is an adapted complex-valued Brownian motion;

(2c) For all $g, h \in H$ and $t, s \geqslant 0$ we have

$$\mathbb{E}(W_H(t)g\overline{W_H(s)h}) = (t \wedge s)[g, h]_H.$$

In this new setting all our results remain true if care is taken with regard to their proper interpretation. For example, Theorem 4.9 holds true, but with κ taking values in $L^2((0,T); L^2(\mu))$.

Let us now consider the Schrödinger equation

$$\begin{aligned} du(t) &= -\frac{i}{2}\,\Delta u(t)\,dt + dW_\mu(t), \qquad t \geqslant 0, \\ u(0) &= 0. \end{aligned} \tag{11.1}$$

The symbol of $A = -(i/2)\Delta$ is given by

$$q(\xi) = \frac{i}{2}\,|\xi|^2, \qquad \xi \in \mathbb{R}^d.$$

For this symbol the assumption **(H)** holds if and only if μ is a finite measure.

This leads to the following result:

Theorem 11.1. *Let $1 \leqslant p < \infty$ and $0 \leqslant \varrho \in L^1_{\rm loc}$ be arbitrary and fixed. The following assertions are equivalent*:

(1) *Problem* (11.1) *admits an $L^p(\varrho)$-valued solution*;

(2) *The measure μ is finite and the weight ϱ is integrable.*

Acknowledgements. We thank Professors Robert Dalang and Jerzy Zabczyk for stimulating discussions.

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